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UNFOLDINGS AND GLOBAL BOUNDS ON THE NUMBER OF COLLISIONS FOR GENERALIZED SEMI-DISPERSING BILLIARDS*

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Abstract. We generalize a global uniform bound on the number of collisions from flat to non-positively curved non-degenerate semi-dispersing billiard systems, and discuss a related problem of gluing a non-positively curved space without boundary out of finitely many copies of a billiard table.

1. Preliminaries. The purpose of this paper is twofold. First, we generalize global estimates on the number of collisions from [B-F-K-1] to arbitrary simply-connected manifolds of non-positive curvature. For instance, this generalization gives global estimates on the number of collisions for a system of hard balls colliding elastically in a hyperbolic space. This generalization involves a new geometric idea, which exploits the fact that the condition "curvature is bounded from above by K=0" can be verified locally for simply-connected Alexandrov spaces (although this fails for bounds K other than zero). This idea was motivated by another problem: given a semi-dispersing billiard table, can one glue together finitely many copies of this table in such a way that each wall participates in exactly one gluing and the resulting boundary-less space has the same curvature bound? Unfortunately, our attempts to describe such gluing have so far failed even for k-dimensional regular simplex. Moreover, it is unlikely that such gluing always exists, and the problem of finding geometric obstructions for its existence seems to be quite intriguing. We will explain, however, how this construction can be carried out in several simple cases.

To explain the connection between the two problems discussed above, we present the following model argument, which served as the starting point for our research. Consider a billiard system in a planar polygon P with curved walls. Assume that the walls are concave but all angles between them are non-zero. We want to show that there exists a number N such that every trajectory of length 1 experiences no more than N collisions. (Note that our goal is to demonstrate our method rather than to prove this simple statement, which can be proved by a ten-line elementary argument.) Using finitely many copies of P and gluing them along the corresponding sides, one can obtain a (singular) surface $\tilde{S}$ with the following property: total angle at each vertex is at least 2π (see Section 6). This implies that locally this surface is an Alexandrov space of non-positive curvature and, therefore, every geodesic of it is the shortest curve in its homotopy class. Notice that billiard trajectories in P can be naturally developed as geodesics in $\tilde{S}$. The space $\tilde{S}$ is paved by (finitely many) copies of P, and each side of these copies is also a shortest curve in $\tilde{S}$. Denote by K the number of edges on $\tilde{S}$ and by r the injectivity radius of $\tilde{S}$, and consider a segment of a trajectory which is shorter than $r/2$ which is, therefore, a shortest curve. This implies that this segment can not intersect the same edge twice, otherwise we would have two shortest curves of length less than $r/2$ and with the same endpoints. Taking into account that a collision of a billiard trajectory in P corresponds to an intersection

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of its development in $\tilde{S}$ with an edge, we conclude that the total number of collisions is less than $N = \frac{2K}{r}$.

We prepare now to deal with the general case, and start with the definitions. Let M be an arbitrary m-dimensional Riemannian manifold without boundary. Consider a collection of n geodesically convex subsets (walls) $B_i \subset M, i = 1, \ldots, n$, (the letter n will be reserved for the number of bodies B_i throughout the paper) in M, such that their boundaries are C^1 submanifolds of codimension one. Let B be one of the connected components of $M \backslash (\bigcup_{i=1}^{n} Int(B_i))$, where $Int(B_i)$ denotes the interior of the set B_i. The set $B \in M$ will be called a billiard table. A semi-dispersing billiard flow $\{T^t\}_{t=-\infty}^{\infty}$ acts on a certain subset $\tilde{TB}$ of full Liouville measure of the unit tangent bundle to B. To be more precise, $\tilde{TB}$ consists of such points $(x, v) \in TM$, $x \in B$, $v \in T_xM$, with the vector v directed "strictly inside of B," and the orbit of (x, v) defined for all $t \in (-\infty, \infty)$ (see, for example, [Bu] for the rigorous definitions and extensive references). The projections of the orbits of that flow to B are called the billiard trajectories. The particle moves inside the set B with unit speed along a geodesic until it reaches one of the sets B_i (collision) where it reflects according to the law "angle of incidence equals angle of reflection." We exclude those trajectories which experience a collision with more than one wall simultaneously. Nevertheless, all the results in this paper (as well as in [B-F-K-1]) remain valid if one introduces any law for the outcome of such collision which agrees with the energy conservation law and the conservation of the projection of the momentum onto the intersection of the walls participating in the collision. (It is relevant to mention here that, due to local uniform estimates on the number of collisions, one can see that there are only finitely many outcomes which can be obtained as limits of actual billiard trajectories.)

