

CONJECTURAL ALGEBRAIC FORMULAS FOR REPRESENTATIONS OF GL_n^\dagger

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0. Introduction.

0.0. Let F be a non-archimedean local field. Due to the recent work of Harris and Taylor [HT], we know that the local Langlands conjecture is true. In other words, for any local field F we know the existence of the one-to-one correspondence $\phi_n: \Pi_n \rightarrow GL_n(F)^\wedge$ between the set Π_n of n -dimensional representations of the Galois group $\mathfrak{G} = \text{Gal}(\overline{F}/F)$ and the set $GL_n(F)^\wedge$ of irreducible nondegenerate representations of the group $GL_n(F)$. In particular, one can associate an irreducible representation π_χ of the group $GL_n(F)$ to a pair (E, χ) , where E is a commutative semisimple algebra over F of degree n and χ is a multiplicative character of the group E^* . However, we do not know any explicit construction for the representation π_χ . In our paper we propose an explicit “algebraic” construction for the representation π_χ at least for $n = 4$.

One can inductively characterize the correspondence ϕ_n in the following way. Suppose that we know the correspondence ϕ_{n-2} . Then for any $\sigma \in \Pi_n$ we can characterize the representation $\phi_n(\sigma)$ as the unique representation of $GL_n(F)$ such that for any representation $\rho \in \Pi_{n-2}$ we have

$$\Gamma(\phi_n(\sigma), \phi_{n-2}(\rho)) = \Gamma(\sigma \otimes \rho^*),$$

where $\Gamma(\phi_n(\sigma), \phi_{n-2}(\rho))$ is the Gamma function of Jacquet, Piatetskiĭ–Shapiro, Shalika [JPS] and $\Gamma(\sigma \otimes \rho^*)$ is the Gamma function of Langlands. More precisely, let $GL_n(F)^\wedge_u \subset GL_n(F)^\wedge$ and $\Pi_{n,u} \subset \Pi_n$ be the subsets of unitary representations. We denote by Γ_n the function on the set $GL_{n-2}(F)^\wedge_u \times \Pi_{n,u}$ defined by $\Gamma_n(\pi, \rho) = \Gamma(\pi, \phi_{n-2}(\rho))$. To any maximal torus T in $GL_n(F)$ and any character χ of T we may associate an n -dimensional representation $\sigma_\chi \in \Pi_n$ and therefore a representation $\pi_\chi = \phi_n(\sigma_\chi) \in GL_n(F)^\wedge$. Let (ρ_{n-2}, Wh_{n-2}) be the Whittaker representation of $GL_{n-2}(F)$. The representation of $GL_{n-2} \times T$ in the space $Wh_{n-2} \otimes L^2(T)$ decomposes in the direct integral

$$(0.1) \quad Wh_{n-2} \otimes L^2(T) = \oplus \int_{GL_{n-2}(F)^\wedge_u \times T} (V_\pi \otimes \chi),$$

where V_π is the space of the representation π . Let A_{n-2}^T be the unitary operator in the space $Wh_{n-2} \otimes L^2(T)$ commuting with $GL_{n-2}(F) \times T$ that in the above decomposition is the multiplication by $\Gamma_n(\pi, \pi_\chi)$. As follows from [JPS], one can write explicit formulas for all π_χ if one knows the operator A_{n-2}^T .

The goal of this paper is to propose an algebraic formula for this operator in the case $n = 4$. More precisely, for any n we construct “algebraic” data $\mu_n = (\mathbf{Z}, \mathbf{Y}, \mathbf{p}, \omega, \mathbf{f})$ that define an operator $A_{n-2}^{T_0}$ corresponding to the maximal split torus

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T_0 in GL_n . In this case the representation π_χ is the induced principal series representation. Next, for $n = 4$, we define an action of the symmetric group S_4 on μ_4 . Then for any maximal torus T in GL_4 we can define the corresponding twist $\mu_{T,4}$ of μ_4 and, therefore, an operator \widehat{A}_2^T on the space $Wh_2 \otimes L^2(T)$. We conjecture that $\widehat{A}_2^T = A_2^T$. Moreover, we conjecture that for any n there exists an action of the symmetric group S_n on μ_n such that $\widehat{A}_{n-2}^T = A_{n-2}^T$ for any maximal torus T in $GL_n(F)$.

Note that for $n = 3$ this conjecture was proved in [K1].

The same formulas work for a real field F .

0.1. Algebraic measures and twisting. We start with the notion of an algebraic measure. For the rest of paper we choose a nontrivial additive character ψ_F of the field F , $\psi_F: F \rightarrow \mathbb{C}^*$. We will denote algebraic varieties over F by bold letters (say, \mathbf{X}) and the sets of F -point by the corresponding italic letters (say, $X = \mathbf{X}(F)$). Similarly, morphisms of algebraic varieties will be denoted by bold letters (say, $\mathbf{f}: \mathbf{X}_1 \rightarrow \mathbf{X}_2$) and the induced mappings of the sets of F -points by the corresponding italic letters (say, $f: X_1 \rightarrow X_2$). For a smooth algebraic variety \mathbf{Y} by $\mathcal{S}(Y)$ we denote the space of locally constant function on Y with compact support.

Let \mathbf{Y} be an algebraic variety over F and μ a complex valued measure on Y . An algebraic presentation of μ is data $(\mathbf{Z}, \mathbf{Y}, \mathbf{p}, \omega, \mathbf{f})$, where \mathbf{Z} is a smooth algebraic variety, $\mathbf{p}: \mathbf{Z} \rightarrow \mathbf{Y}$ a morphism, $\omega \in \Omega^r(\mathbf{Z})$, $r = \dim \mathbf{Z}$, a volume form (i.e., a differential form of the top degree) on \mathbf{Z} , and \mathbf{f} an algebraic function on \mathbf{Z} such that the measure μ is equal to the distribution

$$p_*(|\omega| \cdot (\psi_F \circ f)),$$

where $|\omega|$ is the measure on Z corresponding to ω , see [W]. In other words, for a function $\varphi \in \mathcal{S}(Y)$ we have

$$\int_Y \varphi \mu = \int_Z \varphi(p(z)) \psi_F(f(z)) |\omega|(z).$$

One has to be careful since in cases we are interested in the integral in the right-hand side of the last formula does not converge absolutely. Therefore, we must specify the integration process. We choose the following scheme.

For $a \in F$ let $\mathbf{Z}_a \subset \mathbf{Z}$ be the level variety $\mathbf{Z}_a \stackrel{\text{df}}{=} \{\mathbf{f} = a\}$. The volume form ω on \mathbf{Z} determined the volume form ω_a on \mathbf{Z}_a by the formula

$$\omega_a \stackrel{\text{df}}{=} \text{Res} \frac{\omega}{f - a}.$$

DEFINITION 1. *Algebraic measure* is data $\mu = (\mathbf{Z}, \mathbf{Y}, \mathbf{p}, \omega, \mathbf{f})$ such that the following conditions are satisfied.

(i) For any function $\varphi \in \mathcal{S}(Y)$ and for almost any $a \in F$ the integral

$$I_a(\varphi) \stackrel{\text{df}}{=} \int_{\mathbf{Z}_a} \varphi(p(z)) |\omega_a|(z)$$

converges absolutely.

(ii) $I_a(\varphi)$ is a locally L^1 -function of a .

(iii) The limit

$$I(\varphi) \stackrel{\text{df}}{=} \lim_{n \rightarrow \infty} \int_{|a| \leq p^n} \psi_F(a) I_a(\varphi) |da|$$

exists.

(iv) There exists a complex valued measure μ on Y such that $I(\varphi) = \int_Y \varphi \mu$ for $\varphi \in \mathcal{S}(Y)$.

In the case μ is called the *realization* of $\boldsymbol{\mu}$ and $\boldsymbol{\mu}$ is called an *algebraic presentation* of μ .

Remark. In general, a measure μ can have several nonisomorphic algebraic presentations.

Let Γ be a finite group of F -automorphisms of \mathbf{Y} . A lifting of the action of Γ to $\boldsymbol{\mu}$ is an F -action of Γ on \mathbf{Z} that commutes with \mathbf{p} and preserves ω and \mathbf{f} . If μ is the realization of a Γ -invariant data $\boldsymbol{\mu}$, then μ itself is Γ -invariant.

Given a lifting of the action of Γ to $\boldsymbol{\mu}$, we can construct twisted forms of μ as follows.

Let $\mathfrak{G} = \text{Gal}(\overline{F}/F)$. Elements of $H^1(\mathfrak{G}, \text{Aut } \mathbf{Z})$ correspond to homomorphisms $\alpha: \mathfrak{G} \rightarrow \Gamma$ modulo conjugation by elements of Γ . To any such α we associate the twisted form \mathbf{Z}_α of \mathbf{Z} . This is an algebraic variety over F , which is isomorphic to \mathbf{Z} over \overline{F} . The set of F -points of \mathbf{Z}_α is given by

$$Z_\alpha = \{x \in \mathbf{Z}(\overline{F}) \mid \zeta z = \overline{\alpha}(\zeta)z \text{ for } \zeta \in \mathfrak{G}\}.$$

Similarly, to α we can associate the twisted form \mathbf{Y}_α of \mathbf{Y} . Since the action of Γ on \mathbf{Z} and \mathbf{Y} preserves ω and commutes with \mathbf{p} and \mathbf{f} , we get the twisted data $\boldsymbol{\mu}_\alpha = (\mathbf{Z}_\alpha, \mathbf{Y}_\alpha, \mathbf{p}_\alpha, \omega_\alpha, \mathbf{f}_\alpha)$.

In the case where $\boldsymbol{\mu}_\alpha$ defines an algebraic measure (i.e., the integrals in Definition 1 converge and the limit exists), we define the measure μ_α on Y_α as the realization of $\boldsymbol{\mu}_\alpha$. We emphasize that μ_α depends not just on μ and the action of Γ on \mathbf{Y} , but also on the lifting of this action to $\boldsymbol{\mu}$.

A measure on Y is a linear functional of the space of continuous function. We need a generalization to the case where the function are replaces with sections of a line bundle on \mathbf{Y} . More precisely, we consider the following situation. Let $\tilde{\mathbf{Y}}$ be a variety with the free action $m: \mathbf{U} \times \tilde{\mathbf{Y}} \rightarrow \tilde{\mathbf{Y}}$ of a unipotent group \mathbf{U} , such that $\mathbf{Y} = \tilde{\mathbf{Y}}/\mathbf{U}$, and let $\Psi: \mathbf{U} \rightarrow \mathbf{G}_a$ be a character of \mathbf{U} . Then we can consider the space $\mathcal{S}_\Psi(Y)$ of locally constant functions φ on $\tilde{\mathbf{Y}}$ such that

$$\varphi(u\tilde{y}) = \psi_F(-\Psi(u))\varphi(\tilde{y}), \quad u \in U, \quad \tilde{y} \in \tilde{Y},$$

and the function $\|\varphi\|$ on Y has a compact support.

A Ψ -measure is a linear functional on $\mathcal{S}_\Psi(Y)$ that extends to a continuous functional on a space of continuous Ψ -equivariant functions on $\tilde{\mathbf{Y}}$ that are ‘‘compactly supported’’ (in the above sense). Let $\tilde{\mathbf{Z}}$ be a manifold with a free action of \mathbf{U} , $\tilde{\mathbf{p}}: \tilde{\mathbf{Z}} \rightarrow \tilde{\mathbf{Y}}$ an \mathbf{U} -equivariant map, $\tilde{\omega}$ a \mathbf{U} -invariant volume form on $\tilde{\mathbf{Z}}$, $\tilde{\mathbf{f}}: \tilde{\mathbf{Z}} \rightarrow \mathbb{A}^1$ a function such that

$$\mathbf{f}(u\tilde{z}) = \Psi(u) + \mathbf{f}(\tilde{z}).$$

Let us choose an invariant volume form du on \mathbf{U} .

DEFINITION 2. We say that the data $\tilde{\boldsymbol{\mu}} = (\tilde{\mathbf{Z}}, \tilde{\mathbf{Y}}, \tilde{\mathbf{p}}, \tilde{\omega}, \tilde{\mathbf{f}})$ is an algebraic Ψ -measure if the following conditions are satisfied.

(i') For $\varphi \in \mathcal{S}_\Psi(Y)$ and $a \in F$ denote

$$\tilde{Z}_{a,\varphi} = \{z \in \tilde{\mathbf{Z}} : \tilde{\mathbf{f}}(z)\varphi(\tilde{\mathbf{p}}^*(z)) = a\}.$$

Then $\tilde{Z}_{a,\varphi}$ is invariant under U . Let $Z_{a,\varphi} = \tilde{Z}_{a,\varphi}/U$ and $|\omega_{a,\varphi}|$ the measure on $Z_{a,\varphi}$ induced by $\tilde{\omega}/du$. We assume that for any $\varphi \in \mathcal{S}_\Psi(Y)$ and for almost any $a \in F$ the integral

$$I_a(\varphi) \stackrel{\text{df}}{=} \int_{Z_{a,\varphi}} \tilde{f}(z)\varphi(\tilde{p}^*(z))|\omega_{a,\varphi}|$$

converges absolutely.

(ii') $I_a(\varphi)$ is a locally L^1 -function of a .

(iii') The limit

$$I(\varphi) \stackrel{\text{df}}{=} \lim_{n \rightarrow \infty} \int_{|a| \leq p^n} \psi_F(a) I_a(\varphi) |da|$$

exists.

(iv') There exists an (U, Ψ) -equivariant complex valued measure μ on \tilde{Y} such that $I(\varphi) = \int_Y \varphi \mu$ for $\varphi \in \mathcal{S}_\Psi(Y)$.

As before, we call the data $\tilde{\mu}$ an *equivariant presentation* of μ and μ the *realization* of $\tilde{\mu}$.

Similarly to the above, we can define twisting of equivariant measures.

0.2. Γ -factors. Let \mathbf{U}_- be the lower unipotent subgroup in \mathbf{GL}_ℓ and $\Psi: \mathbf{U}_- \rightarrow \mathbf{G}_a$ the homomorphism given by

$$\Psi(u) = u_{21} + \cdots + u_{\ell, \ell-1}.$$

The *Whittaker representation* (ρ_ℓ, Wh_ℓ) of GL_ℓ is defined by the formula

$$\rho_\ell = \text{Ind}_{\mathbf{U}_-}^{GL_\ell}(\psi).$$

An irreducible representation π of GL_n is called *generic* if it occurs in the decomposition of ρ_n into irreducible components. It is known that any generic unitary π occurs in ρ_ℓ exactly once.

Let T be a maximal torus in $GL_n(F)$ and χ a unitary character of T . The pair (T, χ) determines an n -dimensional representation ρ_χ of \mathfrak{G} , and by the Langlands correspondence (see [HT]), a unitary nondegenerate irreducible representation π_χ of $GL_n(F)$. Let σ be a generic unitary irreducible representation of GL_ℓ . Jacquet, Piatetski-Shapiro, and Shalika [JPS], associated to the pair (π_χ, σ) the number $\Gamma(\pi_\chi, \sigma)$, $|\Gamma(\pi_\chi, \sigma)| = 1$. Using the direct integral decomposition similar to (0.1), we combine the numbers $\Gamma(\pi_\chi, \sigma)$ for all unitary characters $\chi: T \rightarrow \mathbb{C}$ and all generic unitary irreducible representations σ of $GL_\ell(F)$ in a unitary operator A_ℓ^T in the space $Wh_\ell \otimes L^2(T)$ commuting with the action of GL_ℓ on the first factor multiplication by elements of T in the second factor.

