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TWO INTEGRABLE SYSTEMS RELATED TO HYPERBOLIC MONOPOLES*

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Abstract. Monopoles on hyperbolic 3-space were introduced by Atiyah in 1984. This article describes two integrable systems which are closely related to hyperbolic monopoles: a one-dimensional lattice equation (the Braam-Austin or discrete Nahm equation), and a soliton system in (2+1)-dimensional anti-deSitter space-time.

1. Introduction. Sir Michael Atiyah has made important contributions in several areas of mathematical physics, and these include the study of Bogomolny-Prasad-Sommerfield (BPS) monopoles; see, for example, Atiyah and Hitchin (1979). Static BPS monopoles are solutions of a nonlinear elliptic partial differential equation on some three-dimensional Riemannian manifold. Most work on monopoles has dealt with the case when this manifold is Euclidean space R^3: the equations are then completely-integrable, and can be handled by geometrical techniques. But the monopole equations on hyperbolic space H^3 are also integrable, as was pointed out by Atiyah (1984a,b); and in some ways, hyperbolic monopoles are simpler than Euclidean ones. Hyperbolic monopoles tend to Euclidean monopoles as the curvature of the hyperbolic space tends to zero, although this is a delicate fact which was only recently established (Jarvis and Norbury 1997).

This note describes two integrable systems which are intimately related to, and were motivated by, hyperbolic monopoles. The first is a discrete system (or integrable mapping), which in a certain sense is dual to the hyperbolic monopole system; this is the discrete Nahm or Braam-Austin equation. The second comes from replacing the positive-definite space H^3 by a Lorentzian version, namely anti-deSitter space. The Bogomolny equation becomes an evolution equation on this space-time, admitting soliton solutions.

2. Hyperbolic Monopoles and the Discrete Nahm Equations. Motivated by the monad construction for instantons used by Atiyah et al (1978), Nahm (1982) discovered a kind of duality (subsequently called *reciprocity*: see Corrigan and Goddard 1984) between the monopole equations and solutions of a nonlinear ordinary differential equation. This ODE (described below) is called the Nahm equation.

For monopoles on hyperbolic space H^3 of curvature $-C$, a variant of the Nahm construction works (Braam and Austin 1990), at least if C^{-1} is a positive integer. Such "integral" hyperbolic monopoles correspond to certain solutions of a discrete Nahm equation: a nonlinear difference equation defined on C^{-1} lattice sites. This is actually a special case of the ADHM construction (Atiyah et al 1978) for instantons.

One has a picture, therefore, of a correspondence (reciprocity) between monopoles on the 3-space of constant negative curvature $-C$, and solutions of the discrete Nahm equation on a one-dimensional lattice with lattice spacing C (provided C is the inverse of an integer). The limit $C \to 0$ is the continuum limit, in which the discrete Nahm (difference) equation becomes the Nahm (differential) equation. It seems likely on

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general grounds that reciprocity operates, in some sense, for non-integral hyperbolic monopoles; but this remains an open question. In what follows, I shall concentrate just on the discrete Nahm equations, and not say anything about hyperbolic monopoles.

Let k be a fixed positive integer; and let A_j, B_j, C_j and D_j denote $k \times k$ matrices, defined for each value of the integer j. In other words, we have four $k \times k$ matrices on a one-dimensional lattice indexed by j. Let $s_\pm$ denote the forward and backward step operators on this lattice: so $(s_+\Psi)_j = \Psi_{j+1}$ and $(s_-\Psi)_j = \Psi_{j-1}$. For brevity, the subscript j will usually be omitted in what follows; thus A stands for A_j, $A_+ = s_+A$ stands for A_{j+1}, and so forth.