A very natural example of such a billiard system is a system of N balls (of non-zero radii) moving freely in a boundary-less manifold and colliding elastically. A position of the centers of these balls is described by a point in the configuration space which is the N-th Cartesian power of the manifold. (Note that we included in this configuration space "forbidden" positions when some of the balls intersect.) The "walls" are formed by the set of positions where two balls intersect or touch each other (therefore, there are $N(N-1)/2$ walls), and the Riemannian metric is given by the total kinetic energy of the system. The systematic study of such billiards (where the underlying manifold was usually supposed to be a torus or Euclidean space) was initiated by Ya. Sinai and continued by many mathematicians.

In [B-F-K-1] and [B-F-K-2] we established a connection between semi-dispersing billiards and Alexandrov spaces. (For background information on singular Riemannian geometry of non-positive curvature see [Ba], [Gr] and [Re].) Namely, with every billiard trajectory T we associate certain singular space M_T, such that T naturally corresponds to a geodesic in M_T. Moreover, the curvature of the space M_T is bounded from above (in the sense of Alexandrov) by the maximal sectional curvature of M. This allows us to apply the results and techniques of singular geometry to certain billiards problems.

In particular, we obtained a complete solution (see Theorem 1 below) to the problem of the existence of local uniform estimates on the number of collisions for (non-degenerate) semi-dispersing billiards. This problem was first posed by Sinai, who also gave a solution [Si] for billiards in polyhedral angles.

We introduce the following non-degeneracy condition

Definition 1.1. *A billiard B is* **non-degenerate** *in a subset $U \subset M$ (with* *constant $C > 0$), if for every $I \subset \{1, \dots, n\}$ and for every $y \in (U \bigcap B) \backslash (\bigcap_{j \in I} B_j)$,*

$$\frac{dist(y, \bigcap_{j \in I} B_j)}{\max_{k \in I} dist(y, B_k)} \leq C,$$

whenever $\bigcap_{j \in I} B_j$ is non-empty.

A billiard B is called **non-degenerate at a point** *$x \in B$ with constant C if it is non-degenerate in a neighborhood of x with the same constant, and* **locally non-degenerate** *with constant C if it is non-degenerate at every point with constant C.*

We will say that B is **non-degenerate** *if there exist $\delta > 0$ and $C > 0$ such that B is non-degenerate, with constant C, in any δ-ball.*

Roughly speaking, the condition means that if a point is d-close to all the walls from I then it is Cd-close to their intersection.

The definition above is formulated in such a way that its geometric meaning may remain obscure. This definition, however, is not only the most convenient for our applications, but also enables us to verify it for the hard sphere gas model described above. To acquire some geometric insight, we give several equivalent reformulations of the non-degeneracy condition (which will never be used or appear further in this paper).

DEFINITION 1.2. *A billiard table B is non-degenerate if there exists a positive number r such that, at every point, the unit tangent cone to B (which is a subset of an $(m-1)$-dimensional sphere) contains a ball of radius r.*

For flat M this is equivalent to Sinai's "cones condition:" every point of B is a vertex of a round cone of radius r which entirely belongs to B in some neighborhood of its vertex.

Another equivalent reformulation is:

DEFINITION 1.3. *Let B_t be the complement in M to the union of t-neighborhoods of walls. (That is, its boundary is a t-equidistant inward deformation of the boundary of B.) A billiard table B is non-degenerate if $\frac{d}{dt} dist(B, B_t)$ is finite at $t = 0$, where dist means the Hausdorff distance between sets.*

For compact billiard tables, these definitions can also be reformulated in the following way: the operations of taking tangent cone and intersection commute for any collection of the complements to the walls B_i. For non-compact tables, however, this definition guarantees local non-degeneracy only, while the constant C may deteriorate and have no positive lower bound.

In [B-F-K-1] we proved the following

THEOREM 1. *Let a semi-dispersing billiard B with n walls be non-degenerate with constant C at a point x. Then, there exists a neighborhood U_x of x such that every billiard trajectory entering U_x leaves it after making no more than*

$$P(C, n) = (16(C+2))^{2(n-1)}$$

collisions with the walls.

Remark: Note that while $P(C, n)$ does not depend on the curvature of M, the size of U_x does depend on it.

An important fact is that for billiards in $\mathbb{R}^k$ our estimates are global, that is, if all the walls have a non-empty intersection, the total number of collisions (in infinite time) is bounded by a constant $P(C)$. (Non-emptiness of the intersection of the walls rules out a situation in which a particle bounces between two walls for infinitely long.) This allows us to give a complete solution to the problem of finding a uniform estimate on the number of collisions that may occur in a hard ball system in empty Euclidean space.

However, while a local version of the estimate on the number of collisions in a semi-dispersing billiard was proven for arbitrary manifolds, its global analog was established only in $\mathbb{R}^k$. Indeed, the crucial step in the argument was based on some distance comparison lemma for triangles, which is true for all triangles in Euclidean space but only for sufficiently small Riemannian triangles.