Define the action m of the unipotent subgroup $\mathbf{U}_- \subset \mathbf{GL}_\ell$ on the space $\tilde{\mathbf{Y}} = \mathbf{GL}_\ell \times \mathbf{T}$ by left multiplication on the first factor. Let $\mathbf{U}^{(2)} = \mathbf{U}_- \times \mathbf{U}_-$ and let $\Psi^{(2)}: \mathbf{U}^{(2)} \rightarrow \mathbf{G}_a$ be given by

$$\Psi^{(2)}(u, u') = \Psi(u) - \Psi(u').$$

The action m determines the action $m^{(2)}$ of $\mathbf{U}^{(2)}$ on $\tilde{\mathbf{Y}} = \mathbf{GL}_\ell \times \mathbf{T} \times \mathbf{GL}_\ell \times \mathbf{T}$. Define the $\Psi^{(2)}$ -measure $\mu(T, \ell)$ on \tilde{Y} by the formula

$$\int_{\tilde{Y}} f_1(t_1, g_1) \overline{f_2(t_2, g_2)} \mu(T, \ell) = (A_\ell^T f_1, f_2)_{Wh_\ell \otimes L^2(T)}.$$

The first result of the paper (Proposition 5.1) is the construction of an algebraic presentation $\mu_{T_0, n}$ of the measure $\mu(T, \ell)$ in the case where $n = 4$, $T = T_0$ is the split torus in GL_4 , and $\ell = 2$.

0.3. Twisting of the algebraic presentation. For a unitary character $\chi: T_0 \rightarrow \mathbb{C}^*$ let π_χ be the corresponding irreducible unitary representation of the principal series. As is well known, for $w \in S_n$ the representation π_χ and π_{χ^w} are equivalent. Therefore, the constructed $\Psi^{(2)}$ -measure $\mu(T_0, n - 2)$ on Y is S_n -invariant.

Our second result (Proposition 5.2) is the construction of the lifting of the action of S_n on \widehat{Y} to the presentation $\mu_{T_0, n}$ for $n = 4$.

Let T be a maximal torus in GL_4 . Any such torus is obtained from the maximal split torus T_0 by an element of $H^1(\mathfrak{G}, \text{Aut } T_0)$, i.e., by a homomorphism $\alpha: \mathfrak{G} \rightarrow S_n$. Using the lifting of S_4 we can define μ_α as the twisting of $\mu = \mu_{T_0, 4}$.

CONJECTURE 1. *The data μ_α define a $\Psi^{(2)}$ -measure μ_α on $\widetilde{Y} = GL_2 \times T \times GL_2 \times T$.*

The second conjecture is that this measure coincides with the measure defined by the Γ -function. More precisely, let A_α be the operator on the space $Wh_{n-2} \otimes L^2(T)$ corresponding to the measure μ_α .

CONJECTURE 2. *The operator A_α is unitary and in the direct integral decomposition (0.1) is given by the multiplication by $\Gamma(\pi_\chi, \sigma)$*

In other words, $A_\alpha = A_{n-2}^T$ for the operator A_{n-2}^T described in 0.0.

Finally, we conjecture that all of the above remains true for an arbitrary n .

CONJECTURE 3. *For an arbitrary $n \geq 4$ there exists a lifting of the action of S_n to the algebraic presentation $\mu = \mu_{T_0, n-2}$. For a maximal torus T in GL_n corresponding to a homomorphism $\alpha: \mathfrak{G} \rightarrow S_n$, the twisted data μ_α determine a $\Psi^{(2)}$ -measure μ_α on $\widetilde{Y} = GL_{n-2} \times T \times GL_{n-2} \times T$. The corresponding operator in the space $Wh_{n-2} \otimes L^2(T)$ is unitary and in the decomposition (0.1) it is the multiplication by $\Gamma(\pi_\chi, \sigma)$.*

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1. Measures. In this section we present a general result about complex valued measures on vector bundles over smooth varieties over F . This result can be viewed as a formalization of the formula

$$\int_F \psi_F(xy) dy = \delta(x),$$

which is well known in the theory of distributions.

Let \mathbf{M} be an m -dimensional algebraic variety over F . By $\Omega^m(M)$ we will always denote the space of volume forms on \mathbf{M} . Let \mathcal{L} a one-dimensional vector bundle on \mathbf{M} , \mathcal{L}^* the dual bundle, L, L^* the total spaces of \mathcal{L} and \mathcal{L}^* , and $\pi: L \rightarrow M, \pi^*: L^* \rightarrow M$ the corresponding projections. Let also $\zeta: M \rightarrow L^*$ be the zero section of \mathcal{L}^* . For an open set $U \subset M$ we denote by L_U the total space of the restriction of \mathcal{L} to U . Similarly, L_U^* is the total space of the restriction of \mathcal{L}^* to U .

Let $\gamma \in \Gamma(U, \mathcal{L}^*)$ be a section of \mathcal{L}^* and let $N_{\gamma, U} = \{x \in U \mid \gamma(x) = \zeta(x)\}$ be the subvariety of zeros of γ in U . Let $\mathcal{T}_M(N_{\gamma, U})$ and $\mathcal{L}^*(N_{\gamma, U})$ be the restrictions to $N_{\gamma, U}$

of the tangent bundle \mathcal{T}_M and of the vector bundle \mathcal{L}^* . Denote by $T_M(N_{\gamma,U})$ and $L^*(N_{\gamma,U})$ the corresponding total spaces. Let x be a smooth point of $N_{\gamma,U}$, $y = \gamma(x)$ the corresponding point of L^* . The tangent space to L^* at y is canonically represented as the direct sum

$$(1.1) \quad T_{L^*,y} = T_{M,x} \oplus F,$$

where F is the one-dimensional tangent space to the fiber L_x^* of \mathcal{L}^* over x . Therefore the composition of the differential of γ and the projection of $T_{L^*,y}$ to the second summand in (1.1) determines a morphism $\theta_x: T_{M,x} \rightarrow L_x^*$.

DEFINITION 1.1. (i) A section γ is said to be *generic* at a smooth point $x \in N_\gamma$ if θ_x is surjective.

(ii) A section γ of \mathcal{L}^* is said to be *generic in U* if $N_{\gamma,U}$ is smooth and generic at all points of $N_{\gamma,U}$.

If γ is generic at x , then we can identify $\text{Ker } \theta_x$ and the tangent space $T_{N_\gamma,x}$ to N_γ at x .

Let $U \subset M$ be an open set, $\gamma \in \Gamma(U, \mathcal{L}^*)$ a generic section. Our next goal is to construct a morphism $\eta(\gamma): \Omega^{m+1}(L_U) \rightarrow \Omega^{m-1}(N_{\gamma,U})$.

Let γ be generic at a point $x \in N_{\gamma,U}$. Since $T_{L,x} = L_x \oplus T_{M,x}$, for the fiber of $\Omega^{m+1}(L_U)$ at x we have

$$\Omega^{m+1}(L_U)_x = L_x^* \otimes \Lambda^m(T_{M,x}^*).$$

The exact sequence

$$0 \rightarrow T_{N_\gamma,x} \rightarrow T_{M,x} \xrightarrow{\theta_x} L_x^* \rightarrow 0$$

shows that

$$\Lambda^m(T_{M,x}^*) = \Lambda^{m-1}(T_{N_\gamma,x}^*) \otimes L_x = \Omega^{m-1}(N_{\gamma,U})_x \otimes L_x,$$

so that we have

$$\Omega^{m+1}(U)_x = L_x^* \otimes \Omega^{m-1}(N_{\gamma,U})_x \otimes L_x.$$

Using the pairing $L_x^* \otimes L_x \rightarrow F$, we get the map

$$(1.2) \quad \Omega^{m+1}(U)_x \rightarrow \Omega^{m-1}(N_{\gamma,U})_x.$$

The collection of maps (1.2) for all $x \in N_{\gamma,U}$ yields the required map $\eta(\gamma)$.

In coordinates the map $\eta(\gamma)$ is described as follows. Let $U \subset M$ be such that the restriction $\mathcal{L}|_U$ is the trivial line bundle. Choose a trivialization $L_U = U \times F$ and the dual trivialization $L_U^* = U \times F$. Denote by y the coordinate in the fibers of projection $L_U \rightarrow U$ and by y^* be the dual coordinate in the fibers of the projection $L_U^* \rightarrow U$. A section γ of \mathcal{L}^* over U is given by a regular function $\theta(x)$ on U so that $\gamma(x) = (x, \theta(x)) \in U \times F$. For such a section, $N_{\gamma,U} = \{x \in U \mid \theta(x) = 0\}$ and γ is generic at a point $x \in N_{\gamma,U}$ if and only if $d\theta \neq 0$ at x .

A volume form $\omega \in \Omega^{m+1}(U)$ can be written as

$$(1.3) \quad \omega = \ell(p, y) \omega' \wedge dy,$$

where $\omega' \in \Omega^m(U)$, $\ell(p, y)$ is a function on $L_U = U \times F$. For such a form ω we have

$$(1.4) \quad \eta(\gamma)(\omega) = \text{Res}_{N_{\gamma,U}} \left\{ \frac{\ell(p, 0) \omega'}{\theta} \right\}.$$

Clearly, the right-hand side of (1.4) does not depend of the representation of ω in the form (1.3).

DEFINITION 1.2. For an open $U \subset M$, a volume form $\omega \in \Omega^{m+1}(L_U)$ is said to be *fiberwise constant* if $t_b^* \omega = \omega$ for any $b \in \Gamma(U, \mathcal{L})$, where $t_b: L_U \rightarrow L_U$ is the fiberwise addition.

Denote by $\Omega_{fc}^{m+1}(L_U)$ the space of fiberwise constant volume forms on L_U . In coordinates $\omega \in \Omega_{fc}^{m+1}(L_U)$ if and only if in some (hence every) representation of ω in the form (1.3), ℓ does not depend on y . In this case we can take $\ell = \text{const}$.

Let γ be a section of $\mathcal{L}^*|_U$. Define the complex valued function ψ_γ on L_U by the formula

$$\psi_\gamma(z) = \psi_F(\langle \gamma(x), z \rangle),$$

where $z \in L_U$, $x = \pi(y) \in U$, and $\gamma(x) \in \mathcal{L}^*(x)$ is considered as a linear functional on $\mathcal{L}(x)$.

Let \mathcal{L} be as before, $\omega \in \Omega^{m+1}(L)$, and f a complex valued locally constant function on L .

We say that f is locally integrable with respect to ω at a point $x \in M$ if for each sufficiently small compact neighborhood U of x the following condition holds.

- Choose a trivialization $L_U = U \times F$. Let $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \dots \subset F$ be a sequence of open compact subgroups such that $\bigcup \mathcal{O}_i = F$. Denote $V_i = U \times \mathcal{O}_i$. Then the limit

$$(1.5) \quad \lim_{i \rightarrow \infty} \int_{V_i} f|\omega|$$

exists.

It is clear that the limit (1.5) does not depend on the trivialization \mathcal{L} over U in (i) and on the sequence $\{\mathcal{O}_i\}$ in (ii). We denote this limit by

$$\lim_{i \rightarrow \infty} \int_{V_i} f|\omega| = \int_{L_U} f|\omega|.$$

It is also clear that now we can define $\int_{L_U} f|\omega|$ for any open set $U \subset M$ and an arbitrary locally constant function f on L_U such that the projection of $p(\text{supp } f) \subset M$ is compact and f is integrable with respect to ω at each point of $p(\text{supp } f)$.

PROPOSITION 1.1. *Let $U \subset M$ be a compact open set. For any generic section γ of $\mathcal{L}^*|_U$, any locally compact complex valued function f on U , and any fiberwise constant volume form $\omega \in \Omega_{fc}^{d+1}(L_U)$, the function $\pi^*(f)\psi_\gamma$ is locally integrable with respect to ω at all points of U and the integral is given by*

$$\int_{L_U} \pi^*(f)\psi_\gamma|\omega| = \int_{N_{\gamma,U}} f|\eta(\gamma)(\omega)|,$$

where $N_{\gamma,U}$ is the set of zeros of γ in U and in the right hand side we take the restriction of f in $N_{\gamma,U}$.

Proof. First, we consider a case where M is an affine line with the coordinate x , \mathcal{L} is a trivial one-dimensional bundle with the coordinate y along the fibers, and the section γ of \mathcal{L}^* is given by a function $y^* = \theta(x)$, so that $\gamma(x) = (x, \theta(x))$. We have $N_\gamma = \{x \mid \theta(x) = 0\}$ and γ is generic at a point $x \in N_\gamma$ if and only if $\theta'(x) \neq 0$. If γ is generic, then N_γ consist of the finite number of isolated points. Assume that

U is so small that $N_{\gamma,U}$ consists of a single point x_0 . For a fiberwise constant form $a(x) dx dy \in \Omega^2(L_U)$ the value of zero-form $\eta(\gamma)(\omega)$ at $x_0 \in N$ is $a(x_0)/\theta'(x_0)$.

