

BREAKDOWN OF A SHALLOW WATER EQUATION*

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1. Introduction. The 1-dimensional shallow water equation:

$$\begin{aligned}
 1) \quad & \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \frac{\partial p}{\partial x} = 0 \quad \text{with "pressure"} \\
 2) \quad & p(x) = \frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-y|} \left(v^2 + \frac{1}{2} v'^2 \right) dy
 \end{aligned}$$

was noted by FOKAS-FUCHSTEINER [1981] as being formally integrable. CAMASSA-HOLM [1993] rediscovered it from a hydrodynamical point of view and developed it to a large degree, both there and in CAMASSA-HOLM-HYMAN [1994]. Their most striking discovery is that it has (peaked) solitons $v(t, x) = \frac{1}{2} p e^{-|x-q-pt/2|}$ with constant q and p , capable of being superposed in an elementary way, as sums $v(t, x) = \frac{1}{2} \sum_{i=1}^n p_i(t) e^{|x-q_i(t)|}$, where now the q 's and p 's obey the equations $q^\bullet = \partial H / \partial p$, $p^\bullet = -\partial H / \partial q$ of the geodesic flow in the co-tangent bundle of \mathbb{R}^n , equipped with the inverse metric tensor $\frac{1}{2} e^{|q_i - q_j|} : 1 \leq i, j \leq n$, *i.e.*, with Hamiltonian $H = \frac{1}{2} |p|^2 + \frac{1}{4} \sum p_i p_j e^{-|q_i - q_j|}$. From this, it is only a little jump to the realization that the full shallow water flow in, *e.g.*, $C^\infty_{\downarrow}(\mathbb{R})$ is just a transcription of the geodesic flow in the group $D(\mathbb{R})$ of (smooth proper) diffeomorphisms of the line, equipped with an analogous Riemannian geometry; see MISIOLEK [1998] for the more general case $v \in \mathbb{R} + C^\infty_{\downarrow}(\mathbb{R})$ with the attendant BOTT-VISASORO group, and compare ARNOLD-KHESIN [1998] for background and other illustrations of the general principle involved. Not much of that is needed here: it will suffice to observe a) if $v(t)$ obeys the shallow water flow in, *e.g.*, $C^\infty(\mathbb{R})$ and if $Q(t, x)$ solves $Q^\bullet = v(t, Q)$ with $Q_0(x) \equiv x$, then Q is a diffeomorphism of the line¹, imitating the identity near $x = \pm\infty$, and obeying the geodesic flow in $D(\mathbb{R})$, and b) the shallow water flow, expressed by means of $m = v - v''$ as in *) $m^\bullet + (mD + Dm)v = 0$, has first integrals in the form $m(Q) Q'^2(x) \equiv m_0(x)$.² *) shows that $m(Q)$ retains the general shape of $m_0(x)$, independently of $t \geq 0$, illustrating the utility of the diffeomorphism for the study of the flow; indeed, it is the principal tool for the question posed here: *When, and if so how, does the shallow water flow breakdown?*

Now, unlike KdV, breakdown is common enough as already noted in CAMASSA-HOLM [1993]: for example, if $\int_{-\infty}^{\infty} (v^2 + v'^2) dx \equiv H < \infty$, this being a constant of motion, then, from 5) below in the form $\frac{d}{dt} v'(Q) \leq \frac{1}{2} v^2(Q) - \frac{1}{2} v'^2(Q)$, you see that $v'_0(x_0) < -\sqrt{H}$ at any place $x_0 \in \mathbb{R}$ implies $v'_0(t, x_0) \downarrow -\infty$ at some finite time.

To fix ideas, let us suppose that either $v_0 \in C^\infty(\mathbb{R})$ is of period 1 or else $v_0 \in \mathbb{R} + C^\infty_{\downarrow}(\mathbb{R})$, the latter being the most interesting case physically, and let us also simplify life by requiring that $m \equiv v_0 - v''_0$ has only a finite number of (proper) changes of sign (per period). *Then the solution exists in its function class for all time $0 \leq t < \infty$ only if the points x_- where $m_0(x_-) < 0$ lie wholly to the left of the points*

*Received December 8, 1998; accepted for publication February 5, 1999.