In this paper we develop an alternative approach to the problem of finding global estimates, which ideologically is strongly motivated by our attempts to construct "universal unfolding" spaces for billiards (see below).

In Section 5 we prove the following

Theorem 2. *If M is a simply connected manifold of non-positive sectional curvature, $\bigcap_{i=1}^n B_i$ is non-empty, and B is locally non-degenerate with constant C, then every billiard trajectory in B has no more than*

$$K(C, n) = (200(C+2))^{2n^2}$$

collisions in the infinite period of time $(-\infty, \infty)$.

We immediately have the following

Corollary 1.1. *The maximal number of collisions that may occur in a system of N hard elastic balls (of arbitrary masses and radii) moving freely in a simply connected Riemannian space $\mathcal{M}$ of non-positive sectional curvature never exceeds*

$$\left(400N^2\frac{m_{max}}{m_{min}}\right)^{2N^4},$$

where m_{max} and m_{min} are, respectively, the maximal and the minimal masses in the system.

The proof of Corollary 1.1 consists of representing the hard ball system as a certain billiard in $(\mathcal{M})^N$ and verifying that that billiard is non-degenerate in the whole $(\mathcal{M})^N$. The detailed argument is presented in [B-F-K-3].

In this paper we also discuss a problem of constructing an Alexandrov space of non-positive curvature which may serve as a "universal unfolding" space, helping to analyze the dynamics of a given semi-dispersing billiards system on a Riemannian manifold of non-positive curvature. This space would be a multi-dimensional analog of the two-dimensional surface constructed from copies of a billiard table as described in our model argument at the very beginning of the paper. The key feature of this construction is the existence of a natural correspondence between the trajectories of the billiards system and their developments as geodesics in the global model space. We also require that the number of copies of a billiard table used to construct the universal unfolding space be finite. Note that, in both this paper and [B-F-K-1], we use Alexandrov spaces in which these developments only exist for the class of trajectories with the prescribed sequence of collisions. Thus, the problem is to construct a "universal" space where every trajectory can be developed.

This paper is organized as follows. In the next section, we give a sketch of the argument in the proof of a local estimate on the number of collisions, emphasizing the geometric nature of the proof and stressing the point where the global version fails. We also outline our main idea of how to overcome this problem. In Sections 3 and 4 we discuss the problem of constructing universal unfolding spaces and present the construction of the universal unfolding space for billiards with only two walls. In

Section 5, a simple modification of the construction of universal unfolding spaces for billiard tables with two walls proves Theorem 2, our main theorem. Universal unfolding spaces for billiards on compact non-positively curved surfaces are constructed in Section 6.

2. Sketch of the argument. Our exposition in this section has a non-rigorous character. Our goal is to simplify the understanding of what follows and to make this paper more self-contained.

We outline the main steps of the argument from [B-F-K-1], pointing out how we later modify the argument to make it work in the global case. We omit all combinatorial details and deal only with the geometric part of the argument. For simplicity, we assume that M is a non-positively curved manifold. Recall that we intend to prove that trajectories entirely contained in a sufficiently small neighborhood never experience "too many" collisions.

The first step is to prove that the length of a piece of a trajectory T with endpoints x and y, visiting a sequence of walls B_{i_j}, $j = 1, 2, \ldots, k$, with non-empty intersection $\bigcap_{j=1}^{j=k} B_{i_j} = Q$ is no less than the sum of distances $dist(x, z) + dist(y, z)$, for every $z \in Q$.

We consider a sequence of copies M_j, $j = 0, 1, \ldots, k$, of manifold M, and glue M_j and M_{j+1} together along $B_{i_{j+1}}$, where $B_{i_{j+1}}$ is the wall with which our trajectory has experienced its j-th collision. Since all B_{i_j} are convex, by applying Reshetnyak's theorem several times, we conclude that the resulting space M_T also has non-positive curvature.

An important remark: It might seem more natural to glue along the boundaries of B_{i_j} rather than along the whole B_{i_j}. For instance, one would do so thinking of this gluing as "reflecting in a mirror" or by analogy with the usual development of a polygonal billiard. However, gluing along the boundaries will not give us a space with the appropriate curvature bound in any dimension higher than 2.

Note that, the space M_T, in addition to several copies of billiard table B, contains other redundant parts. For example, if we study a billiard in a curved triangle with concave walls, B_i's are not the boundary curves. Instead, we choose as B_i's some convex ovals bounded by extensions of these walls. (One may think of a billiard in a compact component of the complement to three discs.) In this case, these additional parts look like "fins" attached to our space (we borrow the term "fin" from Alexander and Bishop).

