ASIAN J. MATH.
Vol. 2, No. 4, pp. 655–666, December 1998

004

ORDER PARAMETERS, FREE FERMIONS, AND CONSERVATION LAWS FOR CALOGERO-MOSER SYSTEMS*

ERIC D'HOKER† AND D.H. PHONG‡

Abstract. The classical order parameters for the $\mathcal{N}=2$ supersymmetric $SU(N)$ gauge theory with matter in the adjoint representation are exhibited explicitly as conservation laws for the elliptic Calogero-Moser system. Central to the construction are certain elliptic function identities, which arise from considering Feynman diagrams in a theory of free fermions with twisted boundary conditions.

1. Introduction. It has been known for a long time that the elliptic Calogero-Moser system

$$p_i = \dot{x}_i, \qquad \dot{p}_i = m^2 \sum_{j \neq i} \wp'(x_i - x_j), \qquad 1 \leq i, j \leq N, \tag{1.1}$$

is completely integrable, in the sense that it admits a Lax pair of operators $L(z)$, $M(z)$ with a spectral parameter z [1]. Here $\wp(z)$ is the Weierstrass $\wp$-function on a fixed torus $\Sigma = \mathbf{C}/(2\omega_1 \mathbf{Z} + 2\omega_2 \mathbf{Z})$ of modulus $\tau = \omega_2/\omega_1$. The spectral curves

$$\Gamma = \{(k, z);\ \det(kI - L(z)) = 0\}, \tag{1.2}$$

form an N-dimensional family of branched covers of the torus Σ. More recently, in connection with Seiberg-Witten solutions of four-dimensional $SU(N)$ supersymmetric gauge theories [2-5], we have found that the spectral curves (1.2) admit a natural parametrization of the form [5]

$$\det\big(\lambda I - L(z)\big) = \frac{\vartheta_1(\frac{1}{2\omega_1}(z - m\frac{\partial}{\partial k})|\tau)}{\vartheta_1(\frac{z}{2\omega_1}|\tau)} H(k)\Big|_{k=\lambda + m h_1(z)}, \tag{1.3}$$

where $H(k) \equiv \prod_{i=1}^N (k - k_i)$ is a monic polynomial of degree N, and the shift $h_1(z)$ is given by $h_1(z) = \partial_z \log \vartheta_1(\frac{z}{2\omega_1}|\tau)$. From the point of view of four-dimensional gauge theories, the zeroes k_i of $H(k)$ have a very compelling interpretation: they are the classical order parameters of the theory (c.f. (1.5) in [5]). From the point of view of Calogero-Moser systems, they are by construction integrals of motion of the system. However, the derivation of $H(k)$ in [5] did not provide explicit expressions for the k_i's in terms of the Calogero-Moser dynamical variables (x_i, p_i). The goal of the present paper is to solve this problem. In the process, we also find an intriguing link between Calogero-Moser systems and free fermions on the torus Σ.

To state the main result, we require the following notation. Let $\sigma_m(k_1, \cdots, k_N) = \sigma_m(k)$ be the m-symmetric function of the k_i's, as in

$$H(k) = \prod_{i=1}^N (k - k_i) = \sum_{m=0}^N (-)^m \sigma_m(k_1, \cdots, k_N)\ k^{N-m}. \tag{1.4}$$

---

* Received September 28, 1998; accepted for publication December 8, 1998. Research supported in part by the National Science Foundation under grants PHY95-31023 and DMS-98-00783.

† Department of Physics, University of California, Los Angeles, CA 90024, USA and Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA (dhoker@physics.ucla.edu).

‡ Department of Mathematics, Columbia University, New York, NY 10027, USA (phong@math.columbia.edu).

Let $\sigma_m(p)$ be the m-symmetric function of the momenta p_i's, $1 \leq i \leq N$. If S is any subset of $\{1,2,\cdots,N\}$ consisting of $|S|$ numbers, we denote by $\sigma_m(p_S)$ the m-symmetric function of $p_S = \{p_i; i \in S\}$, for any integer m with $m \leq |S|$. The complement of S in $\{1,2,\cdots\}$ is denoted by S^*. If S consists of only two elements $\{i,j\}$, and $f(x)$ is an even function, we shall often write $f(S)$ for $f(x_i - x_j)$. All subsets S, p_S are unordered, unless stated explicitly otherwise. Finally, it is convenient to introduce the following modification of the Weierstrass $\wp$-function

$$\wp^{(0)}(z) = \wp(z) + \frac{\eta_1}{\omega_1}. \tag{1.5}$$

Here, η_1 and η_2 are the periods dual to ω_1 and ω_2. Observe that $\wp^{(0)}(z) \to 0$ as $q \to 0$ and $z \to \infty$. Then