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¹ $Q^\bullet = v'(t, Q) Q'$ and $Q'_0 \equiv 1$ forces $Q'(x) > 0$.

² $\frac{d}{dt} m(Q) Q'^2 = [m^\bullet(Q) + m'(Q) Q^\bullet] Q'^2 + 2mQ^\bullet Q' = 0$, as you will check.

x_+ where $m_0(x_+) > 0$ (which never happens in the periodic case unless m_0 is of one sign). Contrariwise, if m_0 is not of such a shape, then the solution breaks down at some time $T < \infty$: in fact, as $t \uparrow T$, $v'(t, Q) \downarrow -\infty$ at one or more of the (time independent) roots x_0 of $m(Q) = 0$; simultaneously, $Q'(t, x_0) \downarrow 0$, and more— $Q(t, x)$ flattens out in the largest interval $a \leq x \leq b$ containing x_0 in which $m_0(x) \equiv 0$. This is how the diffeomorphism comes to the “edge” of the group; in particular, the geodesic flow in $D(\mathbb{R})$ is incomplete. The existence of $v(t, x) : t < T, x \in \mathbb{R}$ is not discussed here: That is covered by the methods of CONSTANTIN-ESCHER [1998] and CONSTANTIN-McKEAN [1998]. The present paper deals only with breakdown; it is a technical amplification of McKEAN [1999] which reviewed the whole connection with the diffeomorphism group and explained the breaking in a simple case.

Acknowledgment. It is a pleasure to thank A. Constantin for listening with much patience to rough version of the proof described below.

2. Preliminaries. Keep in mind the constants of motion $m(Q)Q'^2 = m_0(x)$ and the moral they convey: that the points x_0 where $m(Q)$ changes sign are fixed in time. Note also the duality: that if $v(t, x)$ is a solution, then so is $-v(t, -x)$. This will cut down the number of cases to be treated. Now come some little tricks which will be continually in use. They obtain up to the breaking time $T \leq \infty$.

$$3) \quad \frac{d}{dt} e^Q (v' - v)(Q) = -\frac{1}{2} e^Q (v' - v)^2(Q) - \frac{1}{2} \int_{-\infty}^Q e^y (v' - v)^2(y) dy.$$

$$4) \quad \frac{d}{dt} e^{-Q} (v' + v)(Q) = -\frac{1}{2} e^{-Q} (v' + v)^2(Q) - \frac{1}{2} \int_Q^{\infty} e^{-y} (v' + v)^2(y) dy.$$

$$5) \quad \frac{d}{dt} v'(Q) = \frac{1}{2} v^2(Q) - \frac{1}{2} v'^2(Q) - \frac{1}{4} \int_{-\infty}^{\infty} e^{-|Q(x)-y|} (v' \pm v)^2(y) dy,$$

in which the ambiguous signature is that of $y - Q(x)$.

$$6) \quad |v| \text{ is bounded, independently of time, by a fixed constant } C.$$

$$7) \quad v' \text{ is bounded above by } 2C + \max v'_0.$$

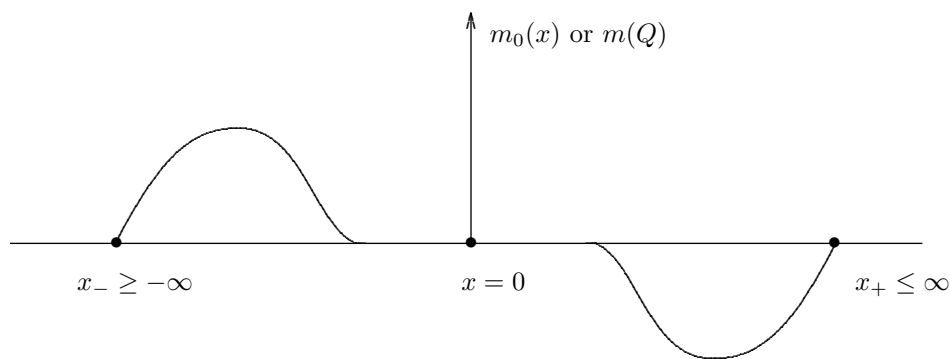
$$8) \quad v' \text{ is bounded below by } -C \text{ if breakdown does not take place.}$$