Let $\{\mathcal{O}_i\}$ be an increasing sequence of open compact subgroups in F , $\cup \mathcal{O}_i = F$, and $V_i = U \times \mathcal{O}_i \subset U \times F = L_U$. Then

$$\begin{aligned} \int_{V_i} \pi^*(f)(z) \psi_\gamma(z) |\omega| &= \int_{U \times \mathcal{O}_i} \psi_F(y\theta(x)) |a(x)| |dx| |dy| \\ &= \int_U f(x) |a(x)| |dx| \int_{\mathcal{O}_i} \psi_F(y\theta(x)) |dy|. \end{aligned}$$

Denote $\mathcal{O}_i^\perp = \{y \in F \mid \psi_F(yu) = 1 \text{ for all } u \in \mathcal{O}_i\}$. Then

$$\int_{\mathcal{O}_i} \psi_F(y\theta(x)) |dy| = \begin{cases} \text{meas } \mathcal{O}_i & \text{if } \theta(x) \in \mathcal{O}_i^\perp, \\ 0 & \text{otherwise.} \end{cases}$$

Next, denote $\widehat{\mathcal{O}}_i^\perp = \{y \in F \mid \theta(y) \in \mathcal{O}_i^\perp\}$. Then

$$\int_{V_i} \pi^*(f)(z) \psi_\gamma(z) |\omega| = (\text{meas } \mathcal{O}_i) \int_{U \cap \widehat{\mathcal{O}}_i^\perp} f(x) |a(x)| |dx|.$$

For a sufficiently large i , $\widehat{\mathcal{O}}_i^\perp$ is a small neighborhood of points $x_0 \in N_{\gamma,U}$ and $\cap_i \widehat{\mathcal{O}}_i^\perp = \{x_0\}$. In particular, $\widehat{\mathcal{O}}_i^\perp \subset U$ and

$$\lim_{i \rightarrow \infty} (\text{meas } \mathcal{O}_i) (\text{meas } \widehat{\mathcal{O}}_i^\perp) = 1/|\theta(x_0)|.$$

Therefore

$$\begin{aligned} \lim_{i \rightarrow \infty} \int_{V_i} \pi^*(z) f(x) \psi_\gamma(z) |\omega| &= \lim_{i \rightarrow \infty} (\text{meas } \mathcal{O}_i) \int_{\widehat{\mathcal{O}}_i^\perp} f(x) |a(x)| |dx| \\ &= \frac{f(x_0) |a(x_0)|}{|\theta'(x_0)|} \end{aligned}$$

and the right-hand side equals

$$\int_{N_{\gamma,U}} f \eta(\gamma)(\omega).$$

In the general case we can argue locally on M . Let $U \subset M$ be such that \mathcal{L} and \mathcal{L}^* are trivial over U . Choose local coordinates x_1, \dots, x_m in U such that γ is given by the equation of the form $y^* = \theta(x_1)$ and $N_{\gamma,U} = \{x_1, \dots, x_m \mid x_1 = x_1^{(0)}\}$ for a single $x_0 \in F$. For a fiberwise constant form $\omega = a(x) dx_1 \wedge \dots \wedge dx_m \wedge dy^* = \omega_1 \wedge dy^*$ on L_U we have

$$\text{Res}_{N_{\gamma,U}} \frac{\omega_1}{\theta} = \sum_{x_1^{(0)} \in N_1} \frac{a(x_1^{(0)}, x_2, \dots, x_m) dx_2 \wedge \dots \wedge dx_m}{\theta'(x_1^{(0)})}.$$

To complete the proof one computes $\int f(z) \psi_\gamma(z) |\omega|$ using the arguments similar to those employed in the case $m = 1$ above.

2. Γ -factors and measures.

2.1. Subgroups of the group GL_n . We recall some notation from the introduction and give also some new ones.

- \mathbf{Q} is the subgroup of \mathbf{GL}_n consisting of the matrices with the first row of the form $(*00\dots 0)$.
- \mathbf{U}_- is the lower unipotent subgroup of \mathbf{GL}_n , $\psi(u) = \psi_F(u_{21} + \dots + u_{nn-1})$ a nondegenerate character (one-dimensional complex representation of U_-).
- $\sigma: \mathbf{GL}_n \rightarrow \mathbf{GL}_n$ is the involution given by the formula $\sigma(g) = (ag^\top a^{-1})^{-1}$, where \top denotes the reflection with respect to the second (nonprincipal) diagonal in \mathbf{GL}_n and $a = \text{diag}(1, -1, 1, \dots, (-1)^{n+1})$. We have $\sigma(\mathbf{U}_-) = U_-$ and $\psi \circ \sigma = \psi$.
- $\mathbf{R} = \mathbf{Q} \cap \mathbf{Q}^\sigma$.
- By Ind we will always understand the *unitary induction*.

2.2. Generic representations of GL_n . Let (π, V) be a unitary representation of GL_n . Let (π^a, V^a) be the smooth model of (π, V) (see [BZ]). Recall that V^a consists of all vectors $v \in V$ such that $\text{Stab } v$ is an open compact subgroup of GL_n , and π^a is the restriction of π to V^a . It is known that V^a is dense in V .

DEFINITION 2.1. An irreducible unitary representation (π, V) of GL_n is said to be *generic* if V^a admits a (U_-, ψ) -equivariant linear functional φ , i.e.,

$$\varphi(u_-v) = \psi(u_-)\varphi(v), \quad v \in V^a, \quad u_- \in U_-.$$

It is known [BZ] that for a generic irreducible unitary representation the functional φ is unique up to a scalar factor.

2.3. Standard representation of Q . Denote by $\mathbf{C} \subset \mathbf{Q} \subset \mathbf{GL}_n$ the center of GL_n . For a unitary character $\mathcal{E}: \mathbf{C} \rightarrow \mathbb{C}^*$ introduce the *standard representation* of Q by the formula

$$(\rho_{Q,\mathcal{E}}, L_{Q,\mathcal{E}}) \stackrel{\text{df}}{=} \text{Ind}_{\mathbf{C}U_-}^Q (\mathcal{E} \cdot \psi).$$

The representation $\rho_{Q,\mathcal{E}}$ is irreducible. The following result proved in [BZ] is the basis of our construction of Γ -factors for GL_n .

THEOREM 2.1. *Let (π, V) be a generic unitary irreducible representation of GL_n , \mathcal{E} the central character of π . The restriction of π to Q is equivalent to $\rho_{Q,\mathcal{E}}$.*

2.4. Two restrictions to Q . Let (π, V) be a generic unitary irreducible representation of GL_n with central character \mathcal{E} . By Theorem 2.1 there exists a (unique up to a factor) unitary operator $\alpha_1: V \rightarrow L_{Q,\mathcal{E}}$ establishing the equivalence $\pi|_Q \simeq \rho_{Q,\mathcal{E}}$.

Similarly, let $\sigma: GL_n \rightarrow GL_n$ be the involution defined in 2.1. Then $\pi^\sigma \stackrel{\text{df}}{=} \pi \circ \sigma$ is also a generic unitary irreducible representation of GL_n . Applying Theorem 2.1 again, we get the unitary operator $\alpha_2: V \rightarrow L_{Q,\mathcal{E}^\sigma}$ establishing the equivalence $\pi^\sigma|_Q \simeq \rho_{Q,\mathcal{E}^\sigma}$, where $\mathcal{E}^\sigma = \mathcal{E} \circ \sigma$ is the central character of π^σ .

Since π and π^σ act in the same space V , we have the operator $\beta_\pi = \alpha_2 \circ \alpha_1^{-1}: L_{Q,\mathcal{E}} \rightarrow L_{Q,\mathcal{E}^\sigma}$. It is a unitary operator satisfying the condition

$$(2.1) \quad \beta_\pi \circ \rho_{Q,\mathcal{E}}(r) = \rho_{Q,\mathcal{E}^\sigma}(r^\sigma) \circ \beta_\pi, \quad r \in R = Q \cap Q^\sigma.$$

Note that the unitary operator β_π satisfying (2.1) is defined uniquely up to a multiplicative factor c_π with $|c_\pi| = 1$.

2.5. An auxiliary operator. We construct the operator $\kappa_\mathcal{E}: L_{Q,\mathcal{E}} \rightarrow L_{Q,\mathcal{E}^\sigma}$ satisfying the intertwining condition similar to (2.1). The operator $\kappa_\mathcal{E}$ will depend on

\mathcal{E} but not on π .

Denote by $w_Q \in Q$ the following permutation matrix:

$$w_Q = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Next, denote $U_Q = w_Q U_- w_Q^{-1}$ and let $\psi_Q: U_Q \rightarrow \mathbb{C}^*$ be given by the formula

$$\psi_Q(u) = \psi(w_Q^{-1} u w_Q), \quad u \in U_Q.$$

Denote also $U_R = U_Q \cap R$, and $\psi_R = \psi_Q|_R: U_R \rightarrow \mathbb{C}^*$.

In this section we will often represent elements of $g \in GL_n$ by block 3×3 -matrices according to the decomposition $n = 1 + (n-2) + 1$ of rows and columns. In such representation, $q \in Q$, $r \in R$, $u_Q \in U_Q$, and $u_R \in U_R$ have the form

$$q = \begin{pmatrix} * & 0 & 0 \\ * & * & * \\ * & * & * \end{pmatrix}, \quad r = \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix},$$

$$u_Q = \begin{pmatrix} 1 & 0 & 0 \\ * & u' & * \\ * & 0 & 1 \end{pmatrix}, \quad u_R = \begin{pmatrix} 1 & 0 & 0 \\ * & u' & 0 \\ * & 0 & 1 \end{pmatrix}.$$

Here $*$ denotes possibly nonzero positions and u' is a lower unipotent matrix of order $n-2$. The characters ψ_Q and ψ_R are given by the formulas

$$\psi_Q(u) = \psi_F(u_{n1} + u_{32} + \dots + u_{n-1, n-2} + u_{2n}),$$

$$\psi_R(u) = \psi_F(u_{n1} + u_{32} + \dots + u_{n-1, n-2}).$$

Introduce also the subgroup A as follows:

$$A = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & E_{n-2} & * \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

where E_{n-2} is the identity matrix of order $n-2$. Notice that $A \subset U_Q$ and $\psi_A(a) = \psi_F(a_{2n})$ for $a \in A$.

Let us define the following representations.

(i) A representation of the group Q :

$$(\widehat{\rho}_{Q, \mathcal{E}}, \widehat{L}_{Q, \mathcal{E}}) \stackrel{\text{df}}{=} \text{Ind}_{CU_Q}^Q(\mathcal{E} \cdot \psi_Q).$$

(ii) A representation of the group R :

$$(2.2) \quad (\rho_{R, \mathcal{E}}, L_{R, \mathcal{E}}) \stackrel{\text{df}}{=} \text{Ind}_{CU_R}^R(\mathcal{E} \cdot \psi_R).$$

Next we introduce the following linear operators between the spaces of these

representations:

$$\begin{aligned} C_1: L_{Q,\mathcal{E}} &\rightarrow \widehat{L}_{Q,\mathcal{E}}, & (C_1 f)(q) &= f(w_Q^{-1}q), \\ C_2: \widehat{L}_{Q,\mathcal{E}} &\rightarrow L_{R,\mathcal{E}}, & C_2(f) &= f|_R. \end{aligned}$$

Finally, define the operator $C_3: L_{R,\mathcal{E}} \rightarrow L_{R,\mathcal{E}^\sigma}$ as follows. Let $B \subset U_Q$ be the $(n-2)$ -dimensional commutative subgroup of matrices with the block representation of the form

$$b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & E_{n-2} & 0 \\ 0 & * & 1 \end{pmatrix},$$

where E_{n-2} is the identity matrix of order $n-2$. Let $f \in L_{R,\mathcal{E}}$ be a smooth (i.e., locally constant) function compactly supported on $CU_R \setminus R$. Denote

$$\widetilde{f}(r) = \int_B f(br^\sigma) |db|$$

one can verify that the integral converges absolutely, that $\widetilde{f} \in L_{R,\mathcal{E}}$, and that the mapping $f \mapsto \widetilde{f}$ defined on smooth compactly supported functions extends to a unitary operator $C_3: L_{R,\mathcal{E}} \rightarrow L_{R,\mathcal{E}^\sigma}$.

Note that all operators C_1, C_2, C_3 depend on \mathcal{E} .

LEMMA 2.1. (i) *The operator C_1 establishes an equivalence of representations $\rho_{Q,\mathcal{E}} \simeq \widehat{\rho}_{Q,\mathcal{E}}$.*

(ii) *The operator C_2 establishes the equivalence of representations of R : $\widehat{\rho}_{Q,\mathcal{E}}|_R \simeq (\rho_{R,\mathcal{E}}, L_{R,\mathcal{E}})$.*

(iii) *The operator C_3 satisfies the condition*

$$\rho_{R,\mathcal{E}^\sigma}(r^\sigma) \circ C_3 = C_3 \circ \rho_{R,\mathcal{E}}(r), \quad r \in R.$$

Proof. (i) and (iii) are clear and (ii) follows from the fact that $U_Q R$ is dense in Q .

DEFINITION 2.1. Define the operator $\kappa_{\mathcal{E}}: L_{Q,\mathcal{E}} \rightarrow L_{Q,\mathcal{E}^\sigma}$ by the formula

$$\kappa_{\mathcal{E}} = C_1^{-1} \circ C_2^{-1} \circ C_3 \circ C_2 \circ C_1.$$

The operator $\kappa_{\mathcal{E}}$ is unitary and one easily verifies that $\kappa_{\mathcal{E}^\sigma} \circ \kappa_{\mathcal{E}} = \text{id}$. By Lemma 2.1, $\kappa_{\mathcal{E}}$ satisfies the condition

$$(2.3) \quad \kappa_{\mathcal{E}} \circ \rho_{Q,\mathcal{E}}(r) = \rho_{Q,\mathcal{E}^\sigma}(r^\sigma) \circ \kappa_{\mathcal{E}} \quad r \in R.$$

Explicit formula for $\kappa_{\mathcal{E}}$ is given as follows. Let $q = w_Q^{-1}ar$, $a \in A$, $r \in R$. Then

$$(2.4) \quad (\kappa_{\mathcal{E}} f)(q) = \int_B f(w_Q^{-1}br^\sigma) |db|.$$

DEFINITION 2.2. Define the operator $\beta_\pi^\kappa: L_{Q,\mathcal{E}} \rightarrow L_{Q,\mathcal{E}}$ by the formula

$$\beta_\pi^\kappa = \kappa_{\mathcal{E}}^{-1} \circ \beta_\pi.$$

By (2.1) and (2.3), the operator β_π^κ commutes with $\rho_{Q,\mathcal{E}}(r)$ for $r \in R$.

2.6. The isomorphism of spaces of operators. Denote by $M \simeq F^* \times GL_{n-2} \subset R$ the subgroup of the matrices m of the form

$$m = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

where $\lambda \in F^*$, $A \in GL_{n-2}$. Denote $U_M = U_- \cap M$, $\psi_M = \psi_R|_M$, and

$$(\rho_{M,\mathcal{E}}, L_{M,\mathcal{E}}) \stackrel{\text{df}}{=} \text{Ind}_{CU_M}^M(\mathcal{E} \cdot \psi_M).$$

For two unitary representations (ρ_1, L_1) , (ρ_2, L_2) of a group G by $\text{Hom}_G(L_1, L_2)$ we denote the space of continuous linear operators $V_1 \rightarrow V_2$ commuting with the action of G .

Our goal in this subsection is to construct an isomorphism of linear spaces

$$(2.5) \quad \alpha_{\mathcal{E}}: \text{Hom}_R(L_{Q,\mathcal{E}}, L_{Q,\mathcal{E}}) \simeq \text{Hom}_M(L_{M,\mathcal{E}}, L_{M,\mathcal{E}}).$$

(i) By Lemma 2.1 (i), (ii), the restriction of $\rho_{Q,\mathcal{E}}$ to R is equivalent to the representation $(\rho_{R,\mathcal{E}}, L_{R,\mathcal{E}})$ with the equivalence established by the operator $C_2 \circ C_1: L_{Q,\mathcal{E}} \rightarrow L_{R,\mathcal{E}}$. Therefore,

$$(2.6) \quad \text{Hom}_R(L_{Q,\mathcal{E}}, L_{Q,\mathcal{E}}) \simeq \text{Hom}_R(L_{R,\mathcal{E}}, L_{R,\mathcal{E}}).$$

(ii) Denote by H the subgroup of the block matrices h of the form

$$(2.7) \quad h = \begin{pmatrix} 1 & 0 & 0 \\ a & E_{n-2} & 0 \\ c & b & \lambda \end{pmatrix},$$

where E_{n-2} is the identity matrix, a and b are $(n-2)$ -dimensional vectors, $c \in F$, and $\lambda \in F^*$. Let $U_H \subset H$ be the subgroup consisting of the matrices $h \in H$ with $b = 0$ and $\lambda = 1$ in (2.7), and $\psi_H: U_H \rightarrow \mathbb{C}^*$ the character given by

$$\psi_H(h) = \psi_F(c).$$

Let (ρ_H, L_H) be the induced representation

$$(\rho_H, L_H) \stackrel{\text{df}}{=} \text{Ind}_{U_H}^H(\psi_H).$$

It is easy to prove that the representation ρ_H is irreducible. Furthermore, since the adjoint action of M on H preserves the subgroup H and the character U_H , we have the representation I of M in L_H given by the formula $(I(m)f)(h) = f(m^{-1}hm)$.