Main Theorem. *The order parameters k_i, $1 \leq i \leq N$, of the gauge theory are related to the Calogero-Moser dynamical variables (x_i, p_i) by the following relations. For any integer K with $0 \leq K \leq N$, we have*

$$\sigma_K(k) = \sigma_K(p) + \sum_{l=1}^{[K/2]} m^{2l} \sum_{\substack{|S_i \cap S_j| = 2\delta_{ij} \\ 1 \leq i,j \leq l}} \sigma_{K-2l}\big(p_{(\cup_{i=1}^l S_i)^*}\big) \prod_{i=1}^{l} \wp^{(0)}(S_i). \tag{1.6}$$

As mentioned earlier, this theorem is partly motivated by current investigation of $\mathcal{N}=2$ supersymmetric four-dimensional gauge theories [6-7]. The Wilson effective action of such theories is dictated by the spectral curves of integrable models (see e.g. [8-10] for reviews). But it is still unclear whether the dynamical variables of the integrable models have any direct interpretation in the context of gauge theories. The preceding theorem can be viewed as a step in addressing this question.

In another direction, spectral curves have recently been obtained for elliptic Calogero-Moser systems defined by general Lie algebras $\mathcal{G}$ [11-13] and supersymmetric $\mathcal{G}$ gauge theories with matter in the adjoint representation [11-15]. However, except in the case of D_n [13] (see also [15]), a convenient parametrization such as (1.3) is still not available. Such a parametrization is for example particularly valuable in evaluating instanton corrections to the prepotential [5]. It is conceivable that a deeper understanding of the order parameters k_i in the above $SU(N)$ case, as well as the elliptic function identities found in the present paper, may shed light on this issue.

Finally, we mention some related problems in the theory of integrable models proper. The symplectic structure of Calogero-Moser systems is attracting considerable attention [16-17]. The integrals of motion (1.6) may be relevant to the well known problem of constructing R-matrices for Calogero-Moser systems (c.f. [18-19]). They may also be of interest in the rational and trigonometric cases [20]. In particular, in the trigonometric case, the gauge order parameters k_i have been useful in the study of Toeplitz determinants, symplectic volumes, and thermodynamic limits [21].

1. Main identities and proof of the theorem. We divide the proof of the Main Theorem into several steps.

(I) In the first step, the defining identity (1.3) for the integrals k_i is rewritten in terms of determinants $D(S)$ similar to det $L(z)$, but with all diagonal entries set to 0. More precisely, recall that the Lax pair $L(z)$, $M(z)$ for the elliptic Calogero-Moser system is given by [1]

$$L_{ij}(z) = p_i\delta_{ij} - m(1-\delta_{ij})\Phi(x_i - x_j, z),$$
$$M_{ij}(z) = m\delta_{ij}\sum_{k\neq i}\wp(x_i - x_k) + m(1-\delta_{ij})\Phi'(x_i - x_j, z), \tag{2.1}$$

with

$$\Phi(x,z) = \frac{\sigma(z-x)}{\sigma(z)\sigma(x)}e^{x\zeta(z)}. \tag{2.2}$$

Here $\sigma(z)$, $\zeta(z)$ are the usual Weierstrass elliptic functions (c.f. Erdelyi [22]). Let $S = \{\alpha(1), \alpha(2), \cdots, \alpha(K)\}$ be a subset of $\{1, 2, \cdots, N\}$ with $|S| = K$ elements. We define $D(S)$ to be the following $K \times K$ determinant

$$D(S) = \det\Big[(1-\delta_{ij})(\Phi(x_{\alpha(i)} - x_{\alpha(j)}, z)\Big]. \tag{2.3}$$

There is no ambiguity in this definition since the right hand side of (2.3) is independent of the ordering of S. The identity (1.3) is then equivalent to

$$\sum_{p+q=N-n}\binom{n+q}{n}m^q A_q(-)^p\sigma_p(k)$$
$$= \sum_{|S|+l=N-n}(-)^l\sigma_l(p_{S^*})m^{|S|}D(S),\ 1\le n\le N. \tag{2.4}$$

In these identities the summation on the right hand side is over all subsets S of $\{1, 2, \cdots, N\}$ and the functions $A_K(z)$ are given explicitly in terms of ϑ-functions

$$A_K(z) = \sum_{n=0}^{K}\binom{K}{n}(-)^n h_n(z)h_1(z)^{K-n},$$
$$h_n(z) = \frac{\partial_z^n\,\vartheta_1(\frac{z}{2\omega_1}|\tau)}{\vartheta_1(\frac{z}{2\omega_1}|\tau)}. \tag{2.5}$$