Consider the two linear operators

$$\begin{aligned} U &:= Cs_+ + \lambda A \\ V &:= D_-s_- - \lambda^{-1}B \end{aligned}$$

(acting on a k-vector Ψ defined on the lattice). Here λ is a constant scalar parameter, and the minus signs are mainly for notational convenience. The eigenvalue equations

$$\begin{aligned} U\Psi &:= C\Psi_+ + \lambda A\Psi = \zeta\Psi, \\ V\Psi &:= D_-\Psi_- - \lambda^{-1}B\Psi = \mu^{-1}\Psi \end{aligned} \tag{1}$$

are difference equations which propagate Ψ forwards and backwards along the lattice. In order for a nontrivial solution (simultaneous eigenfunction of U and V) to exist, we need U and V to commute, and the parameters λ, ζ, μ to satisfy an algebraic relation (which turns out to be the vanishing of a homogeneous polynomial in these three variables). The condition $[U, V] = 0$ gives the discrete Nahm equation; equation (1) is a Lax pair for it; and the algebraic relation defines a spectral curve (from which one can derive conserved quantities, show that the solutions of discrete Nahm correspond to stepping along straight lines on the Jacobian of this curve, etc: see Murray and Singer 1998).

The condition that $[U, V] = 0$ should hold for all λ is equivalent to

$$\begin{aligned} A_+ &= DAD^{-1}, \\ B_+ &= C^{-1}BC, \\ C_+D_+ &= DC + [A_+, B_+]. \end{aligned} \tag{2}$$

These are the discrete Nahm (or Braam-Austin) equations. They consist of three difference equations for four (matrix) functions: the under-determinacy reflects the gauge freedom in (2). Namely, if Λ is a non-singular matrix on the lattice, then the system (2) is invariant under the gauge transformations

$$\begin{aligned} A &\mapsto \Lambda A\Lambda^{-1} \\ B &\mapsto \Lambda B\Lambda^{-1} \\ C &\mapsto \Lambda C\Lambda_+^{-1} \\ D &\mapsto \Lambda_+ D\Lambda^{-1}. \end{aligned} \tag{3}$$

A gauge choice such as $D = C$ converts (2) into a determined system: three matrix difference equations for three matrices.

For completeness, let us note the correspondence with the notation of Braam and Austin (1990), where the equations (2) were first derived. The relation between A, B,

C, D and their matrices β_j, γ_j is given by $A = B^*$, $D = C^*$, and

$$\beta_{2j} = B_j\,,$$
$$\gamma_{2j+1} = C_j\,.$$

I shall not describe the spectral curve and its consequences here; but merely list the eight independent conserved quantities in the $k = 2$ case. They are $\operatorname{tr} A$, $\operatorname{tr} A^2$, $\operatorname{tr} B$, $\operatorname{tr} B^2$, $\operatorname{tr}(CD)$, $\operatorname{tr}(ACD)$, $\operatorname{tr}(BCD)$, and $\operatorname{tr}[CD(CD - 2AB)]$. Each of these expressions is constant on the lattice, by virtue of (2); note that they are also gauge-invariant.

A continuum limit of (2) may be obtained as follows. Replace the integer variable j by $t = jh$, where h is the "lattice spacing", and take the limit $h \to 0$. Write

$$\begin{aligned} C &= (2h)^{-1} I + \tfrac{1}{2}\mathrm{i}T_3 = D, \\ B &= \tfrac{1}{2}(T_1 + \mathrm{i}T_2) = -A^*, \end{aligned} \tag{4}$$

where the T_α are antihermitian $k \times k$ matrices, I denotes the identity matrix, and star denotes complex conjugate transpose. Then the $h \to 0$ limit of (2) is

$$\frac{\mathrm{d}}{\mathrm{d}t} T_\alpha + \tfrac{1}{2}\varepsilon_{\alpha\beta\gamma}\big[T_\beta, T_\gamma\big] = 0, \tag{5}$$

which are the Nahm equations. Similarly, one obtains the standard Lax pair for the Nahm equations as the continuum limit of (1), if ζ and μ are chosen appropriately.