(iii) We have $R = H \rtimes M$ and $U_R = U_H \times U_M$. Also, the restriction of ψ_R to \widehat{H} and to U_M coincides with ψ_H and ψ_M respectively. For each $m \in M$ we have $mU_Hm^{-1} = U_H$ and $\psi_H(mhm^{-1}) = \psi_H(h)$ for all $h \in U_H$. Therefore, regarding $L_H \otimes L_{M,\mathcal{E}}$ as the space of $(U_H \times CU_M, \psi_H \cdot (\mathcal{E} \cdot \psi_M))$ -equivariant functions on $H \times M$, we see that the mapping $f \mapsto (f_1)(h, m) = f(mh)$ establishes an isomorphism of linear spaces

$$(2.8) \quad L_{R,\mathcal{E}} \xrightarrow{\sim} L_H \otimes L_{M,\mathcal{E}}.$$

The group R acts on the spaces on both sides of the last formula: by $\rho_{R,\varepsilon}$ on the left space, and by the formula

$$r = hm \mapsto (\rho_H(h) \circ I(m)) \otimes \rho_{M,\varepsilon}(m)$$

on the right space, and the isomorphism (2.8) intertwines these actions of R . Taking into account that ρ_H is irreducible, we obtain

$$(2.9) \quad \text{Hom}_H(L_{R,\varepsilon}, L_{R,\varepsilon}) \simeq \text{Hom}(L_{M,\varepsilon}, L_{M,\varepsilon}).$$

(iv) The group M acts on both sides of (2.9) by the formula

$$a \mapsto \rho_R(m) \circ a \circ \rho_R(m^{-1}), \quad a: L_{R,\varepsilon} \rightarrow L_{R,\varepsilon}$$

for the left-hand side and a similar formula for right-hand side, and the isomorphism (2.9) intertwines these actions. Taking M -invariant elements, we obtain the isomorphism

$$(2.10) \quad \text{Hom}_R(L_{R,\varepsilon}, L_{R,\varepsilon}) \simeq \text{Hom}_M(L_{M,\varepsilon}, L_{M,\varepsilon}).$$

Combining it with (2.6), we get the required isomorphism α_ε in (2.5).

2.7. The operator Γ_π . Applying the isomorphism α_ε in (2.5) to the operator $\beta_\pi^\kappa: L_{Q,\varepsilon} \rightarrow L_{Q,\varepsilon}$ we obtain an operator $\Gamma_\pi: L_{M,\varepsilon} \rightarrow L_{M,\varepsilon}$ commuting with $\rho_{M,\varepsilon}$. We call it the Γ -operator corresponding to the generic unitary representation π . Let

$$\rho_{M,\varepsilon} = \oplus \int_{\Theta} \theta d\mu(\theta)$$

be the direct integral decomposition of $\rho_{M,\varepsilon}$ into irreducible components, each occurring with multiplicity one. In this decomposition the operator Γ_π is the multiplication by almost everywhere defined function $\Gamma(\pi, \cdot)$ on Θ .

Since $M = C \times GL_{n-2}$, the formula $\theta \mapsto \mathcal{E}^{-1} \otimes \theta$ establishes a bijection between the set Θ of irreducible components of $\rho_{M,\varepsilon}$ and the set of generic unitary irreducible representations of GL_{n-2} .

Denote by $\gamma(\pi, \tau)$ the Gamma factor of [JPS] at the point $s = 1/2$ (see [JPS], (3.1)).

PROPOSITION 2.1. *For any generic unitary representation π of the group GL_n we have $\Gamma(\pi, \theta) = \theta(-1)^{n-1} \gamma(\pi, \mathcal{E}^{-1} \otimes \theta)$ for almost every $\theta \in \Theta$.*

Proof. See Appendix.

3. Γ -measure corresponding to the principal series.

3.1. Standard realization of the principal series. Let $Afl = U_+ \backslash GL_n$ be the affine flag manifold and $(M, \Pi) = \text{Ind}_{U_+}^{GL_n}(1)$ the principal series representation of GL_n . On Afl , we consider the left action of the split torus $T_0 \subset GL_n$ given by the formula $x \mapsto tx$. This formula makes sense because T_0 normalizes U_+ . The action of T_0 commutes with the action of GL_n on Afl , so we can regard Π as a representation of the direct product $GL_n \times T_0$ according to the formula

$$\Pi(g, t)f(x) = f(t^{-1}xg)\Delta_{B_+}(t), \quad x \in Afl, \quad g \in GL_n, \quad t \in T_0,$$

where Δ_{B_+} is the modulus,

$$\Delta_{B_+}(t) = \left| \frac{d(tut^{-1})}{du} \right|^{1/2}, \quad t \in T_0,$$

and du is the invariant volume form on U_+ .

Let $\tilde{B} \subset GL_n \times T_0$ be the following subgroup:

$$\tilde{B} = \{(b, t) : b \in B_+, t \in T_0, t^{-1}b \in U_+\}.$$

Then

$$\Pi \simeq \text{Ind}_{\tilde{B}}^{GL_n \times T_0}(1)$$

(isomorphism of representations of $GL_n \times T_0$).

3.2. Irreducible principal series representations. Let $\chi: T_0 \rightarrow \mathbb{C}$ be a unitary character of T_0 . Regarding χ as a character of B_+ via the isomorphism $T_0 \simeq B_+/U_+$, denote

$$(\pi_\chi, V_\chi) = \text{Ind}_{B_+}^{GL_n}(\chi).$$

Recall that Ind denotes the unitary induction, so that the space of the representation π_χ consists of B_+ -homogeneous functions on GL_n of degree $(\chi\Delta_{B_+})(b)$ for $b \in B_+$.

The following two propositions summarize the well-known results about principal series representations.

PROPOSITION 3.1. (i) *Each π_χ is a generic unitary representation of the group GL_n .*

(ii) *The Weyl group S_n (the symmetric group of order n) acts on T_0 , hence also on the unitary dual $(T_0)^\wedge$ to T_0 . Representations π_{χ_1} and π_{χ_2} are equivalent if and only if $\chi_1 = (\chi_2)^w$ for some $w \in S_n$.*

Together with the representation π_χ we consider the representation $\tilde{\pi}_\chi$ of the group $GL_n \times T_0$ defined by the formula $\tilde{\pi}_\chi = \pi_\chi \otimes \chi^{-1}$.

PROPOSITION 3.2. *We have the direct integral decomposition of representations of $GL_n \times T_0$:*

$$\Pi = \oplus \int_{(T_0)^\wedge} \tilde{\pi}_\chi d\chi,$$

where $d\chi$ is the Haar measure on $(T_0)^\wedge$.

3.3. Restriction to Q . Recall the definition of the representation $\rho_{Q, \mathcal{E}}$ of the group Q (see 2.1). By Proposition 3.1(i) and Theorem 2.1 we have

$$\pi_\chi|_Q \simeq \rho_{Q, \mathcal{E}},$$

where $\mathcal{E}(\lambda) = \chi(\text{diag}(\lambda, \dots, \lambda))$ is the central character of π_χ . Therefore, we have

$$(3.1) \quad \tilde{\pi}_\chi|_{Q \times T_0} \simeq \rho_{Q, \mathcal{E}} \times \chi^{-1}.$$

Introduce the subgroup $C_1 \simeq F^*$ as follows: $C_1 = \{(\lambda E, \lambda E)\} \subset GL_n \times T_0$, where E is the identity element. Clearly, C_1 is in the kernel of the representation Π . Denote by $\tilde{\psi}$ the one-dimensional representation $\tilde{\psi}(\lambda u_-, \lambda E) = \psi(u_-)$ of the group $C_1 U_-$, and let

$$(\tilde{\rho}_Q, \tilde{L}_Q) \stackrel{\text{df}}{=} \text{Ind}_{C_1 U_-}^{Q \times T_0}(\tilde{\psi}).$$

For any $\mathcal{E} \in (F^*)^\wedge$ let

$$(R_{T_0, \mathcal{E}}, L_{T_0, \mathcal{E}}) \stackrel{\text{df}}{=} \text{Ind}_{\{\lambda \mathcal{E}\}}^{T_0}(\mathcal{E})$$

be the \mathcal{E} -homogeneous part of the regular representation of T_0 . Then

$$(3.2) \quad \tilde{\rho}_Q = \oplus \int_{(F^*)^\wedge} (\rho_{Q, \mathcal{E}} \otimes R_{T_0, \mathcal{E}^{-1}}) d\mathcal{E}.$$

On the other hand, by Proposition 3.2 and formula (3.1) we have

$$\Pi|_{Q \times T_0} = \oplus \int_{(F^*)^\wedge} (\rho_{Q, \mathcal{E}} \otimes R_{T_0, \mathcal{E}^{-1}}) d\mathcal{E}.$$

Therefore, the representations $\Pi|_{Q \times T_0}$ and $\tilde{\rho}_Q$ of the group $Q \times T_0$ are equivalent. Since these representations are reducible, there are many isomorphisms of these representations. For our purposes we must choose a particular isomorphism

$$(3.3) \quad \varphi: \Pi|_{Q \times T_0} \rightarrow \tilde{\rho}_Q$$

constructed in [K] using the Jacquet functors. Rather than going into details of Jacquet functors, we present explicit formulas for φ . Before doing this, we need some preparation.

Consider the subgroups \tilde{B} and $\tilde{Q} \stackrel{\text{df}}{=} Q \times T_0$ in $GL_n \times T_0$. Since $\tilde{B}\tilde{Q}$ is dense in $GL_n \times T_0$, the restriction $\Pi|_{\tilde{Q}}$ is equivalent to the induced representation

$$(\Pi_1, M_1) \stackrel{\text{df}}{=} \text{Ind}_{\tilde{B}_1}^{\tilde{Q}}(1), \quad B_1 \stackrel{\text{df}}{=} \tilde{B} \cap \tilde{Q},$$

in the space M_1 of left B_1 -invariant functions on \tilde{Q} . The operator $M \rightarrow M_1$ establishing this equivalence sends a function $f \in M$ on $GL_n \times T_0$ to its restriction to $\tilde{Q} = Q \times T_0$. Hence, both representations $\Pi|_{\tilde{Q}}$ and $\tilde{\rho}_Q$ act in the spaces M_1 and \tilde{L}_Q of functions on the group \tilde{Q} . In these realizations, explicit formulas for the operator φ are described as follows.

Let $L_S \subset \tilde{L}_Q$ be the dense subspace consisting of smooth functions with compact support modulo $C_1 U_-$ (the Schwarz space). Let also $U_1 = U_+ \cap Q$ and $T_0^{\text{diag}} = \{(t, t) : t \in T_0\} \subset Q \times T_0$, so that $B_1 = (U_1 \times \{1\}) \cdot T_0^{\text{diag}} \subset Q \times T_0$.

PROPOSITION 3.3 (see [K], Lemma 3.1.10). (i) For $f \in L_S$ the integral

$$(\varphi f)(q) = \int_{U_1 \times T_0 / C_1} f(utq, t) du dt$$

(which makes sense since f is invariant under C_1) converges absolutely.

(ii) The mapping $f \mapsto \varphi(f)$ extends to a unitary operator $\varphi: \tilde{L}_Q \rightarrow M_1$ intertwining $\tilde{\rho}_Q$ and $\Pi|_{Q \times T_0}$.

(iii) For any $w \in W(T_0) = S_n$ we have

$$\varphi \circ w = \mathcal{F}_w \circ \varphi,$$

where S_n acts on \tilde{L} by the formula

$$wf(q, t) = f(q, t^{w^{-1}}),$$

and \mathcal{F}_w is the Fourier–Weyl operator in the space $M_1 \simeq M$ (for the definition of \mathcal{F}_w , see [GG] or [KL]).

Proof of (i). The proof follows from the directly verified fact that the composition

$$U_1 \hookrightarrow Q \rightarrow U_- \setminus Q$$

is a proper map.

3.4. Representing operators by measures. In this paper we will often replace operators between spaces of induced representations of a group G by the corresponding measures. The general construction is described as follows.

Let G be a topological group, H_1, H_2 two subgroups of G , and θ_1, θ_2 unitary characters of H_1, H_2 respectively. Let

$$(\rho_1, V_1) = \text{Ind}_{H_1}^G(\theta_1), \quad (\rho_2, V_2) = \text{Ind}_{H_2}^G(\theta_2)$$

be two irreducible representations and $E: V_1 \rightarrow V_2$ a linear operator. Define the left- $(H_1 \times H_2, \theta_1^{-1}\theta_2)$ -equivariant complex valued measure μ_E on $G \times G$ by the formula

$$(3.4) \quad \int_{(H_1 \setminus G) \times (H_2 \setminus G)} f_1(x_1) \overline{f_2(x_2)} \mu_E(x_1, x_2) = (Ef_1, f_2)_{V_2},$$

where $(\cdot, \cdot)_{V_2}$ is the inner product in V_2 . Since the linear combinations of the products $f_1(x_1) \overline{f_2(x_2)}$ are dense in the appropriate Hilbert space of left- $(H_1 \times H_2, \theta_1 \theta_2^{-1})$ -equivariant functions on $G \times G$, formula (1.3) determines μ_E uniquely.

If E is an intertwining operator, i.e., commutes with the action of G in V_1 and V_2 , then the measure μ_E is right-invariant with respect to the diagonal action of G on $G \times G$.

In explicit computations it is often convenient to use another realization of the measure associated to an operator. As before, assume that $E: V_1 \rightarrow V_2$ is an operator between the spaces of two induced representations. Choose a section $s_i: X_i \rightarrow G$ of the projection $p_i: G \rightarrow H_i \setminus G$, $i = 1, 2$. Restricting functions in V_i to $X_i \subset G$, we can regard V_i as a space of functions on X_i (with appropriate inner product). Define the measure μ'_E on $X_1 \times X_2$ by the formula similar to (3.4):

$$(3.5) \quad \int_{X_1 \times X_2} f_1(x_1) f_2(x_2) \mu'_E(x_1, x_2) = (Ef_1, f_2)_{V_2}.$$

Again, formula (3.5) determines the measure μ'_E uniquely.

There is an obvious one-to-one correspondence between measures on $X_1 \times X_2$ and left- $(H_1 \times H_2, \theta_1^{-1}\theta_2)$ -equivariant measures on $G \times G$ which sends μ_E to μ'_E .

In what follows we will not distinguish between the measures μ_E and μ'_E . It will be clear from the context (or stated explicitly) which of these two measures is used.