To establish (2.4), we note that

$$\frac{\vartheta_1(\frac{1}{2\omega_1}(z - m\frac{\partial}{\partial k})|\tau)}{\vartheta_1(\frac{z}{2\omega_1}|\tau)}H(k) = \sum_{n=0}^{N}\frac{h_n(z)}{n!}(-m)^nH^{(n)}(k). \tag{2.6}$$

Substituting in $k = \lambda + mh_1(z)$ and expanding at λ, we find

$$\frac{\vartheta_1(\frac{1}{2\omega_1}(z - m\frac{\partial}{\partial k})|\tau)}{\vartheta_1(\frac{z}{2\omega_1}|\tau)}H(k)\bigg|_{k=\lambda+mh_1(z)}$$
$$= \sum_{n=0}^{N}\sum_{q=o}^{N-n}\frac{h_n(z)h_1(z)^q}{n!q!}(-)^nm^{n+q}H^{(n+q)}(\lambda)$$
$$= \sum_{K=0}^{N}A_K(z)\frac{m^KH^{(K)}(\lambda)}{K!}. \tag{2.7}$$

In terms of $\sigma_p(k)$, the polynomials $H^{(K)}(\lambda)$ are just given by

$$H^{(K)}(\lambda) = \sum_{p=0}^{N-K} (-)^m \frac{(N-p)!}{(N-K-p)!} \sigma_p(k) \lambda^{N-K-p}. \tag{2.8}$$

This implies that the right hand side of (1.3) can be rewritten as

$$\begin{aligned} &\frac{\vartheta_1(\frac{1}{2\omega_1}(z - m\frac{\partial}{\partial k})|\tau)}{\vartheta_1(\frac{z}{2\omega_1}|\tau)} H(k)\Big|_{k=\lambda+mh_1(z)} \\ &= \sum_{n=0}^{N} \lambda^n \sum_{p+q=N-n} \binom{n+q}{n} A_q(z) m^q (-)^p \sigma_p(k). \end{aligned} \tag{2.9}$$

On the other hand, the determinant $\det(\lambda I - L(z))$ on the left hand side of (1.3) can be expanded as

$$\det(\lambda I - L(z)) = \sum_{K=0}^{N} m^K \sum_{|S|=K} \big[\prod_{i\in S^*} (\lambda - p_i)\big] D(S). \tag{2.10}$$

Evidently,

$$\prod_{i\in S^*} (\lambda - p_i) = \sum_{l=0}^{N-K} (-)^l \sigma_l(p_{S^*}) \lambda^{N-K-l},$$

so that the left hand side of (1.3) becomes

$$det(\lambda I - L(z)) = \sum_{n=0}^{N} \lambda^n \sum_{l=0}^{N-n} m^{N-l-n} \sum_{|S|=N-l-n} (-)^l \sigma_l(p_{S^*}) D(S). \tag{2.11}$$

Comparing (2.9) with (2.11) gives the desired identities (2.4).

(II) The second step in the proof of the main theorem consists of, in a sense, separating in the determinant $D(S)$ the dependence on the insertion points $x_{\sigma(j)} \in S$ from the dependence on the spectral parameter z. More precisely, we can write

$$D(S) = \sum_{l=0}^{[K/2]} B_{K-2l}(z) \sum_{\substack{|S_i\cap S_j|=2\delta_{ij} \\ 1\le i,j\le l}} \prod_{i=1}^{l} \wp^{(0)}(S_i) \tag{2.12}$$

where $\wp^{(0)}(S_i) = \wp^{(0)}(x_a - x_b)$ if $S_i = \{a, b\}$ is the convention introduced in Section I. To describe the coefficients $B_M(z)$, it is convenient to introduce the notation

$$\wp^{(n)}(z) = \big(\frac{\partial}{\partial z}\big)^n \wp^{(0)}(z), \qquad n \in \mathbf{N}. \tag{2.13}$$

Then the coefficients $B_K(z)$ can be expressed as

$$\begin{aligned} &B_K(z) \\ &= (-)^K \sum_{K=2L_2+3L_3+\cdots} (-)^{\sum_{n=2}^{\infty} L_n} \frac{K!}{\prod_{n=2}^{\infty}(L_n!) \prod_{n=2}^{\infty}(n!)^{L_n}} \prod_{n=2}^{\infty} \big[\wp^{(n-2)}(z)\big]^{L_n}. \end{aligned} \tag{2.14}$$

The proof of the identity (2.14) is the lengthiest part of our argument, and we postpone it until Section III.

