ASIAN J. MATH.
Vol. 2, No. 2, pp. 345–354, June 1998

006

A LINK BETWEEN TWO ELLIPTIC QUANTUM GROUPS*

PAVEL ETINGOF† AND OLIVIER SCHIFFMANN‡

Abstract. We consider the category $\mathcal{C}_B$ of meromorphic finite-dimensional representations of the quantum elliptic algebra $\mathcal{B}$ constructed via Belavin's R-matrix, and the category $\mathcal{C}_F$ of meromorphic finite-dimensional representations of Felder's elliptic quantum group $\mathcal{E}_{\tau,\frac{\gamma}{2}}(\mathfrak{gl}_n)$. For any fixed $c \in \mathbb{C}$, we use a version of the Vertex-IRF correspondence to construct two families of (generically) fully faithful functors $\mathcal{H}_x^c : \mathcal{C}_B \to \mathcal{D}_B$ and $\mathcal{F}_x^c : \mathcal{C}_F \to \mathcal{D}_B$ where $\mathcal{D}_B$ is a certain category of infinite-dimensional representations of $\mathcal{B}$ by difference operators. We use this to construct an equivalence between the abelian subcategory of $\mathcal{C}_B$ generated by tensor products of vector representations and the abelian subcategory of $\mathcal{C}_F$ generated by tensor products of vector representations.

1. Categories of meromorphic representations. In this section, we recall the definitions of various categories of representations of quantum elliptic algebras.

Notations: Let us fix $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$, $\gamma \in \mathbb{R}\backslash\mathbb{Q}$ and $n \geq 2$. Denote by $(v_i)_{i=1}^n$ the canonical basis of $\mathbb{C}^n$ and by $(E_{ij})_{i,j=1}^n$ the canonical basis of $\mathrm{End}(\mathbb{C}^n)$, i.e $E_{ij}v_k = \delta_{jk}v_i$. Let $\mathfrak{h} = \{\sum_i \lambda_i E_{ii} \mid \sum_i \lambda_i = 0\}$ be the space of diagonal traceless matrices. We have a natural identification $\mathfrak{h}^* = \{\sum_i \lambda_i E_{ii}^* \mid \sum_i \lambda_i = 0\}$. In particular, the weight of v_i is $\omega_i = E_{ii}^* - \frac{1}{n}\sum_k E_{kk}^*$.

Classical theta functions: The theta function $\theta_{\kappa,\kappa'}(t;\tau)$ with characteristics $\kappa, \kappa' \in \mathbb{R}$ is defined by the formula

$$\theta_{\kappa,\kappa'}(t;\tau) = \sum_{m\in\mathbb{Z}} e^{i\pi(m+\kappa)((m+\kappa)\tau+2(t+\kappa'))}.$$

It is an entire function whose zeros are simple and form the (shifted) lattice $\{\frac{1}{2} - \kappa + (\frac{1}{2} - \kappa')\tau\} + \mathbb{Z} + \tau\mathbb{Z}$.

Theta functions satisfy (and are characterized up to renormalization by) the following fundamental monodromy relations

$$\theta_{\kappa,\kappa'}(t+1;\tau) = e^{2i\pi\kappa}\theta_{\kappa,\kappa'}(t;\tau), \tag{1}$$

$$\theta_{\kappa,\kappa'}(t+\tau;\tau) = e^{-i\pi\tau-2i\pi(t+\kappa')}\theta_{\kappa,\kappa'}(t;\tau). \tag{2}$$

Theta functions with different characteristics are related to each other by shifts of t:

$$\theta_{\kappa_1+\kappa_2,\kappa_1'+\kappa_2'}(t;\tau) = e^{i\pi\kappa_2^2\tau+2i\pi\kappa_2(t+\kappa_1'+\kappa_2')}\theta_{\kappa_1,\kappa_1'}(t+\kappa_2\tau+\kappa_2';\tau). \tag{3}$$

In particular, we set $\theta(t) = \theta_{\frac{1}{2},\frac{1}{2}}(t;\tau)$.

1.1. Meromorphic representations of the Belavin quantum elliptic algebra. Consider the two $n \times n$ matrices

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \xi & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \xi^{n-1} \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

---

*Received March 20, 1998; accepted for publication June 16, 1998. The authors were supported by the NSF grant DMS-9700477. The second author would like to thank Harvard University Mathematics Department for its hospitality without which this work would not have been possible.

†Department of Mathematics, Harvard University, Cambridge MA 02138 USA (etingof@math.harvard.edu).

‡Department of Mathematics, ENS Paris, 45 rue d' Ulm, 75005 Paris, FRANCE (schiffma@sequoia.ens.fr).

where $\xi = e^{2i\pi/n}$. We have $A^n = B^n = Id$, $BA = \xi AB$, i.e A, B generate the Heisenberg group. Belavin ([3]) introduced the matrix $R^B(z) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n)$, uniquely determined by the following properties:

1. Unitarity: $R^B(z)R^B_{21}(-z) = 1$,
2. $R^B(z)$ is meromorphic, with simple poles at $z = \gamma + \mathbb{Z} + \tau\mathbb{Z}$,
3. $R^B(0) = P : x \otimes y \mapsto y \otimes x$ for $x, y \in \mathbb{C}^n$ (permutation),
4. Lattice translation properties:

$$\begin{aligned} R^B(z+1) &= A_1 R^B(z) A_1^{-1} = A_2^{-1} R^B(z) A_2, \\ R^B(z+\tau) &= e^{-2i\pi\frac{n-1}{n}\gamma} B_1 R^B(z) B_1^{-1} = e^{-2i\pi\frac{n-1}{n}\gamma} B_2^{-1} R^B(z) B_2. \end{aligned}$$

In particular, $R^B(z)$ commutes with $A \otimes A$ and $B \otimes B$. The matrix $R^B(z)$ satisfies the quantum Yang-Baxter equation with spectral parameters:

$$R^B_{12}(z-w)R^B_{13}(z)R^B_{23}(w) = R^B_{23}(w)R^B_{13}(z)R^B_{12}(z-w).$$

The category $\mathcal{C}_B$: Following Faddeev, Reshetikhin, Takhtajan and Semenov-Tian-Shansky, one can define an algebra $\mathcal{B}$ from $R^B(z)$, using the RLL formalism-see [4], [12]. However, we will only need to consider a certain category of modules over this algebra, defined as follows.

