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QUANTUM DETERMINANTS AND QUASIDETERMINANTS*

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Introduction. The notion of a quasideterminant and a quasiminor of a matrix $A = (a_{ij})$ with not necessarily commuting entries was introduced in [GR1-3]. The ordinary determinant of a matrix with commuting entries can be written (in many ways) as a product of quasiminors. Furthermore, it was noticed in [GR1-3, KL, GKLLRT, Mo] that such well-known noncommutative determinants as the Berezinian, the Capelli determinant, the quantum determinant of the generating matrix of the quantum group $U_h(gl_n)$ and the Yangian $Y(gl_n)$ can be expressed as products of commuting quasiminors.

The aim of this paper is to extend these results to a rather general class of Hopf algebras given by the Faddeev-Reshetikhin-Takhtajan type relations – the twisted quantum groups defined in Section 1.4. Such quantum groups arise when Belavin-Drinfeld classical r-matrices [BD] are quantized.

Our main result is that the quantum determinant of the generating matrix of a twisted quantum group equals the product of commuting quasiminors of this matrix.

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1. Twisted quantum groups.

1.1. Quantum gl_n. Consider the quantum universal enveloping algebra $U = U_h(gl_n)$ [Dr]. This is the h-adically complete topological Hopf algebra over $\mathbb{C}[[h]]$ generated by E_i, F_i, $i = 1, ..., n-1$, and H_i, $i = 1, ..., n$ with defining relations

$$[H_i, E_i] = E_i, [H_i, F_i] = -F_i, [H_{i+1}, E_i] = -E_i, [H_{i+1}, F_i] = F_i,$$
$$[H_i, E_j] = [H_i, F_j] = 0 \text{ if } i - j \neq 0, 1; [H_i, H_j] = 0,$$
$$[E_i, F_j] = \delta_{ij} \frac{e^{h(H_i - H_{i+1})} - e^{-h(H_i - H_{i+1})}}{e^h - e^{-h}},$$
$$E_i^2 E_{i\pm1} - (e^h + e^{-h}) E_i E_{i\pm1} E_i + E_{i\pm1} E_i^2 = 0,$$
$$F_i^2 F_{i\pm1} - (e^h + e^{-h}) F_i F_{i\pm1} F_i + F_{i\pm1} F_i^2 = 0, \tag{1.1}$$

The coproduct, counit, and antipode are defined by

$$\Delta(E_i) = E_i \otimes e^{h(H_i - H_{i+1})} + 1 \otimes E_i, \Delta(F_i) = F_i \otimes 1 + e^{-h(H_i - H_{i+1})} \otimes F_i,$$
$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i$$
$$\varepsilon(F_i) = \varepsilon(E_i) = \varepsilon(H_i) = 0,$$
$$S(E_i) = -E_i e^{-h(H_i - H_{i+1})}, S(F_i) = -e^{h(H_i - H_{i+1})} F_i, S(H_i) = -H_i. \tag{1.2}$$

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Let $U_{\geq 0}$ be the subalgebra of U generated by E_i and H_i, $U_{\leq 0}$ be the subalgebra of U generated by F_i and H_i, and U_0 be the subalgebra generated by H_i. They are Hopf subalgebras of U. Let I_+, I_- be the kernels of the natural maps $U_{\geq 0} \to U_0$, $U_{\leq 0} \to U_0$. We also denote by U_+ and U_- the subalgebras of $U_{\geq 0}$ and $U_{\leq 0}$ generated by E_i and F_i respectively.

The Hopf algebra U is quasitriangular: it admits the universal R-matrix

$$\mathcal{R} = e^{h\sum_i H_i\otimes H_i}(1+\sum_{j\geq 1} a_j\otimes a^j) \in U_{\geq 0}\otimes U_{\leq 0}, \tag{1.3}$$

where $a_j \in U_+$ and $a^j \in U_-$, and $\varepsilon(a_j) = \varepsilon(a^j) = 0$.