(III) The third step in the proof of the Main Theorem is to show that the coefficients $A_K(z)$ and $B_K(z)$ are actually equal

$$A_K(z) = B_K(z), \quad K = 1, 2, 3, \cdots . \tag{2.15}$$

Evidently, $A_1(z) = B_1(z) = 0$, while $A_2(z) = h_2(z) - h_1(z)^2$ and $B_2(z) = -\wp^{(0)}(z)$. Since the Weierstrass σ-function and the Jacobi ϑ-function are related by

$$\begin{aligned} \sigma(z) &= 2\omega_1 e^{\frac{\eta_1}{2\omega_1}z^2} \frac{\vartheta_1(\frac{z}{2\omega_1}|\tau)}{\vartheta_1'(0|\tau)}, \\ \wp(z) &= -\partial_z^2 \, log\, \sigma(z) = -\frac{\eta_1}{\omega_1} - \partial_z^2 \log \vartheta_1(\frac{z}{2\omega_1}|\tau), \end{aligned} \tag{2.16}$$

we also have $A_2(z) = B_2(z)$. It suffices then to show that both the $A_K(z)$'s and the $B_K(z)$'s obey the same two-step recursive relation

$$\begin{aligned} A_{K+1}(z) &= -A_K'(z) - K\wp^{(0)}(z)A_{K-1}, \\ B_{K+1}(z) &= -B_K'(z) - K\wp^{(0)}(z)B_{K-1}, \end{aligned} \tag{2.17}$$

for $K \geq 2$. The recursive relation for $A_{K+1}(z)$ is easily established by differentiating $A_K(z)$, and using the fact that

$$\begin{aligned} h_n'(z) &= -h_n(z)h_1(z) + h_{n+1}(z), \\ h_1'(z) &= -h_1(z)^2 + h_2(z) = -\wp^{(0)}(z). \end{aligned} \tag{2.18}$$

To establish the recursive relation for $B_K(z)$, we define $B_0(z)$ to be 1, introduce an additional variable y, and consider the generating function

$$\sum_{K=0}^{\infty} \frac{B_K(z)}{K!} y^K = \exp\left[\sum_{n=2}^{\infty} \frac{(-)^{n+1}}{n!} \wp^{(n-2)}(z) y^n\right]. \tag{2.19}$$

Differentiating with respect to y gives a recursive relation for $B_{K+1}(z)$

$$B_{K+1}(z) = \sum_{n=1}^{K} \binom{K}{n} (-)^n \wp^{(n-1)}(z) B_{K-n}(z), \tag{2.20}$$

while differentiating with respect to z gives a recursive relation for $B_K'(z)$

$$B_K'(z) = \sum_{n=2}^{K} \binom{K}{n} (-)^{n+1} \wp^{(n-1)}(z) B_{K-n}(z). \tag{2.21}$$

Comparing (2.20) with (2.21) gives the desired recursive relation (2.17). The identity $A_K(z) = B_K(z)$ is established.

(IV) With the identities (I-III), we can now prove the theorem. We fix integers N and n, with $n \leq N$. The sum in the right hand side of (2.4) is over all subsets S of

$\{1, 2, \cdots, N\}$. Setting $|S| = K$ and using (II) and (III), we can express it as

$$\sum_{K=0}^{N-n} (-)^{N-n-K} m^K \sum_{S,|S|=K} \sigma_{N-n-K}(p_{S^*}) \sum_{q+2j=K} A_q(z) \sum_{\substack{|S_i \cap S_j| = 2\delta_{ij} \\ S_i \subset S}} \wp^{(0)}(S_1) \cdots \wp^{(0)}(S_j). \tag{2.22}$$

Introduce the index $p = N - n - q$ (not to be confused with the Calogero-Moser momenta!). Then the order of summations in (2.22) can be interchanged to produce

$$\sum_{q=0}^{N-n} m^q A_q(z)(-)^p \sum_{j=0}^{[p/2]} \sum_{S,|S|=q+2j} \sigma_{p-2j}(p_{S^*}) m^{2j} \sum_{\substack{|S_i \cap S_j| = 2\delta_{ij} \\ S_i \subset S}} \wp^{(0)}(S_1) \cdots \wp^{(0)}(S_j). \tag{2.23}$$

