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METRIC FIBRATIONS IN EUCLIDEAN SPACE*

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A systematic study of the local and global geometry of metric foliations in space forms was begun in [5] and [6], primarily for the spherical case. Recall that a *metric foliation* $\mathcal{F}^k$ (without singular focal loci) in a Riemannian manifold Q^{n+k} is a smooth partition into locally everywhere equidistant leaves of dimension k, or equivalently, a Riemannian foliation bundlelike with respect to the fixed ambient metric, so locally given in terms of Riemannian submersions. For generic Q, such foliations do not exist, except with leaves of codimension $n = 1$, i.e., parallel hypersurfaces. However, they are abundant for some of the most important metrics, notably when Q has constant curvature: basic classes of examples here are foliations by principal orbits of isometric group actions and by the exponentials of parallel normals of any submanifold with flat normal bundle.

Understanding metric foliations seems to be most difficult at the local level. Global problems have roots in the classical theory of isoparametric hypersurfaces and isometric group actions. They are also playing an ever increasing role in Riemannian geometry, and arise in particular in connection with rigidity aspects of nonnegatively curved spaces: The extension of the Diameter Sphere Theorem required the classification of metric fibrations of Euclidean spheres [6] in an essential way. The stronger form of the Soul Theorem proved by G. Perelman [11] finally established that all complete manifolds with nonnegative curvature can be constructed as certain metric fibrations over compact ones.

The purpose of this paper is to study metric foliations $\mathcal{F}^k$ of flat Euclidean space $Q = \mathbb{R}^{n+k}$, a border line case which is still fairly rigid and quite interesting in light of the above results, as well as in its own right. Our main results are somewhat similar in scope to those of [6] for spheres, although the key ideas and arguments are often very different. Roughly speaking we show that low dimensional metric foliations of Euclidean spaces are homogeneous, and we describe their classification. It is remarkable and perhaps somewhat surprising that the Soul Theorem enters the arguments when dealing with fibrations. Whether or not our results will extend to higher dimensions along the same lines is not at all clear at this point. Our presentation is organized in a way that we review and discuss some local facts in Section 1 and develop the global techniques needed in Section 2. We can then deal with the stronger results for metric fibrations in Section 3, and finally consider foliations in Section 4.

1. Some local aspects. Let $\mathcal{F}^k$ denote a metric foliation in a complete space Q_c^{n+k} of constant curvature c. The reader is referred to [6] for the terminology and basic facts about metric foliations that will be freely used here, cf. also [9]. X, Y, Z will denote local horizontal fields, T, U, V vertical ones, and lower-case letters refer to individual vectors. We write $e = e^h + e^v \in \mathcal{H} \oplus \mathcal{V}$ for the decomposition of $e \in TQ$

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into its horizontal and vertical parts. Thus, the integrability tensor A and the second fundamental tensor S are given by

$$A_X Y = \frac{1}{2}[X,Y]^v = \overset{v}{\nabla}_X Y, \qquad S_X U = -\overset{v}{\nabla}_U X.$$

The *mean curvature form* of $\mathcal{F}$ is the horizontal 1-form κ on Q with $\kappa(E) = \operatorname{tr} S_{E^h}$. $\mathcal{F}$ is said to be *isoparametric* if the principal curvatures in basic directions are locally constant along leaves. This occurs, for example, when $\mathcal{F}$ is the orbit foliation of an isometric group action. In constant negative curvature c, metric foliations are in general not isoparametric. This can be seen by taking a complete curve without focal points in the hyperbolic plane, and exponentiating parallel sections in its normal bundle at constant distance. When $c \geq 0$ however, the situation is rigid:

THEOREM 1.1. *All metric foliations in complete spaces of constant curvature $c \geq 0$ are isoparametric. In particular, their mean curvature form is basic.*

Proof. Let λ be an eigenvalue of S_x with eigenvector u. It must be shown that if π is a Riemannian submersion locally defining $\mathcal{F}$, and $\tilde{x}$ a horizontal vector with $\pi_* \tilde{x} = \pi_* x$, then λ is an eigenvalue of $S_{\tilde{x}}$ with the same multiplicity. So consider the horizontal geodesic γ with initial velocity x, and the Jacobi field J along γ with $J(0) = u$, $J'(0) = -S_{\dot{\gamma}(0)} u = -\lambda u$. It is straightforward to check that $\pi_* J$ is Jacobi along $\pi \circ \gamma$, with $\pi_* J(0) = 0$, and $(\pi_* J)'(0) = -\pi_* A_x^* u$ (here, A_x^* denotes the adjoint of A_x). Now, if E is the parallel vector field along γ that equals u at 0, then $J(t) = (1 - \lambda t) E(t)$ in the flat case, and $J(t) = (\cos t - \lambda \sin t) E(t)$ in the positive curvature case, which we have normalized so that $c = 1$.
Let us assume for now that if $c = 0$, then $\lambda \neq 0$. Then γ has a focal point, and $\pi \circ \gamma$ has a conjugate point at, say, l. If $\tilde{\gamma}$ is a lift of $\pi \circ \gamma$, then by [10, Lemma 1], there exists a unique Jacobi field $\tilde{J}$ along $\tilde{\gamma}$ with $\tilde{J}(l) = 0$, $\pi_* \tilde{J} = \pi_* J$, and $D(\tilde{J}) = \tilde{J}'^v + S_{\dot{\tilde{\gamma}}} \tilde{J}^v + 2A_{\dot{\tilde{\gamma}}} \tilde{J}^h = 0$. In particular, $\tilde{J}'^v(0) = -S_{\dot{\tilde{\gamma}}(0)} \tilde{J}(0)$. But $\tilde{J}(l) = 0$, so that $\tilde{J} = \phi \tilde{E}$, where $\tilde{E}$ is parallel, and ϕ is the same function that appears in the expression for J. It follows that $\tilde{J}'(0) = \tilde{J}'^v(0) = -S_{\dot{\tilde{\gamma}}(0)} \tilde{J}(0) = \phi'(0) \tilde{J}(0)$, and λ is an eigenvalue of $S_{\dot{\tilde{\gamma}}(0)}$. By [10, Theorem 4], the λ-eigenspaces for $S_{\dot{\tilde{\gamma}}(0)}$ and $S_{\dot{\gamma}(0)}$ have the same dimension. But then this must also be true for $c = 0$ and $\lambda = 0$. ∎

2. The global picture in Euclidean space. Throughout the remainder of the paper, we will assume that the ambient space is Euclidean, and at least in the next two sections, that $\mathcal{F}$ is a fibration determined by a Riemannian submersion $\pi : \mathbb{R}^{n+k} \to M^n$. It follows from the long exact homotopy sequence of π that M is simply connected if and only if the fiber of π is connected. In general, π factors as a fibration over the universal cover of M followed by a covering map. But M has nonnegative sectional curvature by O'Neill's formula [9], and since covering maps of nonnegatively curved spaces are fairly well understood [2], we will assume that the fibers are connected. Using the spectral sequence for the homology of the fibration, it is easily seen that M must be contractible, see also [7]. From [2], one concludes that M is diffeomorphic to Euclidean space, and each soul is a point. Our first objective is to use the soul construction to obtain a totally geodesic fiber. This idea was used in [3] for the first time (unpublished; see also [18] for a generalization to nonnegative curvature).