Proof. $e^Q(v' - v) = -\int_{-\infty}^Q e^y m(y) dy$. Now differentiate by t , using $Q^\bullet = v(Q)$ and $-m^\bullet = (mv)' + \frac{1}{2}(v^2 - v'^2)'$, and integrate by parts to obtain 3). 4) follows by duality. 5) is obtained by combining 3) and 4). 6) is plain if $v \in C^\infty_{\downarrow}(\mathbb{R})$: then $H = \int_{-\infty}^{\infty} (v^2 + v'^2) dx$ is a constant of motion and $v^2 = 2 \int_{-\infty}^x vv' \leq H$. The periodic case is similar with constant of motion $H = \int_0^1 (v^2 + v'^2) dx$ and $v^2 \leq 2H$. Likewise, if $v = c + w \in \mathbb{R} + C^\infty_{\downarrow}(\mathbb{R})$, then c and $H = \int_{-\infty}^{\infty} (w^2 + w'^2) dx$ are constants of motion and $|v| \leq |c| + \sqrt{H}$. 7) now follows from 3), 4), and 6): $e^{\pm Q} (v' \mp v)(Q)$ decreases with time and one of e^Q or e^{-Q} exceeds 1, so $v'(Q) \leq e^{\mp Q} (v'_0 \mp v_0)(Q) \pm v(Q)$, etc. 8) follows from 5) and 6) by the reasoning of sect. 1:

$$\frac{d}{dt} v'(Q) \leq \frac{1}{2} v^2(Q) - \frac{1}{2} v'^2(Q) \leq \frac{1}{2} C^2 - \frac{1}{2} v'^2(Q)$$

drives $v'(Q)$ down to $-\infty$ in finite time at any place where $v'_0(x)$ falls below $-C$. \square

3. Proof of Breakdown. It is to be proved that breakdown takes place if there is any positive “mass” m_0 to the left of some negative mass. Let $x = 0$ be chosen as in the figure so that $m_0(x)$ and so also $m(Q)$ is positive near $x_- < 0$ and negative



near $x_+ > 0$, a single interval containing $x = 0$ being permitted for $m_0(x)$ to vanish. The picture is self-dual. Now if $(v' - v)(Q)$ is ever negative at $x = 0$, then 3) in the form

$$\frac{d}{dt} e^{Q(v' - v)(Q)} \leq -\frac{1}{2} [e^{Q(v' - v)(Q)}]^2 e^{-Q}$$

implies $\int_0^\infty e^{-Q} dt < \infty$ in the absence of breakdown; likewise, if $(v' + v)(Q)$ is ever negative at $x = 0$, then $\int_0^\infty e^Q dt < \infty$; and since $\int_0^\infty e^{\pm Q} dt$ cannot both converge, so you can assume $(v' - v)(Q) \geq 0$ at $x = 0$ at all times $t \geq 0$, by duality. Now distinguish two cases according as $(v' + v)(Q)$ is ever negative at $x = 0$ or not.