3.5. Properties of measures corresponding to operators.

DEFINITION 3.1. Let X be a space with a positive measure ν . A complex valued measure μ on $X \times X$ is said to be a ν -operator measure (or simply an operator measure) if

$$\langle \mu f_1 \times \overline{f_2} \rangle \leq C \|f_1\|_{L_2} \cdot \|f_2\|_{L_2}$$

for all $f_1, f_2 \in L_2(X, \nu)$.

LEMMA 3.1. *Formula (3.4) establishes a one-to-one correspondence between the bounded operators $E: L_2(X, \nu) \rightarrow L_2(X, \nu)$ and the operator measures μ_E on $X \times X$.*

Proof. Clear.

We will need the following properties of operator measures.

I. Let $X' \subset X$ be a subset such that $\nu(X \setminus X') = 0$ and ν' the restriction of ν to X' . For an operator measure μ on $X \times X$ denote by μ' the restriction of μ to $X' \times X'$. Then $\mu \longleftrightarrow \mu'$ is an one-to-one correspondence between operator measures on $X \times X$ and $X' \times X'$.

II. Let $(X, \nu) = (X_1 \times X_2, \nu_1 \times \nu_2)$. Let μ_i be a ν_i -operator measure on $X_i \times X_i$, $i = 1, 2$. Then $\mu = \mu_1 \times \mu_2$ is a ν -operator measure on X .

Similarly we can define operator measures on sections of product line bundles on $X \times X$.

Now we turn to the construction of the Γ -measure corresponding to the principal series representation Π . This construction is similar to the construction of the Γ -factors described in §2, but is performed “for all $\chi \in (T_0)^\wedge$ simultaneously.” At the end of the section we show how this construction is formulated in terms of measures.

3.6. The operator β_Π . Extend this involution σ from 2.1 to $GL_n \times T_0$ by the formula

$$\sigma(g, t) = (g^\sigma, t^\sigma).$$

Since the subgroup $\tilde{B} \subset GL_n \times T_0$ is invariant under σ , we can define the operator $\Pi(\sigma)$ in the space M by the formula

$$\Pi(\sigma)f(x) = f(x^\sigma).$$

Now we set

$$(3.6) \quad \beta_\Pi = \varphi^{-1} \circ \Pi(\sigma) \circ \varphi: \tilde{L}_Q \rightarrow \tilde{L}_Q.$$

Clearly, β_Π is a unitary operator in the space \tilde{L}_Q satisfying the condition

$$(3.7) \quad \beta_\Pi \circ \tilde{\rho}_Q(r, t) = \tilde{\rho}_Q(r^\sigma, t^\sigma) \circ \beta_\Pi, \quad r \in R = Q \cap Q^\sigma, \quad t \in T_0.$$

Formula (3.7) implies, in particular, that the operator β_Π intertwines irreducible principal series representations π_χ and π_{χ^σ} in Π .

PROPOSITION 3.4. *In decomposition (3.2) we have*

$$\beta_\Pi = \oplus \int_{(T_0)^\wedge} (\beta_{\pi_\chi} \otimes \sigma),$$

where the operator β_{π_χ} is defined in 2.5 and $\sigma: L_{T_0, \varepsilon} \rightarrow L_{T_0, \varepsilon^{-1}}$ is induced by the action of σ on T_0 .

Proof. Clear.

3.7. The operator κ . Recall the direct integral decomposition (3.2) of the representation $(\tilde{\rho}_Q, \tilde{L}_Q)$ of the group \tilde{Q} ,

$$(3.8) \quad \tilde{L}_Q = \oplus \int (L_{Q, \varepsilon} \otimes L_{T_0, \varepsilon^{-1}}) d\mathcal{E}.$$

DEFINITION 3.2. In the decomposition (3.8), let us define the operator $\kappa: \tilde{L}_Q \rightarrow \tilde{L}_Q$ by the formula

$$(3.9) \quad \kappa = \oplus \int (\kappa_{\mathcal{E}} \otimes \sigma_{T_0}) d\mathcal{E},$$

where $\sigma_{T_0}: L_{T_0, \mathcal{E}^{-1}} \rightarrow L_{T_0, \mathcal{E}}$ is given by $\sigma F(t) = F(t^\sigma)$.

PROPOSITION 3.5. (i) Formula (3.9) yields a unitary operator $\kappa: \tilde{L}_Q \rightarrow \tilde{L}_Q$ such that $\kappa^2 = \text{id}$.

(ii) The operator κ satisfies the condition

$$(3.10) \quad \kappa \circ \tilde{\rho}_Q(r, t) = \tilde{\rho}_Q(r^\sigma, t^\sigma) \circ \kappa \quad r \in R = Q \cap Q^\sigma, \quad t \in T_0.$$

Proof. Immediately follows from the corresponding properties of the operators $\kappa_{\mathcal{E}}$.

Similarly to formula (2.4), we can give an explicit formula for the operator κ . Let $q = w_Q^{-1}ar$ with $a \in A$, $r \in R$, and let $t \in T_0$. Then

$$(3.11) \quad (\kappa f)(q, t) = \psi_Q(a) \int_B f(w_Q^{-1}br^\sigma, t^\sigma) |db|.$$

3.8. The isomorphism α . In §2 (see 2.7), we have constructed the isomorphisms

$$\alpha_{\mathcal{E}}: \text{Hom}_R(L_{Q, \mathcal{E}}, L_{Q, \mathcal{E}}) \simeq \text{Hom}_M(L_{M, \mathcal{E}}, L_{M, \mathcal{E}}), \quad \mathcal{E} \in (F^*)^\wedge.$$

Here we repeat this construction in the framework of the “tensoring with T_0 ” space \tilde{L}_Q instead of $L_{Q, \mathcal{E}}$. We will use the notation of §2. In addition, we denote $\tilde{R} = R \times T_0$, $\tilde{M} = M \times T_0$. Then $C_1 = \{(\lambda I, \lambda I)\} \subset \tilde{M} \subset \tilde{R}$ and we can define

$$\begin{aligned} (\tilde{\rho}_M, \tilde{L}_M) &\stackrel{\text{df}}{=} \text{Ind}_{C_1 U_M}^{\tilde{M}}(\psi_M), \\ (\tilde{\rho}_R, \tilde{L}_R) &\stackrel{\text{df}}{=} \text{Ind}_{C_1 U_R}^{\tilde{R}}(\psi_R). \end{aligned}$$

Our goal is to construct the isomorphism

$$\alpha: \text{Hom}_{\tilde{R}}(\tilde{L}_Q, \tilde{L}_Q) \simeq \text{Hom}_{\tilde{M}}(\tilde{L}_M, \tilde{L}_M).$$

In addition to the direct integral decomposition (3.11), we have a similar decomposition

$$(3.12) \quad \tilde{M}_Q = \oplus \int (M_{Q, \mathcal{E}} \otimes L_{T_0, \mathcal{E}^{-1}}) d\mathcal{E}.$$

for \tilde{M}_Q . Clearly,

$$\text{Hom}_{\tilde{R}}(\tilde{L}_Q, \tilde{L}_Q) = \oplus \int (\text{Hom}_R(L_{Q, \mathcal{E}}, L_{Q, \mathcal{E}}) \otimes \text{Hom}_{T_0}(L_{T_0, \mathcal{E}}, L_{T_0, \mathcal{E}})) d\mathcal{E},$$

and similarly for \tilde{M}_Q :

$$\text{Hom}_{\tilde{M}}(\tilde{L}_M, \tilde{L}_M) = \oplus \int (\text{Hom}_R(L_{M, \mathcal{E}}, L_{M, \mathcal{E}}) \otimes \text{Hom}_{T_0}(L_{T_0, \mathcal{E}}, L_{T_0, \mathcal{E}})) d\mathcal{E}.$$

In terms of these decompositions, we define the operator α as the direct integral

$$\alpha = \oplus \int (\alpha_{\mathcal{E}} \otimes \text{Id}) d\mathcal{E}.$$

Repeating the construction of 2.6, we see that the isomorphism α is the composition of two isomorphisms,

$$\alpha_{Q \rightarrow R}: \text{Hom}_{\tilde{R}}(\tilde{L}_Q, \tilde{L}_Q) \simeq \text{Hom}_{\tilde{R}}(\tilde{L}_R, \tilde{L}_R)$$

and

$$\alpha_{R \rightarrow M}: \text{Hom}_{\tilde{R}}(\tilde{L}_R, \tilde{L}_R) \simeq \text{Hom}_{\tilde{M}}(\tilde{L}_M, \tilde{L}_M).$$

3.9. Transformation of measures. Let μ_{β} be the $(C_1U_- \times C_1U_-, \psi^{-1} \cdot \psi)$ -equivariant measure on $\tilde{Q} \times \tilde{Q}$ corresponding to the operator β_{Π} and μ_{Γ} be the $(U_M \times U_M, \psi_M^{-1} \cdot \psi_M)$ -equivariant measure on $\tilde{M} \times \tilde{M}$ corresponding to the operator $\alpha(\kappa \circ \beta_{\Pi})$. In this subsection we express μ_{Γ} in terms of μ_{β} . The measure μ_{Γ} is obtained from μ_{β} in the following three steps:

$$(3.13) \quad \mu_{\beta} \implies \mu_{\beta}^{\kappa} \xrightarrow{\alpha_{Q \rightarrow R}} \mu_R \xrightarrow{\alpha_{R \rightarrow M}} \mu_{\Gamma},$$

where μ_{β}^{κ} corresponds to the operator $\kappa \circ \beta_{\Pi}$, μ_R corresponds to the operator $\alpha_{Q \rightarrow R}(\kappa \circ \beta_{\Pi})$, and μ_{Γ} corresponds to the operator $\alpha_{R \rightarrow M}(\alpha_{Q \rightarrow R}(\kappa \circ \beta_{\Pi})) = \alpha(\kappa \circ \beta_{\Pi})$. In the next three lemmas we present the formulas for each of the steps in (3.13).

Recall the definition of the subgroup A in 2.5.

LEMMA 3.2. *The measure μ_{β}^{κ} is the unique $(C_1U_- \times C_1U_-, \psi^{-1} \cdot \psi)$ -equivariant operator measure on $\tilde{Q} \times \tilde{Q}$ whose restriction to the open set*

$$(3.14) \quad Q \times T_0 \times w_Q^{-1}AR \times T_0 \subset Q \times T_0 \times Q \times T_0 = \tilde{Q} \times \tilde{Q}$$

is given by the formula

$$(3.15) \quad \mu_{\beta}^{\kappa}(q, t; w_Q^{-1}ar, t') = \psi_Q^{-1}(a)\mu_{\beta}(q, t; w_Q^{-1}r^{\sigma}, t^{\sigma}).$$

Proof. Formula (3.15) for the restriction of μ_{β}^{κ} to the open set (3.14) follows from (3.11). The uniqueness of the extension follows from Property I in 3.5.

Next we construct μ_R from μ_{β}^{κ} . Consider the auxiliary measure

$$\mu'(q_1, t_1; q_2, t_2) = \mu_{\beta}^{\kappa}(w_Q^{-1}q_1, t_1; w_Q^{-1}q_2, t_2)$$

on $\tilde{Q} \times \tilde{Q}$. This measure is $(C_1U_Q \times C_1U_Q, \psi_Q^{-1} \cdot \psi_Q)$ -equivariant. Therefore, its restriction to the open dense set $AR \times T_0 \times AR \times T_0 \subset \tilde{Q} \times \tilde{Q}$ has the form

$$\mu'(q_1, t_1; q_2, t_2) = \psi_Q^{-1}(a_1)\psi_Q(a_2)da'_1 da'_2 \mu''(r_1, t_1; r_2, t_2),$$

for $q_i = a_i r_i$, $a_i \in A$, $r_i \in R$, $i = 1, 2$, where μ'' is a $(C_1U_R \times C_1U_R, \psi_R^{-1} \cdot \psi_R)$ -equivariant operator measure on $\tilde{R} \times \tilde{R}$.

LEMMA 3.3. *The measure μ_R corresponding to $\alpha_{Q \rightarrow R}(\mu_{\beta}^{\kappa})$ coincides with the measure $\mu''(r_1, t_1; r_2, t_2)$.*

Proof. Clear.

Now we construct μ_Γ from μ_R . We have the semidirect product decompositions $\widetilde{R} = H \ltimes \widetilde{M}$ and $C_1 U_R = U_H \ltimes C_1 U_M$ (see 2.6), hence, in particular, $\widetilde{R} = H \times \widetilde{M}$ as spaces.

Denote by the δ_{H/U_H} the $(U_H \times U_H, \psi_H^{-1} \cdot \psi_H)$ -equivariant measure on $H \times H$ corresponding to the identity operator in the space L_H of (U_H, ψ_H) -equivariant functions on H . Explicitly, δ_{H/U_H} is given as follows. Let

$$V_H = \left\{ v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & E_{n-2} & 0 \\ 0 & b & a \end{pmatrix} \right\} \subset H.$$

Then $H = U_H V_H$ and $U_H \cap V_H = \{1\}$. Let δ_{V_H} be the diagonal measure on $V_H \times V_H$, i.e.,

$$\int_{V_H \times V_H} f(v_1, v_2) \delta_{V_H} = \int_{V_H} f(v, v) |dv|.$$

Then

$$\delta_{H/U_H}(u_1 v_1, u_2 v_2) \stackrel{\text{df}}{=} \psi_H(u_1^{-1} u_2) \delta_{V_H}(v_1, v_2).$$

LEMMA 3.4. *The measure μ_R is of the form $\mu_R = \delta_{U_H} \times \mu'''$ for a unique $(U_M \times U_M, \psi_M^{-1} \cdot \psi_M)$ -equivariant operator measure μ''' on $\widetilde{M} \times \widetilde{M}$, and the measure μ_Γ corresponding to the operator $\alpha(\kappa \circ \beta_\Pi)$ coincides with μ''' .*

Proof. The first statement holds for every operator measure μ corresponding to an operator $\Phi \in \text{Hom}_H(\widetilde{L}_R, \widetilde{L}_R)$. The second statement follows from the explicit construction of the isomorphism $\alpha_{R \rightarrow M}$ (see 2.6).

Finally, we combine Lemmas 3.2–3.4 and give an expression for μ_Γ . Consider the map

$$S \stackrel{\text{df}}{=} (A \times H \times M) \times T_0 \times (A \times H \times M) \times T_0 \rightarrow \widetilde{Q} \times \widetilde{Q}$$

given by the formula

$$(a_1, u_1, m_1; t_1; a_2, u_2, m_2; t_2) \mapsto (w_Q^{-1} a_1 h_1 m_1, t_1; w_Q^{-1} a_2 h_2 m_2, t_2) \in Q \times T_0 \times Q \times T_0 = \widetilde{Q} \times \widetilde{Q}.$$

This map is an embedding of S into an open dense subset $\widetilde{S} \subset \widetilde{Q} \times \widetilde{Q}$. Lemmas 3.2–3.4 immediately show that the restriction $\mu' = \mu_\beta|_{\widetilde{S}}$ of the operator measure μ_β to $\widetilde{S} \simeq S$ is given by the formula

$$\mu'(a_1, u_1, m_1; t_1; a_2, u_2, m_2; t_2) = \delta_{H/U_H}(h_1, h_2) \psi_Q^{-1}(a_1) \psi_Q(a_2) \widehat{\mu}(m_1, t_1; m_2, t_2)$$

for some $(U_M \times U_M, \psi_M^{-1} \cdot \psi_M)$ -equivariant operator measure $\widehat{\mu}$ on $\widetilde{M} \times \widetilde{M}$, and that the measure μ_Γ corresponding to the operator $\alpha(\kappa \circ \beta_\Pi)$, equals $\widehat{\mu}$.