Recall from [2] that if γ is a ray in M, and B_t^γ denotes the ball of radius t around $\gamma(t)$, then $C_\gamma = M \setminus B_\gamma$, where $B_\gamma = \cup_{t>0} B_t^\gamma$, is a totally convex subset of M. Intersecting over all rays γ emanating from a fixed point p yields a compact totally convex set (t.c.s.) C. Such sets are submanifolds of M with totally geodesic interior and possibly empty or nonsmooth boundary ∂C. In the latter case, the subset $C(1)$ of C consisting of points at maximal distance from the boundary is again a t.c.s. of strictly lower dimension. Iterating this procedure finitely many times yields in our case a one-point set $C(k)$ called a *soul* of M.

Although convexity is not, in general, a property that is preserved under Riemannian submersions, the contracting procedure described above does ensure that preimages are convex: First of all, these submersions are *submetries* [1], i.e., maps f which take a ball of radius t around p onto the ball of same radius around $f(p)$. Thus, with the above notation, if $\tilde{\gamma}$ is a horizontal lift of γ to $\mathbb{R}^{n+k}$, then π maps the open half-space $B_{\tilde{\gamma}}$ onto B_γ. Denote by $\tilde{B}_\gamma$ the union of all $B_{\tilde{\gamma}}$, where $\tilde{\gamma}$ is a horizontal lift of γ, and by $\tilde{C}_\gamma$ its complement in $\mathbb{R}^{n+k}$.

LEMMA 2.1. *$\tilde{C}_\gamma$ is a closed convex set in Euclidean space, and $\pi(\tilde{C}_\gamma) = C_\gamma$.*

Proof. $\tilde{C}_\gamma$ is an intersection of closed half-spaces, so the first part is clear. The inclusion $C_\gamma \subset \pi(\tilde{C}_\gamma)$ is obvious. If the other inclusion were false, there would exist some $q \in \tilde{C}_\gamma$ with $\pi(q) \in B_\gamma$, so that $d(\gamma(t_0), \pi(q)) < t_0$ for some t_0. Lift a minimal geodesic between $\pi(q)$ and $\gamma(t_0)$ to q. Its endpoint is $\tilde{\gamma}(t_0)$ for some lift $\tilde{\gamma}$ of γ, and since $d(q, \tilde{\gamma}(t_0)) = d(\pi(q), \gamma(t_0)) < t_0$, $q \in B_{\tilde{\gamma}}$, which is impossible. ∎

If $\tilde{C}$ denotes the intersection of $\tilde{C}_\gamma$, where γ ranges over all rays in M emanating from p, then $\tilde{C}$ is a closed, convex subset of Euclidean space, and one readily checks that $\tilde{C} = \pi^{-1}(C)$, $\partial\tilde{C} = \pi^{-1}(\partial C)$ (here C is the intersection of the C_γ as described in the soul construction above). If the boundary of C is empty, i.e., if C consists of a point, then the fiber $\tilde{C}$ over C is totally geodesic. Otherwise, for $0 \le a \le a_0 = \max\{d(q, \partial C) \mid q \in C\}$, consider the sets $C^a = \{q \in C \mid d(q, \partial C) \ge a\}$, and $\tilde{C}^a = \{q \in \tilde{C} \mid d(q, \partial\tilde{C}) \ge a\}$. Both are closed, and the first one is a t.c.s. in M, whereas the second one is convex in $\mathbb{R}^{n+k}$. Now, given a Riemannian submersion π and any two points p, q in the base, one always has that $d(p, q) = d(\pi^{-1}(p), \pi^{-1}(q)) = d(\tilde{p}, \pi^{-1}(q))$ for any $\tilde{p}$ in the fiber over p. Using this observation, it is easy to see that $\tilde{C}^a = \pi^{-1}(C^a)$. In particular, if $C(1) = C^{a_0}$ is the lower-dimensional t.c.s. in the soul construction and $\tilde{C}(1) = \tilde{C}^{a_0}$, then $\tilde{C}(1) = \pi^{-1}(C(1))$ is a closed convex subset of Euclidean space. Iterating this procedure until the base set consists of a single point yields a fiber over this point which is a closed, connected, convex submanifold without boundary of $\mathbb{R}^{n+k}$, i.e., an affine subspace. We have proved

THEOREM 2.2. *If $\pi : \mathbb{R}^{n+k} \to M^n$ is a Riemannian submersion, then the fiber $\pi^{-1}(p)$ over a soul $\{p\}$ of M is an affine subspace.*

The fact that $\pi^{-1}(p)$ is totally geodesic implies that the soul $\{p\}$ is a pole (i.e., the exponential map of M at p is a diffeomorphism), and the geodesic reflection in p is an isometry. Notice also that if π is not an orthogonal projection $\mathbb{R}^{n+k} \to 0 \times \mathbb{R}^n$, then the soul is essentially unique. More precisely, if it is not unique, then M splits isometrically as $M_0 \times \mathbb{R}^l$, and π splits as $\pi_0 \times 1_{\mathbb{R}^l} : \mathbb{R}^{n+k-l} \times \mathbb{R}^l \to M_0 \times \mathbb{R}^l$, where M_0 has a unique soul: This follows from the fact that the totally geodesic fiber $\pi^{-1}(q)$ over any other soul $\{q\}$ must be equidistant from $\pi^{-1}(p)$, so that the minimal connections between these two fibers induce a parallel section in the normal bundle of either. Exponentiating this section yields a totally geodesic affine subspace that projects down to a geodesic through p and q. Since the preimage is totally convex, this geodesic must be a line.

We now use the above result to give a strengthened version of Theorem 1.1 for the fibration case. Recall that a foliation is said to be *taut* if there exists a metric on the ambient space for which the leaves become minimal submanifolds, cf. [13], [14], [15], [16]. It is known that when the ambient space is a Riemannian manifold and the foliation is metric, one criterion for tautness is the requirement that the mean curvature form κ be basic and exact [17].

