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ON THE MOTIVE OF G AND THE PRINCIPAL HOMOMORPHISM $SL_2 \to \hat{G}$ *

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Abstract. Let k be a field, and let G be a connected reductive group over k. Let $\hat{G}$ be the Langlands dual group, which is a reductive group over $\mathbf{C}$. In [Gr] we attached a motive M of Artin-Tate type to G. In this paper, we relate M to the principal homomorphism $SL_2 \to \hat{G}$, which was introduced by de Siebenthal [deS] and Dynkin [D], and studied extensively by Kostant [K]. As a corollary, we relate the L-function of the dual motive $M^\vee$, when k is a local non-Archimedean field, to the Langlands L-function of the Steinberg representation of $G(k)$, with respect to the adjoint representation of the L-group. We also construct an involution θ of $\hat{\mathfrak{g}} = \mathrm{Lie}(\hat{G})$ when $k = \mathbf{R}$.

1. The L-group (cf.[Ko], [S2]).

Let k^s be a separable closure of k, and $\Gamma = \mathrm{Gal}(k^s/k)$. We define the root datum $\psi = (X^\bullet, \Delta^\bullet, X_\bullet, \Delta_\bullet)$ of G as a projective limit, as in Kottwitz [Ko, pg. 614]; the group Γ acts naturally on ψ.

A dual group $\hat{G}$ for G is, by definition, a reductive group over $\mathbf{C}$ which is furnished with

$$\text{a pinning } (\hat{G} \supset \hat{B} \supset \hat{T};\ e_\alpha : \mathbf{G}_a \simeq \hat{U}_\alpha) \tag{1.1}$$

$$\text{an isomorphism } i : \psi(\hat{T}, \hat{B}) \simeq \psi^\vee. \tag{1.2}$$

The description of a pinning, and the dual root datum $\psi^\vee$ can be found in Springer's survey, where a pinning is called a splitting [S2, pg. 10].

The dual group $\hat{G}$ exists, and is unique up to a unique isomorphism, by results of Chevalley (cf. [S2, pg. 9]). More precisely, if $\hat{G}'$ is another dual group with pinning $(\hat{G}' \supset \hat{B}' \supset \hat{T}'; e'_\alpha)$ and isomorphism $i' : \psi(\hat{T}', \hat{B}') \simeq \psi^\vee$, there is a unique pinned isomorphism $f : \hat{G} \to \hat{G}'$ such that the following diagram commutes

$$\begin{tikzcd} i: \psi(\hat{T}, \hat{B}) \arrow[r, phantom, "\simeq"] \arrow[d, "\psi(f)"] & \psi^\vee \arrow[d, equal] \\ i': \psi(\hat{T}', \hat{B}') \arrow[r, phantom, "\simeq"] & \psi^\vee \end{tikzcd} \tag{1.3}$$

As a consequence of (1.3), the group $\mathrm{Aut}(\psi) = \mathrm{Aut}(\psi^\vee)$ acts as pinned automorphisms of the group $\hat{G}$, as it acts on the set Isom $(\psi(\hat{T}, \hat{B}), \psi^\vee)$.

Since Γ acts on ψ, it acts as pinned automorphisms of $\hat{G}$. The L-group ${}^L G$ is defined as the semi-direct product

$${}^L G = \hat{G} \rtimes \Gamma. \tag{1.4}$$

2. The principal homomorphism (cf. [K], [Se], [S1]).

Let Δ be the root basis determined by the pinning $\hat{T} \subset \hat{B} \subset \hat{G}$. For each α in Δ, we have a fixed isomorphism $e_\alpha : \mathbf{G}_a \simeq \hat{U}_\alpha$. Let

$$X_\alpha = \mathrm{Lie}(e_\alpha)(1) \quad \text{in } \mathrm{Lie}(\hat{U}_\alpha) \tag{2.1}$$

$$X = \sum_{\alpha \in \Delta} X_\alpha \qquad \text{in } \hat{\mathfrak{g}} = \mathrm{Lie}(\hat{G}). \tag{2.2}$$

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Then X is a principal nilpotent element in $\hat{\mathfrak{g}}$.