Let $\mathcal{C}_B$ be the category whose objects are pairs $(V, L(z))$ where V is a finite dimensional vector space and $L(z) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(V)$ is an invertible meromorphic function (the L-operator) such that $L(z+n) = L(z)$ and $L(z+n\tau) = L(z)$, satisfying the following relation in the space $\mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(V) \otimes \mathrm{End}(V)$:

$$R^B_{12}(z-w)L_{13}(z)L_{23}(w) = L_{23}(w)L_{13}(z)R^B_{12}(z-w) \tag{4}$$

(as meromorphic functions of z and w); morphisms $(V, L(z)) \to (V', L'(z))$ are linear maps $\varphi : V \to V'$ such that $(1 \otimes \varphi)L(z) = L'(z)(1 \otimes \varphi)$ in the space $\mathrm{Hom}(\mathbb{C}^n \otimes V, \mathbb{C}^n \otimes V')$. The quantum Yang-Baxter relation for R^B implies that $(\mathbb{C}^n, \chi(z)R^B(z-w)) \in \mathcal{O}b(\mathcal{C}_B)$ for all $w \in \mathbb{C}$, where we set $\chi(z) = \frac{\theta(z-(1-\frac{1}{n})\gamma)}{\theta(z)}$. This object is called the vector representation and will be denoted simply by $V_B(w)$.

The category $\mathcal{C}_B$ is naturally a tensor category with tensor product

$$(V, L(z)) \otimes (V', L'(z)) = (V \otimes V', L_{12}(z)L'_{13}(z)) \tag{5}$$

at the level of objects and with the usual tensor product at the level of morphisms.

There is a notion of a dual representation in the category $\mathcal{C}_B$: the (right) dual of $(V, L(z))$ is $(V^*, L^*(z))$ where $L^*(z) = L^{-1}(z)^{t_2}$ (first apply inversion, then apply the transposition in the second component t_2). If $V, W \in \mathcal{O}b(\mathcal{C}_B)$ and $\varphi \in \mathrm{Hom}_{\mathcal{C}_B}(V, W)$ then $\varphi^t \in \mathrm{Hom}_{\mathcal{C}_B}(W^*, V^*)$, and the functor $\mathcal{C}_B \to \mathcal{C}_B$, $V \mapsto V^*$ is a contravariant equivalence of categories. Moreover, for $V, W, Z \in \mathcal{O}b(\mathcal{C}_B)$, we have canonical isomorphisms $(V \otimes W)^* \simeq (W^* \otimes V^*)$ and $\mathrm{Hom}_{\mathcal{C}_B}(V \otimes W, Z) \simeq \mathrm{Hom}_{\mathcal{C}_B}(V, Z \otimes W^*)$.

We will also need an extended category $\mathcal{C}^x_B$ defined as follows: objects of $\mathcal{C}^x_B$ are objects of $\mathcal{C}_B$ but we set

$$\mathrm{Hom}_{\mathcal{C}^x_B}(V, V') = \mathrm{Hom}_{\mathcal{C}_B}(V, V') \otimes M_{\mathbb{C}}$$

where $M_{\mathbb{C}}$ is the field of meromorphic functions of a complex variable x. In other words, morphisms in $\mathcal{C}^x_B$ are meromorphic 1-parameter families of morphisms in $\mathcal{C}_B$.

The category $\mathcal{D}_B$: We now define a difference-operator variant of the categories $\mathcal{C}_B, \mathcal{C}_B^x$. Let us denote by $M_{\mathfrak{h}^*}$ the field of $(n\omega_i)$-periodic meromorphic functions $\mathfrak{h}^* \to \mathbb{C}$ and by $D_{\mathfrak{h}^*}$ the $\mathbb{C}$-algebra generated by $M_{\mathfrak{h}^*}$ and shift operators $T_\mu : M_{\mathfrak{h}^*} \to M_{\mathfrak{h}^*}, f(\lambda) \mapsto f(\lambda+\mu)$ for $\mu \in \mathfrak{h}^*$. If V is a finite-dimensional vector space, we set $V_{\mathfrak{h}^*} = M_{\mathfrak{h}^*} \otimes_{\mathbb{C}} V$, and $D(V) = D_{\mathfrak{h}^*} \otimes_{\mathbb{C}} \mathrm{End}(V)$. Let $\mathcal{D}_B$ be the category whose objects are pairs $(V, L(z))$ where V is a finite-dimensional $\mathbb{C}$-vector space and $L(z) \in \mathrm{End}(\mathbb{C}^n) \otimes D(V)$ is an invertible operator with meromorphic coefficients satisfying (4) in $\mathrm{End}(\mathbb{C}^n) \otimes D(V) \otimes D(V)$; morphisms $(V, L(z)) \to (V', L'(z))$ are $(n\omega_i)$-periodic meromorphic functions $\varphi : \mathfrak{h}^* \to \mathrm{Hom}(V, V')$ such that $(1 \otimes \varphi)L(z) = L(z)(1 \otimes \varphi)$ in $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n \otimes V_{\mathfrak{h}^*}, \mathbb{C}^n \otimes V'_{\mathfrak{h}^*})$ (i.e morphisms are $M_{\mathfrak{h}^*}$-linear).

The category $\mathcal{D}_B$ is a right-module category over $\mathcal{C}_B$, i.e we have a (bi)functor $\otimes : \mathcal{D}_B \times \mathcal{C}_B \to \mathcal{D}_B$ defined by (5), and for any $V, W \in \mathcal{O}b(\mathcal{D}_B)$, $Z \in \mathcal{O}b(\mathcal{C}_B)$, we have a canonical isomorphism $\mathrm{Hom}_{\mathcal{D}_B}(V \otimes Z, W) \simeq \mathrm{Hom}_{\mathcal{D}_B}(V, W \otimes Z^*)$. The category $\mathcal{D}_B^x$ is defined in an analogous way: objects are pairs $(V, L(z,x))$ as in $\mathcal{D}_B$ but the L-operator is now a meromorphic function of z and x, and morphisms $(V, L(z,x)) \to (V', L'(z,x))$ are meromorphic maps $\varphi(\lambda, x) : \mathfrak{h}^* \times \mathbb{C} \to \mathrm{Hom}_{\mathbb{C}}(V, V')$ satisfying $(1 \otimes \varphi)L(z,x) = L(z,x)(1 \otimes \varphi)$.