1.2. Twists.

DEFINITION 1.1. *(Drinfeld) We say that an element $\mathcal{J} \in U\otimes U$ is a twist if $\mathcal{J} = 1$ mod h, and*

$$(\varepsilon\otimes 1)(\mathcal{J}) = (1\otimes\varepsilon)(\mathcal{J}) = 1,\ (\Delta\otimes 1)(\mathcal{J})\mathcal{J}_{12} = (1\otimes\Delta)(\mathcal{J})\mathcal{J}_{23}. \tag{1.4}$$

REMARK. In this definition and below, $\otimes$ denotes the tensor product completed with respect to the h-adic topology.

DEFINITION 1.2. *We say that a twist $\mathcal{J}$ is upper triangular if $\mathcal{J} = \mathcal{J}^0\mathcal{J}'$, where $\mathcal{J}^0 = e^{h\sum_{ij} a_{ij} H_i\otimes H_j} \in U_0\otimes U_0$ ($a_{ij} \in \mathbb{C}[[h]]$), and $\mathcal{J}' \in 1 + I_+\otimes I_-$.*

1.3. The twisted coproduct. Given any twist $\mathcal{J}$, we define a new coproduct $\Delta_{\mathcal{J}}(x) := \mathcal{J}^{-1}\Delta(x)\mathcal{J}$ on U (from now on we do not use the coproduct Δ and therefore denote $\Delta_{\mathcal{J}}$ simply by Δ). This coproduct defines a new Hopf algebra structure on U. We denote the obtained Hopf algebra by $U_{\mathcal{J}}$.

The Hopf algebra $U_{\mathcal{J}}$ is quasitriangular with the universal R-matrix

$$\mathcal{R}_{\mathcal{J}} = \mathcal{J}_{21}^{-1}\mathcal{R}\mathcal{J}. \tag{1.5}$$

Since $U_{\mathcal{J}}$ coincides with U as an algebra, it has the same representations. Let V be the n-dimensional (vector) representation, with the standard basis v_i such that $H_i v_j = \delta_{ij} v_j$, and $E_i v_{i+1} = v_i$. Let $R_{\mathcal{J}} : V\otimes V \to V\otimes V$ be defined by

$$R_{\mathcal{J}} = \mathcal{R}_{\mathcal{J}}|_{V\otimes V}. \tag{1.6}$$

1.4. The twisted quantum group. Define the quantum function algebra $A_{\mathcal{J}}$. This is the h-adically complete algebra over $\mathbb{C}[[h]]$ which is generated by $T, T^{-1} \in Mat_n(\mathbb{C})\otimes A_{\mathcal{J}}$ with the Faddeev-Reshetikhin-Takhtajan defining relations

$$TT^{-1} = T^{-1}T = 1,\ R_{\mathcal{J}}^{12}T^{13}T^{23} = T^{23}T^{13}R_{\mathcal{J}}^{12}. \tag{1.7}$$

This algebra is a Hopf algebra with $\Delta(T) = T^{12}T^{13}, \varepsilon(T) = 1, S(T) = T^{-1}$. We call it *the twisted quantum group*.

Let $(,) : A_{\mathcal{J}}\times U_{\mathcal{J}} \to \mathbb{C}[[h]]$ be the bilinear form defined by the formula $(T,x) = \pi_V(x)$ and the properties $(1,x) = \varepsilon(x)$, $(ab,x) = (a\otimes b, \Delta(x))$. It is easy to see that this form is well defined and satisfies the equation $(a,xy) = (\Delta(a), x\otimes y)$. This implies that $(,)$ defines a Hopf algebra homomorphism $\theta : U_{\mathcal{J}} \to A_{\mathcal{J}}^*$ and a Hopf algebra homomorphism $\theta' : A_{\mathcal{J}} \to U_{\mathcal{J}}^*$.

One can show that the map θ' is injective. This property is proved by considering the quasiclassical limit, and will be used in Section 3.