However, for each j, we have the following combinatorial identity

$$\begin{aligned} &\sum_{S,|S|=q+2j} \sigma_{p-2j}(p_{S^*})\, m^{2j} \sum_{\substack{|S_i \cap S_j| = 2\delta_{ij} \\ S_i \subset S}} \wp^{(0)}(S_1) \cdots \wp^{(0)}(S_j) \\ &\qquad = \binom{n+q}{n} \sum_{|S_i \cap S_j| = 2\delta_{ij}} \sigma_{p-2j}(p_{(\cup_{i=1}^j S_i)^*}) \wp^{(0)}(S_1) \cdots \wp^{(0)}(S_j). \end{aligned} \tag{2.24}$$

for $p - 2j \geq 0$. In fact, by permutation invariance, the expressions on the two sides of (2.24) are proportional. To determine the coefficient of proportionality, we compare the coefficients of the term $\wp^{(0)}(x_1 - x_2) \cdots \wp^{(0)}(x_{2j-1} - x_{2j}) p_{2j+1} \cdots p_p$ which occurs on both sides. In the sum on the right hand side of (2.24), such a term occurs exactly once. On the other hand, such a term occurs in the sum on the left hand side whenever we can choose a subset S of size $q + 2j$, containing $\{1, \cdots, 2j\}$, and not containing $\{2j+1, \cdots, p\}$. This means that S consists of $\{1, \cdots, 2j\}$, together with q more elements in $\{p+1, \cdots, N\}$. There are exactly

$$\binom{N-p}{q} = \binom{n+q}{q} = \binom{n+q}{n} \tag{2.25}$$

such choices. Thus the expression (2.23) becomes

$$\sum_{q=0}^{N-n} m^q A_q(z)(-)^p \binom{n+q}{n} \sum_{|S_i \cap S_j| = 2\delta_{ij}} \sigma_{p-2j}(p_{(\cup_{i=1}^j S_i)^*}) \wp^{(0)}(S_1) \cdots \wp^{(0)}(S_j), \tag{2.26}$$

with $p = N - n - q$. Comparing with the left hand side of (2.4) gives the theorem.

3. The determinant $D(S)$ and free fermions. It remains to establish the identities in (II) of Section II for the determinants $D(S)$. For this we need the notion of "k-cycle", which we now describe. Let $\{1, 2, \cdots, k\}$ be any set of k indices, which we choose to be the first k integers just for notational convenience. Then the expression

$$\begin{aligned} &\Phi_{12}\Phi_{23} \cdots \Phi_{(k-1)k}\Phi_{k1} \\ &\equiv \Phi(x_1 - x_2, z)\Phi(x_2 - x_3, z) \cdots \Phi(x_{k-1} - x_k, z)\Phi(x_k - x_1, z), \end{aligned} \tag{3.1}$$

is a single-valued, meromorphic function of all insertion points $x_1, \cdots, x_k$, as well as of the spectral parameter z. Here we have made use of the monodromy properties of the function $\Phi(x, z)$ as a function of x

$$\begin{aligned} \Phi(x+2\omega_a, z) &= \Phi(x, z) e^{2\omega_a \zeta(z) - 2\eta_a z}, \\ \frac{1}{2\pi} \partial_{\bar{x}} \Phi(x-y, z) &= \delta(x-y). \end{aligned} \tag{3.2}$$

(As a function of z, $\Phi(x, z)$ is already by itself single-valued on the torus Σ.) It is useful to note that in expressions such as (3.1), the function $\Phi(x, z)$ can be effectively replaced by $\sigma(z-x)/\sigma(z)\sigma(x)$. We define a k-cycle to be the sum of all inequivalent expressions (3.1) under permutations of the indices $1, 2, \cdots, k$. Since (3.1) is evidently invariant under shifts in the indices $1, 2, \cdots, k$, this sum corresponds to a sum over $\mathbf{S}_k/\mathbf{Z}_k$, where $\mathbf{S}_k$ is the group of permutations of k elements. Equivalently, we can fix an index, say k, and write a k-cycle as

$$C_k(x_1, \cdots, x_k; z) = \sum_{\alpha \in \mathbf{S}_{k-1}} \Phi_{k\alpha(1)} \Phi_{\alpha(1)\alpha(2)} \cdots \Phi_{\alpha(k-1)k}, \tag{3.3}$$

identifying in effect $\mathbf{S}_k/\mathbf{Z}_k$ with the group $\mathbf{S}_{k-1}$ of permutations of $k-1$ elements. It is easy to verify that, for $k \geq 3$, the k-cycle $C_k(x_1, \cdots, x_k; z)$ is actually a function $C_k(z)$ independent of the insertion points $\{x_1, x_2, \cdots, x_k\}$. Indeed, viewed as a function of say x_1, it is meromorphic and has simple poles at the other insertion points $x_2, \cdots, x_k$. The residues at each of these poles however cancel out between the various terms in (3.2), so that the k-cycle is actually constant in each x_j. The main problem is then to determine the dependence on z of k-cycles. The identity central to our approach is the following

$$C_k(z) = \sum_{\alpha \in \mathbf{S}_{k-1}} \Phi_{k\alpha(1)} \Phi_{\alpha(1)\alpha(2)} \cdots \Phi_{\alpha(k-1)k} = \wp^{(k-2)}(z), \qquad k = 3, 4, \cdots. \tag{3.4}$$

(For $k = 2$ the 2-cycle is not independent of the insertion points. In fact, we have

$$\Phi_{12}\Phi_{21} = \wp(z) - \wp(x_1 - x_2) = \wp^{(0)}(z) - \wp^{(0)}(x_1 - x_2), \tag{3.5}$$

a well-known and basic identity in the theory of elliptic Calogero-Moser systems.) Postponing for the moment the proof of (3.4), we return to the study of the determinants $D(S)$.