1.2. Meromorphic representations of the elliptic quantum group $\mathcal{E}_{\tau,\gamma/2}(\mathfrak{gl}_n)$.

Felder's dynamical R-matrix: let us consider the functions of two complex variables

$$\alpha(z,l) = \frac{\theta(l+\gamma)\theta(z)}{\theta(l)\theta(z-\gamma)}, \qquad \beta(z,l) = \frac{\theta(z-l)\theta(\gamma)}{\theta(l)\theta(z-\gamma)}.$$

As functions of z, α and β have simple poles at $z = \gamma + \mathbb{Z} + \tau\mathbb{Z}$ and satisfy

$$\begin{aligned} \alpha(z+1,l) &= \alpha(z,l), \qquad & \alpha(z+\tau,l) &= e^{-2i\pi\gamma}\alpha(z,l), \\ \beta(z+1,l) &= \beta(z,l), \qquad & \beta(z+\tau,l) &= e^{-2i\pi(\gamma-l)}\beta(z,l). \end{aligned}$$

Felder introduced in [5] the matrix $R^F(z,\lambda) : \mathbb{C} \times \mathfrak{h}^* \to \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n)$:

$$R^F(z,\lambda) = \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} \alpha(z, \lambda_i - \lambda_j) E_{ii} \otimes E_{jj} + \sum_{i \neq j} \beta(z, \lambda_i - \lambda_j) E_{ji} \otimes E_{ij}$$

where $\lambda = \sum_i \lambda_i E_{ii}^* \in \mathfrak{h}^*$.

This matrix is a solution of the quantum dynamical Yang-Baxter equation with spectral parameters

$$\begin{aligned} &R_{12}^F(z-w, \lambda - \gamma h_3) R_{13}^F(z,\lambda) R_{23}^F(w, \lambda - \gamma h_1) \\ &\qquad = R_{23}^F(w,\lambda) R_{13}^F(z, \lambda - \gamma h_2) R_{12}^F(z-w, \lambda) \end{aligned}$$

where we have used the following convention: if V_i are diagonalizable $\mathfrak{h}$-modules with weight decomposition $V_i = \bigoplus_\mu V_i^\mu$ and $a(\lambda) \in \mathrm{End}(\bigotimes_i V_i)$ then

$$a(\lambda - \gamma h_l)_{|\bigotimes_i V_i^{\mu_i}} = a(\lambda - \gamma \mu_l).$$

As usual, indices indicate the components of the tensor product on which the operators act.

In addition, $R^F(z,\lambda)$ satisfies the following two conditions:

1. Unitarity: $R_{12}^F(z,\lambda) R_{21}^F(-z,\lambda) = Id$,

2. Weight zero: $\forall h \in \mathfrak{h}$, $[h_1 + h_2, R^F(z,\lambda)] = 0$.

The category $\mathcal{C}_F$: It is possible to use $R^F(z,\lambda)$ to define an algebra by the RLL-formalism (see [5]): the elliptic quantum group $\mathcal{E}_{\tau,\gamma/2}(\mathfrak{gl}_n(\mathbb{C}))$. However, we will only need the following category of its representations $\mathcal{C}_F$, introduced by Felder in [5] and studied by Felder and Varchenko in [6]: objects are pairs $(V, L(z,\lambda))$ where V is a finite-dimensional diagonalizable $\mathfrak{h}$-module and $L(z,\lambda) : \mathbb{C}\times\mathfrak{h}^* \to \mathrm{End}(\mathbb{C}^n)\otimes\mathrm{End}(V)$ is an invertible meromorphic function which is $(n\omega_i)$-periodic in λ and which satisfies the following two conditions:

$$[h_1 + h_2, L(z,\lambda)] = 0,$$

$$\begin{aligned} &R^F_{12}(z-w,\lambda-\gamma h_3)L_{13}(z,\lambda)L_{23}(w,\lambda-\gamma h_1)\\ &\qquad = L_{23}(w,\lambda)L_{13}(z,\lambda-\gamma h_2)R^F_{12}(z-w,\lambda). \end{aligned} \tag{6}$$

Morphisms $(V, L(z,\lambda)) \to (V', L'(z,\lambda))$ are $(n\omega_i)$-periodic meromorphic weight zero maps $\varphi(\lambda) : V \to V'$ such that $L'(z,\lambda)(1\otimes\varphi(\lambda-\gamma h_1)) = (1\otimes\varphi(\lambda))L(z,\lambda)$. The quantum dynamical Yang-Baxter relation for $R^F(z,\lambda)$ implies that we have $(\mathbb{C}^n, R^F(z-w,\lambda)) \in \mathcal{O}b(\mathcal{C}_F)$ for all $w\in\mathbb{C}$. This is the vector representation and it will be denoted by $V_F(w)$.

The category $\mathcal{C}_F$ is naturally equipped with a tensor structure: it is defined on objects by

$$(V, L(z,\lambda)) \otimes (V', L'(z,\lambda)) = (V\otimes V', L_{12}(z,\lambda-\gamma h_3)L'_{13}(z,\lambda)),$$

and if $\varphi\in\mathrm{Hom}_{\mathcal{C}_F}(V,W)$, $\varphi'\in\mathrm{Hom}_{\mathcal{C}_F}(V',W')$ then

$$(\varphi\otimes\varphi')(\lambda) = \varphi(\lambda-\gamma h_2)\otimes\varphi'(\lambda)\in\mathrm{Hom}_{\mathcal{C}_F}(V\otimes V', W\otimes W').$$

There is a notion of a dual representation in the category $\mathcal{C}_F$: the (right) dual of $(V, L(z,\lambda))$ is $(V^*, L^*(z,\lambda))$ where $L^*(z,\lambda) = L^{-1}(z,\lambda+\gamma h_2)^{t_2}$ (apply inversion, shifting and then apply the transposition in the second component t_2). If $V, W \in \mathcal{O}b(\mathcal{C}_F)$ and $\varphi(\lambda)\in\mathrm{Hom}_{\mathcal{C}_F}(V,W)$ then $\varphi^*(\lambda) := \varphi(\lambda+\gamma h_1)^t \in \mathrm{Hom}_{\mathcal{C}_F}(W^*,V^*)$, and the functor $\mathcal{C}_F\to\mathcal{C}_F$, $V\mapsto V^*$ is a contravariant equivalence of categories. Moreover, for any $V,W\in\mathcal{O}b(\mathcal{C}_F)$, there is a canonical isomorphism $(V\otimes W)^*\simeq(W^*\otimes V^*)$.

The extended category $\mathcal{C}_F^x$ is defined by $\mathcal{O}b(\mathcal{C}_F^x) = \mathcal{O}b(\mathcal{C}_F)$ and

$$\mathrm{Hom}_{\mathcal{C}_F^x}(V,V') = \mathrm{Hom}_{\mathcal{C}_F}(V,V')\otimes M_{\mathbb{C}}$$

i.e morphisms in $\mathcal{C}_F^x$ are meromorphic 1-parameter families of morphisms in $\mathcal{C}_F$.

2. The functor $\mathcal{F}_x^c : \mathcal{C}_F \to \mathcal{D}_B$. In this section, we define a family of functors from meromorphic (finite-dimensional) representations of $\mathcal{E}_{\tau,\frac{\gamma}{2}}(\mathfrak{gl}_n(\mathbb{C}))$ to infinite-dimensional representations of the quantum elliptic algebra $\mathcal{B}$.