1.5. Quantum determinant. We have $T = \sum E_{ij} \otimes t_{ij}$, where E_{ij} are elementary matrices and $t_{ij} \in A_{\mathcal{J}}$. So we can think of T as the matrix (t_{ij}) over $A_{\mathcal{J}}$. Let us define the quantum determinant of this matrix.

It is known that the Hopf algebra $A_{\mathcal{J}}$ is a flat deformation of the function algebra $\mathcal{O}(GL_n)$. Moreover, it is isomorphic to $\mathcal{O}(GL_n)[[h]]$ as a coalgebra (by a map that equals 1 modulo h). So right $A_{\mathcal{J}}$-comodules correspond to left GL_n-modules. Let Det be the 1-dimensional $A_{\mathcal{J}}$-comodule corresponding to the determinant character of GL_n. Let v be a generator of Det, and $\pi^* : \mathrm{Det} \to \mathrm{Det} \otimes A_{\mathcal{J}}$ the coaction. We have $\pi^*(v) = v \otimes D$. The element D is obviously grouplike. This element is called the quantum determinant of T. It equals the ordinary determinant modulo h.

2. Quasideterminants and the main theorem.

2.1. Quasideterminants. Quasideterminants were introduced in [GR1], as follows. Let X be an $m \times m$-matrix over an algebra A. For any $1 \le i, j \le m$, let $r_i(X)$, $c_j(X)$ be the i-th row and the j-th column of X. Let X^{ij} be the submatrix of X obtained by removing the i-th row and the j-th column from X. For a row vector r let $r^{(j)}$ be r without the j-th entry. For a column vector c let $c^{(i)}$ be c without the i-th entry. Assume that X^{ij} is invertible. Then the quasideterminant $|X|_{ij} \in A$ is defined by the formula

$$|X|_{ij} = x_{ij} - r_i(X)^{(j)}(X^{ij})^{-1}c_j(X)^{(i)}, \tag{2.1}$$

where x_{ij} is the ij-th entry of X.

For any $n \times n$-matrix $X = (x_{ij})$ over an algebra A and any permutation $\sigma \in S_n$, denote by $\det_\sigma(X)$ the expression

$$\det_\sigma(X) = \mu_\sigma(|X|_{nn}, |X^{nn}|_{n-1,n-1}, ..., |X^{n..i,n..i}|_{i-1,i-1}, ..., x_{11}), \tag{2.2}$$

where $X^{n...i,n...i}$ is the matrix obtained from X by erasing rows and columns with numbers $i, ..., n$, and $\mu_\sigma(a_1, ..., a_n) = a_{\sigma 1} ... a_{\sigma n}$.

It is easy to see that if Y is an upper triangular matrix with ones on the diagonal and Z a lower triangular matrix with ones on the diagonal then $\det_\sigma(ZXY) = \det_\sigma(X)$.

2.2. The main theorem. The main result of this paper is the following theorem.

Main Theorem. *Let $\mathcal{J}$ be an upper triangular twist. Then the factors in (2.2) commute with each other, and for any $\sigma \in S_n$*

$$D = det_\sigma(T). \tag{2.3}$$

This theorem for $\mathcal{J} = 1$ was formulated in [GR1-3] (see Theorem 4.2 in [GR3]) and proved in [KL] (Theorem 3.1).

Concrete examples of upper triangular twists are contained in [Ho]. A review of these examples can be found in Section 4.

3. Proof of the main theorem. First of all, the first statement of the theorem (commutativity of the factors) follows from the second one, so it is enough to prove the second statement (formula (2.3)).

Let $L^+_{\mathcal{J}} = (\pi_V \otimes 1)(\mathcal{R}_{\mathcal{J}})$, where $\pi_V : U_{\mathcal{J}} \to \mathrm{End}(V)$ defines the vector representation of $U_{\mathcal{J}}$. Let $L^-_{\mathcal{J}} = (\pi_V \otimes 1)(\mathcal{R}^{-1}_{\mathcal{J},21})$. If $\mathcal{J} = 1$, we will denote $L^\pm_{\mathcal{J}}$ simply by

$L^{\pm}$.