3. Reduction to a Discrete Toda System. It has long been known that the Nahm equation reduces to the Toda lattice. The T_α take values in a Lie algebra, which (in the simplest case that we are considering here) is $su(k)$. To reduce to Toda, one takes T_3 in a Cartan subalgebra, and $T_1 \pm \mathrm{i}T_2$ corresponding to $\pm$ a set of simple roots. What follows is the discrete version of this reduction.

We express A, B, C and D in terms of $2k$ lattice functions $f_a = f_{aj}$, $p_a = p_{aj}$, where $a = 1, 2, \ldots, k$, as follows: $C = D = \operatorname{diag}(f_1, f_2, \ldots, f_k)$ and

$$B = \begin{pmatrix} 0 & p_2 & 0 & \ldots & 0 \\ 0 & 0 & p_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & p_k \\ p_1 & 0 & 0 & \ldots & 0 \end{pmatrix} = -A^*\,. \tag{6}$$

Then the discrete Nahm equations (2) reduce to

$$\begin{aligned} (p_a)_+ &= p_a f_a / f_{a-1}\,, \\ f_a^2 &= (f_a^2)_- + p_{a+1}^2 - p_a^2\,, \end{aligned} \tag{7}$$

where f_0 is interpreted as f_k, and p_{k+1} as p_1 (in other words, the index a is periodic with period k). The equations (7) constitute a discrete-time Toda lattice. The first example of such a system was that of Hirota (1977), and many other examples have been described more recently.

Notice that the quantity $\Sigma = \sum_{a=1}^{k} f_a^2$ is conserved; in other words, $\Sigma_+ = \Sigma$. Let us define a parameter h by $\Sigma = k/h^2$, and assume that $f_a h \to 1$ as $h \to 0$, for each a.

Then the $h \to 0$ limit of (7) is the differential equation

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \log p_a^2 = p_{a+1}^2 - 2p_a^2 + p_{a-1}^2, \tag{8}$$

which is the Toda lattice.

Let us look in detail at the $k = 2$ case. Rewrite f_a and p_a in terms of three functions u, v, w, and the constant h, according to

$$\begin{aligned} f_1 &= h^{-1}\sqrt{1+2hw}, \\ f_2 &= h^{-1}\sqrt{1-2hw}, \\ p_1 &= u - v, \\ p_2 &= u + v. \end{aligned}$$

Then (7) is

$$\begin{aligned} w_+ &= w + 2hu_+v_+ \\ u_+ &= (u - 2hwv)/\sqrt{1-4h^2w^2} \\ v_+ &= (v - 2hwu)/\sqrt{1-4h^2w^2}, \end{aligned} \tag{9}$$

which are a discrete-time version of Euler's equations for a spinning top (with an appropriate choice of moments of inertia). Indeed, in the continuum limit $h \to 0$, (9) becomes $\mathrm{d}w/\mathrm{d}t = 2uv$, $\mathrm{d}u/\mathrm{d}t = -2vw$, $\mathrm{d}v/\mathrm{d}t = -2wu$. These are Euler's equations, which can be solved in terms of elliptic functions; and (9) can be solved likewise, as follows.

Notice that (9) admits two independent conserved quantities, namely

$$\begin{aligned} \Theta &= u^2 - v^2, \\ \Omega &= 2w^2 + u^2 + v^2 - 4hwuv. \end{aligned} \tag{10}$$

This enables one to express u and v in terms of w (and Θ, Ω); and then the first equation in (9) leads to a difference equation for w alone. Its solution is

$$w_j = \frac{k}{2h} \operatorname{sn}(bh) \operatorname{sn}(bjh + c), \tag{11}$$

where k, b and c are "constants of integration". Here k denotes the modulus of the elliptic functions. The conserved quantities Θ and Ω are related to k and b as follows:

$$\begin{aligned} \Theta &= (2h^2)^{-1}\big[\operatorname{cn}(bh) - \operatorname{dn}(bh)\big], \\ \Omega &= (2h^2)^{-1}\big[1 - \operatorname{cn}(bh)\operatorname{dn}(bh)\big]. \end{aligned}$$