2.1. Twists by difference operators. For any finite-dimensional diagonalizable $\mathfrak{h}$-module V, let $e^{\gamma D}\in\mathrm{End}(V_{\mathfrak{h}^*})$ denote the shift operator: $e^{\gamma D}.\sum_\mu f_\mu(\lambda)v_\mu = \sum_\mu f(\lambda+\gamma\mu)v_\mu$, $v_\mu\in V_\mu$. Now let $(V, L(z,\lambda))\in\mathcal{C}_F$, and let $S(z,\lambda), S'(z,\lambda) : \mathbb{C}\times\mathfrak{h}^*\to\mathrm{End}(\mathbb{C}^n)$ be meromorphic and nondegenerate. Define the difference-twist of $(V, L(z,\lambda))$ to be the pair $(V, L^{S,S'}(z))$ where

$$L^{S,S'}(z) = S_1(z,\lambda-\gamma h_2)L(z,\lambda)e^{-\gamma D_1}S'_1(z,\lambda)^{-1}\in\mathrm{End}(\mathbb{C}^n)\otimes D(V). \tag{7}$$

This is a difference operator acting on $\mathbb{C}^n \otimes V_{\mathfrak{h}^*}$.

LEMMA 1. *The difference operator* $L^S(z,\lambda)$ *satisfies the following relation in* $\mathrm{End}(\mathbb{C}^n) \otimes D(V) \otimes D(V)$*:*

$$T_{12}(z,w,\lambda-\gamma h_3)L_{13}^{S,S'}(z)L_{23}^{S,S'}(w) = L_{23}^{S,S'}(w)L_{13}^{S,S'}(z)T'_{12}(z,w,\lambda)$$

where

$$T(z,w,\lambda) = S_2(w,\lambda)S_1(z,\lambda-\gamma h_2)R_{12}^F(z-w,\lambda)S_2(w,\lambda-\gamma h_1)^{-1}S_1(z,\lambda)^{-1}, \quad (8)$$

$$T'(z,w,\lambda) = S'_1(z,\lambda)S'_2(w,\lambda+\gamma h_1)R_{12}^F(z-w,\lambda)S'_1(z,\lambda+\gamma h_1)^{-1}S'_2(w,\lambda)^{-1}. \quad (9)$$

Proof. The proof is straightforward, using relation (6) for $L(z,\lambda)$ and the weight zero property of $R^F(u,\lambda)$ and $L(u,\lambda)$. □

2.2. The Vertex-IRF transform. Let $\phi_l(u) = e^{2i\pi(\frac{l^2\tau}{n}+\frac{lu}{n})}\theta_{0,0}(u+l\tau;n\tau)$ for $l = 1,\dots n$. Then the vector $\Phi(u) = (\phi_1(u),\dots,\phi_n(u))$ is, up to renormalization, the unique holomorphic vector in $\mathbb{C}^n$ satisfying the following monodromy relations:

$$\Phi(u+1) = A\Phi(u), \quad (10)$$

$$\Phi(u+\tau) = e^{-i\pi\frac{\tau}{n}-2i\pi\frac{u}{n}}B\Phi(u). \quad (11)$$

Now let $S(z,\lambda) : \mathbb{C} \times \mathfrak{h}^* \to \mathrm{End}(\mathbb{C}^n)$ be the matrix whose columns are $(\Phi_1(z,\lambda),\dots,\Phi_n(z,\lambda))$ where $\Phi_j(z,\lambda) = \Phi(z-n\lambda_j)$. Using (10)-(11), it is easy to see that we have $\det(S(z,\lambda)) = \mathrm{Const}(\lambda)\theta(z)$ where $\mathrm{Const}(\lambda) \neq 0$ and hence that $S(z,\lambda)$ is invertible for $z \neq 0$ and generic λ.

LEMMA 2. *We have*

$$R^B(z-w)S_1(z,\lambda)S_2(w,\lambda-\gamma h_1) = S_2(w,\lambda)S_1(z,\lambda-\gamma h_2)R^F(z-w,\lambda),$$

$$R^B(z-w)S_2(w,\lambda)S_1(z,\lambda+\gamma h_2) = S_1(z,\lambda)S_2(w,\lambda+\gamma h_1)R^F(z-w,\lambda).$$

Proof. The first relation is equivalent to the following identities for $i,j = 1,\dots n$:

$$R^B(z-w)\Phi_i(z,\lambda)\otimes\Phi_i(w,\lambda-\gamma\omega_i) = \Phi_i(z,\lambda-\gamma\omega_i)\otimes\Phi_i(w,\lambda),$$

$$\begin{aligned}R^B(z-w)\Phi_i(z,\lambda)\otimes\Phi_j(w,\lambda-\gamma\omega_i) &= \alpha(z-w,\lambda_i-\lambda_j)\Phi_i(z,\lambda-\gamma\omega_j)\otimes\Phi_j(w,\lambda)\\ &\quad+\beta(z-w,\lambda_i-\lambda_j)\Phi_j(z,\lambda-\gamma\omega_i)\otimes\Phi_i(w,\lambda).\end{aligned}$$

These identities are proved by comparing poles and transformation properties under lattice translations as functions of z and w, and using the uniqueness of Φ. The second relation of the lemma is proved in the same way. These identities are essentially the Vertex/Interaction-Round-a-Face transform of statistical mechanics (see [9],[11] and [7] for the case $n = 2$). □

The Vertex-IRF transform first appeared in the work of Baxter [1] and was subsequently generalized to the Belavin R-matrix by Jimbo, Miwa and Okado in [10].

2.3. Construction of the functor $\mathcal{F}_x^c : \mathcal{C}_F \to \mathcal{D}_B$. Let us fix some $c \in \mathbb{C}$. We now define the family of functors $\mathcal{F}_x^c : \mathcal{C}_F \to \mathcal{C}_B$ indexed by $x \in \mathbb{C}$: for $(V, L(z,\lambda)) \in \mathcal{C}_F$, set $\mathcal{F}_x^c((V, L(z,\lambda))) = (V, L^{S_x,S_{x+c}}(z))$ with $S_u(z,\lambda) = S(z-u,\lambda)$ as above and let $\mathcal{F}_x^c$ be trivial at the level of morphisms.