Let $f_{\pm} : A_{\mathcal{J}} \to U_{\mathcal{J}}$ be the algebra homomorphisms defined by the formula $f_{\pm}(T) = L^{\pm}_{\mathcal{J}}$. It is easy to check that they are well defined and are coalgebra antihomomorphisms.

Let $f : A_{\mathcal{J}} \to U_{\mathcal{J}} \otimes U_{\mathcal{J}}$ be the algebra homomorphism defined by

$$f(x) = (f_+ \otimes f_-)(\Delta(x)).$$

PROPOSITION 3.1. *f is injective.*

Proof. It is easy to see that $f_+(x) = (x, \mathcal{R}_{\mathcal{J}})$, and $f_-(x) = (x, \mathcal{R}^{-1}_{\mathcal{J},21})$ (this notation means that we take the inner product of x with the first component of $\mathcal{R}_{\mathcal{J}}$ and $\mathcal{R}^{-1}_{\mathcal{J},21}$ and leave the second component intact). Therefore, $f(x) = (x, \mathcal{R}_{\mathcal{J},12}\mathcal{R}^{-1}_{\mathcal{J},31})$. This implies that if $x \in \mathrm{Ker}(f)$ then $(x, y) = 0$ for any $y \in U'$, where $U' \subset U_{\mathcal{J}}$ is the saturated subalgebra generated by the left and right components of $\mathcal{R}_{\mathcal{J}}$.

We claim that $U' = U_{\mathcal{J}}$. Indeed, $U_{\mathcal{J}}$ is a quantization of the quasitriangular Lie bialgebra (gl_n, r), where r is a classical r-matrix on gl_n such that $r^{21} + r = 2\sum_{ij} E_{ij} \otimes E_{ji}$. Thus, the components of r generate gl_n, i.e. (gl_n, r) is a minimal quasitriangular Lie bialgebra. This implies that $U_{\mathcal{J}}$ is a minimal quasitriangular Hopf algebra, i.e. $U' = U_{\mathcal{J}}$.

To conclude the argument, we recall that the map θ' is injective. This implies that if $(x, y) = 0$ for all $y \in U_{\mathcal{J}}$ then $x = 0$. Thus, $\mathrm{Ker}(f) = 0$, as desired. □

Proposition 3.1 shows that it is enough to prove (2.3) after applying f. Therefore, the main theorem is a consequence of the following two propositions.

PROPOSITION 3.2. *$f(D) = Pe^{hH} \otimes Pe^{-hH}$, where $H = \sum_{i=1}^{N} H_i$, and $P = e^{h\sum_{ij}(a_{ji}-a_{ij})H_i}$.*

PROPOSITION 3.3. *For any $\sigma \in S_n$ one has $f(det_{\sigma}(T)) = Pe^{hH} \otimes Pe^{-hH}$.*

Proof of Proposition 3.2. Since D is grouplike, it is enough for us to show that $f_{\pm}(D) = Pe^{\pm hH}$.

Define a functor F from right $A_{\mathcal{J}}$-comodules to left $U_{\mathcal{J}}$-modules, as follows. Any right $A_{\mathcal{J}}$-comodule W is also a left $A^*_{\mathcal{J}}$-module, hence the pullback $\theta^*(W)$ is a left $U_{\mathcal{J}}$-module. We set $F(W) := \theta^*(W)$.