COROLLARY 2.3. *Any metric fibration of Euclidean space is taut.*

Proof. κ is basic already for foliations by 1.1. Since $\kappa(E) = \operatorname{tr} S_{E^h}$, $d\kappa(U,V) = 0$ for vertical U, V. If X is basic, then the bracket $[X,U]$ is vertical, so that $d\kappa(X,U) = X\kappa(U) - U\kappa(X) - \kappa[X,U] = 0$. It remains to show that $d\kappa(X,Y) = 0$ for basic X, Y. This expression can also be written as $-2 \operatorname{div} A_X Y$, cf. e.g. [4]. To establish that each integrability field $A_X Y$ is divergence-free, we first observe that this divergence is the one induced by the metric on the corresponding fiber (since for basic Z, $\langle \nabla_Z A_X Y, Z\rangle = -\langle A_X Y, \nabla_Z Z\rangle = 0$), and that $\operatorname{div} A_X Y$ is constant along a fiber (because κ, and hence also $d\kappa$, is basic). So consider an arbitrary fiber L, and a minimal segment c of length l from the totally geodesic fiber F to L. The horizontal lifts of $\pi \circ c$ induce a diffeomorphism $h^c : F \to L$. Its derivative is given by $h^c_* u = J(l)$, where J is the holonomy Jacobi field (i.e., J is a vertical Jacobi field satisfying $J' = -S_{\dot c} J - A^*_{\dot c} J$) with $J(0) = u$. Since F is totally geodesic, the Jacobi equations in Euclidean space imply that

$$|h^c_* u|^2 = |u|^2 + l^2 |A^*_X u|^2, \tag{2-1}$$

where X is the basic field along F that projects down to $\pi_* \dot c(0)$. But since the ambient space has constant curvature, $|A_X Y|^2$ is constant along fibers for basic X, Y, and $|h^c_*|$, which is bounded below by 1, must also be bounded above on F. It follows that if B_r denotes the diffeomorphic h^c-image of the ball of radius r in F around some fixed point, then $\operatorname{vol} B_r \geq a \cdot r^k$ and $\operatorname{vol} \partial B_r \leq b \cdot r^{k-1}$ for some constants a and b. Now we apply Stokes' Theorem and obtain

$$a \cdot |\operatorname{div} A_X Y| \cdot r^k \leq \left| \int_{B_r} \operatorname{div} A_X Y \right| = \left| \int_{\partial B_r} \langle A_X Y, N_r\rangle \right| \leq b \cdot |A_X Y| \cdot r^{k-1},$$

(with N_r denoting the outward unit normal to ∂B_r) to conclude that $\operatorname{div} A_X Y$ must be zero on L if the above inequality is to hold for all $r > 0$. □

Up to congruence, the totally geodesic fiber F of $\pi : \mathbb{R}^{n+k} \to M^n$ constructed in 2.2 is $\mathbb{R}^k = \mathbb{R}^k \times 0 \subset \mathbb{R}^k \times \mathbb{R}^n$. The normal bundle ν of $\mathbb{R}^k$ in $\mathbb{R}^{n+k}$ comes with an induced flat connection which is just the horizontal component of the standard Euclidean one. ν also has a Bott connection given by

$$\overset{B}{\nabla}_U X = [\tilde U, \tilde X]^h, \qquad U \in \chi F, \quad X \in \Gamma\nu, \tag{2-2}$$

where $\tilde U$, $\tilde X$ denote extensions of U, X with $\tilde U$ vertical. Observe that the Bott-parallel fields are precisely the basic ones: In fact, the connection difference 1-form is the form ω on F with values in the skew-symmetric endomorphism bundle of ν given by

$$\omega(U)X = \overset{h}{\nabla}_U X - \overset{B}{\nabla}_U X = -A^*_X U. \tag{2-3}$$

In particular, the curvature tensors of the two connections are related by

$$R^h = R^B + d_{\nabla^B}\omega + [\omega, \omega], \tag{2-4}$$

where d_{∇^B} is the exterior derivative operator associated with the connection ∇^B; i.e., $d_{\nabla^B}\omega(U,V) = \nabla^B_U\omega(V) - \nabla^B_V\omega(U) - \omega[U,V]$, cf. [12]. But the Bott connection is also flat because of the Jacobi identity for brackets (or more directly because it admits global parallel sections), so that

$$d_\nabla\omega = -d_{\nabla^B}\omega = -[\omega,\omega], \tag{2-5}$$

where the first equality follows from interchanging the two connections in (2-4).

Notice that ω is Bott-parallel iff $\omega(U)X = -A^*_X U$ is basic along F for basic X and parallel U, or equivalently, iff $A_X Y$ is parallel along F for basic X and Y. In this case, the fibration is actually homogeneous, and one can explicitely describe the group action:

THEOREM 2.6. *Let $\pi : \mathbb{R}^{n+k} \to M^n$ be a metric fibration, F the totally geodesic fiber over the soul of M, and ω the connection difference form along F. Then ω is closed iff it is Bott-parallel. In this case,*

(1) *ω induces a Lie algebra homomorphism $\omega : \mathbb{R}^k \to \mathfrak{so}(n)$;*

(2) *π is the orbit fibration of the free isometric group action ψ of $\mathbb{R}^k$ on $\mathbb{R}^{n+k} = \mathbb{R}^k \times \mathbb{R}^n$ given by*

$$\psi(v)(u,x) = (u+v, \phi(v)x), \qquad u,v \in \mathbb{R}^k, \quad x \in \mathbb{R}^n,$$

where $\phi : \mathbb{R}^k \to SO(n)$ is the representation of $\mathbb{R}^k$ induced by ω.

Proof. Clearly, ω is Bott-closed if it is Bott-parallel, and is then also closed by (2-5). For the converse, observe that if X, Y are basic along F,

$$\begin{aligned}\overset{v}{\nabla}_U(A_XY) &= \overset{v}{\nabla}_U\overset{v}{\nabla}_XY = \overset{v}{\nabla}_U\nabla_XY = \overset{v}{\nabla}_X\nabla_UY + \overset{v}{\nabla}_{[U,X]}Y = \overset{v}{\nabla}_X\nabla_UY \\ &= \overset{v}{\nabla}_X(-S_YU - A^*_YU) = \overset{v}{\nabla}_X(-S_YU) + A_XA^*_YU,\end{aligned}$$

so that

$$\langle\overset{v}{\nabla}_U(A_XY),V\rangle = -X\langle S_YU,V\rangle + \langle\omega(U)Y,\omega(V)X\rangle.$$

If α denotes the one-form metrically dual to A_XY, then

$$\begin{aligned}d\alpha(U,V) &= \langle\overset{v}{\nabla}_U(A_XY),V\rangle - \langle\overset{v}{\nabla}_V(A_XY),U\rangle \\ &= -\langle[\omega(U),\omega(V)]X,Y\rangle = \langle d_\nabla\omega(U,V)X,Y\rangle,\end{aligned}$$

where we have used (2-5). Thus, if ω is closed, then so is α, and A_XY is a gradient. But A_XY has constant norm, and must therefore be parallel along F. As observed earlier, this means that ω is Bott-parallel.