PROPOSITION 1. $\mathcal{F}_x^c : \mathcal{C}_F \to \mathcal{D}_B$ *is a functor.*

Proof. It follows from Lemma 2 that $(V, L^{S_x, S_{x+c}}(z)) \in \mathcal{O}b(\mathcal{D}_B)$. Furthermore, if $\varphi(\lambda) \in \mathrm{Hom}_{\mathcal{C}_F}((V, L(z,\lambda)), (V', L'(z,\lambda)))$ then by definition we have $L'(z,\lambda)(1 \otimes \varphi(\lambda - \gamma h_1)) = (1 \otimes \varphi(\lambda))L(z,\lambda)$, so that

$$\begin{aligned} S_1(z-x,&\lambda-\gamma h_2)L'(z,\lambda)e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1}(1\otimes\varphi(\lambda)) \\ &= S_1(z-x,\lambda-\gamma h_2)L'(z,\lambda)(1\otimes\varphi(\lambda-\gamma h_1))e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1} \\ &= S_1(z-x,\lambda-\gamma h_2)(1\otimes\varphi(\lambda))L'(z,\lambda)e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1} \\ &= (1\otimes\varphi(\lambda))S_1(z-x,\lambda-\gamma h_2)L'(z,\lambda)e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1} \end{aligned}$$

since $\varphi(\lambda)$ is of weight zero. Thus $\mathcal{F}_x^c(\varphi(\lambda))$ is an intertwiner in the category $\mathcal{D}_B$. □

We can also think of the family of functors $\mathcal{F}_x^c$ as a single functor $\mathcal{F}^c : \mathcal{C}_F^x \to \mathcal{D}_B^x$.

Remark. We can think of the difference-twist and the relations in Lemma 2 as a dynamical analogue of the notion of equivalence of R-matrices due to Drinfeld and Belavin-see [2].

3. The image of the trivial representation and the functor $\mathcal{H}_x^c : \mathcal{C}_B \to \mathcal{D}_B$. Applying the functor $\mathcal{F}_x^c$ to the trivial representation $(\mathbb{C}, \mathrm{Id}) \in \mathcal{O}b(\mathcal{C}_F)$ yields

$$\mathcal{F}_x^c((\mathbb{C},\mathrm{Id})) = (\mathbb{C}, S(z-x,\lambda)e^{-\gamma D_1}S(z-x-c,\lambda)^{-1}).$$

We will denote this object by I_x^c. For instance, when $n = 2$, we obtain a representation of the Belavin quantum elliptic algebra as difference operators acting on the space of periodic meromorphic functions in one variable λ, i.e given by an L-operator

$$L(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

where $a(z), b(z), c(z), d(z)$ are operators of the form $f(z)T_{-\gamma} + g(z)$ where $T_{-\gamma}$ is the shift by $-\gamma$.

Such representations of $\mathcal{B}$ by difference operators already appeared in the work of Krichever, Zabrodin ([11]) (for $n = 2$) and Hasegawa ([8],[9])(for the general case), where they were also derived by some Vertex-IRF correspondence.

Definition. Let $c \in \mathbb{C}$ and let $\mathcal{H}_x^c : \mathcal{C}_B \to \mathcal{D}_B$ be the functor defined by the assignment $V \to I_x^c \otimes V$ and which is trivial at the level of morphisms. The family of functors $\mathcal{H}_x^c$ gives rise to a functor $\mathcal{H}^c : \mathcal{C}_B^x \to \mathcal{D}_B^x$.

4. Full faithfulness of the functor $\mathcal{H}_x^c : \mathcal{C}_B \to \mathcal{D}_B$. In this section, we prove the following result

Proposition 2. *Let $V, V' \in \mathcal{O}b(\mathcal{C}_B)$. Then for all but finitely many values of $x \bmod \mathbb{Z} + \mathbb{Z}\tau$, the map*

$$\mathcal{H}_x^c : \mathrm{Hom}_{\mathcal{C}_B}(V, V') \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}_B}(\mathcal{H}_x^c(V), \mathcal{H}_x^c(V'))$$

is an isomorphism.

Proof. Since $\mathrm{Hom}_{\mathcal{C}_B}(V, V') \simeq \mathrm{Hom}_{\mathcal{C}_B}(\mathbb{C}, V' \otimes V^*)$, $\mathrm{Hom}_{\mathcal{D}_B}(I_x^c \otimes V, I_x^c \otimes V') \simeq \mathrm{Hom}_{\mathcal{D}_B}(I_x^c, I_x^c \otimes V' \otimes V^*)$, it is enough to show that the map $\mathcal{H}_x^c : \mathrm{Hom}_{\mathcal{C}_B}(\mathbb{C}, W) \to \mathrm{Hom}_{\mathcal{D}_B}(I_x^c, I_x^c \otimes W)$ is an isomorphism for all $W \in \mathrm{O}b(\mathcal{C}_B)$. Since $\mathcal{H}_x^c$ is trivial at the level of morphisms, this map is injective. Now let $W \in \mathcal{O}b(\mathcal{C}_B)$ and let $\varphi(\lambda) \in \mathrm{Hom}_{\mathcal{D}_B}(I_x^c, I_x^c \otimes W)$, that is, $\varphi(\lambda)$ is a $(n\omega_i)$-periodic meromorphic function $\mathfrak{h}^* \to W$ satisfying the equation

$$\begin{aligned} \varphi_2(\lambda)&S_1(z-x,\lambda)e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1} \\ &= S_1(z-x,\lambda)e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1}L_{12}(z)\varphi_2(\lambda) \end{aligned}$$

where $L(z)$ is the L-operator of W. This is equivalent to

$$L_{12}(z)\varphi_2(\lambda) = S_1(z-x-c,\lambda)\varphi_2(\lambda+\gamma h_1)S_1(z-x-c,\lambda)^{-1} \tag{12}$$

Now $L(z)$ is an elliptic function (of periods n and $n\tau$) so it is either constant or it has a pole. Restricting W to the subrepresentation $\mathrm{Span}(\varphi(\lambda), \lambda \in \mathfrak{h}^*)$, we see that the latter case is impossible for generic x as the RHS of (12) has a pole at $z = x + c$ only; hence $L(z)$ is constant. Furthermore, from (12) we see that the matrix

$$M(\lambda) = S_1(z-x-c,\lambda)^{-1}L_{12}S_1(z-x-c,\lambda)$$

is independent of z. In particular, setting $z \mapsto z+1$ and using the transformation properties (10) of $S(z,\lambda)$, we obtain $[A_1, L_{12}] = 0$. This implies that $L = \sum_i E_{ii} \otimes D_i$ for some $D_i \in \mathrm{End}(W)$.

LEMMA 3. *Let U be a finite dimensional vector space, let $T \in \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(U)$ be an invertible solution of the equation*

$$R^B_{12}(z)T_{13}T_{23} = T_{23}T_{13}R^B_{12}(z) \tag{13}$$

such that $T = \sum_i E_{ii} \otimes D_i$ for some $D_i \in \mathrm{End}(U)$. Then $[D_i, D_j] = 0$ for all i, j and there exists $X \in \mathrm{End}(U)$ such that $X^n = 1$ and $D_{i+1} = XD_i$ for all $i = 1, \dots n$.