4. The measure μ_β .

4.1. The measure μ_β . In this section we compute explicitly the measure μ_β corresponding to the operator $\beta_\Pi: \widetilde{L}_Q \rightarrow \widetilde{L}_Q$ defined by formula (3.6). Hence, μ_β is a $(C_1 U_- \times C_1 U_-, \psi \cdot \psi^{-1})$ -equivariant complex valued measure on $\widetilde{Q} \times \widetilde{Q}$ given by the

formula

$$(4.1) \quad \langle \mu_\beta, f_1 \times \overline{f_2} \rangle = (\beta_\Pi f_1, f_2)_{\tilde{L}_Q}$$

for $f_1, f_2 \in \tilde{L}_Q$.

Since φ and $\Pi(\sigma)$ in the definition (3.6) of β_Π are unitary operators, and $\Pi(\sigma)$ is an involution, we have

$$(4.2) \quad (\beta_\Pi f_1, f_2)_{\tilde{L}_Q} = (\varphi f_1, \Pi(\sigma)\varphi f_2)_{M_1}, \quad f_1, f_2 \in \tilde{L}_Q$$

Let $B_- \subset Q \subset GL_n$ be the subgroup of lower triangular matrices. Denote by the same letter the corresponding subgroup $B_- \times \{1\}$ in $\tilde{Q} \subset GL_n \times T_0$. We have $B_- \cap B_1 = \{1\}$ and $B_- B_1$ is dense in \tilde{Q} . Therefore, the inner product in M_1 can be written in the form

$$(4.3) \quad (F_1, F_2)_{M_1} = \int_{B_-} F_1(b_-) \overline{F_2(b_-)} |db_-|,$$

where db_- is a fixed left invariant volume form on B_- and the integral converges absolutely.

Denote $Z = U_1 \times T_0/C \times U_1 \times T_0/C \times B_-$. Recall that by $L_S \subset \tilde{L}_Q$ we denoted the space of smooth $(C_1 U_-)$ -equivariant functions on \tilde{Q} that are compact modulo $C_1 U_-$. Taking into account the formulas for the operator φ (Proposition 3.3(i, ii)) we formally obtain from (4.1) and (4.2) that

$$(4.4) \quad \langle \mu_\beta, f_1 \times \overline{f_2} \rangle = \int_Z f_1(u_1 t_1 b_-, t_1) \overline{f_2(u_2 t_2 b_-^\sigma, t_2)} |du_1 dt_1 du_2 dt_2 db_-|,$$

where $f_1, f_2 \in L_S$ are considered as functions on $\tilde{Q} = Q \times T_0$ invariant under C_1 .

Since the integral (4.4) does not converge absolutely, we must specify the order of integration. Fix $f_1, f_2 \in L_S$ and denote the integrand in (4.4) by $F = F_{f_1, f_2}(u_1, t_1; u_2, t_2; b_-)$. Then Proposition 3.3(i)–(ii) show that for a fixed $b_- \in B_-$ there exists a compact set $K(b_-) = K_{f_1, f_2}(b_-) \subset U_1 \times T_0/C \times U_1 \times T_0/C$ (depending on the supports of f_1 and f_2) such that $F(u_1, t_1; u_2, t_2; b_-)$ vanishes for $(u_1, t_1; u_2, t_2) \notin K(b_-)$.

PROPOSITION 4.1. *For $f_1, f_2 \in L_S$ we have*

$$(4.5) \quad \langle \mu_\beta, f_1 \times \overline{f_2} \rangle = \int_{B_-} db_- \int_{K(b_-)} F_{f_1, f_2}(u_1, t_1; u_2, t_2; b_-) |du_1 dt_1 du_2 dt_2 db_-|$$

and the integral over B_- converges absolutely.

Proof. Formula (4.5) follows from Proposition 3.3 (i)–(ii). The absolute convergence of the integral over B_- is the absolute convergence of the integral in (4.3).

4.2. Reduction of the measure μ_β . In this subsection we use the construction described in §1 (Proposition 1.1) to show that the measure μ_β is supported on a of codimension one submanifold $Y_1 \subset Y$ and compute the restriction of μ_β to Y_1 .

In the group B_- , introduce the following subsets:

$$B' = \left\{ r(a) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & E & 0 \\ a & 0 & 1 \end{pmatrix}, a \in F \right\},$$

$$B'' = \{b_- \in B_- \mid (b_-)_{n1} = 0\}.$$

Then $B_- = B'B''$ with the unique decomposition and for $b_- = r(a)b''$ we have $db_- = da db''$ for a unique volume form db'' on B'' . With this decomposition of B_- we will regard $Z = U_1 \times T_0/C \times U_1 \times T_0/C \times B_-$ as the total space of the trivial line bundle over $Z_0 = U_1 \times T_0/C \times U_1 \times T_0/C \times B''$ with a fixed trivialization. We identify each fiber with $B' \simeq F$, so that $Z = Z_0 \times B'$.

Fix $f_1, f_2 \in L_S$ and recall that by $F(z) = F_{f_1, f_2}(z)$ we denoted the integrand in (4.4) and in (4.5), so that the right-hand side of (4.5) is

$$(4.6) \quad \int_{B_-} db_- \int_{K_{f_1, f_2}(b_-)} F_{f_1, f_2}(u_1, t_1; u_2, t_2; b_-) |\omega(z)|,$$

where $\omega = du_1 dt_1 du_2 dt_2 db_-$ is a volume form on Z . Using the intertwining property (3.7) of the operator β_{Π} with $r = r_a \in B'$, we easily see that for $z = (z_0, r_a) \in Z_0 \times B'$ we have

$$F(z) = \psi_F(\gamma(z_0)a)F_0(z_0),$$

where F_0 is the restriction of F to $Z_0 = Z_0 \times \{0\} \subset Z$ and the function γ (a section of the trivial bundle) is given by the formula

$$\gamma(z_0) = (u_1)_{2n}(t_1)_n(t_1)_1^{-1} + (-1)^{n+1}(u_2)_{2n}(t_2)_n(t_2)_1^{-1}$$

for $z_0 = (u_1, t_1; u_2, t_2; b'') \in Z_0$.

Denote by $N = N_\gamma \subset Z_0$ the set of zeros of γ .

LEMMA 4.1. *N is a smooth subvariety of Z_0 , the section γ is generic at all points of N , and the form ω in (4.6) is fiberwise constant.*

Proof. Direct verification.

Denote by $\omega_1 = \eta(\gamma)\omega$ the induced form on N (see (1.4)), and by F_1 the restriction of F to N . We want to prove that

$$(4.7) \quad \int_Z F(z)|\omega| = \int_N F_1(z)|\omega_1|.$$

However, we cannot apply Proposition 1.1 directly since Z_0 and N are not compact and the integral on the left-hand side of (4.7) is a iterated integral as explained in 4.1. Nevertheless, formula (4.7) holds if the integral on the right-hand side is interpreted as an iterated integral similarly to the integral on the left, but “intersected with N ”.

Namely, for $b'' \in B''$ denote

$$M(b'') = \{(u_1, t_1; u_2, t_2) \text{ such that } (u_1, t_1; u_2, t_2; b'') \in N\}$$

and let $\omega_{b''} = \omega_1/db''$ be the corresponding volume form on $M(b'')$. Denote the integral (4.6) by I .

PROPOSITION 4.2. *We have*

$$(4.8) \quad I = \int_{B''} |db''| \int_{M(b'')} F_1(u_1, t_1; u_2, t_2; b'') |\omega_{b''}|$$

and both the inner and the outer integral on the right-hand side converge absolutely.

Proof. Recall that for a fixed $b_- \in B_-$ the inner integral in (4.6) is taken over the compact set $K(b_-) \subset U_1 \times T_0/C \times U_1 \times T_0/C$. It is easy to see that if $b_- = r_a b''$, then the set $K(b_-)$ can be taken not depending on a . Therefore, we denote it by $K(b'')$.

Now we take an arbitrary sequence of open compact sets $\mathcal{B}_n \subset B''$ such that

$$(4.9) \quad \mathcal{B}_1 \subset \cdots \subset \mathcal{B}_n \subset \cdots, \quad \bigcup \mathcal{B}_n = B''.$$

Let also $\mathcal{O}_m \subset B' = F$ be given by

$$\mathcal{O}_m = \{r_a \mid |a| \leq p^m\}.$$

Denote by $\Phi(b_-)$ the inner integral in (4.6) and let

$$I_{mn} = \int_{\mathcal{O}_m \times \mathcal{B}_n} \Phi(b_-) |db_-|.$$

Since the integral over B_- in (4.6) converges absolutely, we have

$$I = \lim_{m, n \rightarrow \infty} I_{mn} = \lim_{n \rightarrow \infty} \left(\lim_{m \rightarrow \infty} I_{mn} \right).$$

Denote

$$K_n = \bigcup_{b'' \in \mathcal{B}_n} K(b'') \subset U_1 \times T_0/C \times U_1 \times T_0/C.$$

This is a compact set and

$$\begin{aligned} I_{mn} &= \int_{\mathcal{O}_m \times \mathcal{B}_n} da db'' \int_{K_n} F(z_0, r_a) |du_1 dt_1 du_2 dt_2| \\ &= \int_{\mathcal{O}_m} da \int_{K_n \times \mathcal{B}_n} F(z_0, r_a) |\omega_0|. \end{aligned}$$

Since $K_n \times \mathcal{B}_n$ is a compact set, we can apply Proposition 1.1 and obtain

$$\lim_{m \rightarrow \infty} I_{mn} = \int_{(K_n \times \mathcal{B}_n) \cap N} F_1(z) |\omega_1|.$$

Since $\{\mathcal{B}_n\}$ is an arbitrary sequence of open sets satisfying (4.9), Proposition 4.2 is proved.

COROLLARY. *The restriction $\mu_\beta^{(1)}$ of the measure μ_β to Y_1 is given by formula (4.8).*

4.3. Another presentation of the measure $\mu_\beta^{(1)}$. To construct an algebraic presentation we need to transform formula (4.8) for the measure $\mu_\beta^{(1)}$.

Let us choose an arbitrary semialgebraic section $X \hookrightarrow \tilde{Q}$ of the projection $\tilde{Q} \rightarrow C_1 U_- \setminus \tilde{Q}$, so that each $\tilde{q} \in \tilde{Q}$ we have a unique decomposition

$$\tilde{q} = u_-(\tilde{q})x(\tilde{q}), \quad u_-(\tilde{q}) \in U_-, \quad x(\tilde{q}) \in X.$$

According to the remark at the end of 3.4, we can define the measure μ_β as a measure on the set $Y = X \times X$.

Denote by $Y_1 \subset Y$ the subvariety of those $y = (q_1, t_2; q_2, t_2)$ for which

$$(4.10) \quad \tilde{\gamma}(y) \stackrel{\text{df}}{=} (q_1)_{11}^{-1}(q_1)_{2n} + (-1)^{n+1}(q_2)_{11}^{-1}(q_2)_{2n} = 0.$$

Then the measure μ_β is supported on Y_1 (see 4.2) and $\mu_\beta^{(1)}$ is the restriction of μ_β to Y_1 , i.e.,

$$\int_Y F \mu_\beta = \int_{Y_1} F|_{Y_1} \mu_\beta^{(1)}$$

for smooth compactly supported functions F on Y .

First of all, formula (4.4) immediately shows that the measure μ_β is supported on the subset $Y_0 \subset Y$ consisting of the quadruples $(q_1, t_1; q_2, t_2)$ such that

$$(4.11) \quad \det q_1 \det q_2 = \det t_1 \det t_2.$$

For two elements $g_1, g_2 \in GL_n$ we write $g_1 \stackrel{U_-}{\sim} g_2$ if g_1 and g_2 belong to the same double coset modulo U_- , i.e.,

$$g_1 = u_{-1} g_2 u_{-2}, \quad u_{-1}, u_{-2} \in U_-.$$

For $y = (q_1, t_2; q_2, t_2) \in Y$ denote

$$(4.12) \quad b(y) = q_1 q_2^\top \in GL_n, \quad t(y) = t_1 t_2^\top \in T_0.$$

Note that $b(y)_{1n} = 0$ for $y \in Y$ and, by (4.11), for $y \in Y_0$ we have

$$\det b(y) = \det t(y).$$

Now we define Z' as a subset of the direct product $Y_0 \times U_+$ consisting of the pairs (y, u) such that

$$(4.13) \quad b(y) \stackrel{U_-}{\sim} t(y)u,$$

i.e.,

$$(4.14) \quad u_{-1} b(y) u_{-2}^\top = t(y)u, \quad u_{-1}, u_{-2} \in U_-.$$

For $(u_1, t_1; u_2, t_2; b_-) \in Z = U_1 \times T_0/C \times U_- \times T_0/C \times B_-$ we define $\theta(z) \in Y \times U_+$ as follows:

$$\theta(z) = (q_1, t_1; q_2, t_2; u) \in X \times X \times U_+,$$

where

$$(4.15) \quad \begin{aligned} (q_1, s_1) &= x(t_1 u_1 b_-, t_1), \\ (q_2, s_2) &= x(t_2 u_2 b_-^\sigma, t_2), \\ u &= t(y)^{-1} u_1 t(y) u_2^\top. \end{aligned}$$

Clearly, $u \in U_+$.

PROPOSITION 4.3. *We have $\theta(z) \in Z'$.*

Proof. Define $(c_1 u_{-1}, c_1), (c_2 u_{-2}, c_2) \in C_1 U_-$ by the formulas

$$(c_1 u_{-1}, c_1) = u_-(t_1 u_1 b_-, t_1), \quad (c_2 u_{-2}, c_2) = u_-(t_2 u_2 b_-^\sigma, t_2),$$

so that in $\tilde{Q} = Q \times T_0$ we have

$$(4.16) \quad (t_1 u_1 b_-, t_1) = (c_1 u_{-1}, c_1)(q_1, s_1),$$

$$(4.17) \quad (t_2 u_2 b_-^\sigma, t_2) = (c_2 u_{-2}, c_2)(x_2, s_2).$$

Applying the antiinvolution \top to (4.17), multiplying by (4.16), and using formulas (4.11) and (4.15), we get

$$(4.18) \quad \begin{aligned} t_1 t_2^\top &= ct(y) \\ t_1 t_2^\top u &= cu_{-1} b(y) u_{-2}^\top, \end{aligned}$$

where $c = c_1 c_2$. Therefore, so that $b(y) \stackrel{U_-}{\sim} t(y)u$. Since $u \in U_+$, we have $q(z) \in Z'$. Proposition 4.3 is proved.