Our trajectory T can be naturally developed in the space M_T as a geodesic, and, therefore, it is the distance minimizer between its endpoints $\tilde{x}$ and $\tilde{y}$ (since M_T is a space of non-positive curvature). The path xzy (recall that z belongs to the common intersection of all walls which participate in the gluings producing M_T) can also be lifted into M_T with the same endpoints $\tilde{x}$ and $\tilde{y}$. Hence, it is longer than T, which proves the first step.

Now we continue our local argument, proceeding with the second step. Reasoning by contradiction and using induction and combinatorics, one can reduce the problem to the situation where all B_i have a non-empty intersection $Q = \bigcap B_i$ and, given any number N, one can find a (piece of a) trajectory which contains at least N non-overlapping segments with the following properties:

1. Each segment has both endpoints in the same B_i. Let us denote these endpoints by x and y.
2. There is a point z between x and y in the trajectory, such that one can find a point $\tilde{z} \in Q$ such that $C \cdot dist(z, B_i) \geq dist(\tilde{z}, B_i)$.

The second condition ensures that B_i is the wall which satisfies the inequality (for z) from Definition 1.1 (the non-degeneracy condition).

A standard elementary geometric argument shows that there is a constant S, such that $S \cdot (dist(xz) + dist(yz) - dist(xy)) \geq dist(x\tilde{z}) + dist(y\tilde{z}) - dist(xy)$. This constant S depends only on C when M is a Euclidean space, and on the geometry of the neighborhood in question for general M. This is precisely the ingredient of the proof which fails in the global case. Indeed, imagine that triangle xyz has a long side $dist(xy) = L$ and a short altitude h that falls in the middle of the side xy, and suppose that the triangle xyz belongs to a submanifold of zero curvature. Then the excess $(dist(xz) + dist(yz) - dist(xy))$ is approximately h^2/L. However, if the triangle $x\tilde{z}y$ belongs to a submanifold of curvature -1, we can approximate its excess $dist(x\tilde{z}) + dist(y\tilde{z}) - dist(xy)$ only as $2h$, which leaves no chances of finding the uniform constant S.

Now we can finish the local argument. We are leading to a contradiction of the statement obtained in the first step. That is, provided the number of segments described above is sufficiently large, we will shorten our trajectory in the class of curves with the same endpoints and also make it visit Q. Indeed, if the number of such segments is greater than $S + 1$, we choose one of them with the smallest excess $d = dist(xz) + dist(yz) - dist(xy)$ in the triangle xyz and replace this segment with two shortest curves, $x\tilde{z}$ and $\tilde{z}y$. We increased the length of our trajectory by no more than Sd. For all the others segments, we replace them with the shortest curves connecting their endpoints. In each case we gain at least d (which, by construction, is the smallest excess), and the number of such segments is at least $S + 1$. Therefore, our new curve visits Q and is shorter than T, which contradicts the conclusion in the first step.

Notice that the same "length shortening" argument can be repeated for any geodesic in M_T which visits the interiors of sufficiently many copies of the billiard table B. More generally, geodesics in T_M, being projected to M, can be regarded as trajectories of a billiard system when some walls may "become transparent for the particle." In other words, such a trajectory may collide with a wall, or may just go through it.

We now introduce an overly simplistic version of the main idea for handling the global case. We can try to modify M_T in the following way: instead of attaching the last copy of M, we can attach the first one, "closing up" the space M_T. If we could show that this new space also has non-positive curvature, we would immediately have a contradiction: in this space, a development of T must be the shortest between its endpoints, but since the endpoints belong to the same copy of M, they can be connected by a shortest curve inside of this copy. This last gluing, however, means an identification in our space, and we certainly can not apply Reshetnyak's theorem to it.

We recall that a space has non-positive curvature iff, for every triangle, its angles are no bigger than the corresponding angles of the comparison triangle in the Euclidean plane. Since non-positiveness of curvature is a local property, it is sufficient to verify the angle comparison condition for small triangles only. However, using the correspondence between geodesics and billiard trajectories, one can conclude (reasoning exactly as in the proof of the *local* estimates on the number of collisions, presented in [B-F-K-1] and sketched below) that each side of a small triangle can not intersect interiors of too many copies of the billiard table B. Assume that the sequence of collisions that determines the gluing of M_T is sufficiently long. Then, for every small triangle for which we want to verify the angle comparison property, most of the copies of B are irrelevant. In particular, we can undo one of the gluings without changing the small triangle, and find ourselves in a situation in which we "have broken the cycle of copies" and may now apply Reshetnjak's theorem.