Exact Formulas for $D(S)$

Since the diagonal elements of the matrix $D(S)$ all vanish, the determinant can be expanded as

$$D(S) = \sum_{\alpha} (-)^{\alpha} \Phi_{1\alpha(1)} \Phi_{2\alpha(2)} \cdots \Phi_{K\alpha(K)}, \tag{3.6}$$

where the summation is only over permutations α without any fixed point. But it is readily seen that any permutation α without fixed point corresponds to a decomposition of the index set $\{1, 2, \cdots, K\}$ into disjoint subsets S_j of at least two elements, in each of which α acts as a shift. Since the sign of a shift on N elements is $(-)^{N+1}$, the sign of α is $(-)^{K+l}$, where l is the number of subsets S_j. All permutations without fixed points can be generated this way, by following up the decomposition of the index set into smaller sets S_j with permutations within each smaller set S_j. Taking these "internal" permutations into account, the contribution of each decomposition $S = \cup_{j=1}^{l} S_j$ to the determinant (3.6) is a product of k-cycles

$$\prod_{j=1}^{l} \big[|S_j| - \text{cycles} \big]. \tag{3.7}$$

In view of (3.4) and (3.5), this establishes the fact that the determinant $D(S)$ must be of the form (2.12), for some as yet complicated coefficients $B_{K-2l}(z)$, $1 \leq l \leq [K/2]$.

To determine $D(S)$, it suffices to determine the "constant term" $B_K(z)$. This is because identities of the form (2.12) will be established inductively, by examining the poles of both sides of the equation in each of the variables x_i. For example, consider the double pole in the variable x_1, near the value x_2. For the left hand side, it is

$$-\Phi_{12}\Phi_{21} \times D(S \setminus \{1,2\}) = [\wp^{(0)}(x_1 - x_2) - \wp^{(0)}(z)] \times D(S \setminus \{1,2\}). \tag{3.8}$$

For the right hand side, it is

$$\Bigg[\sum_{l=1}^{[K/2]} B_{K-2l}(z) \sum_{\substack{|S_i \cap S_j| = 2\delta_{ij} \\ S_i \subset S \setminus \{1,2\}}} \wp^{(0)}(S_2) \cdots \wp^{(0)}(S_l) \Bigg] \wp^{(0)}(x_1 - x_2). \tag{3.9}$$

By induction, the expression between brackets is indeed $D(S \setminus \{1,2\})$. Similarly, the simple poles cancel. This shows that $D(S)$ is determined up to an additive function of z only.

We can derive now the explicit formula (2.14) for $B_K(z)$. Since we are restricting our attention to the constant term $B_K(z)$ in the expansion (3.6-3.7) for $D(S)$, we can replace even the 2-cycles in (3.7) by $\wp^{(0)}(z)$. With this simplification, the contribution of (3.7) to $B_K(z)$ is just

$$\prod_{j=1}^{l} \wp^{(|S_j|-2)}(z). \tag{3.10}$$

Now the exact subsets S_j themselves no longer matter, and the only relevant information is their size $|S_j|$. For each partition of K into L_2 subsets of 2 elements, L_3 subsets of 3 elements, etc.

$$K = 2L_2 + 3L_3 + 4L_4 + \cdots = \sum_{n=2}^{\infty} n L_n, \tag{3.11}$$

the expression (3.10) becomes

$$\prod_{n=2}^{\infty} \big[\wp^{(n-2)}(z) \big]^{L_n}. \tag{3.12}$$

Now the number of ways of selecting L (unordered) sets of n elements each from an ensemble of N elements is

$$\frac{1}{L!} \frac{N!}{(n!)^L (N - nL)!}. \tag{3.13}$$

Thus the total number of terms of the form (3.12) is

$$\begin{aligned}&\frac{1}{L_2!}\frac{N!}{(2!)^{L_2}(N-2L_2)!}\times\frac{1}{L_3!}\frac{(N-2L_2)!}{(3!)^{L_3}(N-2L_2-3L_3)!}\\&\times\frac{1}{L_4!}\frac{(N-2L_2-3L_3)!}{(4!)^{L_4}(N-2L_2-3L_3-4L_4)!}\times\cdots=\frac{N!}{\prod_{n=2}^{\infty}L_n!\prod_{n=2}^{\infty}(n!)^{L_n}}.\end{aligned}\tag{3.14}$$

Altogether, this establishes the formula (2.14).