The functions u and v are then given by

$$\begin{aligned} (u+v)^2 &= (\Omega - 2w^2 + \Delta)/(1-2hw), \\ (u-v)^2 &= (\Omega - 2w^2 - \Delta)/(1+2hw), \end{aligned} \tag{12}$$

where $\Delta_j = h^{-1}\operatorname{sn}(bh)\operatorname{cn}(bjh+c)\operatorname{dn}(bjh+c)$. To see that u and v are well-defined (and real) for all b, c and $0 < k < 1$, note first that $1 \pm 2hw$ is positive. Secondly, $\Omega - 2w^2 \geq 0$, from the inequality $1 - \operatorname{cn}\operatorname{dn} \geq k^2 \operatorname{sn}^2$. Finally, $(\Omega - 2w^2)^2 - \Delta^2 = (1 - 4h^2w^2)\Theta^2 \geq 0$.

As a final remark, note that in the limiting case $k = 1$, the solution is

$$w = (2h)^{-1} \tanh(bh) \tanh(bt),$$
$$v = u = (2h)^{-1} \tanh(bh) \operatorname{sech}(bt) \big/ \sqrt{1 + \tanh(bh) \tanh(bt)}.$$

It seems likely that this elliptic $k = 2$ solution corresponds to hyperbolic 2-monopoles with gauge group SU(2), via the Braam-Austin construction. More generally, for $k > 2$ one may speculate that discrete-Toda solutions correspond to hyperbolic k-monopoles with C_k cyclic symmetry, since this is what happens for Euclidean monopoles (Sutcliffe 1996).

4. Solitons in (2+1)-Dimensional Anti-deSitter Space-Time. Let M be a three-dimensional Riemannian manifold, with metric g and volume element η. We are interested in Yang-Mills-Higgs fields on M, with gauge group SU(2) (for simplicity). So we have a Higgs field $\Phi = \Phi(x^\mu)$ taking values in the Lie algebra su(2); here $x^\mu = (x^0, x^1, x^2)$ are local coordinates on M. A gauge potential (connection) $A_\mu(x^\nu)$ determines the covariant derivative $D_\mu \Phi = \partial\Phi/\partial x^\mu + [A_\mu, \Phi]$. The gauge field (curvature) is the su(2)-valued 2-form $F_{\mu\nu} = [D_\mu, D_\nu]$. And the Bogomolny equations for (A_μ, Φ) are

$$D\Phi = *F, \tag{13}$$

or, in index notation,

$$D_\mu \Phi = \tfrac{1}{2} g_{\mu\nu} \eta^{\nu\alpha\beta} F_{\alpha\beta}. \tag{13'}$$

These are coupled nonlinear partial differential equations which, in general, are not completely integrable. But (13) *is* an integrable system (in the sense that a Lax pair exists) if the metric g has constant curvature. For example, if (M, g) is Euclidean space R^3 or hyperbolic space H^3, then (13) is the equation for Euclidean or hyperbolic BPS monopoles, respectively.

Another possibility is for g to have Lorentzian signature $- + +$, and then (13) are evolution equations in the space-time (M, g). Soliton solutions in the case of flat space-time have been studied in some detail: see Ward (1988, 1990, 1998). The aim here is to describe an example in curved space-time.