Proof. Let us write $R^B(z) = \sum_{p,q,r,s} R_{p,q,r,s}(z)E_{pq} \otimes E_{rs}$. Then equation (13) is equivalent to $R_{p,q,r,s}(z)D_p \times D_q = R_{p,q,r,s}(z)D_sD_r$ for all p, q, r, s. But it follows from the general formula for $R^B(z)$ that $R_{p,q,r,s}(z) \neq 0$ if and only if $p+q \equiv r+s \pmod{n}$. Thus we have $[D_i, D_j] = 0$ for all i, j and $X := D_iD_{i+1}^{-1}$ is independent of i, and satisfies $X^n = 1$. □

By the above lemma, there exists $X \in \mathrm{End}(W)$ such that $X^n = 1$ and $D_{i+1} = XD_i$. Suppose that $X \neq 1$ and choose $e \in W$ such that $X(e) = \xi^k e$ with $\xi^k \neq 1$. Now we apply the transformation $z \mapsto z + \tau$ to the matrix $M(\lambda)$. Noting that, by (11), $S(z-x-c+\tau,\lambda) = e^{-i\pi\tau/2-2i\pi(z-x-c)/n}BS(z-x-c,\lambda)F(\lambda)$ where $F(\lambda) = \mathrm{diag}(e^{-2i\pi\lambda}, \dots e^{-2i\pi\lambda_n})$, we obtain the equality

$$\begin{aligned}F(\lambda)^{-1}S_1(z-x-c,\lambda)^{-1}B_1^{-1}L_{12}B_1S_1(z-x-c,\lambda)F(\lambda)\\ = S_1(z-x-c,\lambda)^{-1}L_{12}S_1(z-x-c,\lambda).\end{aligned}$$

Applying this to the vector e yields $\mathrm{Ad}F(\lambda)(M(\lambda))(e) = \xi^{-k}M(\lambda)(e)$. This is possible for all λ only if $k \equiv 0 \pmod{n}$. Hence $X = 1$ and (12) reduces to the equation $D\varphi_2(\lambda) = \varphi_2(\lambda+\gamma h_1)$. In particular $\varphi(\lambda)$ is $\gamma(\omega_i-\omega_j)$-periodic. But by our assumption, $\varphi(\lambda)$ is $(n\omega_i)$-periodic and γ is real and irrational. Therefore $\varphi(\lambda)$ is constant and it is a morphism in the category $\mathcal{C}_B$. □

Now, considering x as a parameter, we obtain:

COROLLARY 1. *The functor $\mathcal{H}^c : \mathcal{C}^x_B \to \mathcal{D}^x_B$ is fully faithful.*

REMARK. Equation (12) shows that $\mathrm{Hom}_{\mathcal{D}_B}(I^c_x, I^c_x \otimes V) = \mathrm{Hom}_{\mathcal{D}_B}(V^*, I^0_{x+c})$. Thus the above proposition states that for any finite-dimensional representation $V \in \mathcal{O}b(\mathcal{C}_F)$ and for all but finitely many x mod $\mathbb{Z} + \tau\mathbb{Z}$, we have $\mathrm{Hom}_{\mathcal{D}_B}(V^*, I^0_x) = \mathrm{Hom}_{\mathcal{C}_B}(V^*, \mathbb{C})$, where the isomorphism is induced by the embedding $\mathbb{C} \subset I^0_x$ (constant functions). However, for finitely many values of x mod $\mathbb{Z} + \tau\mathbb{Z}$, this may not be true: see [11] and [9] where some finite-dimensional subrepresentations of I^0_x are considered.

5. Full faithfullness of the functor $\mathcal{F}_x^c : \mathcal{C}_F \to \mathcal{D}_B$. In this section, we prove the following result:

PROPOSITION 3. *The functor $\mathcal{F}_x^c : \mathcal{C}_F \to \mathcal{D}_B$ is fully faithful.*

Proof. We have to show that for any two objects V, V' in $\mathcal{C}_F$ there is an isomorphism $\mathcal{F}_x^c : \mathrm{Hom}_{\mathcal{C}_F}(V, V') \to \mathrm{Hom}_{\mathcal{D}_B}(\mathcal{F}_x^c(V), \mathcal{F}_x^c(V'))$. Since $\mathcal{F}_x^c$ is trivial at the level of morphisms, this map is injective. Now let $V, W \in \mathcal{O}b(\mathcal{C}_F)$ and let $\varphi(\lambda) \in \mathrm{Hom}_{\mathcal{D}_B}(\mathcal{F}_x^c(V), \mathcal{F}_x^c(W))$. By definition, $\varphi(\lambda) : V \to W$ satisfies the relation

$$\begin{aligned}\varphi_2(\lambda)S_1(z-x, \lambda-\gamma h_2)L_{12}^V(z,\lambda)e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1}&\\ = S_1(z-x,\lambda-\gamma h_2)L_{12}^W(z,\lambda)e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1}\varphi_2(\lambda)&\end{aligned}$$

where $L^V(z,\lambda)$ (resp. $L^W(z,\lambda)$) is the L-operator of V (resp. W). This is equivalent to

$$\varphi_2(\lambda)S_1(z-x,\lambda-\gamma h_2)L_{12}^V(z,\lambda) = S_1(z-x,\lambda-\gamma h_2)L_{12}^W(z,\lambda)\varphi_2(\lambda-\gamma h_1) \qquad (14)$$

Introduce the following notations: write $W = \bigoplus_\xi W_\xi$, $V = \bigoplus_\mu V_\mu$, $\varphi(\lambda) = \sum_\nu \varphi_\nu(\lambda)$ for the weight decompositions (so that $\varphi_\nu : V_\xi \to W_{\xi+\nu}$). Also let $S(z-x,\lambda) = \sum_{i,j} S^{ij}(z-x,\lambda)E_{ij}$, $L_{12}^V(z,\lambda) = \sum_{i,j} E_{ij} \otimes L_V^{ij}(z,\lambda)$ and use the same notation for $L^W(z,\lambda)$. Applying (14) to $v_i \otimes \zeta_\mu$ for some i and $\zeta_\mu \in V_\mu$ yields

$$\begin{aligned}\sum_{j,k,\nu} S^{kj}(z-x,\lambda-\gamma(\mu+\omega_i-\omega_j))v_k \otimes \varphi_\nu(\lambda)(L_V^{ji}(z,\lambda)\zeta_\mu)&\\ = \sum_{l,k,\sigma} S^{kl}(z-x,\lambda-\gamma(\mu+\omega_i-\omega_l+\sigma))v_k \otimes L_W^{li}(z,\lambda)\varphi_\sigma(\lambda-\gamma\omega_i)\zeta_\mu&\end{aligned} \qquad (15)$$