3. "Universal unfolding" spaces. By universal unfolding space we mean a space $\tilde{M}$ which results from gluing together (along the sets B_i, $i = 1, \dots, n$) a finite number of copies of M so that

1. Every copy is glued with exactly n other copies along each of the bodies B_i, $i = 1, \dots, n$. More precisely, for every copy M_j
   (a) there are n distinct copies M_j^i, $i = 1, \dots, n$, such that $M_j \bigcap M_j^i = B_i$;
   (b) if for some copy M_k, $M_j \bigcap M_k = B_i$ then $M_k = M_j^i$;
   (c) for any M_k, $M_k \bigcap M_j \subset B_i$, for some $i \in \{1, \dots, n\}$;
2. $\tilde{M}$ has curvature bounded by the maximal sectional curvature of M (we always assume that M has non-positive curvature, so $\tilde{M}$ must have non-positive curvature as well).

The space $\tilde{M}$ will then become a universal unfolding space for trajectories of the billiard flow. Namely, we will no longer have to construct a space M_T for each trajectory T like we did in [B-F-K-1]: for each copy M_k of M in $\tilde{M}$ every trajectory will have a unique lift to a geodesic in $\tilde{M}$ that starts in M_k. Moreover, we will be able to view the geodesic flow on $\tilde{M}$ as a finite cover of the billiard flow on M.

Due to the non-uniqueness of extensions for geodesics in singular spaces, the use of the term "geodesic flow on $\tilde{M}$" requires some clarification. The extension is not unique at the points where the geodesic g^t that belongs to some copy M_1 of M in $\tilde{M}$ encounters another copy M_2. At such points g^t may be continued in M_1 or in M_2. Let us call the continuation in M_2 the **regular continuation.** Let us call a geodesic in $\tilde{M}$ **regular** if it never crosses more then two copies of M at a time, and whenever it encounters a new copy of M it continues in a regular way (that is, changes the copy). Let us call a tangent vector v at a point $x \in \tilde{M}$ **regular** if the corresponding geodesic is regular, and x belongs to $\tilde{B}$ – the union of the images of B under the canonical embeddings. Then, we can correctly define a geodesic flow on the set of all regular vectors, which is a full measure subset of the unit tangent bundle to $\tilde{B}$.

Denote by $\pi : \tilde{M} \to M$ the natural projection from $\tilde{M}$ onto M that maps each copy M_k isometrically on M. Then the derivative $D\pi$ of π (defined almost everywhere on the tangent bundle to $\tilde{M}$) projects the geodesic flow on $\tilde{M}$ onto the billiard flow in B. Thus, once $\tilde{M}$ is constructed, virtually any problem about the billiard flow on M may be restated as a problem about the geodesic flow on $\tilde{M}$. In particular, the ergodicity of the billiard flow is equivalent to the ergodicity of the geodesic flow.

It is easy to show that $\tilde{M}$ can be constructed only if B is non-degenerate.

We will show how to construct a universal unfolding space for a billiard with only two walls and for arbitrary non-degenerate semi-dispersing billiards on surfaces of non-positive curvature. Unfortunately, the problem of how to determine if universal unfolding spaces exist for a given billiard table remains open.

4. Billiards with two walls. In this section we will present a construction of $\tilde{M}$ for billiards with two walls. There is also an alternative construction presented at the end of this section. The advantage of the alternative construction is that it does not use the local estimates and also has a nice Corollary 4.1. The reason we chose to explain in detail the construction based on local estimates is that its modification leads to the proof of Theorem 2 (see Section 5) .

Let K be an even number. Consider K copies $M_1, M_2, \dots, M_K$ of M. For $L > K$

let $M_L = M_l$, where $1 \le l \le K$ and $l = L(\text{mod}\, K)$. Let us glue M_i, $i = 1, 2, \dots, K$, with M_{i+1} by the body B_1 if i is even, and by B_2 if i is odd. The result of the gluing is a simply connected space $\tilde{M}$. Notice that by construction, for each i, there is a canonical isometric embedding $E_i : M \to \tilde{M}$, which is an isometry between M and M_i and maps the subsets B_1, B_2 in M into corresponding subsets B_1, B_2 in M_i.

Now, we will prove that, if K is large enough, then $\tilde{M}$ has non-positive curvature in the sense of Alexandrov.

Proof. Using the methods of [B-F-K-1], it is easy to prove that there exists a number P such that for any $y \in \tilde{M}$ there exists a neighborhood U_y such that every geodesic in U_y is contained in a union of at most P copies of M.

Let $K > 3P$. Then, every geodesic triangle Δ in U_y is contained in a union M_U of at most $3P$ copies of M. Let $M_{i_1}, \dots, M_{i_k}$, $1 \le i_{k+1} = i_1 < i_2 < \dots < i_k \le K$, be these copies. Then, since $k < K$, there exists j such that $i_{j+1} \ne i_j + 1(\text{mod}\, K)$. Without loss of generality we may assume that $j = k$.