We now proceed to show the second part of the statement. $\mathbb{R}^k$ will be identified with its tangent space at any point via parallel translation, and similarly, sections of the normal bundle of F will be viewed as maps $\mathbb{R}^k \to \mathbb{R}^n$. The restriction of ω to $0 \in \mathbb{R}^k$ then defines a linear map $\omega : \mathbb{R}^k \to \mathfrak{so}(n)$. By (2-5), ω is also a Lie-algebra homomorphism. Let $\phi : \mathbb{R}^k \to SO(n)$ denote the corresponding group homomorphism. Observe that the section X given by $X_u = \phi(u)x$ is the basic field with $X_0 = x$: Indeed,

$$(\nabla_wX)_v = \tfrac{d}{dt}_{|0}(t \mapsto \phi(v+tw)x) = \tfrac{d}{dt}_{|0}(t \mapsto \phi(tw))\cdot\phi(v)x = \omega(w)X_v.$$

In particular, the fiber $F_{(u,x)}$ of π through any point (u,x) can be described as follows: Let X be the basic field with $X_u = x$. Then $F_{(u,x)}$ is the set of all $(u+v, X_{u+v})$ as v

ranges over $\mathbb{R}^k$. On the other hand, the free action ψ from the statement satisfies

$$\psi(v)(u,x) = (u+v, \phi(v)x) = (u+v, \phi(u+v)\phi(-u)x) = (u+v, X_{u+v}),$$

since $X_0 = \phi(-u)x$. This establishes the claim.

Notice that since $|A_X Y|$ is constant fiberwise, Theorem 2.6 immediately implies homogeneity of one-dimensional fibrations. This is actually already true for foliations, cf. [5] and Section 4.

3. The fibrations of rank ≤ 3. Consider the vector space $\mathcal{A}$ of integrability fields spanned by all fields $A_X Y$ along the totally geodesic fiber F^k from Section 2. The *rank* of the fibration π along F is defined to be the maximal dimension of the space $\mathcal{A}_p = \operatorname{span}\{U_p \mid U \in \mathcal{A}\}$ as p ranges over F. Observe that the rank is always $\leq k$, whereas the dimension of $\mathcal{A}$ is $\leq n(n-1)/2$. In this section, we show that fibrations of rank no larger than 3 are homogeneous in the sense of Proposition 2.6. This classifies, in particular, all metric fibrations in Euclidean spaces of dimension at most 7.

Although the above bound suggests that the dimension of $\mathcal{A}$ is in general quite large, there are indications to the contrary: As observed in [6], O'Neill's identity for horizontal curvatures implies that $\langle A_X Y, A_X Z\rangle$ is constant along fibers. Equivalently, $A_X^* A_X$ preserves basic fields, and by polarization, so does $A_X^* A_Y + A_Y^* A_X$. Our main goal is to establish that π is homogeneous if $A_X^* A_Y$ preserves basic fields, and from now on, we operate under this assumption. Notice that the rank of π now equals the dimension of $\mathcal{A}$.

LEMMA 3.1. *$\mathcal{A}$ together with the bracket operation $\{,\}$, where $\{U,V\}$ denotes the orthogonal projection onto $\mathcal{A}$ of $[U,V]$, is a (metric) Lie algebra isomorphic to a subalgebra of $\mathfrak{so}(n)$.*

Proof. If ω denotes the connection difference form from Section 2, then, by hypothesis, $\omega(V)$ preserves basic fields for $V \in \mathcal{A}$. Thus, $-\omega$ induces a linear isomorphism between $\mathcal{A}$ and a subspace $\omega(\mathcal{A})$ of the Lie algebra of $SO(n)$ by setting $\langle -\omega(V)x, y\rangle = -\langle \omega(V)X, Y\rangle$ for basic extensions X, Y of x, y. Moreover, $\nabla_U(\omega(V))X = \nabla_U(\omega(V)X) - \omega(V)(\nabla_U X) = [\omega(V), \omega(U)]X$ for $U, V \in \mathcal{A}$ and basic X, so that $d\omega(U,V) = \nabla_U \omega(V) - \nabla_V \omega(U) - \omega[U,V] = -2[\omega(U), \omega(V)] - \omega[U,V]$, where d is an abbreviation for d_∇. Comparing with (2-5), we deduce that $-\omega[U,V] = [\omega(U), \omega(V)]$. It follows that $\omega(\mathcal{A})$ is a Lie algebra and $-\omega$ induces a Lie algebra structure on $\mathcal{A}$ whose bracket is the one in the claim.

LEMMA 3.2. *The distribution* $\ker\omega = \mathcal{A}^\perp$ *generates a Riemannian foliation $\mathcal{F}$ of F^k.*

Proof. Since $\omega[T_1, T_2] = -d\omega(T_1, T_2) = [\omega T_1, \omega T_2] = 0$ for $T_i \in \ker\omega$, $\ker\omega$ is integrable and generates a foliation $\mathcal{F}$. Similarly, if $T \in \ker\omega$, then $\omega[T, A_X Y] = -d\omega(T, A_X Y) = [\omega(T), \omega(A_X Y)] = 0$. In general, a foliation is Riemannian iff the Bott connection on the normal bundle of the leaves is Riemannian, i.e., $U\langle X, Y\rangle = \langle [U,X], Y\rangle + \langle X, [U,Y]\rangle$ for vertical U and horizontal X, Y. It is easy to see that this is equivalent to the requirement that $\langle [U,X], X\rangle = 0$ for vertical U, and horizontal X of constant norm, cf. also [17]. Thus, in our case, $\mathcal{F}$ is Riemannian, and in fact $A_X Y$ is basic for $\mathcal{F}$. □

Observe that $\langle \nabla_U V, W\rangle$ is constant for $U, V, W \in \mathcal{A}$ by 3.2. Let κ denote the mean curvature form of $\mathcal{F}$. Then by the proof of 2.3,

$$\kappa(A_XY) = -\operatorname{div} A_XY + \sum_i \langle U_i, \nabla_{U_i}(A_XY)\rangle = \sum_i \langle U_i, \nabla_{U_i}(A_XY)\rangle$$

for any orthonormal basis U_i of $\mathcal{A}$. In other words, κ is constant on each element of a basic spanning set for $\mathcal{A}$. It follows that $d\kappa = 0$, and $\kappa = df$ for some function f whose gradient ∇f is a global basic parallel vector field, since κ has constant norm. Consider a (necessarily geodesic) integral curve c of ∇f. Then $\kappa(\dot c) = |\kappa|^2$ is a constant function. But the proof of Lemma 3.3 below implies that $|S_{\dot c(t)}| \to 0$ as $t \to \infty$ for any metric foliation of Euclidean space. Thus, $\kappa = 0$ and $\mathcal{F}$ has minimal leaves.

