

## THE HARNACK ESTIMATE FOR THE RICCI FLOW ON A SURFACE – REVISITED\*

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1. The Harnack estimate for the Ricci flow on a surface of positive curvature occurs in [H]. This was generalized to the case with some negative curvature by Ben Chow [C]. However, he introduced the global potential function  $\varphi$  solving

$$\Delta\varphi = R - r,$$

where  $R$  is the scalar curvature and  $r$  is the mean scalar curvature. Since this does not lend itself well to three dimensions, we rederive a Harnack estimate for surfaces with some negative curvature using only local quantities. The idea of use square root was used by the second author in [Y1] and [Y2] (note that an error of [Y1] is corrected in this paper).

**THEOREM. 1.1.** *For any constants  $K$  and  $L$  we can find (positive) constants  $A$ ,  $B$ ,  $C$  and  $D$  with the following property. If we have any solution to the Ricci flow on a compact surface such that at the initial time  $t = 0$  we have*

$$R \geq 1 - K$$

and

$$\frac{1}{R+K} \frac{\partial R}{\partial t} - \frac{1}{(R+K)^2} |\nabla R|^2 \geq -L,$$

then for all  $t \geq 0$  we have

$$\frac{1}{R+K} \frac{\partial R}{\partial t} - \frac{1}{(R+K)^2} |\nabla R|^2 + F\left(\frac{|\nabla R|^2}{(R+K)^2}, R+K\right) \geq 0,$$

where

$$F(X, Y) = A + \sqrt{2B(X+Y) + C} + D \log Y.$$

*Proof.* First note if  $\sqrt{C} \geq L$  the inequality holds at  $t = 0$ , we only need to show it is preserved by the maximum principle. To simplify the notation we let

$$\square = \frac{\partial}{\partial t} - \Delta$$

and

$$L = \log(R + K).$$

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Note  $R + K \geq 1$  so  $L \geq 0$ . We compute

$$\begin{aligned}\square R &= R^2, \\ \square L &= |\nabla L|^2 + e^L - 2K + K^2 e^{-L}, \\ \square e^L &= e^{2L} - 2K e^L + K^2, \\ \square |\nabla L|^2 &= 2\nabla L \cdot \nabla |\nabla L|^2 - 2|\nabla^2 L|^2 + 2e^L |\nabla L|^2 - 2K^2 e^{-L} |\nabla L|^2, \\ \square \Delta L &= 2\nabla L \cdot \nabla \Delta L + 2|\nabla^2 L|^2 + (2e^L - K - K^2 e^{-L}) \Delta L \\ &\quad + (2e^L - K + K^2 e^{-L}) |\nabla L|^2.\end{aligned}$$

We define the Harnack expression:

$$H = \Delta L + e^L.$$

Then we have

$$\begin{aligned}\square H &= 2\nabla L \cdot \nabla H + 2|\nabla^2 L|^2 + (2e^L - K - K^2 e^{-L}) \Delta L \\ &\quad + (-K + K^2 e^{-L}) |\nabla L|^2 + e^{2L} - 2K e^L + K^2.\end{aligned}$$

Let  $X = |\nabla L|^2$  and  $Y = e^L$  (note  $Y \geq 1$ ). Then we have

$$\begin{aligned}\square X &= 2\nabla L \cdot \nabla X - 2|\nabla^2 L|^2 + 2XY - 2K^2 X/Y, \\ \square Y &= 2\nabla L \cdot \nabla Y - 2XY + Y^2 - 2KY + K^2, \\ \square H &= 2\nabla L \cdot \nabla H + 2|\nabla^2 L|^2 + (2Y - K - K^2/Y) \Delta L \\ &\quad + (-K + K^2/Y) X + Y^2 - 2KY + K^2.\end{aligned}$$

Let  $F = F(X, Y)$  so  $\nabla F = F_X \nabla X + F_Y \nabla Y$ . Then

$$\square F = F_X \square X + F_Y \square Y - F_{XX} |\nabla X|^2 - 2F_{XY} \nabla X \cdot \nabla Y - F_{YY} |\nabla Y|^2.$$

Assume  $F$  is *concave*, i.e.

$$\nabla^2 F = \begin{pmatrix} F_{XX} & F_{XY} \\ F_{XY} & F_{YY} \end{pmatrix} \leq 0.$$

Then

$$\square F \geq F_X \square X + F_Y \square Y$$

and we get

$$\begin{aligned}\square F &\geq 2\nabla L \cdot \nabla F + F_X \left[ -2|\nabla^2 L|^2 + 2e^L |\nabla L|^2 - 2K^2 e^{-L} |\nabla L|^2 \right] \\ &\quad + F_Y \left[ -2e^L |\nabla L|^2 + e^{2L} - 2K e^L + K^2 \right] \\ &= 2\nabla L \cdot \nabla F + F_X \left[ -2|\nabla^2 L|^2 + 2XY - 2K^2 X/Y \right] \\ &\quad + F_Y \left[ -2XY + Y^2 - 2KY + K^2 \right].\end{aligned}$$