Consider the pushforward functors $f_{\pm *}$ from right $A_{\mathcal{J}}$-modules to left $U_{\mathcal{J}}$-comodules. Consider also the functors $F_{\pm}$ from left $U_{\mathcal{J}}$-modules to left $U_{\mathcal{J}}$-comodules given by $\pi^*_{F_+(W)}(w) = (\pi_W \otimes 1)(\mathcal{R}_{\mathcal{J}})w^{(1)}$, and $\pi^*_{F_-(W)}(w) = (\pi_W \otimes 1)(\mathcal{R}^{-1}_{\mathcal{J},21})w^{(1)}$ (here $w^{(1)}$ means w in the first component). It is easy to see that $F_{\pm} \circ F = f_{\pm *}$.

Let $\chi : U \to \mathbb{C}[[h]]$ be the character defined by $\chi(E_i) = \chi(F_i) = 0, \chi(H_i) = 1$ (the determinant character). It is easy to see that $F(\mathrm{Det}) = \chi$. Indeed, if $\tilde{V}$ is the standard comodule over $A_{\mathcal{J}}$, then $F(\tilde{V}) = V$, and Det, χ are the unique 1-dimensional subobjects in $\tilde{V}^{\otimes n}$ and $V^{\otimes n}$, respectively.

Now, $f_{\pm}(D)$ is the element of $U_{\mathcal{J}}$ which corresponds to the 1-dimensional comodule $f_{\pm}(\mathrm{Det}) = F_{\pm}(\chi)$. This implies that

$$f_+(D) = (\chi \otimes 1)(\mathcal{R}_{\mathcal{J}}), f_-(D) = (\chi \otimes 1)(\mathcal{R}^{-1}_{\mathcal{J},21}). \tag{3.1}$$

Now the Proposition follows from formula (1.3). □

Proof of Proposition 3.3. We have

$$f(T) = (f_+ \otimes f_-)(T^{12}T^{13}) = \pi^1_V(\mathcal{J}^{-1}_{21}\mathcal{R}_{12}\mathcal{J}_{12}\mathcal{J}^{-1}_{31}\mathcal{R}^{-1}_{31}\mathcal{J}_{13}), \tag{3.2}$$

where π_V^1 is π_V evaluated in the first component. By (1.4), we have

$$\mathcal{J}_{12}\mathcal{J}_{31}^{-1} = \mathcal{J}_{3,12}^{-1}\mathcal{J}_{31,2}. \tag{3.3}$$

(Here $\mathcal{J}_{3,12}$ means that the first component of $\mathcal{J}$ acts in the third component of the tensor product, and the second component of $\mathcal{J}$ acts in the first two components of the tensor product, and $\mathcal{J}_{31,2}$ is defined similarly). Thus, (3.2) implies

$$\begin{aligned} f(T) = \pi_V^1(\mathcal{J}_{21}^{-1}\mathcal{R}_{12}\mathcal{J}_{3,12}^{-1}\mathcal{J}_{31,2}\mathcal{R}_{31}^{-1}\mathcal{J}_{13}) = \\ \pi_V^1(\mathcal{J}_{21}^{-1}\mathcal{J}_{3,21}^{-1}\mathcal{R}_{12}\mathcal{R}_{31}^{-1}\mathcal{J}_{13,2}\mathcal{J}_{13}). \end{aligned} \tag{3.4}$$

It is easy to see that $(\pi_V \otimes 1)(\mathcal{J}')$ is an upper triangular matrix with ones on the diagonal, and $(\pi_V \otimes 1)((\mathcal{J}'_{21})^{-1})$ is a lower triangular matrix with ones on the diagonal.

Taking this into account, we obtain

$$f(\det_\sigma(T)) = \det_\sigma[\pi_V^1((\mathcal{J}^0_{21})^{-1}(\mathcal{J}^0_{3,21})^{-1}\mathcal{R}_{12}\mathcal{R}_{31}^{-1}\mathcal{J}^0_{13,2}\mathcal{J}^0_{13})]. \tag{3.5}$$