LEMMA 3.3. *Any Riemannian foliation of Euclidean space by minimal leaves is locally congruent to a metric product foliation.*

Proof. For any metric foliation on a Riemannian manifold, differentiation of the holonomy Jacobi fields along a horizontal geodesic c yields

$$S'^{v}_{\dot c} = S^2_{\dot c} - A_{\dot c}A^*_{\dot c} + R^v_{\dot c},$$

where $R_{\dot c} = R(\cdot, \dot c)\dot c$, see for example [19]. In Euclidean space, we obtain, by taking traces, the Riccati-type equation

$$f' = f^2 + \tfrac{1}{k}|S_{\dot c} - fI|^2 - \tfrac{1}{k}|A_{\dot c}|^2,$$

where $f = \frac{1}{k}\operatorname{trace} S_{\dot c}$ and k is the dimension of the foliation, cf. also [8]. Thus, $|S_{\dot c}|^2 \equiv |A_{\dot c}|^2$ if the leaves are minimal. Notice that the assertion immediately follows in the case of a fibration (since then A is zero along the totally geodesic fiber over the soul, and so vanishes everywhere). For foliations, we will argue that in general,

$$\frac{|A_{\dot c(t)}|}{|S_{\dot c(t)}|} \to 0 \quad \text{as} \quad t \to \infty.$$

Observe first that $\ker(S_{\dot c} + A^*_{\dot c})$ is always a parallel subspace along c. In fact, if u belongs to the kernel at, say, $t = 0$, then the holonomy Jacobi field J with $J(0) = u$ is parallel (since $J'(0) = 0$) and belongs to the kernel for all t. Let f_i be an orthonormal basis of the image of $S_{\dot c(0)} + A^*_{\dot c(0)}$, and e_i vectors such that $(S_{\dot c(0)} + A^*_{\dot c(0)})e_i = -f_i$. Denote by E_i (resp. F_i) the parallel vector fields along c that equal e_i (resp. f_i) at 0. Then the holonomy fields J_i with $J_i(0) = e_i$ are given by $J_i(t) = E_i(t) + tF_i(t)$. Notice that

$$\left(\frac{J_i}{|J_i|} - F_i\right)(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Since the F_i are orthonormal, it suffices to show that $|A^*_{\dot c}U|^2/|S_{\dot c}U|^2 \to 0$ as $t \to \infty$ for $U = J/|J|$, where $J = E + tF$ is a holonomy field. Now,

$$|S_{\dot c}U|^2 = |S_{\dot c}(\tfrac{J}{|J|})|^2 \geq \left(\frac{\langle J', J\rangle}{|J|^2}\right)^2 = \frac{(\langle E, F\rangle + t|F|^2)^2}{(|E|^2 + 2t\langle E, F\rangle + t^2|F|^2)^2},$$

so that $t^2|S_{\dot c}U|^2 \to 1$. Similarly,

$$t^2\frac{|J'|^2}{|J|^2} = \frac{t^2|F|^2}{|J|^2} \to 1.$$

Since

$$|A^*_{\dot c}U|^2 + |S_{\dot c}U|^2 = \frac{|J'|^2}{|J|^2},$$

we have that $t^2|A^*_{\dot c}U|^2 \to 0$, and the lemma follows. □

By 3.3, the bracket operation $\{,\}$ in $\mathcal{A}$ coincides with the ordinary vector field bracket. By 3.1, $\mathcal{A}$ is then isometric to a Lie subalgebra of $\mathfrak{so}(n)$ with a flat metric. Such an algebra is known to be necessarily solvable. But the only solvable subalgebras of $\mathfrak{so}(n)$ are the abelian ones, so that $d\omega = -[\omega,\omega] = 0$ on $\mathcal{A}$, and therefore on all of F. 2.6 now implies:

Proposition 3.4. *Let $\mathcal{B}$ denote the space of basic sections of the normal bundle of F. If $A^*_X A_Y(\mathcal{B}) \subset \mathcal{B}$ for all $X, Y \in \mathcal{B}$, then the fibration is homogeneous.*

Theorem 3.5. *Any metric fibration of Euclidean space with rank ≤ 3 is homogeneous.*

Proof. According to 3.4, it only remains to show that $\langle A_X Y, A_Z W\rangle$ is constant along F for $X, Y, Z, W \in \mathcal{B}$. We may assume that the rank of A_W is less than $r = \operatorname{rank}\mathcal{F}$, for otherwise inner products are constant by [6, Lemma 3.1]. A_W then has nullity $> n - r \ge n - 3$, and its kernel must intersect $\operatorname{span}\{X, Y, Z\}$ nontrivially, except perhaps if X, Y, Z are linearly dependent. But in the latter case, $\langle A_X Y, A_Z W\rangle$ may be expressed as a linear combination of terms each of which involves only three basic vectors, and these are constant by skew-symmetry of each A_X together with the fact that $A^*_X A_X(\mathcal{B}) \subset \mathcal{B}$. Suppose therefore that $aX + bY + cZ \in \ker A_W$, with, say, $a \ne 0$. Then $a\langle A_W X, A_Y Z\rangle + b\langle A_W Y, A_Y Z\rangle + c\langle A_W Z, A_Y Z\rangle = 0$, where the last two terms are constant because they each involve only three basic vectors. Thus, $\langle A_W X, A_Y Z\rangle = \langle A_X W, A_Z Y\rangle$ is constant, and since $A^*_Z A_X + A^*_X A_Z$ preserves basic sections, so is $\langle A_X Y, A_Z W\rangle$. The cases when b or c are nonzero are handled in a similar way. □

Theorem 3.5 completely describes metric fibrations in Euclidean spaces of dimension ≤ 7. More generally, we have:

Corollary 3.6. *k-dimensional metric fibrations of $\mathbb{R}^{n+k}$ are homogeneous for $n \le 3$ or $k \le 3$, and up to congruence, in 1-1 correspondence with equivalence classes of representations $\mathbb{R}^k \to SO(n)$.*

4. The two-dimensional foliations. The classification of foliations is a more delicate problem than the one for fibrations, since for instance, one can no longer rely on the soul construction and the totally geodesic fiber guaranteed by Theorem 2.2. One-dimensional metric foliations of Euclidean space were completely described in [5]. They are always generated by Killing fields, and thus fall under the jurisdiction of 3.7. In this section, we show that two-dimensional foliations are homogeneous, and that they are in fact fibrations, so that here too, the classification results of section 3 apply.

Recall that a k-dimensional metric foliation $\mathcal{F}^k$ is said to be *substantial* along a leaf L if there is a basic X for which A_X is onto $\mathcal{V}$ at some $q \in L$. The condition is independent of q, and if $\mathcal{F}^k$ (with $k \le 3$) is substantial along some leaf, then it is homogeneous [6]. In 4.1, we deal with non-substantial foliations, and in 4.2, conclude that the substantial ones also are fibrations.