Introduce the compensated Harnack expression

$$\tilde{H} = H + F = \Delta L + e^L + F(|\nabla L|^2, e^L),$$

where  $X = |\nabla L|^2$  and  $Y = e^L$ . We compute

$$\begin{aligned} \square \tilde{H} &\geq 2\nabla L \cdot \nabla \tilde{H} + 2|\nabla^2 L|^2 + 2Y\Delta L + Y^2 - K(\Delta L + X + 2Y) \\ &\quad + K^2(-\Delta L/Y + X/Y + 1) + F_X[-2|\nabla^2 L|^2 + 2XY - 2K^2X/Y] \\ &\quad + F_Y[-2XY + Y^2 - 2KY + K^2]. \end{aligned}$$

Thus we have

$$\begin{aligned} \square \tilde{H} &\geq 2\nabla L \cdot \nabla \tilde{H} + 2(1 - F_X)|\nabla^2 L|^2 + 2Y\Delta L + 2XY(F_X - F_Y) \\ &\quad + Y^2(1 + F_Y) - K[\Delta L + X + 2Y(1 + F_Y)] \\ &\quad + K^2[-\Delta L/Y + (1 - 2F_X)X/Y + (1 + F_Y)]. \end{aligned}$$

Use

$$2|\nabla^2 L|^2 \geq (\Delta L)^2$$

and assume  $F_X \leq 1$  (which happens if  $B \leq \sqrt{C}$ ).

At  $\tilde{H} = \min$ , we have

$$\nabla \tilde{H} = 0$$

and at  $\tilde{H} = 0$  we have

$$\Delta L = -(Y + F).$$

To keep  $\tilde{H} \geq 0$  by the maximum principle we need

$$\begin{aligned} &(1 - F_X)(Y + F)^2 - 2Y(Y + F) + 2XY(F_X - F_Y) + Y^2(1 + F_Y) \\ &\quad - K[X + Y + 2YF_Y - F] + K^2[2 + F_Y + F/Y + (1 - 2F_X)X/Y] \\ &\geq 0. \end{aligned}$$

If in addition  $F_Y \geq 0$  and  $F_X \leq 1/2$  (which happens if  $B \leq \sqrt{C}/2$ ) we only need

$$\begin{aligned} &(1 - F_X)F^2 - 2YFF_X + (Y^2 - 2XY)(F_Y - F_X) \\ &\geq K[X + Y + 2YF_Y - F]. \end{aligned}$$

Look for  $F(X, Y)$  in the form

$$F = A + \sqrt{2B(X + Y) + C} + D \log Y.$$

For ease write

$$\begin{aligned} Q &= 2B(X + Y) + C, \\ F &= A + \sqrt{Q} + D \log Y. \end{aligned}$$

Compute

$$F_X = \frac{B}{\sqrt{Q}}, \quad F_Y = \frac{B}{\sqrt{Q}} + \frac{D}{Y}, \quad F_Y - F_X = \frac{D}{Y}.$$

We have

$$\begin{aligned}
& (1 - F_X) F^2 - 2Y F F_X + (Y^2 - 2XY) (F_Y - F_X) \\
&= \left(1 - \frac{B}{\sqrt{Q}}\right) (A + \sqrt{Q} + D \log Y)^2 - 2Y (A + \sqrt{Q} + D \log Y) \frac{B}{\sqrt{Q}} \\
&\quad + Y (Y - 2X) \frac{D}{Y} \\
&= 2(B - D) X + DY + (C - 2AB) + (A - B) \sqrt{Q} + D \log Y (\sqrt{Q} - 2B) \\
&\quad + (A + D \log Y) \left(\sqrt{Q} - \frac{2BY}{\sqrt{Q}}\right) + \left(1 - \frac{B}{\sqrt{Q}}\right) (A + D \log Y)^2 \\
&\geq 2(B - D) X + DY,
\end{aligned}$$

provided  $C \geq 2AB$ ,  $A \geq B$ ,  $C \geq 4B^2$  (so  $\sqrt{Q} \geq 2B$ ), and using  $\sqrt{Q} \geq 2BY/\sqrt{Q}$ .

On the other hand if  $B \leq \sqrt{C}/2$  then

$$2Y F_Y = \frac{2BY}{\sqrt{Q}} + 2D \leq A + \sqrt{Q} \leq F$$

provided  $A \geq 2D$  since  $2BY \leq Q$ . Thus we now only need

$$2(B - D) X + DY \geq K(X + Y),$$

which happens if

$$D \geq K \text{ and } 2(B - D) \geq K.$$

To summarize, we need

$$D \geq K,$$

$$B \geq D + \frac{1}{2}K,$$

$$A \geq B \text{ and } A \geq 2D,$$

$$C \geq L^2, \quad C \geq 4B^2 \text{ and } C \geq 2AB,$$

all of which is easily done by choosing them in this order. This completes the proof.  $\square$

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#### REFERENCES

- [C] B. CHOW, *The Ricci flow on the 2-sphere*, J. Diff. Geom., **33**(1991), 325-334.
- [H] R.S. HAMILTON, *The Ricci flow on surfaces*, Math. and Gen. Rel., Contemporary Mathematics **71**(1988), 237-262.
- [Y1] S.T. YAU, *On the harnack inequality of partial differential equations*, Comm. Anal. Geom., **2**(1994).
- [Y2] S.T. YAU, *Harnack inequality for non-self adjoint evolution equations*, Math. Res. Letters, **2**(1995).