Case 1: $(v' + v)(Q) \geq 0$ at $x = 0$ at all times $t \geq 0$. $(d/dQ) e^{Q(v' - v)(Q)} = -e^Q m(Q)$ is ≤ 0 between x_- and $x = 0$, so $(v' - v)(Q) \geq 0$ there, and $(v' + v)(Q)$ likewise; in particular, $v'(Q) \geq |v(Q)|$ for $x_- \leq x \leq 0$, and, in the absence of breakdown, so that $v'(Q) \geq -C$ by 8), 5) implies

$$+\infty > \int_0^\infty dt e^{-Q(x)} \int_0^{Q(x)} e^y (v' - v)^2(y) dy \text{ between } x = x_- \text{ and } x = 0.$$

Fix $x_- < a < c < b < 0$ with $c = \frac{1}{2}(a + b)$ and $m_0(x) > 0$ between a and b . Then with $x = c$ in the last display,

$$\begin{aligned} +\infty &> \int_0^\infty dt e^{-Q(c)} \int_{Q(a)}^{Q(c)} [e^y (v' - v)]^2 e^{-y} dy \\ &> \int_0^\infty dt e^{-Q(c)} [e^{Q(v' - v)} \text{ at } x = c]^2 \int_{Q(a)}^{Q(c)} e^{-y} dy \\ &= \int_0^\infty dt [(v' - v)(Q) \text{ at } x = c]^2 \times [e^{Q(c) - Q(a)} - 1]. \end{aligned}$$

This contradicts the fact that $m_0(x) > 0$ for $a \leq x \leq b$, as will be seen in a few easy steps.

STEP 1 uses $(v' - v)(Q) \geq 0$ to obtain the simple bound:

$$\begin{aligned} \left[\int_c^b \sqrt{m_0(x)} dx \right]^2 &= \left[\int_c^b \sqrt{m(Q)} dQ \right]^2 \\ &\leq \int_c^b m(Q) dQ \times [\Delta Q \equiv Q(b) - Q(c)] \\ &= \int_c^b (v - v' + v' - v'')(Q) dQ \times \Delta Q \\ &\leq [(v' - v)(Q) \text{ at } x = c] \times \Delta Q. \end{aligned}$$

STEP 2. Now let $v(Q)$ be negative at $x = b$. Then $v'(Q) \geq |v(Q)|$ implies $v(Q) \leq 0$ down to $x = a$, so from $Q^\bullet = v(Q)$ and $Q'^\bullet = v'(Q) Q'$, it appears that

$$Q'' = Q' \int_0^t v''(Q) Q' dt' = Q' \int_0^t (v - m)(Q) Q' dt' < 0,$$

whence

$$\begin{aligned} \left[\int_c^b \sqrt{m_0} \right]^4 &\leq [(v' - v)^2(Q) \text{ at } x = c] \times 2 [e^{Q(b)-Q(c)} - 1] \\ &\leq [(v' - v)^2(Q) \text{ at } x = c] \times 2 [e^{Q(c)-Q(a)} - 1], \end{aligned}$$

and now the summability of the last line forces $\text{meas } (t \geq 0 : v(Q) \leq 0 \text{ at } x = b)$ to be finite; in particular, $\int_0^\infty v_-(Q) dt > -\infty$ at $x = b$, by 6). Obviously, there is nothing special about $x = b$: the same is true at $x = a$ by a self-evident reprise. This was the goal of Step 2.

STEP 3 will confirm that $\int_0^\infty v_+(Q) dt$ is also finite at $x = a$: $(v' - v)(Q)$ is positive between x_- and 0 and decreases there in view of $v'' - v' = -m + v - v' < 0$, so

$$\exp \left[\int_0^t v'(Q) - \int_0^t v(Q) \right] = Q' e^{x-Q} \text{ is } \geq 1 \text{ and decreases, too.}$$

Then

$$\begin{aligned} 1 \leq Q'(b) e^{b-Q(b)} &\leq \frac{1}{b-a} \int_a^b e^{x-Q(x)} dQ \\ &\leq \frac{e^b}{b-a} \times e^{-Q(a)} \end{aligned}$$

and now this estimate, controlling $Q(a) = a + \int_0^t v_-(Q) + \int_0^t v_+(Q)$ from above, confirms $\int_0^\infty v_+(Q) dt < \infty$.

STEP 4. But this means that Q is bounded, not only if $x = a$, but at $x = c$ as well, by a self-evident reprise, and now $\left(\int_c^b \sqrt{m_0} \right)^4$ as seen in step 2, is over-estimated

³ v_- is the negative part of v .