For any root α of $\hat{T}$, let H_α in $\hat{\mathfrak{g}}$ be the vector determined by the co-root $\alpha^\vee : \mathbf{G}_m \to \hat{T}$, and let

$$H = \sum_{\alpha > 0} H_\alpha = \sum_{\alpha \in \Delta} c_\alpha \cdot H_\alpha \qquad \text{in } \hat{\mathfrak{g}}. \tag{2.3}$$

The coefficients c_α are integers, which are given in the tables of Bourbaki for simple $\hat{G}$ [Bo].

Finally, for each $\alpha \in \Delta$, let Y_α be the unique basis of $\mathrm{Lie}(\hat{U}_{-\alpha})$ which satisfies:

$$[X_\alpha, Y_\alpha] = H_\alpha. \tag{2.4}$$

Let

$$Y = \sum_{\alpha \in \Delta} c_\alpha \cdot Y_\alpha \quad \text{in } \hat{\mathfrak{g}}. \tag{2.5}$$

Then (X, H, Y) is an $s\ell_2$-triple; the brackets are given by

$$[H, X] = 2X, \quad [X, Y] = H, \quad [H, Y] = -2Y. \tag{2.6}$$

Hence there is a homomorphism of Lie algebras over $\mathbf{C}$

$$\phi : s\ell_2 \to \hat{\mathfrak{g}} \tag{2.7}$$

mapping $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ to X, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ to H, and $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ to Y. Since SL_2 is simply-connected, there is a homomorphism of reductive groups over $\mathbf{C}$

$$\varphi : SL_2 \to \hat{G} \tag{2.8}$$

with $\mathrm{Lie}(\varphi) = \phi$. This is the principal homomorphism, which is completely determined by the data defining the dual group.

The Galois group Γ acts on $\hat{G}$, and permutes the elements $\alpha \in \Delta$. Hence Γ fixes X. Since $c_\alpha = c_{\gamma\alpha}$, Γ also fixes Y. Hence Γ fixes the triple (X, H, Y), and the principal homomorphism φ of (2.8) extends to a homomorphism:

$$\varphi : SL_2 \times \Gamma \to {}^L G. \tag{2.9}$$

3. The centralizer of X (cf. [K], [S2]). Let $\hat{P} \subset \hat{\mathfrak{g}}$ be the centralizer of X:

$$\hat{P} = \{A \in \hat{\mathfrak{g}} : [A, X] = 0\}. \tag{3.1}$$

This is a Lie sub-algebra. Since X is a regular nilpotent element, $\hat{P}$ is abelian, of dimension equal to the rank of $\hat{G}$.

Since $[H, X] = 2 \cdot X$, the adjoint action of H on $\hat{\mathfrak{g}}$ preserves $\hat{P}$. This gives a $\mathbf{Z}$-grading

$$\hat{P} = \bigoplus_n \hat{P}_n \tag{3.2}$$

where

$$\hat{P}_n = \{A \in \hat{P} : [H, A] = n \cdot A\}. \tag{3.3}$$

Since we have a direct sum decomposition

$$\hat{P} = Z(\hat{\mathfrak{g}}) \oplus \{A \in \hat{U} : [A, X] = 0\} \tag{3.4}$$

we have $\hat{P}_n = 0$ for $n < 0$, and $\hat{P}_0 = Z(\hat{\mathfrak{g}})$. Since $[H, X_\alpha] = 2X_\alpha$ for all $\alpha \in \Delta$, the eigenvalues of H on $\hat{U}$ are all even integers, so $\hat{P}_n = 0$ unless $n = 2m \geq 0$, and

$$\hat{P} = \bigoplus_{m \geq 0} \hat{P}_{2m}. \tag{3.5}$$

The Galois group Γ fixes both X and H, so acts on the complex vector spaces $\hat{P}_{2m}$. We may therefore view $\hat{P}_{2m}$ as an Artin motive for k, with coefficients in $\mathbf{C}$.

Proposition 3.6. *The Artin-Tate motive* $\bigoplus_{m \geq 0} \hat{P}_{2m}(-m)$ *is isomorphic to the motive* M *of* G *with coefficients in* $\mathbf{C}$.