Recall that $\mathcal{J}^0 = e^{h\sum_{ij} a_{ij}H_i\otimes H_j}$. Substituting this into (3.5), we get

$$\begin{aligned} f(\det_\sigma(T)) = \det_\sigma\Big[\mathrm{diag}(e^{-h\sum_i a_{ij}(H_i\otimes 1 + 1\otimes H_i)})e^{-h\sum_{ij} a_{ij}H_j\otimes H_i} \times \\ L^+_{12}L^-_{13}\mathrm{diag}(e^{h\sum_i a_{ji}(H_i\otimes 1+1\otimes H_i)})e^{h\sum_{ij} a_{ij}H_j\otimes H_i}\Big]. \end{aligned} \tag{3.6}$$

Using the fact that all diagonal quasiminors of $L^+_{12}L^-_{13}$ are of weight zero, we obtain from (3.6):

$$f(\det_\sigma(T)) = (P\otimes P)e^{-h\sum_{ij} a_{ij}H_i\otimes H_j}\det_\sigma(L^+_{12}L^-_{13})e^{h\sum_{ij} a_{ij}H_i\otimes H_j}. \tag{3.7}$$

By the Main theorem for $\mathcal{J} = 1$, we have $\det_\sigma(L^+_{12}L^-_{13}) = e^{hH}\otimes e^{-hH}$. This implies that $f(\det_\sigma(T)) = Pe^{hH}\otimes Pe^{-hH}$, as desired. □

4. Construction of triangular twists. In this section we will explain a construction of triangular twists following the paper of Hodges [Ho].

Let Γ_1, Γ_2 be disjoint subsets of $\{1, ..., n-1\}$, and $\tau : \Gamma_1 \to \Gamma_2$ a bijection such that $|a-b| = 1$ iff $|\tau(a) - \tau(b)| = 1$. We denote by $U^m_{\geq 0}$ the algebra generated by H_j and $E_i, i \in \Gamma_m$ $(m = 1, 2)$, and by $U^m_{\leq 0}$ the algebra generated by H_j and $F_i, i \in \Gamma_m$. We also denote by U^m the algebra generated by H_j and $E_i, F_i, i \in \Gamma_m$.

Let $\mathfrak{h}$ be the linear span of H_j. We have $\mathfrak{h} = \mathfrak{h}_m \oplus \mathfrak{h}_m^\perp$, where $\mathfrak{h}_m$ is the span of $H_i - H_{i+1}$ for $i \in \Gamma_m$ and $\mathfrak{h}_m^\perp$ is the orthogonal complement of $\mathfrak{h}_m$ with respect to the standard inner product. Slightly abusing notation, we denote by τ the linear map $\mathfrak{h} \to \mathfrak{h}$ such that $\tau(H_i - H_{i+1}) = H_{\tau(i)} - H_{\tau(i)+1}$, for $i \in \Gamma_1$, and $\tau(\mathfrak{h}_1^\perp) = 0$. Let $f_\tau : U^1 \to U^2$ be the homomorphism of Hopf algebras defined by the formula $f_\tau(E_i) = E_{\tau(i)}$, $f_\tau(F_i) = F_{\tau(i)}$, $f_\tau(H_i) = \tau(H_i)$.

Let $\mathbb{R} = e^{h\sum_i H_i\otimes H_i}(1 + \sum_{j\geq 1} a_j \otimes a^j)$ be the universal R-matrix of U_1. Here as before $a_j \in U_+, a^j \in U_-$, and $\varepsilon(a_j) = \varepsilon(a^j) = 0$.

Let $\Theta \in \mathfrak{h}\otimes\mathfrak{h}$ be a tensor. Let

$$\mathcal{J} = e^{-h\Theta}(f_\tau \otimes 1)(\mathbb{R}) \in U^2_{\geq 0} \otimes U^1_{\leq 0}. \tag{4.1}$$

Proposition 4.1. *Let* $Z=\sum_i \tau(b_i)\otimes b_i$*, where* b_i *is an orthonormal basis of* $\mathfrak{h}_1$*. Suppose that* Θ *satisfies the following conditions:*

(i) $(x\otimes 1, Z-\Theta)=(1\otimes\tau(x), Z-\Theta)=0$, $x\in\mathfrak{h}_1$*;*

(ii) $(\tau(x)\otimes 1+1\otimes x,\Theta)=0$, $x\in\mathfrak{h}_1$.