It is easy to see that if $i_{j+1} = i_j + 1$ then $M_{i_{j+1}} \bigcap M_{i_j}$ is equal to one of the bodies of B_1, B_2, but if $i_{j+1} \ne i_j + 1(\text{mod}\, K)$ then $M_{i_{j+1}} \bigcap M_{i_j}$ is equal to $B_1 \bigcap B_2$. Thus, M_U is the result of the following gluing of k copies $M_{i_1}, \dots, M_{i_k} : M_{i_j}$ is glued with $M_{i_{j+1}}$ by either B_1, or B_2 or $B_1 \bigcap B_2$, for $i = 1, 2, ..., k-1$. (There is no need to glue together M_{i_k} and M_{i_1}, since they are already glued along $B_1 \bigcap B_2$, as a result of the previous $k - 1$ gluings!) Applying Reshetnyak's theorem [Re] $k - 1$ times, we see that M_U is a singular space of non-positive curvature. Since M_U has non-positive curvature, Δ has non-positive defect. Note that the angles of Δ as a triangle in M_U are correctly defined and not smaller than the angles of Δ in the whole space. (Indeed, removing a subset can not decrease distances between points on the walls of triangles which participate in the definition of angles.) Thus, $\tilde{M}$ has non-positive curvature in the neighborhood U_y of y.

Since, $\tilde{M}$ is a simply connected space with locally non-positive curvature, due to Alexandrov's theorem [Re], it has non-positive curvature globally as well. □

Thus, $\tilde{M}$ can serve as a universal unfolding space for the billiard in the outside of $B_1 \bigcup B_2$. In particular, we proved Theorem 2 for $n = 2$. (For any trajectory the number of collisions can not be bigger than the number of copies of M used to construct $\tilde{M}$.)

Alternative construction. Let α be the smallest angle between the tangent planes to ∂B_1 and ∂B_2 over all the points of $\partial B_1 \bigcap \partial B_2$. Construct $\tilde{M}$ as before, with $K = K(\alpha) \ge \frac{2\pi}{\alpha}$. Now the fact that $\tilde{M}$ has non-positive curvature can be verified just by looking at the tangent cones of its points and showing that they are all $CAT(1)$. Also, note that there is an involutive isometry I of $\tilde{M}$ such that $I(M_1) = M_K$, $I(M_2) = M_{K-1}$, $I(M_3) = M_{K-2}$, $\dots$, $I(M_{K/2}) = M_{K/2+1}$. Now, we immediately see that no geodesic corresponding to a billiard trajectory can intersect more than half of the number of the copies of M in $\tilde{M}$. Therefore, we have the following

Corollary 4.1. *For a billiard outside of two bodies B_1 and B_2 on a simply connected manifold of non-positive curvature, the maximal number of collisions for any trajectory in the infinite period of time $(-\infty, \infty)$ is not greater than $\left[\frac{\pi}{\alpha}\right]$, where α is the smallest angle between the tangent planes to ∂B_1 and ∂B_2 over all the points of $\partial B_1 \bigcap \partial B_2$.*

5. Proof of Theorem 2. Let $\bar{M}$ be a space formed by several copies M_1, $M_2, \dots, M_p$ of M which are glued together in a circular order along some of the sets B_i, $i = 1, \dots, n$. To be more precise, $\bar{M}$ is such that

1. for $i = 1, \dots, p-1$, M_i is glued to M_{i+1};

2. M_p is glued to M_1;
3. each of the gluings is along one of the sets B_i, $i = 1, \dots, n$.

We will refer to the gluing of M_j and M_{j+1} as the **j-th gluing**, and we will say that it is **of type** i if the gluing is along the body B_i, $i = 1, \dots, n$.

Let $S = M_{j(1)} \bigcup M_{j(2)} \bigcup \cdots \bigcup M_{j(k)}$, $1 \leq j(1) < j(2) < \cdots < j(k) \leq p$, be some subset of $\bar{M}$ formed by several of the sets $M_1, M_2, \dots, M_p$. We will say that the collection $M_{j(1)}, M_{j(2)}, \dots, M_{j(k)}$ is **disconnected** if there exists $1 \leq m \leq k$ such that all n possible types of gluings appear among the gluings with numbers between $j(m)$ and $j(m+1) - 1$ (in the circular order), where we assume that $j(k+1) = j(1)$. Then we immediately have the following

LEMMA 5.1. *If the collection $M_{j(1)}, M_{j(2)}, \dots, M_{j(k)}$ is disconnected then $S = M_{j(1)} \bigcup M_{j(2)} \bigcup \cdots \bigcup M_{j(k)}$ is a space of non-positive curvature.*