Since the motive M of G was defined [G, §1] as

$$\bigoplus_{d \geq 1} V_d(1-d), \tag{3.7}$$

where V_d was the space of primitive invariants of degree d for the Weyl group of G over k^s, in its action on the symmetric algebra on $E = X^\bullet \otimes \mathbf{Q}$, this proposition is equivalent to the statement that for all $m \geq 0$

$$\hat{P}_{2m} \simeq V_{m+1} \otimes \mathbf{C}, \tag{3.8}$$

as representations of $\Gamma = \mathrm{Gal}(k^s/k)$. We will prove this in the next section. We note that we could also define the representations $\hat{P}_{2m}$ over $\mathbf{Q}$, by defining $\hat{G}$ as a split reductive group over $\mathbf{Q}$.

4. The proof of Proposition 3.6. To prove (3.8), we reduce to some simple cases, using the following

Lemma 4.1. *a) If* G *and* G' *are isogenous over* k, *then* $\hat{P}_{2m} \simeq \hat{P}'_{2m}$ *(as complex representations of* Γ*).*

b) If G *and* G' *are inner twistings over* k, *then* $\hat{P}_{2m} \simeq \hat{P}'_{2m}$.

c) If $G = G' \times G''$, *then* $\hat{P}_{2m} \simeq \hat{P}'_{2m} \oplus \hat{P}''_{2m}$.

d) If K *is a finite extension of* k *contained in* k^s, *and* $G = \mathrm{Res}_{K/k}(G')$, *then* $\hat{P}_{2m} \simeq \mathrm{Ind}_{\Gamma_K}^{\Gamma}(\hat{P}'_{2m})$.

Proof. In case a), we have an isomorphism $\hat{\mathfrak{g}} \simeq \hat{\mathfrak{g}}'$ of dual algebras, which commutes with the action of Γ. This gives an isomorphism $\hat{P}_{2m} \simeq \hat{P}'_{2m}$, for all $m \geq 0$. In case b), we have an isomorphism $\hat{G} \simeq \hat{G}'$ of dual groups, which commutes with the action of Γ [S2, pg. 12]. Hence we get an isomorphism of dual algebras, and the result follows.

In part c), we have $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}' \oplus \hat{\mathfrak{g}}''$ as Lie algebras with Γ-action. Since $X = X' + X''$ and $H = H' + H''$, the result on $\hat{P}$ follows.

In part d), let $\Sigma = \mathrm{Hom}(K, k^s) \simeq \Gamma/\Gamma_K$. Then $\hat{\mathfrak{g}} = \mathrm{Funct}(\Sigma, \hat{\mathfrak{g}}')$, and $\hat{P}_{2m} = \mathrm{Funct}(\Sigma, \hat{P}'_{2m})$ with Γ-action $\sigma f(s) = f(\sigma s)$ [Sp, pg. 12]. Hence $\hat{P}_{2m} \simeq \underset{s\in\Sigma}{\oplus} s\hat{P}'_{2m} = \mathrm{Ind}^{\Gamma}_{\Gamma_K}(\hat{P}'_{2m})$. $\square$

The analogous results to those in Lemma 4.1 also hold for the Γ-modules V_d, V'_d [G, §2]. Using b) we may reduce (3.8) to the case when G is quasi-split over k. Using a), we may assume that G is the product of a central torus and a simply-connected derived group.

The result of (3.8) is true when $G = T$ is a torus. Then $\hat{P} = \hat{P}_0 = \hat{\mathfrak{g}} = X_\bullet(\hat{T}) \otimes \mathbf{C} = X^\bullet(T) \otimes \mathbf{C} = V_1 \otimes \mathbf{C}$. Hence it suffices to prove (3.8) when G is quasi-split and simply-connected. Since every such group is the product of groups $\mathrm{Res}_{K/k}(G_K)$ with G_K absolutely quasi-simple over K [T, pg. 46], by c) and d) we are reduced to proving (3.8) when G is quasi-split, simply-connected, and absolutely quasi-simple over k.

In this case, the equality

$$\dim \hat{P}_{2m} = \dim V_{m+1} \tag{4.2}$$

is the well-known relation ($d = m+1$) between the degrees d of the primitive invariants and the exponents m of the Weyl group [H, §3.20]. This completes the proof when G is split, so Γ acts trivially on $\hat{G}$.