4.1. Homogeneity of nonsubstantial foliations. Let $\mathcal{F}$ denote a nonsubstantial metric foliation of $\mathbb{R}^{n+2}$ with 2-dimensional leaves. By [5], we may assume that A does not vanish anywhere, and it easily follows that the rank of A is everywhere

1. Let U denote a (local) unit field spanning the image of A, T a unit vertical field orthogonal to U. Thus,

$$\overset{h}{\nabla}_X T = \overset{h}{\nabla}_T X = -A^*_X T = 0 \tag{4.1-1}$$

for basic X. Recall from [6] that if Y is horizontally parallel along a horizontal geodesic c, then $(A_{\dot c}Y)'^v = 2S_{\dot c}A_{\dot c}Y$. It follows that if $A_x \neq 0$,

$$\overset{v}{\nabla}_X U = 2\langle S_X U, T\rangle T, \qquad \overset{v}{\nabla}_X T = \nabla_X T = -2\langle S_X U, T\rangle U. \tag{4.1-2}$$

Equation (4.1-2) is also valid when $A_x = 0$: If v is an eigenvector of S_x, then the holonomy Jacobi field J with $J(0) = u$ along the line in direction x must be parallel since it never vanishes. Thus, $S_x = 0$ whenever $A_x = 0$. Next, choose Y, Z so that $A_Y Z \neq 0$, and extend them to horizontally parallel fields along $t \mapsto \exp tX$. Then, again by [6], $\overset{v}{\nabla}_X A_Y Z = S_X A_Y Z = 0$, so $\overset{v}{\nabla}_X U = 0$. Now (4.1-2) in turn implies

$$\begin{aligned} [X,T] &= \langle S_X T, T\rangle T - \langle S_X T, U\rangle U \\ [X,U] &= \langle S_X U, U\rangle U + 3\langle S_X T, U\rangle T. \end{aligned} \tag{4.1-3}$$

LEMMA 4.1.4. *The mean curvature form κ is closed.*

Proof. Recall from the proof of 2.3 that κ is closed as soon as $d\kappa(X,Y) = 0$. Now, if $\{V_1, V_2\}$ is an orthonormal basis of $\mathcal{V}$, then

$$\begin{aligned} d\kappa(X,Y) &= X\langle Y, \sum \nabla_{V_i}V_i\rangle - Y\langle X, \sum \nabla_{V_i}V_i\} - \langle [X,Y], \sum \nabla_{V_i}, V_i\rangle \\ &= \sum_i \left(\langle Y, \nabla_X \nabla_{V_i}V_i\rangle - \langle X, \nabla_Y \nabla_{V_i}V_i\rangle\right). \end{aligned}$$

It remains to show that $\langle \nabla_X \nabla_T T, Y\rangle$ and $\langle \nabla_X \nabla_U U, Y\rangle$ are symmetric in X and Y. The first expression can be written

$$\langle \nabla_X \nabla_T T, Y\rangle = \langle R(X,T)T, Y\rangle + \langle \nabla_T \nabla_X T, Y\rangle + \langle \nabla_{[X,T]}T, Y\rangle,$$

where

$$\langle \nabla_T \nabla_X T, Y\rangle = T\langle \nabla_X T, Y\rangle - \langle \nabla_X T, \nabla_T Y\rangle = -4\langle S_X U, T\rangle\langle S_Y U, T\rangle$$

(using (4.1-2)) is symmetric in X and Y, and

$$\begin{aligned} \langle \nabla_{[X,T]}T, Y\rangle &= \langle S_X T, T\rangle\langle \nabla_T T, Y\rangle - \langle S_X T, U\rangle\langle \nabla_U T, Y\rangle \\ &= \langle S_X T, T\rangle\langle S_Y T, T\rangle - \langle S_X T, U\rangle\langle S_Y T, U\rangle \end{aligned}$$

(using (4.1-3)) is also symmetric. Similarly,

$$\begin{aligned} \langle \nabla_U \nabla_X U, Y\rangle &= U\langle \nabla_X U, Y\rangle - \langle \nabla_X U, \nabla_U Y\rangle \\ &= -U\langle U, A_X Y\rangle - \langle \overset{h}{\nabla}_X U, \overset{h}{\nabla}_U Y\rangle - \langle \overset{v}{\nabla}_X U, \overset{v}{\nabla}_U Y\rangle \\ &= -\langle A^*_X U, A^*_Y U\rangle + 2\langle S_X U, T\rangle\langle S_Y U, T\rangle, \end{aligned}$$

and

$$\begin{aligned} \langle \nabla_{[X,U]}U, Y\rangle &= \langle S_X U, U\rangle\langle \nabla_U U, Y\rangle + 3\langle S_X T, U\rangle\langle \nabla_T U, Y\rangle \\ &= \langle S_X U, U\rangle\langle S_Y U, U\rangle + 3\langle S_X T, U\rangle\langle S_Y T, U\rangle. \end{aligned}$$

This establishes the lemma.

LEMMA 4.1.5. *For basic X, Y, $\operatorname{div} A_X Y \equiv 0$. Thus, T is auto-parallel on each leaf, or equivalently, U generates a metric foliation on each leaf.*

Proof. $A_X Y$ is divergence-free by 4.1.4, as explained in 2.3. For the second part, choose X, Y so that $A_X Y \neq 0$. Since $A_X Y$ has constant norm along leaves, we have

$$\langle \nabla_T T, U\rangle = -\operatorname{div} U = 0, \tag{4.1-6}$$

and the claim follows. □

Let $R^\perp$ denote the curvature tensor of the normal bundle of a leaf. By the Ricci equation for constant curvature,

$$\langle R^\perp(U,T)X, Y\rangle = \langle [S_X, S_Y]U, T\rangle.$$

Now, $R^\perp(U,T)X = \overset{h}{\nabla}_U \overset{h}{\nabla}_T X - \overset{h}{\nabla}_T \overset{h}{\nabla}_U X - \overset{h}{\nabla}_{[U,T]} X$. Recalling that $\overset{h}{\nabla}_T X = 0$, that $\overset{h}{\nabla}_U X = -A_X^* U$ is basic, and that $\langle [U,T], U\rangle = -\langle T, \nabla_U U\rangle$, we obtain

$$-\langle T, \nabla_U U\rangle \langle A_X^* U, Y\rangle = \langle [S_X, S_Y]U, T\rangle. \tag{4.1-7}$$

THEOREM 4.1.8. *$\mathcal{F}$ is homogeneous. In fact, a nonsubstantial $\mathcal{F}$ is a global fibration generated by a Killing field and a parallel field.*