Free Fermions and k-Cycles as Feynman Diagrams

Finally, we turn to the proof of the fundamental identity (3.4). The main idea is to view k-cycles $C_k(x_1,\cdots,x_k;z)$ as the one-loop amplitude in a theory of free fermions with twisted boundary conditions on a torus and fermion propagator $\Phi(x-y,z)$. Here z is viewed as a fixed parameter.

First, since $C_k(x_1,\cdots,x_k;z)$ is independent of x_i anyway, we may just as well integrate each x_i over the torus Σ with area τ_2,

$$\begin{aligned}C_k(z)&=\int_\Sigma\frac{d^2x_1}{\tau_2}\cdots\int_\Sigma\frac{d^2x_k}{\tau_2}C_k(x_1,\cdots,x_k;z)\\&=(k-1)!\int_\Sigma\frac{d^2x_1}{\tau_2}\cdots\int_\Sigma\frac{d^2x_k}{\tau_2}\Phi(x_1-x_2,z)\cdots\Phi(x_k-x_1,z)\end{aligned},\tag{3.15}$$

where the factor $(k-1)!$ comes out since integration in each variable automatically takes care of symmetrization. In this integrated form, the one-loop amplitude $C_k(z)$ has an even simpler interpretation, which we now develop. Starting from the free massless fermion propagator,

$$\Phi(x-y,z),\tag{3.16}$$

we may construct the "full" propagator of a fermion in the presence of a constant (background) gauge potential with strength m, by summing up the effects of repeated gauge potential coupling operator insertions. The "full" fermion propagator may thus be defined by the geometric series

$$S(x-y|z,m)=\Phi(x-y,z)+\frac{m}{\tau_2}\int_\Sigma d^2y_1\Phi(x-y_1,z)\Phi(y_1-y,z)+\cdots\tag{3.17}$$

or in terms of the recursive relation

$$S(x-y|z,m)=\Phi(x-y,z)+\frac{m}{\tau_2}\int_\Sigma d^2y_1\Phi(x-y_1,z)S(y_1-y|z,m).\tag{3.18}$$

The k-cycles $C_k(z)$ are now easily gotten as the $k-1$ derivatives with respect to m of the propagator $S(x-y|z,m)$ at coincident points $S(0|z,m)$, as we shall use below in (3.26).

Equivalently, we may characterize $S(x-y|z,m)$ by its monodromy and differential equation, which follow from the analogous properties of the propagator $\Phi(x-y,z)$, and the definitions (3.17) and (3.18),

$$\begin{aligned}S(x+2\omega_a|z,m)&=S(x|z,m)e^{2\omega_a\zeta(z)-2\eta_a z},\\\bar{D}S(x-y|z,m)&=\delta(x-y),\end{aligned}\tag{3.19}$$

where we introduce the $\bar{D}$ and D operators by

$$\bar{D} = \frac{1}{2\pi}\partial_{\bar{x}} - \frac{m}{\tau_2}, \qquad D = \frac{1}{2\pi}\partial_x - \frac{\bar{m}}{\tau_2}. \tag{3.20}$$

These operators are precisely the Dirac operators on the torus Σ in the presence of a constant gauge potential with strength m for left- and right-movers respectively.

Now we need $C_k(z)$, gotten by one closed loop, i.e. by $S(0|z,m)$. Thus it suffices to evaluate the determinant, since

$$\frac{\partial}{\partial m}\log \mathrm{Det}\bar{D}D = \frac{\partial}{\partial m}\mathrm{Tr}\log \bar{D} = -\frac{1}{\tau_2}\mathrm{Tr}\bar{D}^{-1} = -S(0|z,m). \tag{3.21}$$

The eigenvalues of $\bar{D}$ on the space of functions with monodromy as in (3.19) can be determined as usual. They are given by

$$\lambda_{n_1 n_2} = \frac{1}{\tau_2}(n_1\tau - n_2 + m + z), \qquad n_1, n_2 \in \mathbf{Z}. \tag{3.22}$$

We recall from [23], p. 1002, that the determinant of a Dirac operator $\bar{D}$ is given by

$$\mathrm{Det}\bar{D} = \frac{\vartheta_{\nu_1\nu_2}(0|\tau)}{\eta(\tau)}, \tag{3.23}$$

if its eigenvalues are of the form

$$\lambda_{n_1 n_2} = \frac{1}{\tau_2}\big[(n_1 + \frac{1}{2} - \frac{1}{2}\nu_1)\tau - (n_2 + \frac{1}{2} - \frac{1}{2}\nu_2)\big]. \tag{3.24}$$