Now assume G is quasi-split, and split by the Galois extension K, so Γ acts on $\hat{G}$ through the quotient $\mathrm{Gal}(K/k)$. A computation of the invariant subalgebra $(\hat{\mathfrak{g}})^\Gamma$ is given in Bourbaki [Bo, Ch. VIII, Ex. 5.13]: if $\hat{\mathfrak{g}}$ is of type A_ℓ ($\ell \geq 2$) and $[K : k] = 2$, $(\hat{\mathfrak{g}})^\Gamma$ is of type $B_{\ell/2}$ if ℓ is even and type $C_{(\ell+1)/2}$ if ℓ is odd, if $\hat{\mathfrak{g}}$ is of type D_ℓ ($\ell \geq 4$) and $[K : k] = 2$, $(\hat{\mathfrak{g}})^\Gamma$ is of type $B_{\ell-1}$, if $\hat{\mathfrak{g}}$ is of type E_6 and $[K : k] = 2$, $\hat{\mathfrak{g}}^\Gamma$ is of type F_4, and if $\hat{\mathfrak{g}}$ is of type D_4 and $[K : k] = 3$ or 6, $\hat{\mathfrak{g}}^\Gamma$ is of type G_2. This completely determines the action of Γ on the spaces $\hat{P}_{2m}$, and this agrees with the action of Γ on the spaces V_{m+1} [G, 2.3]. (A simple recipe to remember is that Γ acts nontrivially on V_d when d is odd; when G is of type ${}^rD_{2k}$, Γ also acts nontrivially on V_{2k}.)

5. The adjoint representation. The group ${}^LG = \hat{G} \rtimes \Gamma$ acts linearly on the complex vector space $\hat{\mathfrak{g}}$ by the adjoint representation. Via the homomorphism $\varphi : SL_2 \times \Gamma \to {}^LG$ of (2.9), we can restrict this representation to the product $SL_2 \times \Gamma$. A fundamental result, due to Kostant [K], is the following.

PROPOSITION 5.2. *We have a direct sum decomposition*

$$\hat{\mathfrak{g}} \simeq \underset{m\geq 0}{\oplus} \mathrm{Sym}^{2m}(\mathbf{C}^2) \otimes \hat{P}_{2m}$$

as a representation of $SL_2 \times \Gamma$.

6. The Steinberg representation. In this section, we assume that k is a non-Archimedean local field, with finite residue field of order q. The Langlands parameter of the Steinberg representation of $G(k)$ is the unramified homomorphism:

$$St : W \xrightarrow[f]{} SL_2 \times \Gamma \xrightarrow[\varphi]{} {}^LG, \tag{6.1}$$

where

$$f(w) = \left(\begin{pmatrix} \|w\|^{1/2} & 0 \\ 0 & \|w\|^{-1/2} \end{pmatrix}, w \right). \tag{6.2}$$

Here W is the Weil group of k (a dense subgroup of Γ); the parameter also includes the nilpotent element $N = X$ in $\hat{\mathfrak{g}} = \mathrm{Lie}(\hat{G})$, which satisfies

$$ad\ St(w) \cdot N = \|w\| \cdot N. \tag{6.3}$$

Let $L(St, ad, s)$ be the Langlands L-function of the Steinberg representation, with respect to the adjoint representation ${}^LG \to GL(\hat{\mathfrak{g}})$.

Proposition 6.4. *We have an equality of local L-functions*

$$L(St, ad, s) = L(M^\vee, s),$$

where $M^\vee = \bigoplus_{m \geq 0} V_{m+1}(m)$ *is the dual motive of* M.

Proof. By definition [Ta, pg. 21]

$$L(St, ad, s) = \det(1 - F \cdot q^{-s} | \hat{\mathfrak{g}}^I_{N=0})^{-1}.$$

Here I is the inertia subgroup of W, and F is a geometric Frobenius with $\|F\| = q^{-1}$, which generates W/I. Since $N = X$, the space $\hat{\mathfrak{g}}_{N=0}$ is the centralizer $\hat{P}$ studied in §3. By Proposition 3.6, $\hat{P} \simeq \bigoplus_{m \geq 0} V_{m+1}$ as a representation of Γ.