Proof. Observe first that since the leaves are two-dimensional and $\mathcal{F}$ is isoparametric, each leaf has constant curvature. This curvature is nonpositive by 4.1.5. Let $C = \{p \in \mathbb{R}^{n+2} \mid \langle \nabla_U U, T\rangle_p = 0\}$. We will later see that C has nonempty interior. Assuming this for the moment, let $O \subset C$ be open. Notice that $[S_x, S_y]$, being skew-symmetric, vanishes at any $p \in O$ by (4.1-7). It follows that $x \mapsto S_x$ has rank ≤ 1 for $x \in \mathcal{H}_p$: If not, there is an x for which $S_x U = \lambda U$ with $\lambda \neq 0$. This implies $S_x T = 0$, which contradicts the rank assumption. Next, we claim that $ST \equiv 0$ on $\mathbb{R}^{n+2}$. To establish this, we may assume the rank equals 1, for otherwise there is a totally geodesic leaf, and (4.1-9) below then implies the claim. Now, any leaf L intersecting O is flat because the restrictions of T, U to $O \cap L$ are parallel by (4.1-6) and (4.1-7), and leaves have constant curvature. For $p \in O$, choose a unit $x \in \mathcal{H}_p$ orthogonal to $\ker(x \mapsto S_x)$, and extend it to a basic field X along the leaf through p. By the Gauss equations, S_X has 0 as eigenvalue. Thus, there exists a unit vertical field V on O with $SV \equiv 0$. If c is a horizontal geodesic, then the Riccati equation for S from 3.3 yields $0 = \langle S_{\dot c} V, V\rangle' = |S_{\dot c} V|^2 - |A_{\dot c}^* V|^2$, so $V \equiv T$ on O. ST must then vanish on any leaf intersecting O: indeed, the trace of S_X is constant along a leaf, and so is $\langle S_X U, U\rangle$, since $X|A_X Y|^2 = 4\langle S_X A_X Y, A_X Y\rangle$ for X the tangent field of a horizontal geodesic c and Y horizontally parallel along c. Thus, $\langle S_X T, T\rangle$ is constant along a leaf, which implies that $ST = 0$ on the leaf. By (4.1-9), ST vanishes everywhere. This implies $\nabla_X T \equiv 0$ for basic X. Furthermore, $C = \mathbb{R}^{n+2}$ by (4.1-7), and $\nabla_U T$ is horizontal everywhere. Finally, $\langle \nabla_U T, X\rangle = \langle S_X T, U\rangle = 0$, so T is globally parallel, and $\mathcal{F}$ splits off a line. The remaining foliation is one-dimensional, hence homogeneous.

To complete the proof of 4.1.8, it remains to rule out the possibility that C has empty interior. Suppose this were actually the case. For $p \notin C$, let $H_p := \{\, x \in \mathcal{H}_p \mid U$ is an eigenvector of $S_x \,\}$. Since the subspace of symmetric operators of $\mathbb{R}^2$ that have a fixed eigenvector is a hyperplane in the space of all symmetric operators, $\dim H_p \geq n-1$, hence equals $n-1$ (otherwise the right side of (4.1-7) is 0). Thus, $A_x y = 0$ for all x, y in H_p. Let z be a nonzero horizontal vector orthogonal to H_p, x a unit vector spanning the image of A_z^*. For any $y \in H_p$ orthogonal to x,

$$\langle [S_y, S_z]U, T\rangle = -\langle A_y z, U\rangle\langle T, \nabla_U U\rangle = 0.$$

Now, $S_y U = \lambda U$, $S_y T = 0$, so $\langle [S_y, S_z]U, T\rangle = -\langle S_z S_y U, T\rangle = -\lambda\langle S_z U, T\rangle$. Thus, $\lambda = 0$, i.e., $S_y = 0$, and there exists a $(n-2)$-dimensional subspace $H_p^0 \subset H_p \subset \mathcal{H}_p$ on which S and A identically vanish. Since C has empty interior, H_p^0 is defined for all $p \in \mathbb{R}^{n+2}$. We claim this distribution $\mathcal{H}^0$ is globally parallel. It suffices to show this outside C. Now, $\mathcal{H}^0$ is basic along a leaf by the isoparametric property and the constancy of $|A_X Y|$ for basic X, Y. By its definition, $\mathcal{H}^0$ is then parallel along leaves. Let γ_e denote the geodesic in direction e. We only need to show that $\mathcal{H}^0$ is parallel along γ_e for $e \in \mathcal{H}^0$, and for $e = z, x$, where z, x are as above. For $e = z$, denote by X the unit field spanning the image of $A^*_{\dot\gamma_z}$ with $X(0) = x$. Then X is horizontally parallel along γ_z. If Y is a horizontally parallel field orthogonal to X, γ_z, then by [6, (2.6)],

$$(A_X Y \circ \gamma_z)' = S_{\dot\gamma_z} A_X Y - S_Y A_{\dot\gamma_z} X.$$

Observe that $S_Y = 0$ whenever $A_X Y = 0$. Since $A_X Y(0) = 0$, $A_X Y$ and S_Y vanish identically. Thus, Y is actually parallel along γ_z, and is tangent to $\mathcal{H}^0$. The argument for $e = x$ is similar, so we consider the case $e \in \mathcal{H}^0$. More generally, let c be a horizontal geodesic such that $U \circ c$ is an eigenvector field of $S_{\dot c}$, and let Y be horizontally parallel along c. Then

(4.1-9) $\qquad \langle S_Y T, U \circ c\rangle' = 0$

(4.1-10) $\qquad \langle S_Y U, U \circ c\rangle' = \langle S_{\dot c} U, U \circ c\rangle\langle S_Y U, U \circ c\rangle - \langle A^*_Y U, A^*_{\dot c} U\rangle.$

The claim clearly follows from the above two equations. We prove (4.1-10), since (4.1-9) is similar. Consider basic extensions X (of $\dot c$) and Y. Since $(U \circ c)'$ is horizontal,

$$\begin{aligned}\langle S_Y U, U \circ c\rangle' &= \langle (S_Y U \circ c)', U \circ c\rangle = -\langle \nabla_X \overset{v}{\nabla}_U Y, U\rangle \circ c \\ &= -\langle \nabla_X \nabla_U Y, U\rangle \circ c + \langle \nabla_X \overset{h}{\nabla}_U Y, U\rangle \circ c \\ &= -\langle \nabla_U \nabla_X Y, U\rangle \circ c - \langle \nabla_{[X,U]} Y, U\rangle \circ c - \langle \overset{h}{\nabla}_U Y, \overset{h}{\nabla}_X U\rangle \circ c,\end{aligned}$$