There are two curved space-times with constant curvature: deSitter space with positive scalar curvature R, and anti-deSitter space with $R < 0$ (Hawking and Ellis 1973). I shall deal here with the latter case only, namely anti-deSitter space (AdS). By definition, (2+1)-dimensional anti-deSitter space is the universal covering space of the hyperboloid $\mathcal{H}$ with equation

$$U^2 + V^2 - X^2 - Y^2 = 1, \tag{14}$$

and with metric induced from

$$ds^2 = -dU^2 - dV^2 + dX^2 + dY^2. \tag{15}$$

If, for example, we parametrize the hyperboloid $\mathcal{H}$ by

$$\begin{aligned} U &= \sec\rho\cos\theta \\ V &= \sec\rho\sin\theta \\ X &= \tan\rho\cos\varphi \\ Y &= \tan\rho\sin\varphi \end{aligned} \tag{16}$$

with $0 \le \rho < \pi/2$, then we get the metric

$$ds^2 = \sec^2\rho(-d\theta^2 + d\rho^2 + \sin^2\rho\, d\varphi^2). \tag{17}$$

At this stage, the space-time contains closed timelike curves, because of the periodicity of θ. Anti-deSitter space is the universal cover of $\mathcal{H}$, in which θ is unwound (so that $\theta \in R$). Consequently, AdS, as a manifold, is the product of an open spatial disc (on which ρ and φ are polar coordinates) with time $\theta \in R$. It is a space of constant curvature, with scalar curvature equal to -6. Null/spacelike infinity $\mathcal{I}$ consists of the timelike cylinder $\rho = \pi/2$; this surface is never reached by timelike geodesics.

In what follows below, we shall also use Poincaré coordinates t, x and $r > 0$. They are defined by

$$\begin{aligned} t &= -V/(U+X) \\ r &= 1/(U+X) \\ x &= Y/(U+X), \end{aligned} \tag{18}$$

in terms of which the metric is

$$ds^2 = r^{-2}(-dt^2 + dr^2 + dx^2). \tag{19}$$

But these only cover a small part of AdS, corresponding to half $U + X > 0$ of the hyperboloid $\mathcal{H}$. The surface $r = 0$ is part of infinity $\mathcal{I}$.

The minitwistor space corresponding to AdS, or rather to the Poincaré space (19), is $CP^1 \times CP^1$, which we visualize as a quadric Q in CP^3 (*cf.* Hitchin 1982). The points of space-time correspond to certain plane sections (conics) of Q. The space of all planes is a CP^3. But the relevant conics have to be real (which in this case means that their defining planes have real coefficients), and nondegenerate. So the space of these acceptable conics is the "top half" of RP^3, parametrized by the real homogeneous coordinates (U, V, X, Y) with $U^2 + V^2 - X^2 - Y^2 > 0$. This RP^3_+ is double-covered by the original hyperboloid $\mathcal{H}$, and is essentially the Poincaré space (19). More accurately, the coordinates (t, r, x) cover all of RP^3_+ except for a set of measure zero. If ω and ζ are standard coordinates on the two CP^1 factors of Q, then the conics are

$$\omega = \omega(\zeta) = \frac{v\zeta - (uv + r^2)}{\zeta - u} = \frac{(Y+V)\zeta + (X-U)}{(U+X)\zeta + (V-Y)}, \tag{20}$$

where $u = x + t$ and $v = x - t$. Eqn (20) expresses the correspondence between space-time and twistor space Q.

The idea now is that holomorphic vector bundles V over Q (saisfying some mild conditions) determine multi-soliton solutions of (13) in anti-deSitter space, via the usual Penrose transform. In our case, the relevant vector bundles are stable bundles of rank 2, with Chern numbers $c_1 = 0$ and $c_2 = 2n$, n being a positive integer. In the simplest case $n = 1$, the moduli space of such bundles is 5-complex-dimensional (*cf.* Hurtubise 1986, Buchdahl 1987). When we impose reality conditions, which amounts to taking the gauge group to be SU(2) rather than SL(2,C), the moduli space becomes 5-real-dimensional. So we expect, in this simplest case, to get a five-parameter family of soliton solutions, exactly as for the flat-space-time system (Ward 1988, 1990, 1998). This is exactly what happens (Hickin 1998).