Proof. Without loss of generality we may assume that m, from the definition of disconnectedness, is equal to k. Then, exactly as in the construction of the universal unfolding space for billiards with two walls, we see that S is a result of gluing $M_{j(l)}$ to $M_{j(l+1)}$, for $l = 1, \dots, k - 1$, along the intersection of all bodies B_i, $i = 1, \dots, n$, that correspond to the gluings with numbers from $j(l)$ to $j(l+1) - 1$. (After we have performed these $k-1$ gluings, there is no need to glue together $M_{j(k)}$ and $M_{j(1)}$ since, as a result of the previous gluings the set $\bigcap_{i=1}^{n} B_i$ is already a subset of $M_{j(k)} \bigcap M_{j(1)}$.) Thus, applying Reshetnyak's theorem [Re] $k - 1$ times, we see that S is a space of non-positive curvature. □

Now we will prove Theorem 2 by induction on n.

Theorem 2 is trivially true for $n = 1$. Assume that it is proved for $n - 1$ bodies. Let us prove it for n bodies. We will show that

$$K(C, n) \leq (3P(C, n) + 1)(K(C, n - 1) + 2). \tag{1}$$

Let $T(x_0, x_1, \dots, x_L, x_{L+1})$ be a billiard trajectory with starting point x_0, end point x_{L+1}, and consecutive points of collisions $x_1, \dots, x_L$. Assume that

$$L > (3P(C, n) + 1)(K(C, n - 1) + 2).$$

Let us glue together L copies $M_1, \dots, M_L$ of M in a circular order, so that

1. for $j = 1, \dots, L - 1$, M_j is glued to M_{j+1} along that body B_i for which $x_j \in B_i$;
2. M_L is glued to M_1 along that body B_i for which $x_L \in B_i$.

Denote the result of the gluings by $\bar{M}(T)$.

We will show that $\bar{M}(T)$ has non-positive curvature. This will immediately lead to a contradiction, since there will be two different geodesics connecting $x_0 \in M_1$ and $x_{L+1} \in M_1$: the geodesic in $\bar{M}(T)$ corresponding to the trajectory T, and the geodesic in M_1 connecting x_0 and x_{L+1}.

Let $y \in \bar{M}(T)$. Then there exists a neighborhood U_y of y such that every geodesic triangle Δ in U_y is contained in a union S of at most $3P(C, n)$ copies of M. Since we assume Theorem 2 to be true for $n-1$ bodies, we see that among any $K(C, n-1)+1$ consecutive collisions of the trajectory T, there must be at least one collision with each body B_i, $i = 1, \dots, n$. Therefore, any collection of no more than $3P(C, n)$ copies of M in $\bar{M}(T)$ is disconnected. Thus, by Lemma 5.1, S has non-positive curvature. Hence, as in the construction of $\tilde{M}$ in Section 4 we conclude that Δ has non-positive defect, and $\bar{M}(T)$ has non-positive curvature at U_y.

Since, $\bar{M}(T)$ is a simply connected space with locally non-positive curvature, due to Alexandrov's theorem [Re], it has non-positive curvature globally as well.

To get explicit estimates for $K(C,n)$ we use equation (1) and the estimate $P(C,n) = (16(C+2))^{2(n-1)}$ from Theorem 1 to get

$$K(C,n) \leq 12P(C,n)K(C,n-1) \leq (12P(C,n))^n < (200(C+2))^{2n^2}.$$

Theorem 2 is proven.

6. Universal unfolding spaces for semi-dispersing billiards on surfaces of non-positive curvature. If M is a surface of non-positive curvature we may assume without a loss of generality that for any $i,j,k \in \{1,\dots,n\}$, $i \neq j$, $j \neq k$, $k \neq i$ the intersection $B_i \bigcap B_j \bigcap B_k$ is empty. (That is, intersections of more than two walls are empty.)

Let $\alpha = \min_{i \neq j}\ \alpha(B_i, B_j)$. Let $K = K(\alpha)$ be as in Section 4.

Let Γ be a finite group with n generators γ_i, $i = 1,\dots,n$, such that if a relation of the form $\gamma_{i_1}^{k_1}\dots\gamma_{i_l}^{k_l} = e$, $i_m \neq i_{m+1}$, $l \in \mathbb{N}, m = 1,\dots,(l-1)$ holds, then necessarily $|k_1| + \dots + |k_l| > K$. An explicit example of such a group can be found in [S].

Consider $|\Gamma|$ copies of M, and denote them as M_g, $g \in \Gamma$. Consider another $|\Gamma|$ copies of M, and denote them as M^g, $g \in \Gamma$.

Now, let us glue together these $2|\Gamma|$ copies of M by performing the following operations: if $g_1 = \gamma_i g_2$, then we glue together M_{g_1} and M^{g_2} along the body B_i. Denote by $\tilde{M}$ the result of all these gluings.