In the present case, $\nu_1 = 1$, $\nu_2 = 1 - 2(m+z)$, and we obtain

$$\mathrm{Det}\bar{D} = \frac{\vartheta_1(z+m|\tau)}{\eta(\tau)}. \tag{3.25}$$

Returning to the k-cycles $C_k(z)$, we can write

$$C_k(z) = (\frac{\partial}{\partial m})^{k-1} S(0|z,m)\Big|_{m=0} = -(\frac{\partial}{\partial m})^k \, log\, \frac{\vartheta_1(z+m|\tau)}{\eta(\tau)}\Big|_{m=0}. \tag{3.26}$$

The desired identity (3.4) follows now from the elliptic function identity (2.16). The proof of the main theorem is complete.

REFERENCES

[1] I. Krichever, *Elliptic solutions of the Kadomtsev-Petviashvili equation and integrable systems of particles*, Funct. Anal. Appl., 14 (1980), pp. 282–290.

[2] R. Donagi and E. Witten, *Supersymmetric Yang-Mills theory and integrable systems*, Nucl. Phys. B, 460 (1996), pp. 299–334, hep-th/9510101.

[3] E. Martinec and N. Warner, *Integrable models and supersymmetric gauge theory*, Nucl. Phys. B, 459 (1996), pp. 97–112, hep-th/9609161.

[4] E. Martinec, *Integrable structures in supersymmetric gauge and string theory*, hep-th/9510204.

[5] E. D'Hoker and D.H. Phong, *Calogero-Moser systems in SU(N) Seiberg-Witten theory*, hep-th/9709053, Nucl. Phys. B, 513 (1998), pp. 405.

[6] N. Seiberg and E. Witten, *Electro-magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory*, Nucl. Phys. B, 426 (1994), pp. 19, hep-th/9407087.

[7] N. Seiberg and E. Witten, *Monopoles, duality, and chiral symmetry breaking in N=2 supersymmetric QCD*, Nucl. Phys. B, 431 (1994), pp. 484–550, hep-th/9410167.

[8] W. Lerche, *Introduction to Seiberg-Witten theory and its stringy origins*, Proceedings of the *Spring School and Workshop in String Theory* (1996, hep-th/9611190), ICTP, Trieste.

[9] R. Donagi, *Seiberg-Witten integrable systems*, alg-geom/9705010.

[10] R. Carroll, *Prepotentials and Riemann surfaces*, hep-th/9802130.

[11] E. D'Hoker and D.H. Phong, *Calogero-Moser Lax pairs with spectral parameters for general Lie algebras*, hep-th/9804124.

[12] E. D'Hoker and D.H. Phong, *Calogero-Moser and Toda systems for twisted and untwisted affine Lie algebras*, hep-th/9804125.

[13] E. D'Hoker and D.H. Phong, *Spectral curves for super Yang-Mills with adjoint hypermultiplet for general Lie algebras*, hep-th/9804126.

[14] A. M. Uranga, *Towards mass-deformed N=4 SO(N) and Sp(K) gauge theories from brane configurations*, hep-th/9803054.

[15] T. Yokono, *Orientifold four plane in brane configurations and N=4 USp(2N) and SO(2N) theory*, hep-th/9803123.

[16] A. Gorsky and N. Nekrasov, *Elliptic Calogero-Moser systems from two-dimensional current algebras*, ITEP-NG/1-94, hep-th/9401021.

[17] I. M. Krichever and D. H. Phong, *Symplectic forms in the theory of solitons*, hep-th/9708170, to appear in Surveys in Differential Geometry, III.

[18] M. A. Olshanetsky and A. Perelomov, *Classical integrable finite-dimensional systems related to Lie algebras*, Phys. Reports, 71 (1981), pp. 313–400.

[19] H. W. Braden, *R-matrices, Generalized Inverses, and Calogero-Moser-Sutherland Models*, Proc. Conference on Calogero-Moser-Sutherland Models, Montreal, 1997.

[20] A. J. Bordner, E. Corrigan, and R. Sasaki, *Calogero-Moser Models : A New Formulation*, hep-th/9805106.

[21] K. L. Vaninsky, *On explicit parametrization of spectral curves for Moser-Calogero particles and its application*, June 1998 preprint, Kansas State University.

[22] A. Erdelyi (ed.), *Higher Transcendental Functions*, R.E. Krieger Publishing Co., Florida, 1981.

[23] E. D'Hoker and D.H. Phong, *The geometry of string perturbation theory*, Rev. Modern Physics, 80 (1988), pp. 917–1065.