One explicit way of seeing how solutions arise is as follows: it involves a Lax pair for the integrable system (13). Define two operators ∇_1 and ∇_2 by

$$\begin{aligned}\nabla_1 &= r\partial_r - 2(\zeta - u)\partial_u \\ \nabla_2 &= 2\partial_v + r^{-1}(\zeta - u)\partial_r.\end{aligned} \tag{21}$$

Notice that ∇_1 and ∇_2 both annihilate the expression (20). This is related to the fact that twistor space Q is the quotient of the distribution $\{\nabla_1, \nabla_2\}$ (on the four-dimensional correspondence space whose local coordinates are (t, r, x, ζ)). The Lax pair involves the gauge-covariant version of (21), and consists of the pair of equations

$$\begin{aligned}\big[rD_r + \Phi - 2(\zeta - u)D_u\big]\psi &= 0 \\ \big[2D_v + r^{-1}(\zeta - u)(D_r - r^{-1}\Phi)\big]\psi &= 0,\end{aligned} \tag{22}$$

where $\psi = \psi(t, r, x, \zeta)$ is a 2×2 matrix. The consistency condition for this overdetermined system is exactly (13).

The functions ψ corresponding to $n = 1$ bundles can be taken to have the rational form

$$\psi = I - \frac{(\bar{\zeta}_0 - \zeta_0)}{(\zeta - \zeta_0)} \frac{p^* \otimes p}{p \cdot p^*}, \tag{23}$$

where I denotes the identity 2×2 matrix, ζ_0 is a complex constant, $p(t, r, x)$ is a row 2-vector of linear functions of $\omega_0 = \omega(\zeta_0)$, and p^* denotes its complex conjugate transpose. So p has the form

$$p = (a\omega_0 + b, c\omega_0 + d), \tag{24}$$

where a, b, c, d are complex constants with $ad - bc \neq 0$. The Yang-Mills-Higgs fields (Φ, A_μ) can then be read off from (22–24), and they will automatically satisfy (13). The parameters ζ_0, a, b, c, d are not all significant: it is clear from (23) that an overall complex scaling of p will not change ψ, and furthermore that multiplying p on the right by a constant SU(2) matrix will induce a gauge transformation on (Φ, A_μ). Removing this freedom leaves us with a five-real-parameter family of solutions.

Each of these solutions represents a single soliton (lump), and the five parameters describe the location (2), velocity (2) and size (1) of this soliton. It is straightforward to write down the fields (Φ, A_μ) explicitly as rational functions of t, r, x (or U, V, X, Y) — but these expressions are not immediately transparent, and we shall make do with the following remarks. The solitons are spatially localized, in the sense that $\Phi \to 0$ and $F_{\mu\nu} \to 0$ as $r \to 0$, *i.e.*. as one approaches null/spacelike infinity $\mathcal{I}$. To see a simple example, one may set $p = (\omega_0, 1)$. Then the positive-definite gauge-invariant quantity $-\operatorname{tr}\Phi^2$ (which is a good one for visualizing the field) is given by

$$-\operatorname{tr}\Phi^2 = 8r^4 \big/ \big[(r^2 + x^2 - t^2)^2 + 2x^2 + 2t^2\big]^2. \tag{25}$$

The graph of this function is a single lump, with its maximum along the timelike geodesic $x = 0$, $r^2 = t^2 + 1$: a soliton in free fall.

5. Concluding Remarks. Many of the ramifications of Atiyah's (1984b) work on hyperbolic monopoles are only now being addressed. In the positive-definite case, a study of the relation between hyperbolic monopoles (and their symmetries) and the corresponding spectral curves is currently underway (Murray and Singer 1998). In the Lorentzian case, soliton solutions and their corresponding vector bundles are being investigated (Hickin 1998); specific questions include multi-soliton ($n > 1$) dynamics, and what happens in deSitter (rather than anti-deSitter) space.

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