We claim that, locally, $\tilde{M}$ has non-positive curvature in the sense of Alexandrov.

Proof. Let $x \in \tilde{M}$. Then we have to consider the following three possibilities.

1. $y = \pi(x)$ does not belong to any of the bodies B_i, $i = 1,\dots,n$. Then, clearly, a small neighborhood of x in $\tilde{M}$ is isometric to a small neighborhood of y in M and, thus, $\tilde{M}$ has non-positive curvature at x.
2. $y = \pi(x)$ belongs to only one of the bodies B_i, $i = 1,\dots,n$. Assume it is B_1. Then a small neighborhood of x in $\tilde{M}$ is isometric to a small neighborhood of the image of y under the canonical maps of M into M^2, where M^2 is the result of gluing together two copies of M along the set B_1. Since by Reshetnyak's theorem, M^2 has non-positive curvature, we see that $\tilde{M}$ also has non-positive curvature at x.
3. $y = \pi(x)$ belongs to the intersection of two of the bodies B_i, $i = 1,\dots,n$. Let these be B_1 and B_2. Then x belongs to $2K'$ copies M_{g_p}, M^{g_q}, $p,q \in \{1,\dots K'\}$, of M in $\tilde{M}$, and it is clear that we can rearrange the indices p and q so that
   (a) $M_{g_{p_i}}$ is glued to $M^{g_{q_i}}$ along B_1, for all $i = 1,\dots,K'$;
   (b) $M^{g_{q_i}}$ is glued to $M_{g_{p_{i+1}}}$ along B_2, for all $i = 1,\dots,K'$ (as in Section 4 we assume that $M_{g_{p_{K'+1}}} = M_{g_{p_1}}$).

   The two conditions above imply that $g_{p_i} = \gamma_1 g_{q_i}$ and $g_{q_i} = \gamma_2^{-1} g_{p_{i+1}}$, for all $i = 1,\dots,K'$. Thus, we see that the word $(\gamma_1\gamma_2^{-1})^{K'}$ is a relation in Γ. Therefore, by the construction of $\tilde{M}$, $2K' \geq K(\alpha)$. Thus, using exactly the same argument as in Section 4 we see that $\tilde{M}$ has non-positive curvature at x.

The three cases considered above exhaust all the possibilities, since we assumed that all the intersections of more than two bodies B_i are empty. Thus, we proved that $\tilde{M}$ has non-positive curvature in the sense of Alexandrov. □

Concluding this section we would like to mention again that, for billiards on surfaces, there is an alternative construction of a universal unfolding space described at the beginning of this paper. At first glance it may seem that $\tilde{S}$ is a much more natural object than $\tilde{M}$, because the geodesic flow on $\tilde{S}$ is a finite cover of the billiard flow, and also the geodesic flow is defined on a *full measure* subset of the whole tangent bundle to $\tilde{S}$ (whereas for $\tilde{M}$ we have to consider a part of the flow "trapped in $\tilde{B}$"). Furthermore, the proof of the fact that $\tilde{S}$ has non-positive curvature is much easier than the proof that $\tilde{M}$ has non-positive curvature.

However, this simplified construction of $\tilde{S}$ has one serious flaw: it does not allow a generalization to any higher dimension. If we glue together two copies of a more-than-two-dimensional semi-dispersing billiard table by their isometric co-dimension one "walls", we may expect to get a space of non-positive curvature only if the "walls" are flat. Thus, while the construction of a higher-dimensional generalization of $\tilde{M}$ seems to be a difficult problem, the construction of $\tilde{S}$ cannot be generalized at all.

7. The problem about simplexes. Finally, we would like to state an open problem, closely related to the problem of constructing the universal unfolding spaces for semi-dispersing billiards, but which is formulated in purely geometric terms. We feel that, by itself, it is an intriguing problem.

Let S be some n dimensional simplex (or, more generally, a polyhedron) in $\mathbb{R}^n$, $n > 2$ (the problem is obviously easy for $n = 2$). Is it possible to glue a compact space of non-positive curvature without a boundary by using a finite number of isometric copies of S? (By gluing we mean that two copies may be glued together along their isometric faces.)

Clearly this problem is a particular case of the universal unfolding space problem, i.e., the solution to this problem may serve as a universal unfolding space for the billiard inside of S.

In a forthcoming paper [B-F-Kl-K] we plan to present a result which, in particular, will yield a positive answer to this question in dimension three. Contrary to what could be anticipated, the construction is not at all elementary. In fact, it is essentially based on Thurston's Theory of hyperbolic structures on $3-$dimensional manifolds.

A closely related question is: When is it possible to glue a *manifold* of non-positive curvature and without boundary by using a finite number of isometric copies of S? In [B-F-Kl-K] we plan to give the necessary and sufficient conditions in dimension three.

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