where we used the weight-zero property of $L^V(z,\lambda)$ and $L^W(z,\lambda)$. Applying v_k^* to (15) and projecting on the weight space $W_{\mu+\omega_i+\xi}$ gives the relation

$$\begin{aligned}\sum_{\substack{\nu,j\\ \nu-\omega_j=\xi}} S^{kj}(z-x,\lambda-\gamma(\mu+\omega_i-\omega_j))\varphi_\nu(\lambda)(L_V^{ji}(z,\lambda)\zeta_\mu)&\\ = \sum_{\substack{\sigma,l\\ \sigma-\omega_l=\xi}} S^{kl}(z-x,\lambda-\gamma(\mu+\omega_i-\omega_j+\sigma))L_W^{li}(z,\lambda)(\varphi_\sigma(\lambda-\gamma\omega_i)\zeta_\mu)&\end{aligned} \qquad (16)$$

for any i, k, ξ and $\zeta_\mu \in V_\mu$. Now let $A = \{\chi \mid \varphi_\chi(\lambda) \neq 0\}$. Fix some j and let $\beta \in A$ be an extremal weight in the direction $-\omega_j$ (i.e $\beta - \omega_j + \omega_k \notin A$ for $k \neq j$). Then (16) for $\xi = \beta - \omega_j$ reduces to

$$\begin{aligned}S^{kj}(z-x,\lambda-\gamma(\mu+\omega_i-\omega_j))\varphi_\beta(\lambda)(L_V^{ji}(z,\lambda)\zeta_\mu)&\\ = S^{kj}(z-x,\lambda-\gamma(\mu+\omega_i-\omega_j+\beta))L_W^{ji}(z,\lambda)\varphi_\beta(\lambda-\gamma\omega_i)\zeta_\mu&\end{aligned} \qquad (17)$$

CLAIM. *There exists $i \in \{1, \ldots n\}$, μ and $\zeta_\mu \in V_\mu$ such that $\varphi_\beta(\lambda)(L_V^{ji}(z,\lambda)\zeta_\mu) \neq 0$ for generic z and λ.*

Proof. Recall the central element $\mathrm{Qdet}(z,\lambda) \in \mathcal{E}_{\tau,\frac{\gamma}{2}}(\mathfrak{gl}_n)$. By definition, its action on V is invertible. Expanding $\mathrm{Qdet}(z,\lambda)$ along the j^{th}-line, we have $\mathrm{Qdet}(z,\lambda) = \sum_i L_V^{ji}(z,\lambda)P_i(z,\lambda)$ for some operators $P_i(z,\lambda) \in \mathrm{End}(V)$. In particular,

$$\sum_i \mathrm{Im}\, L^{ji}(z,\lambda) = V,$$

and the claim follows.

Thus, the ratio $S^{kj}(z-x, \lambda - \gamma(\mu + \omega_i - \omega_j + \beta))/S^{kj}(z-x, \lambda - \gamma(\mu + \omega_i - \omega_j))$ is independent of k. This is possible only if $\beta \in \sum_{r \neq j} \mathbb{C} E^*_{rr}$. Applying this to $j = 1, \ldots n$, we see that $A = \{0\}$. Hence $\varphi(\lambda)$ is an $\mathfrak{h}$-module map. But then relation (14) reduces to $\varphi_2(\lambda) L^V_{12}(z,\lambda) = L^W_{12}(z,\lambda)\varphi_2(\lambda - \gamma h_1)$, and $\varphi(\lambda)$ is an intertwiner in the category $\mathcal{C}_F$. □

Corollary 2. *The functor $\mathcal{F}^c : \mathcal{C}^x_F \to \mathcal{D}^x_B$ is fully faithful.*

6. The image of the vector representation. Let us denote

$$\tilde{V}_F(w) = (\mathbb{C}^n, \chi(w) R^F(w, \lambda)).$$

It is an object of $\mathcal{C}_F$ which equals the tensor product of the vector representation $V_F(w)$ by the one-dimensional representation $(\mathbb{C}, \chi(z))$.

Proposition 4. *For any x, w, $x + c \not\equiv w \pmod{\mathbb{Z} + \tau\mathbb{Z}}$, we have $\mathcal{F}^c_x(\tilde{V}_F(w)) \simeq \mathcal{H}^c_x(V_B(w))$.*

Proof. By definition, we have

$$\mathcal{F}^c_x(\tilde{V}_F(w)) = (\mathbb{C}^n, \chi(z) S_1(z-x, \lambda - \gamma h_2) R^F(z-w,\lambda) e^{-\gamma D_1} \times S_1(z-x-c,\lambda)^{-1}),$$

$$I^c_x \otimes V_B(w) = (\mathbb{C}^n, \chi(z) S_1(z-x,\lambda) e^{-\gamma D_1} S_1(z-x-c,\lambda) R^B(z-w))$$

We claim that the map $\varphi(\lambda) = e^{-\gamma D}(S(w-x-c,\lambda)^{-1}) e^{\gamma D} \in \mathrm{End}(\mathbb{C}^n)$ is an intertwiner $\mathcal{H}^c_x(V_B(w)) \simeq I^c_x \otimes V_B(w) \xrightarrow{\sim} \mathcal{F}^c_x(\tilde{V}_F(w))$. Indeed, we have

$$\begin{aligned}
& S_1(z-x, \lambda - \gamma h_2) R^F(z-w,\lambda) e^{-\gamma D_1} S_1(z-x-c,\lambda)^{-1} (1 \otimes \varphi(\lambda)) \\
&\quad = e^{-\gamma D_2} S_1(z-x,\lambda) e^{\gamma D_2} R^F(z-w,\lambda) e^{-\gamma(D_1+D_2)} \\
&\qquad\qquad S_1(z-x-c,\lambda+\gamma h_2)^{-1} S_2(w-x-c,\lambda)^{-1} e^{\gamma D_2} \\
&\quad = e^{-\gamma D_2} S_1(z-x,\lambda) e^{-\gamma D_1} R^F(z-w,\lambda) \\
&\qquad\qquad S_1(z-x-c,\lambda+\gamma h_2)^{-1} S_2(w-x-c,\lambda)^{-1} e^{\gamma D_2} \\
&\quad = e^{-\gamma D_2} S_1(z-x,\lambda) e^{-\gamma D_1} S_2(w-x-c,\lambda+\gamma h_1)^{-1} \\
&\qquad\qquad S_1(z-x-c,\lambda)^{-1} R^B(z-w) e^{\gamma D_2} \\
&\quad = e^{-\gamma D_2} S_1(z-x,\lambda) S_2(w-x-c,\lambda) e^{-\gamma D_1} S_1(z-x-c,\lambda)^{-1} R^B(z-w) e^{\gamma D_2} \\
&\quad = (1 \otimes \varphi(\lambda)) S_1(z-x,\lambda) e^{-\gamma D_1} S_1(z-x-c,\lambda)^{-1} R^B(z-w)
\end{aligned}$$

where we used Lemma 2 and the zero-weight property of $R^F(u,\lambda)$. □

Lemma 4. *Let $V, V' \in \mathcal{O}b(\mathcal{C}_F)$, $W, W' \in \mathcal{O}b(\mathcal{C}_B)$ and suppose that $\mathcal{F}^c_x(V) \simeq \mathcal{H}^c_x(W)$ and $\mathcal{F}^c_x(V') \simeq \mathcal{H}^c_x(W')$. Then $\mathcal{F}^c_x(V \otimes V') \simeq \mathcal{H}^c_x(W \otimes W')$.*