Denote by $p': Z' \rightarrow Y_0$ the projection and let $Z'_1 \subset Z'$ be the preimage of Y_1 , so that Z'_1 consists of those $z = (q_1, t_1; q_2, t_2; u)$ for which

$$\tilde{\gamma}(p(z')) = 0.$$

Now we define open dense subset $Y_2 \subset Y_1$, $Z_2 \subset Z_1$, and $Z'_2 \subset Z'_1$ by the conditions:

$$(4.19) \quad \begin{aligned} Y_2 &= \{y \in Y_1 \text{ such that } (b(y))_{1n-1} = (b(y))_{2n} \neq 0\} \\ Z_2 &= \{(u_1, t_1; u_2, t_2; b_-) \in Z_1 \text{ such that } (u_1)_{2n} \neq 0, (u_2)_{2n} \neq 0\}; \\ Z'_2 &= \{(y, u) \in Z'_1 \text{ such that } y \in Y_2.\} \end{aligned}$$

Clearly, θ maps Z_2 to Z'_2 .

PROPOSITION 4.4. *$\theta: Z_2 \rightarrow Z'_2$ is a bijection.*

Proof. We construct the inverse map $\theta': Z'_2 \rightarrow Z_2$. In this construction we assume that $B_- \hookrightarrow GL_n \times T_0$ is a genuine section of the projection $GL_n \times T_0 \rightarrow B_1 \setminus GL_n \times T_0$ (for the notation, see 3.1). To make all the arguments precise we must replace B_- by a semialgebraic section $\hat{B}_- \subset GL_n \times T_0$ such that \hat{B}_- is invariant under the left multiplication by elements of B' .

Let $z = (q_1, t_1; q_2, t_2; u) \in Z'_2$, i.e., conditions (4.11), (4.13) and (4.19) hold. For each pair (u_{-1}, u_{-2}) satisfying (4.14) we define $u_1 \in U_1$ and $b_- \in B_-$ from the (unique) decomposition

$$u_1 t_1 b_- = u_{-1} q_1$$

(with t_1, u_{-1}, q_1 known). From condition (4.19) one easily gets that the family of pairs (u_{-1}, u_{-2}) satisfying (4.14) is one-dimensional and there exists exactly one pair for which the corresponding b_- is in B'' . Below by (u_{-1}, u_{-2}) we will mean this particular pair.

Now we set

$$(4.20) \quad u_2 = (u')^\top (t(y)^{-1} u_1 t(y))^\sigma.$$

From the definition (4.15) of u we get

$$u_2 t_2 b_-^\sigma = u_{-2} x_2.$$

In particular, $u_2 \in Q$ and $u_2 \in U' \subset U_+$, so that $u_2 \in Q \cap U_+ = U_1$. Therefore,

$$\theta'(z') \stackrel{\text{df}}{=} z = (u_1, t_1; u_2, t_2, b_-) \in Z_1$$

and $p(z) = y$. Hence, $z \in Z_2$. It is also clear that $q \circ \theta' = \text{id}_{Z'_2}$.

For $y \in Y'_2$ we denote by $U_y \subset U_+$ the fiber of the projection $p': Z'_2 \rightarrow Y_2$. This is a dense open subset in a closed codimension n subvariety \overline{U}_y in U_+ given by the n equations

$$(4.21) \quad \Phi_1^{(y)}(u) = \dots = \Phi_n^{(y)}(u) = 0, \quad u \in U_+.$$

The first two of these equations are

$$u_{1n} = 0, \quad u_{1n-1} = u_{2n}.$$

Introduce the volume form $\omega_{Y_1} = \text{Res} \frac{\omega_{\tilde{Y}}}{\tilde{\gamma}}$ ($\tilde{\gamma}$ is defined by (4.10)) and let ω_{Y_2} be the restriction of ω_{Y_1} to Y_2 . Introduce also a volume form ω_{U_y} on each fiber U_y by the formula

$$\omega_{U_y} = \text{Res} \frac{\bigwedge_{i < j} du_{ij}}{\Phi_1^{(y)} \dots \Phi_n^{(y)}}.$$

The forms ω_{Y_2} and ω_{U_y} , $y \in Y_2$, determine a volume form ω' on Z'_2 .

Finally, introduce an algebraic function f' on Z'_2 by the formula

$$f'(y, u) = \Psi(u_{-1} u_{-2}),$$

where u_{-1} , u_{-2} are taken from (4.14). Note that although u_{-1} and u_{-2} in (4.14) are not unique, the condition $\tilde{\gamma}(y) = 0$ implies that f' is well defined on Z'_2 .

CONJECTURE. Denote by $\mu_\beta^{(2)}$ the restriction of $\mu_\beta^{(1)}$ to the open dense subset $Y_2 \subset Y_1$. Then the data

$$\boldsymbol{\mu} = (\mathbf{Z}'_2, \mathbf{Y}_2, \mathbf{p}', \omega', \mathbf{p}')$$

is an algebraic presentation of $\mu_\beta^{(2)}$.

Remarks. 1. If the conjecture is true, than it is easy to prove that the measure realized by the data $\boldsymbol{\mu}$ coincides with $\mu_\beta^{(2)}$.

2. In all examples intersection of the level sets of f' in \mathbf{Z}'_2 with the fibers of \mathbf{p}' are open subsets of Calabi–Yau manifolds and the form ω' extends to a regular form on the completion.

For $n = 3$ the above conjecture is obviously true because each fiber U_y consists of just one point. In the next section we prove this conjecture for $n = 4$.

5. Explicit formulas for GL_4 .

5.1. The measure $\mu_\beta^{(2)}$. Now we consider the case $n = 4$. We write a matrix $u \in U$ in the form

$$u = \begin{pmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_4 & a_5 \\ 0 & 0 & 1 & a_6 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

so that $\omega_U(u) = \bigwedge_{i=1}^6 da_i$.

Fix $y = (q_1, t_1; q_2, t_2) \in Y_2$. Let the Bruhat decomposition of $b(u) \in GL_4$ be $b(y) = u'_{-1} \Lambda u'_{-2}$ with $u'_{-1}, u'_{-2} \in U_-$,

$$\Lambda = \begin{pmatrix} 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \\ 0 & \nu_1 & 0 & 0 \\ \nu_2 & 0 & 0 & 0 \end{pmatrix}.$$

Let $t(y) \in T_0$ be the diagonal matrix $t(y) = \text{diag}(\tau_1, \tau_2, \tau_3, \tau_4)$.

The condition (4.13) is equivalent to the following equalities:

$$a_3 = 0, \quad \tau_1 a_2 = \tau_2 a_5 = \lambda, \quad \tau_1 \tau_2 \tau_3 (a_1 a_4 a_6 - a_1 a_6 - a_2 a_5) = \lambda^2 \nu_1,$$

so that the fiber $U_y \subset U_+$ over y is the affine plane, and we can take $s_1 = \tau_1 a_1$ and $s_2 = \tau_3 a_6$ as coordinates on this plane.

The function $f'(z')$ on Z'_2 is given by the formula

$$(5.1) \quad f'(z') = \Psi(u_{-1} u_{-2}^\top)(R(s_1, s_2; \Lambda, t)),$$

where $R(s_1, s_2; \Lambda, t)$ is a rational function in s_1, s_2 depending on parameters Λ, t . Explicitly,

$$R(s_1, s_2, \Lambda, t) = \lambda^{-1} [s_1 + s_2 + \lambda(\tau_1 + \tau_2)s_1^{-1} + \lambda(\tau_3 + \tau_4)s_2^{-1} \\ + \nu_1^{-1} \tau_3 \tau_4 s_1 s_2^{-1} + \nu_1^{-1} \tau_1 \tau_2 s_2 s_1^{-1} + \lambda^2 \nu_1 s_1^{-1} s_2^{-1}].$$

Finally, the form ω_{U_y} on the fiber $U_y \subset U_+$ is given by

$$\omega_{U_y} = \frac{da_1 da_4 da_6}{d(\tau_1 \tau_2 \tau_3 (a_1 a_4 a_6 - a_1 a_6 - a_2 a_5))} = \frac{ds_1 ds_2}{s_1 s_2}.$$

PROPOSITION 5.1. *The data $\boldsymbol{\mu} = (\mathbf{Z}'_2, \mathbf{Y}_2, \mathbf{p}', \omega', \mathbf{f}')$ give an algebraic presentation of the measure $\mu_\beta^{(2)}$.*

Proof. We must prove that conditions (i)–(iv) of Definition 0.1 are satisfied.

Instead of proving (i)–(iii) we prove a stronger result that similar properties hold at each fiber U_y of $p': Z'_2 \rightarrow Y_2$. Take a point $y \in Y_2$.

The factor $\Psi(u_{-1} u_{-2}^\top)$ in formula (5.1) is constant on the fiber U_y , so we can ignore it.

For almost each $\xi \in F$ the level set $R + \xi$ of the function R inside the affine plane \mathbb{A}^2 with coordinates s_1, s_2 is (the affine part of) a smooth elliptic curve E_ξ given by the equation $F_\xi(s_1, s_2; \Lambda, t, \xi) = 0$, where

$$F_\xi(s_1, s_2; \Lambda, t, \xi) \stackrel{\text{df}}{=} s_1 s_2 (R(s_1, s_2; \Lambda, t) + \xi) = s_1^2 s_2 + s_1 s_2^2 + \nu_1^{-1} \tau_3 \tau_4 s_1^2 \\ + \xi s_1 s_2 + \nu_1^{-1} \tau_1 \tau_2 s_2^2 + \lambda(\tau_3 + \tau_4)s_1 + \lambda(\tau_1 + \tau_2)s_2 + \lambda^2 \nu_1.$$

The form ω_ξ on the level set E_ξ is given by the formula

$$\omega_\xi = \text{Res} \frac{ds_1 ds_2}{F_\xi(s_1, s_2; \Lambda, t, \xi)}.$$

It is easy to see that ω_ξ is a (unique up to a scalar factor) regular differential on E_ξ . Therefore, for almost all $\xi \in F$ the integral

$$I(\xi) \stackrel{\text{df}}{=} \int_{E_\xi} |\omega_\xi|$$

converges.

Next we must prove that $I(\xi)$ is a locally L^1 -function of ξ . The singularities of $I(\xi)$ occur at the point where E_ξ becomes a singular curve. It is easy to see from explicit expression for ω_ξ that for each such point $\bar{\xi}$ the form $\omega_{\bar{\xi}}$ is nonsingular at each generic point of the (possibly reducible) curve $E_{\bar{\xi}}$. By a general theorem (see [S]), this implies that

$$I(\xi) = O(|\log(\xi - \bar{\xi})|^k)$$

for some k as $\xi \rightarrow \bar{\xi}$ and $I(\xi)$ is an L^1 -function near $\bar{\xi}$.

Finally, as $|\xi| \rightarrow \infty$, the curve E_ξ degenerate into the curve E_∞ given by the equation $s_1 s_2 = 0$, and the form ω_∞ on E_∞ is regular at both generic points of E_∞ . The same general result easily implies that as $|\xi| \rightarrow \infty$, the function $I(\xi)$ depends only on $|\xi|$. This implies that the sequence of integrals

$$\int_{|\xi| \leq p^n} I(\xi) |d\xi|.$$

stabilizes.

So, we proved that “fiberwise versions” of properties (i)–(iii) of Definition 0.1 hold. In particular, this implies that if $f_1, f_2 \in L_S$, then the integral

$$(5.2) \quad \int_{Z'_2} \psi_F(f'(z'))(p')^*(f_1 \times \overline{f_2})(z') |\omega'(z')|$$

converges in the sense of Definition 0.1. Using the convergence of the integrals in formula (4.8) one can see that the integral (5.2) equals $\langle \mu_\beta^{(1)}, f_1 \times \overline{f_2} \rangle$. Proposition 5.1 is proved.

5.2. S_4 -invariance. In this section we construct an action of the group S_4 on the data $\boldsymbol{\mu} = (\mathbf{Z}'_2, \mathbf{Y}_2, \mathbf{p}', \omega', \mathbf{f}')$.

Recall that the action of S_4 on Y (and the induced action on Y_2) is given by $w(x_1, t_2; x_2, t_2) = (x_1, t_1^w; x_2, t_2^{w^\sigma})$. We must construct the action of S_4 on Z'_2 compatible with the action of S_4 on Y_2 and preserving the function f' and the form ω' .

Let $\Sigma_i(\tau_1, \tau_2, \tau_3, \tau_4)$, $i = 1, 2, 3, 4$, be the i -th elementary symmetric function. For the proof, it suffice to rewrite the formulas the curve E_ξ and for the form ω_{E_ξ} in terms of S_4 -invariant combinations Σ_i of τ_i .

We pass from the coordinates s_1, s_2 on the plane to homogeneous coordinates S_1, S_2, S_3 such that $s_1 = S_1/S_3$, $s_2 = (S_2 - S_1)/S_3$. In these coordinates we have

$$S_3^3 F_\xi = P_1(S_2, S_3) S_1^2 + P_2(S_2, S_3) S_1 + P_3(S_2, S_3),$$

where

$$\begin{aligned} P_1(S_2, S_3) &= -S_2 + (\nu_1^{-1}\tau_1\tau_2 + \nu_1^{-1}\tau_3\tau_4 - \xi)S_3, \\ P_2(S_2, S_3) &= S_2^2 + (-\xi + 2\nu_1^{-1}\tau_1\tau_2)S_2S_3 + S_3^2\lambda(-\tau_1 - \tau_2 + \tau_3 + \tau_4), \\ P_3(S_2, S_3) &= \nu_1S_2(\nu_1^{-1}\tau_1S_2 + \lambda S_3)(\nu_1^{-1}\tau_2S_2 + \lambda S_3). \end{aligned}$$

Direct computations show that the discriminant $\Delta = P_2^2 - 4P_1P_3$ is a symmetric polynomial in $\tau_1, \tau_2, \tau_3, \tau_4$. Expressed in terms of elementary symmetric functions Σ_i , it is

$$\begin{aligned} \Delta &= S_2^4 + 2\xi S_2^3 S_3 + (\xi^2 + 2\Sigma_1 + 4\rho\Sigma_4)S_2^2 S_3^2 \\ &\quad + (2\xi\Sigma_1 + 4\rho^{-1} - 4\rho\Sigma_3)S_2 S_3^3 + (2\xi\rho^2 + \Sigma_1^2 - 4\Sigma_2)S_3^4, \end{aligned}$$

where $\rho = \lambda^2\nu_1$.

Taking instead of S_1 the variable S given by

$$(5.3) \quad S = 2P_1(S_2, S_3)S_1 - P_2(S_2, S_3),$$

we obtain that the equation of E_ξ is

$$S^2 - \Delta(S_2, S_3) = 0.$$

Hence, E_ξ is invariant under the action of S_4 .

Using the change of variables (5.3) one gets, in an obvious way, the action of S_4 on \mathbf{Z}'_2 .