⁴ v_+ is the positive part of v .

by a constant multiple of $e^Q (v' - v)^2(Q)$ at $x = c$, this being summable by 3) and the fact that $(v' - v)(Q) \geq 0$. That is the contradiction.

Case 2: $(v' - v)(Q) \geq 0$ at $x = 0$ for all $t \geq 0$ and $(v' + v)(Q) < 0$ at $x = 0$ for, e.g., $t = 0$. Then 4) shows that, in the absence of breakdown, $\int_0^\infty e^Q dt < \infty$ at $x = 0$ and so also for $x \leq 0$, Q being increasing in x , in which case, $\int_0^\infty e^{-Q} dt = +\infty$ and $e^Q (v' - v) \geq 0$ for every $x \leq 0$, by 3). Now $\int_0^\infty e^Q dt < \infty$ for $x \leq 0$ only if

$$\infty > \int_0^\infty e^Q Q' dt = \int_0^\infty \exp \left[\int_0^t (v' + v)(Q) dt' \right] dt \quad a.e.,$$

from which you learn that $(v' - v)(Q)$ turns strictly negative, and stays that way, by 4), at sometime $T(x) < \infty$, for almost all $x \leq 0$; moreover, the fact that $e^{-Q} (v' + v)(Q)$ decreases for $x_- \leq x \leq 0$ implies that $T(x)$ is finite and decreasing in the half-open interval $x_- < x \leq 0$. Two subcases are now distinguished according to the behavior of $\Delta Q = Q(b) - Q(a)$ for $x_- < a < b < 0$.

Case 2.1: ΔQ is bounded from above for every choice of $x_- < a < b < 0$ (as it must be in the periodic case when $\Delta Q \leq Q(1) - Q(0) = 1$). Now $\infty > T(a) = 0$, say, in which case $(v' - v)(Q) > 0$, $(v' + v)(Q) \leq 0$, and $v(Q) < 0$ for all $a \leq x \leq b$ and $t \geq 0$. Now comes a familiar type of trick:

$$\begin{aligned} \left[\int_a^b \sqrt{m_0} dx \right]^2 &= \left[\int_a^b \sqrt{m} dQ \right]^2 \leq \int_a^b m(Q) dQ \times \Delta Q \\ &\leq \int_a^b -v''(Q) dQ \times \Delta Q \\ &= -v'(Q)|_a^b \times \Delta Q \end{aligned}$$

from which it appears that, in the absence of breakdown, ΔQ is also bounded from below, by 7) and 8). But also $Q'' < 0$ as in Step 2 of Case 1, v being negative, from which you learn that $0 < Q'(b)/Q'(a) < 1$ and that

$$\frac{d}{dt} \frac{Q'(b)}{Q'(a)} = v'(Q)|_a^b \times \frac{Q'(b)}{Q'(a)} < 0 \text{ is summable.}$$

$Q'' < 0$ and the boundedness of ΔQ for every choice of $x_- < a < b < 0$ come into play once more to check that $Q'(b)/Q'(a)$ is bounded below, and now a contradiction is obtained: $v'(Q)|_a^b$ is summable, but

$$\left[\int_a^b \sqrt{m_0} dx \right]^2 \leq v'(Q)|_a^b \times \text{the upper bound of } \Delta Q \text{ is not.}$$

Case 2.2: $Q(b') - Q(a')$ is unbounded above for some $x_- < a' < b' < 0$. Now pick $x_- < c < a < b < a'$ with $T(c) = 0$ for simplicity and $m_0(x) > 0$ between c and b . Note that $v(Q)$ is negative, as before, so that $v''(Q) < v(Q) < 0$, $m(Q)$ being positive, and also that $Q'' < 0$. A variant of the familiar $\left[\int_a^b \sqrt{m_0} dx \right]^2 \leq \int_a^b m(Q) dQ \times \Delta Q$ is