Proof. If $\varphi(\lambda) : V \to W$ and $\varphi'(\lambda) : V' \to W'$ are intertwiners then it is easy to check using the methods above that $\varphi'_2(\lambda - \gamma h_1)\varphi_1(\lambda) : V \otimes V' \to W \otimes W'$ is an intertwiner. □

Applying this to tensor products of the vector representations, we obtain

Corollary 3. *For any $x \in \mathbb{C}$ and $w_1, \ldots, w_r \in \mathbb{C} \backslash \{x + c + \mathbb{Z} + \tau\mathbb{Z}\}$, we have*

$$\mathcal{F}^c_x(\tilde{V}_F(w_1) \otimes \ldots \tilde{V}_F(w_r)) \simeq \mathcal{H}^c_x(V_B(w_1) \otimes \ldots V_B(w_r)).$$

Corollary 4. *For any $w_1, \dots, w_r \in \mathbb{C}$, we have*

$$\mathcal{F}^c(\tilde{V}_F(w_1) \otimes \dots \tilde{V}_F(w_r)) \simeq \mathcal{H}^c(V_B(w_1) \otimes \dots V_B(w_r)).$$

Notice that in this case, we have a canonical intertwiner, given by the formula

$$\varphi_{1\dots r}(\lambda, w_1, \dots, w_r) = \tilde{S}_r^{-1}(w_r - x - c, \lambda - \gamma \sum_{i=1}^{r-1} h_i) \dots \tilde{S}_1^{-1}(w_1 - x - c, \lambda),$$

where we set $\tilde{S}(z, \lambda) = e^{-\gamma D} S(z, \lambda) e^{\gamma D}$.

7. Equivalence of subcategories. Let us summarize the results of sections 4-8. By proposition 2, we can identify $\mathcal{C}_B^x$ with a full subcategory $\mathcal{D}_1^x$ of $\mathcal{D}_B^x$. By proposition 3, we can identify $\mathcal{C}_F^x$ with a full subcategory $\mathcal{D}_2^x$ of $\mathcal{D}_B^x$. Moreover, $\mathcal{D}_1^x$ and $\mathcal{D}_2^x$ intersect (at least if we replace $\mathcal{D}_B^x$ by the equivalent category $\widetilde{\mathcal{D}_B^x}$ whose objects are isomorphism classes of objects of $\mathcal{D}_B^x$), and the intersection contains objects of the form $\mathcal{F}^c(\bigotimes_i \tilde{V}_F(w_i)) \simeq \mathcal{H}^c(\bigotimes_i V_B(w_i))$, where $i = 1, \dots r$ and $w_i \in \mathbb{C}$. Hence,

Theorem *The abelian subcategory $\mathcal{V}_B^x$ of $\mathcal{C}_B^x$ generated by objects $\bigotimes_i V_B(w_i)$ for $i = 1, \dots r$, $r \in \mathbb{N}$ and $w_i \in \mathbb{C}$ and the abelian subcategory $\mathcal{V}_F^x$ of $\mathcal{C}_F^x$ generated by objects $\bigotimes_j \tilde{V}_F(w_j)$ for $j = 1, \dots s$, $s \in \mathbb{N}$ and $w_j \in \mathbb{C}$ are equivalent.*

Note that for numerical values of x, $\mathcal{F}_x^c : \mathcal{C}_F \to \mathcal{D}_B$ is always fully faithful, and $\mathcal{F}_x^c(\mathcal{C}_F)$ a full subcategory of $\mathcal{D}_B$, but this is not true of $\mathcal{H}_x^c$, because of the existence of nontrivial finite-dimensional subrepresentations of I_x^0.

REFERENCES

[1] Baxter R.J, *Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. I*, Ann. Phys. **76** (1973) 1-24, *II*, *ibid.* 25-47, *III*, *ibid.* 48-71.

[2] Belavin A.A, and Drinfeld V.G, *Triangle equation and simple Lie algebras*, Soviet Sci. reviews, Sect C **4** 93-165 (1984).

[3] Belavin A. A., *Dynamical symmetries of integrable quantum systems*, Nucl. Phys. B, **180**, 189-200 (1981).

[4] Faddeev L., Reshetikhin N., and Takthadjan L., *Quantization of Lie groups and Lie algebras*, Algebraic Analysis, Vol **1** 129-139 Acad. Press (1988).

[5] Felder G., *Elliptic quantum groups*, preprint hep-th/9412207, to appear in the Proceedings of the ICMP, Paris 1994.

[6] Felder G., and Varchenko A., *On representations of the elliptic quantum group $E_{\tau,\eta}(\mathfrak{sl}_2)$*, Comm. Math. Phys. **181** (1996), 746-762.

[7] Felder G., and Varchenko A., *Algebraic Bethe Ansatz for the elliptic quantum group $E_{\tau,\eta}(\mathfrak{sl}_2)$*, preprint q-alg/9605024. Nuclear Physics B **480** (1996), 485-503.

[8] Hasegawa K., *Crossing symmetry in elliptic solutions of the Yang-Baxter equation and a new L-operator for Belavin's solution*, J, Phys. A: Math. Gen. **26** (1993) 3211-3228.

[9] Hasegawa K., *L-operator for Belavin's R-matrix acting on the space of theta functions*, J. Math. Phys. **35**(4) (1994) 6158-6171.

[10] Jimbo M., Miwa T., and Okado M., *Local state probabilities of solvable lattice models: an $A_n^{(1)}$ family*, Nucl. Phys. B300 [FS22] 74-108 (1988).

[11] Krichever I., and Zabrodin A., *Spin generalization of the Ruijsenaars-Schneider model, non-abelian 2D Toda chain and representations of the Sklyanin algebra*, preprint hep-th/9505039.

[12] Reshetikhin N.Y., and Semenov-Tian-Shansky M.A., *Central extensions of quantum current groups*, Lett. Math. Phys. **19** (1990) 133-142.