PROPOSITION 5.2. *The described action of S_4 on \mathbf{Z}'_2 determines the action of S_4 on the data $\boldsymbol{\mu} = (\mathbf{Z}'_2, \mathbf{Y}_2, \mathbf{p}', \omega', \mathbf{f}')$.*

Proof. By construction, the action of S_4 commutes with \mathbf{p}' and \mathbf{f}' . The invariance of the form ω_ξ on E_ξ under the action of S_4 is easy, and it implies the invariance of ω' .

Using Proposition 5.2, one can easily write down the twisted data $\boldsymbol{\mu}_\alpha$ that conjecturally determine the measure μ_α defined by the Γ -function for GL_4 (see Conjectures 1 and 2 in the introduction).

Appendix. Comparison with the JPS gamma factor.

by J. Cogdell

In this appendix the proof of Proposition 2.1 is given.

PROPOSITION A.1. *Let $\tau = \mathcal{E}^{-1} \otimes \theta$ be a generic irreducible unitary representation of GL_{n-2} . Then*

$$\Gamma(\pi, \theta) = \omega_{\tilde{\tau}}(-1)^{n-1} \gamma(\pi \times \tilde{\tau}, 1/2, \psi),$$

where $\omega_{\tilde{\tau}}$ is the central character of $\tilde{\tau}$, $\tilde{\tau}$ is the contragredient of τ (which, since τ is unitary, is $\bar{\tau}$), and γ is the gamma factor of [JPS].

In this appendix we use a slightly different set up from the one in the main body of the paper (see [JPS]). In particular, we will use the following notation.

- w_n is the $n \times n$ permutation matrix $w_n = \begin{pmatrix} & & 1 \\ & \cdot & \\ 1 & & \end{pmatrix}$ with 1's along the

skew diagonal.

- For the outer involution of GL_n in this appendix we take the involution given by $g^\sigma = w_n {}^t g^{-1} w_n$.

- Q is the stabilizer of the point $(0 : \dots : 0 : 1) \in \mathbb{P}^n$ and so consists of matrices whose last row is $(0, \dots, 0, *)$.

- $R = Q \cap Q^\sigma$ is then the standard parabolic subgroup associated to the partition $n = 1 + (n - 2) + 1$ containing the upper triangular unipotent subgroup U .

- $\psi = \psi_n$ is the standard non-degenerate character of U .

Let $w_Q = \begin{pmatrix} 0 & 1 & 0 \\ I_{n-2} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then

$$U_Q = w_Q U w_Q^{-1} = \left\{ \begin{pmatrix} 1 & 0 & c \\ a' & u' & a \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

with $\psi_Q(u) = \psi_0(u_{1,n} + u_{2,3} + \dots + u_{n-2,n-1} + u_{n-1,1})$

$$U_R = U_Q \cap R = \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & u' & a \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

with $\psi_R = \psi_Q|_R$. Here, the matrices are in block form associated to the partition $(1, n - 2, 1)$ of n .

The induced representations are all as in the main text, with the replacement of U_- by the upper unipotent subgroup U .

We want to analyze a particular element of $\text{Hom}_R(\rho_{R,\mathcal{E}}, \rho_{R,\mathcal{E}})$ beginning with a irreducible unitary generic representation (π, V_π) of GL_n with central character \mathcal{E} .

Step 1: To get our first realization of $\rho_{R,\mathcal{E}}$ we do the following. We first pass from V_π to its Whittaker model $\mathcal{W}(\pi, \psi)$ and then restrict these functions to Q :

$$v \in V_\pi \mapsto W_v(q)$$

where W_v is the Whittaker function associated to v and $q \in Q$. The space

$$\{W_v(q) \mid W_v \in \mathcal{W}(\pi, \psi), q \in Q\}$$

gives a realization of $\rho_{Q,\mathcal{E}}$.

To get a realization of $\rho_{R,\mathcal{E}}$ we apply your maps C_1 and then C_2 :

$$W_v(q) \mapsto W_v(w_Q^{-1}q) \mapsto W_v(w_Q^{-1}r)$$

with now $r \in R$. So our first realization of $\rho_{R,\mathcal{E}}$ is on the space

$$\{W_v(w_Q^{-1}r) \mid W_v \in \mathcal{W}(\pi, \psi), r \in R\}$$

with R acting by right translation.

Step 2: The element of $\text{Hom}_R(\rho_{R,\mathcal{E}}, \rho_{R,\mathcal{E}})$ that we want to analyze is, using the notation in Section 2,

$$C_3 \circ C_2 \circ C_1 \circ \beta_\pi \circ C_1^{-1} \circ C_2^{-1}.$$

If we begin with $W_v(w_Q^{-1}r)$, then applying $C_1^{-1} \circ C_2^{-1}$ brings us back to $W_v(q)$. The map β_π in these models is the map

$$\beta_\pi : W_v(q) \mapsto \widetilde{W}_v(qw_n) = \widetilde{W}_{\pi(w_n)v}(q),$$

where, as in [JPS], we have set $\widetilde{W}(g) = W(w_n {}^t g^{-1})$. Applying C_1 and then C_2 to this gives

$$\widetilde{W}_{\pi(w_n)v}(q) \mapsto \widetilde{W}_{\pi(w_n)v}(w_Q^{-1}q) \mapsto \widetilde{W}_{\pi(w_n)v}(w_Q^{-1}r).$$

Now, applying the map C_3 gives

$$\widetilde{W}_{\pi(w_n)v}(w_Q^{-1}r) \mapsto \int_B \widetilde{W}_{\pi(w_n)v}(w_Q^{-1}\underline{b}r^\sigma) db,$$

where we now have

$$B = \left\{ \underline{b} = \begin{pmatrix} 1 & b & 0 \\ 0 & I_{n-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid b \in F^{n-2} \right\}.$$

So, our element of $\text{Hom}(\rho_{R,\mathcal{E}}, \rho_{R,\mathcal{E}})$ is

$$W_v(w_Q^{-1}r) \mapsto \int_B \widetilde{W}_{\pi(w_n)v}(w_Q^{-1}\underline{b}r^\sigma) db.$$

Step 3: We now pass from $\text{Hom}(\rho_{R,\mathcal{E}}, \rho_{R,\mathcal{E}})$ to $\text{Hom}(\rho_{M,\mathcal{E}}, \rho_{M,\mathcal{E}})$ using $\rho_{R,\mathcal{E}} \simeq \rho_H \otimes \rho_{M,\mathcal{E}}$. This isomorphism is effected by restricting the functions in $\rho_{R,\mathcal{E}}$ to M . To pass from $\rho_{R,\mathcal{E}}$ to $\rho_{M,\mathcal{E}}$ we must then twist these restrictions by the action of M on ρ_H . Let us write $M = C_n \times GL_{n-2}$ and correspondingly $m = cm'$ with $m' \in GL_{n-2}$ embedded as the center block in GL_n . Then the unitary action of m on ρ_H , if we realize this as functions on an appropriate space X , is $\varphi(x) \mapsto |\det(m')|^{1/2} \varphi(xm')$. This gives, in essence, $\rho_{M,\mathcal{E}} = |\det(m')|^{-1/2} \rho_{R,\mathcal{E}}|_M$.

So, our element of $\text{Hom}(\rho_{M,\mathcal{E}}, \rho_{M,\mathcal{E}})$, in our models, takes the form

$$\mathcal{E}(c) |\det(m')|^{-1/2} W_v(w_Q^{-1}m') \mapsto \mathcal{E}(c) |\det(m')|^{-1/2} \int_B \widetilde{W}_{\pi(w_n)v}(w_Q^{-1}\underline{b}(m')^\sigma) db.$$

Step 4: This morphism should act as a scalar $\Gamma(\pi, \theta)$ on each irreducible component θ of $\rho_{M,\mathcal{E}}$. Each such component is of the form $\theta = \mathcal{E} \otimes \tau$ with τ an irreducible unitary generic representation of GL_{n-2} . To compute this scalar we want to project into the $\mathcal{E} \otimes \tau$ component by pairing $\rho_{M,\mathcal{E}}$ with the contragredient $(\mathcal{E} \otimes \tau)^\sim = \mathcal{E}^{-1} \otimes \tilde{\tau} = \tilde{\mathcal{E}} \otimes \tilde{\tau}$. In this pairing, the central characters cancel. So we can effect the pairing by taking $\tilde{\tau}$ in its ψ^{-1} -Whittaker model and integrating over $U_{n-2} \backslash GL_{n-2}$. (We do not worry about convergence of the integrals.)

Let $W_{\tilde{\tau}}(g) \in \mathcal{W}(\tilde{\tau}, \psi^{-1})$. Before applying the morphism we have

$$\begin{aligned} I &= \langle |\det(m')|^{-1/2} W_v(w_Q^{-1} m'), W_{\tilde{\tau}}(m') \rangle \\ &= \int_{U_{n-2} \backslash GL_{n-2}} W_v \left(w_Q^{-1} \begin{pmatrix} 1 & & \\ & m' & \\ & & 1 \end{pmatrix} \right) W_{\tilde{\tau}}(m') |\det(m')|^{-1/2} dm' \\ &= \int_{U_{n-2} \backslash GL_{n-2}} W_{\pi(w_Q^{-1})v} \left(\begin{pmatrix} m' & & \\ & I_2 & \\ & & 1 \end{pmatrix} \right) W_{\tilde{\tau}}(m') |\det(m')|^{1/2-1} dm' \\ &= \Psi(W_{\pi(w_Q^{-1})v}, W_{\tilde{\tau}}, 1/2), \end{aligned}$$

where $\Psi(W_{\pi(w_Q^{-1})v}, W_{\tilde{\tau}}, 1/2)$ is as in [JPS].

After applying the morphism, we should get

$$\tilde{I} = \langle |\det(m')|^{-1/2} \int_B \tilde{W}_{\pi(w_n)v}(w_Q^{-1} \underline{b}(m')^\sigma) db, W_{\tilde{\tau}}(m') \rangle$$

which, if the morphism is to act by the scalar $\Gamma(\pi, \theta)$ on this piece, should give

$$\begin{aligned} \tilde{I} &= \Gamma(\pi, \theta) \langle |\det(m')|^{-1/2} W_v(w_Q^{-1} m'), W_{\tilde{\tau}}(m') \rangle \\ &= \Gamma(\pi, \theta) \Psi(W_{\pi(w_Q^{-1})v}, W_{\tilde{\tau}}, 1/2). \end{aligned}$$

Step 5: The final step is to identify \tilde{I} with the right-hand side of the $GL_n \times GL_{n-2}$ functional equation. If we write the integral \tilde{I} out it is

$$\begin{aligned} \tilde{I} &= \int_{U_{n-2} \backslash GL_{n-2}} \int_B \tilde{W}_{\pi(w_n)v}(w_Q^{-1} \underline{b}(m')^\sigma) db W_{\tilde{\tau}}(m') |\det(m')|^{-1/2} dm' \\ &= \int \int \tilde{W}_v \left(w_Q^{-1} \underline{b} \begin{pmatrix} 1 & & \\ & m' & \\ & & 1 \end{pmatrix}^\sigma w_n \right) db W_{\tilde{\tau}}(m') |\det(m')|^{-1/2} dm'. \end{aligned}$$

We next have a few elementary calculations:

$$\begin{aligned} \tilde{W}_v(g) &= \tilde{W}_{\pi(w_Q^{-1})v}(gw_Q) \\ \begin{pmatrix} 1 & & \\ & m' & \\ & & 1 \end{pmatrix}^\sigma &= \begin{pmatrix} 1 & & \\ & (m')^\sigma & \\ & & 1 \end{pmatrix} \\ w_Q^{-1} \begin{pmatrix} 1 & b & \\ & I_{n-2} & \\ & & 1 \end{pmatrix} w_Q &= \begin{pmatrix} I_{n-2} & & \\ & b & 1 \\ & & 1 \end{pmatrix} = \tilde{b} \\ w_Q^{-1} \begin{pmatrix} 1 & & \\ & m' & \\ & & 1 \end{pmatrix} w_Q &= \begin{pmatrix} m' & & \\ & I_2 & \\ & & 1 \end{pmatrix} \\ \begin{pmatrix} w_{n-2} & & \\ & I_2 & \\ & & 1 \end{pmatrix} w_Q^{-1} w_n w_Q &= \begin{pmatrix} I_{n-2} & & \\ & & w_2 \end{pmatrix} = w_{n,n-2} \end{aligned}$$

If we now use these calculations in our expression for \tilde{I} , and set $v' = \pi(w_Q^{-1})v$, we obtain

$$\tilde{I} = \int \int \widetilde{W}_{v'} \left(\tilde{b} \begin{pmatrix} w_{n-2}^t(m')^{-1} & \\ & I_2 \end{pmatrix} w_{n,n-2} \right) db W_{\tilde{\tau}}(m') |\det(m')|^{-1/2} dm'.$$

Now we change of variables $m' \mapsto w_{n-2}^t(m')^{-1}$ and note that $W_{\tilde{\tau}}(w_{n-2}^t(m')^{-1}) = \widetilde{W}_{\tilde{\tau}}(m')$. Then our expression can be written

$$\tilde{I} = \int \int (\rho(w_{n,n-2}) \widetilde{W}_{v'}) \begin{pmatrix} m' & & \\ bm' & 1 & \\ & & 1 \end{pmatrix} db \widetilde{W}_{\tilde{\tau}}(m') |\det(m')|^{1/2} dm'.$$

Making the change of variables $b \mapsto b(m')^{-1}$ we finally obtain

$$\begin{aligned} \tilde{I} &= \int \int (\rho(w_{n,n-2}) \widetilde{W}_{v'}) \begin{pmatrix} m' & & \\ b & 1 & \\ & & 1 \end{pmatrix} db \widetilde{W}_{\tilde{\tau}}(m') |\det(m')|^{-1/2} dm' \\ &= \int \int (\rho(w_{n,n-2}) \widetilde{W}_{v'}) \begin{pmatrix} m' & & \\ b & 1 & \\ & & 1 \end{pmatrix} db \widetilde{W}_{\tilde{\tau}}(m') |\det(m')|^{(1-1/2)-1} dm' \\ &= \Psi(\rho(w_{n,n-2}) \widetilde{W}_{v'}, \widetilde{W}_{\tilde{\tau}}, 1 - 1/2; 1). \end{aligned}$$

Thus we arrive at

$$\Gamma(\pi, \theta) \Psi(W_{v'}, W_{\tilde{\tau}}, 1/2) = \Psi(\rho(w_{n,n-2}) \widetilde{W}_{v'}, \widetilde{W}_{\tilde{\tau}}, 1 - 1/2; 1).$$

By the local functional equation of [JPS] we have

$$\omega_{\tilde{\tau}}(-1)^{n-1} \gamma(\pi \times \tilde{\tau}, 1/2, \psi) \Psi(W_{v'}, W_{\tilde{\tau}}, 1/2) = \Psi(\rho(w_{n,n-2}) \widetilde{W}_{v'}, \widetilde{W}_{\tilde{\tau}}, 1 - 1/2; 1).$$

Hence we have

$$\Gamma(\pi, \theta) = \omega_{\tilde{\tau}}(-1)^{n-1} \gamma(\pi \times \tilde{\tau}, 1/2, \psi)$$

as claimed.

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