now used. It reads

$$\begin{aligned} \left[\int_a^b \sqrt{m_0(x)} dx \right]^2 &\leq \int_a^b m(Q) [Q(x) - Q(c)] dQ \int_a^b \frac{dQ}{Q(x) - Q(c)} \\ &< \int_{Q(c)}^{Q(b)} -v''(x) [x - Q(c)] dx \lg \frac{Q(b) - Q(a)}{Q(a) - Q(c)} \\ &< \left. \begin{aligned} & - [v'(Q) \text{ at } x = b] \times Q(b) - Q(c) \\ & - [v(Q) \text{ at } x = c] \end{aligned} \right\} \times \lg \frac{b - a}{a - c}, \end{aligned}$$

$Q'' < 0$ being used to appraise the logarithm. I propose to show that $v'(Q) > 0$ at $x = b$ and that $v(Q) = o(1)$ at $x = c$ for suitable $t = t_1 < t_2 < \dots \uparrow \infty$. This will be contradictory. Chose these "special" times so that $Q(b') - Q(a') \uparrow +\infty$. Then $Q(a') - Q(b) \uparrow +\infty$, too, since $Q'' < 0$, and you can also assume $(d/dt) [Q(a') - Q(b)] > 0$ at special times. But

$$\frac{d}{dt} [Q(a') - Q(b)] = \int_b^{a'} v'(Q) dQ \leq [v'(Q) \text{ at } x = b] \times [Q(a') - Q(b)]$$

so $v'(Q)$ is positive at $x = b$ for special times and stays positive down to x_- in view of $v''(Q) < 0$. This is half the battle. Next, fix $d < c$. Then $-v(Q)$ decreases between d and c , so

$$0 < - [v(Q) \text{ at } x = c] \times [Q(c) - Q(d)] < - \int_d^c v''(Q) dQ,$$

in which the right hand side is bounded in the absence of breaking, and since $Q(c) - Q(d) \uparrow +\infty$ by reason of $Q'' < 0$, you may conclude that $v(Q) = o(1)$ at $x = c$ for special times. The proof is finished.

4. How It Breaks Down. The conditions for breaking are now established, but *how does it happen?*

Return to fig. 1 with $m(Q) > 0$ for $x_- < x < a$, $m(Q) = 0$ for $a \leq x \leq b$, and $m(Q) < 0$ for $b < x < x_+$; necessarily, $a \leq 0 \leq b$. $T < \infty$ is the breaking time, *i.e.*, the moment when $v'(Q)$ gets out of hand, assuming that this takes place between x_- and x_+ . Breakdown can happen only in this way, as will be seen.

Item 1. $v'(Q) \equiv w$ is bounded above, by 7), and 5) in the form $m^\bullet \leq \frac{1}{2}C^2 - \frac{1}{2}w^2$ shows that it can get out of hand only by an ultimate decrease to $-\infty$, in the style of $w =$ a negative constant $\times (T-t)^{-1}$ or worse, in which case $Q'(t, x) = \exp \int_0^t v'(Q) \downarrow 0$ as $t \uparrow T$, and conversely: if $Q' \downarrow 0$ as $t \uparrow T < \infty$, then $w = v'(Q)$ is ultimately very negative and 5) drives it down to $-\infty$.

Item 2. $Q(T-, x)$ exists everywhere, v being bounded, and $e^Q (v' - v)(Q)$ decreases/increases where $m(Q)$ is positive/negative, from which you see that breakdown occurs, if at all, at points where $m(Q)$ changes sign, from positive to negative: $x = 0$ is such a point.

Item 3. $v'(Q)$ at $x = 0$ now decreases to $-\infty$ as $t \uparrow T$ and carries with it the slope $v'(Q)$ at the general point x of the (maximal) interval $[a, b]$ where $m(Q) = 0$, $v'' = v$ being bounded there; in particular, $Q(x)$ flattens out in the whole interval $a \leq x \leq b$, by item 1.

Item 4 is the converse: if $Q(x)$ flattens out in any interval $a \leq x \leq b$, then $m_0(x)$ and so also $m(Q)$ vanishes there; simultaneously, $v'(Q) \downarrow -\infty$.