and (4.1-10) follows from (4.1-3). This establishes that $\mathcal{H}^0$ is parallel and induces a splitting $\mathbb{R}^{n+2} = \mathbb{R}^{n-2} \times \mathbb{R}^4$, with $\mathcal{F}$ tangent to $\mathbb{R}^4$. Thus, we have a metric foliation on $\mathbb{R}^4$ with the property that at every point there exists a unique direction X such that S_X has U as eigenvector. We will show this cannot be. Observe first that X is auto-parallel. Next, we claim that X is basic: if, say, X is not basic along the leaf L, then, since $\mathcal{F}$ is isoparametric and $S_X T = 0$, there must exist $p \in L$ and two independent horizontal directions $x := X_p$ and x' such that 0 is an eigenvalue of both S_x and $S_{x'}$. It then follows that S_y has nontrivial kernel for any horizontal y. To see this, choose an orthogonal operator P of $\mathcal{V}_p$ that maps T_p to the 0-eigenvector V of $S_{x'}$—notice that we may assume $S_{x'}$ is nontrivial, for otherwise the image of $y \mapsto S_y$ is one-dimensional, and the statement is clear. Then $S_x = a S_{x'} \circ P$ for some $a \in \mathbb{R}$. Thus, for $y = \alpha x + \beta x'$, $S_y = \alpha S_x + \beta S_{x'} = S_{x'} \circ (a\alpha P + \beta I)$ has rank at most that of $S_{x'}$, and hence nontrivial kernel. Now, for any horizontal geodesic c starting from L, X cannot be basic along the leaves through $c(t)$ for small t, since otherwise X would be basic along L by a limiting argument. Thus, there exists a unit vertical field V along c such that $S_{\dot c(t)} V = 0$ for small t. The Riccati equation then implies $A^*_{\dot c} V = 0$,

so that $V = T$. Hence T, and therefore also U, is an eigenvector of S_y for any $y \perp L$, contradicting the definition of X. This establishes the claim that X is basic.
Finally, let Y be a unit horizontal field orthogonal to X. Since X is basic, so is Y. Moreover, both $[Y, U]$ and $[Y, T]$ are vertical, so that Y, U, and T generate a codimension one foliation on $\mathbb{R}^4$. Since $\nabla_X X = 0$, this foliation is Riemannian, and therefore parallel. Thus, X is a globally parallel vector field, which is clearly impossible, since for example $\overset{h}{\nabla}_U X = -A^*_X U \neq 0$.

4.2. Global conclusions. We now proceed to show that 2-dimensional Riemannian foliations on $\mathbb{R}^{n+2}$ are fibrations. In light of 4.1.8, the substantial case is essentially the one that remains to be considered. We shall nevertheless treat both cases simultaneously. Thus, by [6], $\mathcal{F}$ is the orbit foliation of a 2-dimensional connected Lie subgroup G of isometries of R^{n+2} acting locally freely. Recall that the full isometry group $\mathbb{E}^{n+2} = O(n+2) \ltimes \mathbb{R}^{n+2}$. If (M_i, b_i), $i = 1, 2$, form a basis for the Lie algebra $\mathfrak{g}$ of G, then $[(M_1, b_1), (M_2, b_2)] = ([M_1, M_2], M_1 b_2 - M_2 b_1) = (0, M_1 b_2 - M_2 b_1)$, since a 2-dimensional subalgebra of the orthogonal algebra is always abelian. It easily follows that $\mathfrak{g}$ is abelian: This is clear if M_1, M_2 are linearly independent. Otherwise, we may assume $\mathfrak{g}$ has as basis $(0, a)$, (M, b), in which case $[(0, a), (M, b)] = (0, -Ma)$, so that a is an eigenvector of M. Since M is skew-symmetric, the bracket is 0. Thus, we have a representation

$$\mathbb{R}^2 = \mathfrak{g} \xrightarrow{\exp} G \hookrightarrow \mathbb{E}^{n+2}.$$

LEMMA 4.2.1. *There exists a totally geodesic leaf.*

Proof. We may assume that, with the above notation, M_1 and M_2 are linearly independent, for otherwise $\mathcal{F}$ is generated by a parallel field and a Killing field, and the statement is clear: The leaf through any point where the Killing field has minimal norm will be totally geodesic. Let $(A_i, a_i) = \exp(M_i, b_i)$, and change the origin, if necessary, so that $A_1 a_1 = a_1$, i.e., $b_1 = a_1$, and $M_1 b_1 = 0$. Observe that $M_2 b_1$ also vanishes: Indeed, $0 = M_2 M_1 b_1 = M_1 M_2 b_1 = M_1^2 b_2$. Thus, $M_1 b_2 = 0$ because M_1 is skew-symmetric, and therefore also $M_2 b_1 = 0$. Now, the square of the norm of the Killing field generated by (M_i, b_i) is given by $\phi_i(p) = |M_i p + b_i|^2$, which is easily seen to assume a minimum at those p satisfying $M_i^2 p + M_i b_i = 0$. Thus, both Killing fields have minimal norm at p when $p \in \ker M_1$ and $M_2^2 p + M_2 b_2 = 0$, where $b_2 \notin \operatorname{im} M_2$. Since M_1 and M_2 commute, M_2 maps the kernel of M_1 into itself. Moreover, b_2 lies in the kernel of M_1 as observed earlier. Decomposing

$$b_2 = b_2^0 + M_2 b_2^1 \in \ker M_{2|\ker M_1} \oplus \operatorname{im} M_{2|\ker M_1},$$

we see that the set where both norms are simultaneously minimal is the affine subspace

$$-b_2^1 + (\ker M_1 \cap \ker M_2).$$

If p belongs to this subspace, then the orbit of p consists of straight lines through p. Moreover, $p + t b_1$ also belongs to this subspace for any $t \in \mathbb{R}$. Since these points lie in the orbit of p, this orbit is a plane.

PROPOSITION 4.2.2. *The action of $\mathbb{R}^2$ on $\mathbb{R}^{n+2}$ is free. In particular, $\mathcal{F}$ is a fibration.*

Proof. If L denotes the totally geodesic orbit from the lemma, we claim that the restricted action ϕ of $\mathbb{R}^2$ on L is free: indeed, up to an isomorphism of $\mathbb{R}^2$,

$$\phi(s,t)p = \exp(s(M_1,b_1) + t(M_2,b_2))p = p + t(M_2p + b_2) + sb_1$$

for all $p \in L$. Thus, if $\phi(s,t)p = p$, then $t(M_2p + b_2) + sb_1 = 0$, and s, t must both vanish if the action is to be locally free. For arbitrary $q \notin L$, consider the minimal geodesic $c : [0,l] \to \mathbb{R}^{n+2}$ from q to L. If $g(q) = q$, then $g \circ c$ is again a minimal geodesic from q to L, so $(g \circ c)(l) = c(l)$. By the above, g must then be the identity.

The results from section 3 now immediately imply

THEOREM 4.2.3. *The 2-dimensional metric foliations of* $\mathbb{R}^{n+2}$ *are, up to congruence, homogeneous fibrations in 1-1 correspondence with representations* $\mathbb{R}^2 \to SO(n)$.

Foliations of codimension one are metric product foliations, and those of dimension one are always homogeneous [5].

COROLLARY 4.2.4. *Any metric foliation of* $\mathbb{R}^n$ *is a homogeneous fibration if* $n < 5$.

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