On the other hand, by Proposition 5.2, the element

$$\begin{pmatrix} \|F\|^{1/2} & 0 \\ 0 & \|F\|^{-1/2} \end{pmatrix} = \begin{pmatrix} q^{-1/2} & 0 \\ 0 & q^{1/2} \end{pmatrix}$$

of $SL_2(\mathbf{C})$ acts as q^{-m} on $\hat{P}(2m) = V_{m+1}$. Hence

$$\begin{aligned} L(St, ad, s) &= \prod_{m \geq 0} \det(1 - F \cdot q^{-s} | V_{m+1}(m)^I)^{-1} \\ &= L(M^\vee, s) \end{aligned}$$

as claimed. □

Corollary 6.5. *The value* $L(St, ad, 1) = L(M^\vee(1))$ *is positive and nonzero in* $\mathbf{Q}$.

Indeed, this is proved for $L(M^\vee(1))$ in [G, 5.1]. The fact that the adjoint L-function of the Steinberg representation is regular at $s = 1$ is consistent with [GP, Conj. 2.6], as the Steinberg representation of a quasi-split group is generic.

7. Involutions in ${}^LG$.

In this section, we assume that k is the field $\mathbf{R}$ of real numbers, so $\Gamma = \mathrm{Gal}(\mathbf{C}/\mathbf{R}) = \{1, \tau\}$, with τ = complex conjugation. We also assume the group G is simply-connected, and has a compact inner form. Then $\hat{G}$ is of adjoint type over $\mathbf{C}$, and the action of τ on $\hat{G}$ induces the opposition involution in $\mathrm{Out}(\hat{G})$ [T, 1.5.1].

Since $\hat{G}$ is an adjoint group, the principal homomorphism $\varphi : SL_2 \to \hat{G}$ factors through the quotient $PGL_2 = SL_2/\langle \pm 1 \rangle$ [Se, pg. 533]. We obtain a homomorphism

$$\varphi : PGL_2 \times \Gamma \to {}^LG. \tag{7.1}$$

Proposition 7.2. *Let* $\theta = \varphi\big(\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right) \times \tau\big)$ *in* ${}^LG$. *Then* $\theta^2 = 1$ *and*

$$\mathrm{Tr}(\theta|\hat{\mathfrak{g}}) = -\mathrm{rank}(\hat{G}).$$

Proof. Clearly $\theta^2 = 1$. To compute the trace of θ, we use the decomposition for $SL_2 \times \Gamma$

$$\hat{\mathfrak{g}} = \bigoplus_{m \geq 0} \mathrm{Sym}^{2m}(\mathbf{C}^2) \otimes \hat{P}_{2m}$$

of Proposition 5.2. The trace of $\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$ on the representation $\mathrm{Sym}^{2m}(\mathbf{C}^2) \otimes (\Lambda^2\mathbf{C}^2)^{-m}$ of PGL_2 is equal to $(-1)^m$. The involution τ acts on $\hat{P}_{2m} = V_{m+1}$ by multiplication by $(-1)^{m+1}$. Hence

$$\begin{aligned}
\mathrm{Tr}(\theta|\mathrm{Sym}^{2m}(\mathbf{C}^2) \otimes \hat{P}_{2m}) &= -\dim(\hat{P}_{2m})\\
\mathrm{Tr}(\theta|\hat{\mathfrak{g}}) = \sum_{m \geq 0} -\dim(\hat{P}_{2m}) &= -\dim(\hat{P})\\
&= -\mathrm{rank}(\hat{G}). \qquad \square
\end{aligned}$$

We note that E. Cartan proved the inequality

$$-\mathrm{rank}(\hat{\mathfrak{g}}) \leq \mathrm{Tr}(\theta|\hat{\mathfrak{g}}) \leq \dim(\hat{\mathfrak{g}}) \tag{7.3}$$

for an arbitrary involution θ of a complex semi-simple Lie algebra $\hat{\mathfrak{g}}$. Hence the involutions constructed in Proposition 7.2 are as negative as possible. They form a single conjugacy class, and correspond to the Cartan involutions of the split real form of $\hat{\mathfrak{g}}$.

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