Proof. Let $Q(x)$ flatten out in $a \leq x \leq b$ with $x_- < a < b \leq 0$, say, and take $c < d$ properly between a and b . Then

$$\begin{aligned} \left[\int_c^d \sqrt{m_0} dx \right]^2 &\leq \int_c^d m(Q) [Q(b) - Q(x)] dQ \int_c^d \frac{dQ}{Q(b) - Q(x)} \\ &\leq \int_c^b (v - v'')(Q) [Q(b) - Q(x)] dQ \times \ell g \frac{Q(b) - Q(c)}{Q(b) - Q(d)} \\ &\leq \left. \begin{aligned} &C \times \frac{1}{2} [Q(b) - Q(a)]^2 \\ &+ v'(Q) \text{ at } x = a \times [Q(b) - Q(a)] \\ &- v(Q)|_a^b \end{aligned} \right\} \times \ell g \frac{Q(b) - Q(c)}{Q(b) - Q(d)} \\ &\leq \text{a constant multiple of } \sqrt{Q(b) - Q(a)} \times \ell g \frac{Q(b) - Q(a)}{Q(b) - Q(d)} \end{aligned}$$

owing to the control of $v'(Q)$ above, by 7), and to the elementary estimate $|v(b) - v(a)|^2 \leq \int_a^b (v')^2 \times (b - a)$, the integral being controlled by the proper constant of motion in all function classes. Now $Q'' = Q' \int_0^t v''(Q) Q'$ is bounded above by a multiple K of Q' for $a \leq x \leq b$ in view of $m = v - v'' > 0$ and 7); in particular, $Q' e^{-KQ}$ is decreasing in x , so the flattening out of Q in $[a, b]$ implies that, for $t \uparrow T$, $Q' \downarrow 0$ and so also $v'(Q) \downarrow -\infty$, uniformly in $[a, b]$.

Now use 5) for $w = v'(Q)$ in the form $w^\bullet = -\frac{1}{2} w^2 + 2V$, with $V = \frac{1}{4} v^2(Q)$ etc., and write $w = 2\psi^\bullet/\psi$ with $\psi(0) = 1$, say, and $\psi^\bullet(0) < 0$, fixing the variable x between a and b . Then $\psi^{\bullet\bullet} = V\psi$, $\psi(T-) = 0$, and $\psi^\bullet(T-) < 0^5$, reflecting the fact that $w \downarrow -\infty$ as $t \uparrow T$. It follows that $v'(Q)$ behaves like $-2/(T - t)$ or nearly so as $t \uparrow T$, so Q' is no worse than $(T - t)^{3/2}$ for $a \leq x \leq b$. But now $\int_c^d \sqrt{m_0}$, as seen in the big display, is over-estimated by $(T - t)^{3/4} \ell g(T - t)$ and so vanishes, and $m_0(x) = 0$ in whole of $[a, b]$, the subinterval $[c, d]$ having been chosen as you will.

5. A Little Example. This falls outside the function classes admitted above but never mind. $v(t, x)$ is soliton/anti-soliton pair

$$\frac{1}{2} p e^{-|x+q|} - \frac{1}{2} p e^{-|x-q|},$$

symmetric about $x = 0$, with positive $q = q(t)$ and $p = p(t)$, $q^\bullet = -p(1 - e^{-2q})$, $p^\bullet = p^2 e^{-2q}$, and the constant of motion $H = \frac{1}{2} p^2 (1 - e^{-2q})$. With $\Theta = T - t$, you find $q = \ell g ch \Theta$, $p = ch \Theta / sh \Theta$ and, for $x > \ell g ch T = q(0)$, $e^Q = e^x + ch \Theta - ch T$, as you may check. In particular, Q flattens out and $v'(Q) \downarrow -\infty$ in the interval $|x| \leq q(0) = \ell g ch T$, precisely.

⁵ $\psi^\bullet(T-) = 0$ implies $\psi \equiv 0$ which is not the case.

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