Then the element $\mathcal{J}$ *is a upper triangular twist.*

Proof. The properties $(\varepsilon\otimes 1)(\mathcal{J})=1$, $(1\otimes\varepsilon)(\mathcal{J})=1$ and the triangularity are obvious, so it suffices to prove the second relation in (1.4).

Denote $(f_\tau\otimes 1)(\mathbb{R})$ by $\hat{\mathbb{R}}$. The hexagon relations for the R-matrix give $(\Delta\otimes 1)(\mathbb{R})=\mathbb{R}_{13}\mathbb{R}_{23}$, and $(1\otimes\Delta)(\mathbb{R})=\mathbb{R}_{13}\mathbb{R}_{12}$. From them we get

$$(\Delta\otimes 1)(\mathcal{J})\mathcal{J}_{12}=e^{-h(\Theta_{13}+\Theta_{23})}\hat{\mathbb{R}}_{13}\hat{\mathbb{R}}_{23}e^{-h\Theta_{12}}\hat{\mathbb{R}}_{12},$$

$$(1\otimes\Delta)(\mathcal{J})\mathcal{J}_{23}=e^{-h(\Theta_{13}+\Theta_{12})}\hat{\mathbb{R}}_{13}\hat{\mathbb{R}}_{12}e^{-h\Theta_{23}}\hat{\mathbb{R}}_{23}. \tag{4.2}$$

Now, identity (i) implies that $[e^{-h\Theta_{12}}\hat{\mathbb{R}}_{12}, e^{-h\Theta_{23}}\hat{\mathbb{R}}_{23}]=0$ (here it is also used that the sets Γ_1,Γ_2 are disjoint). Therefore, the second identity of (1.4) is equivalent to the equation

$$e^{-h\Theta_{23}}\hat{\mathbb{R}}_{13}e^{h\Theta_{23}}=e^{-h\Theta_{12}}\hat{\mathbb{R}}_{13}e^{h\Theta_{12}}. \tag{4.3}$$

The last equation is equivalent to $[\Theta_{12}-\Theta_{23},\hat{\mathbb{R}}_{13}]=0$, which is equivalent to identity (ii). The proposition is proved. □

Proposition 4.2. *Equations (i) and (ii) have a solution.*

Proof. Make a change of variable $Y=Z-\Theta$. The obtained equations with respect to Y are:

(i) $(x\otimes 1,Y)=(1\otimes\tau(x),Y)=0$, $x\in\mathfrak{h}_1$;

(ii) $(\tau(x)\otimes 1+1\otimes x,Y)=x+\tau(x)$, $x\in\mathfrak{h}_1$

(here we use that $(x,y)=(\tau(x),\tau(y)), x,y\in\mathfrak{h}_1$). The set of solutions of equation (i) is the space $\mathfrak{h}_1^\perp\otimes\mathfrak{h}_2^\perp$. Let Y be any vector in this space. Define operators $a:\mathfrak{h}_1\to\mathfrak{h}_1^\perp$, $b:\mathfrak{h}_1\to\mathfrak{h}_2^\perp$ defined by $a(x)=(1\otimes x,Y)$, $b(x)=(\tau(x)\otimes 1,Y)$. Then equation (ii) is equivalent to

$$a(x)+b(x)=x+\tau(x). \tag{4.4}$$

Now we will use the following easy lemma.

Lemma. Let $a:\mathfrak{h}_1\to\mathfrak{h}_1^\perp$, $b:\mathfrak{h}_1\to\mathfrak{h}_2^\perp$ be any linear maps. Then the equations $a(x)=(1\otimes x,Y), b(x)=(\tau(x)\otimes 1,Y)$ have a solution in $\mathfrak{h}_1^\perp\otimes\mathfrak{h}_2^\perp$ if and only if

$$(a(x),\tau(y))=(b(y),x). \tag{4.5}$$

for any $x,y\in\mathfrak{h}_1$.

Proof of the Lemma. Since $\mathfrak{h}_1\cap\mathfrak{h}_2=0$, the maps $\mathfrak{h}_1\to(\mathfrak{h}_2^\perp)^*$, $\mathfrak{h}_2\to(\mathfrak{h}_1^\perp)^*$ given by $z\to(z,*)$ are injective. The Lemma easily follows from this observation.

The Lemma implies that for proving the Proposition it suffices to show that equations (4.4),(4.5) have a solution. Substituting (4.4) into (4.5), we get

$$(a(x),\tau(y))=(y+\tau(y)-a(y),x)=(y,x)+(\tau(y),x), \tag{4.6}$$

since $(a(y),x)=0$. Thus, it suffices to show that there exists $a:\mathfrak{h}_1\to\mathfrak{h}_1^\perp$ such that $(a(x),\tau(y))=(x,y+\tau(y))$. This is obvious, since, as we mentioned, the natural map $\mathfrak{h}_2\to(\mathfrak{h}_1^\perp)^*$ is injective. □

Thus, can construct an upper triangular twist $\mathcal{J}$ corresponding to any triple $(\Gamma_1, \Gamma_2, \tau)$. From one such twist one may obtain an affine space of twists using the following proposition.

PROPOSITION 4.3. *Let $\mathfrak{h}_0$ be the space of all $y \in \mathfrak{h}$ such that $(y, x) = (y, \tau(x))$, $x \in \mathfrak{h}_1$. Let $\beta \in \Lambda^2\mathfrak{h}_0$. Let $\mathcal{J}$ be the upper triangular twist constructed above. Then $\mathcal{J}_\beta = \mathcal{J}e^{h\beta}$ is also an upper triangular twist.*

Proof. As before, the only thing that requires a proof is that $\mathcal{J}_\beta$ is a twist. This is equivalent to saying that $e^{h\beta}$ is a twist for $U_{\mathcal{J}}$. This follows from the fact that elements of $\mathfrak{h}_0$ are primitive in $U_{\mathcal{J}}$, as the twist $\mathcal{J}$ has weight 0 with respect to $\mathfrak{h}_0$. □

In conclusion we discuss the connection of the above constructions with the Belavin-Drinfeld classification of quasitriangular structures on a simple Lie algebra [BD]. This classification states that the quasitriangular structures on a simple Lie algebra are labeled by two types of data – discrete data and continuous data. The discrete data is a triple $(\Gamma_1, \Gamma_2, \tau)$, where Γ_1, Γ_2 are Dynkin subdiagrams of the Dynkin diagram of the Lie algebra (not necessarily connected), and $\tau : \Gamma_1 \to \Gamma_2$ is a Dynkin diagram isomorphism such that for any $\alpha \in \Gamma_1$ there exists k such that $\tau^k(\alpha) \notin \Gamma_1$. The continuous data is a point of a certain affine space hanging over any fixed discrete data. The algebras $U_{\mathcal{J}_\beta}$ for various $\mathcal{J}, \beta$ constructed above provide quatizations of all quasitriangular structures on gl_n corresponding to triples $(\Gamma_1, \Gamma_2, \tau)$ with Γ_1, Γ_2 being disjoint.

If Γ_1, Γ_2 are not disjoint, the above method of constructing a twist does not work, since the left and right components of $e^{-h\Theta}\hat{\mathbb{R}}$ no longer commute. However, by [EK], any quasitriangular structure can be quantized by means of a suitable twist. We expect that such a twist can be chosen to be upper triangular. In this case, the Main theorem will generalize to twisted quantum groups corresponding to all triples $(\Gamma_1, \Gamma_2, \tau)$.

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