

MANIFOLDS CARRYING LARGE SCALAR CURVATURE*

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Abstract. Let $W = S \otimes E$ be a complex spinor bundle with vanishing first Chern class over a simply connected spin manifold M of dimension ≥ 5 . Up to connected sums we prove that W admits a twisted Dirac operator with positive order-0-term in the Weitzenböck decomposition if and only if the characteristic numbers $\hat{A}(TM)[M]$ and $\text{ch}(E)\hat{A}(TM)[M]$ vanish. This is achieved by generalizing [2] to twisted Dirac operators.

1. Introduction. A key point in the Lichnerowicz argument, showing that the \hat{A} genus is an obstruction to the existence of a metric with positive scalar curvature, is the fact that the scalar curvature appears as the order-0-term in the Weitzenböck decomposition of the ordinary Dirac Laplacian \mathcal{D}^2 . It was shown in [2], [8] that positive scalar curvature can be preserved under surgeries in codimension ≥ 3 . Within the class of simply connected spin manifolds of dimension ≥ 5 the cobordism relation is generated by surgeries of this type. Therefore all such manifolds admitting a metric of positive scalar curvature could be determined by computations in the spin cobordism ring (see [2], [8], [6], [7]).

Here we extend this to general Dirac operators (see [1], [4]). The role of scalar curvature is taken by the order-0-term in the Weitzenböck decomposition of a twisted Dirac operator. This term is positive if the scalar curvature is larger than a certain norm of the curvature endomorphism of the coefficient bundle. First we prove a surgery theorem for the order-0-term in the the Weitzenböck decomposition of twisted Dirac Laplacians \mathcal{D}_{∇}^2 (Theorem 1). Next we consider complex spinor bundles with trivial first Chern class over simply connected spin manifolds of dimension ≥ 5 . Up to connected sums, we determine all spinor bundles within this class, which admit a Dirac operator with positive order-0-term in its Weitzenböck decomposition (Theorem 2). This is done by a computation in the cobordism ring $\sum_{n,k} \Omega_n^{spin}(BSU(k)) \otimes \mathbb{Q}$.

2. Statement of Results. Let W be a complex spinor bundle over a spin manifold M . Then W is a twisted spinor bundle $W = S \otimes E$, where S is the spinor bundle associated to the irreducible representation of the Clifford algebra and E is a complex vector bundle, see [1], [4]. To a Riemannian metric g on M and a Hermitian connection ∇ on E there is naturally associated the twisted Dirac operator \mathcal{D}_{∇} acting on sections of W . The Weitzenböck decomposition of its Dirac laplacian \mathcal{D}_{∇}^2 reads ([1], [4])

$$\mathcal{D}_{\nabla}^2 = D^*D + \frac{1}{4}s + \sum_{i,j} e_i e_j \otimes R_{e_i, e_j},$$

the sum being taken over an orthonormal basis $\{e_i\}$ of the tangent space of M . Here D is the covariant derivative on W induced from the connection ∇ and the Levi-Civita connection on M . By s we denote the scalar curvature of M and by R the curvature tensor of ∇ . We also define $\mathcal{E}(\nabla) := 4 \sum_{i,j} e_i e_j \otimes R_{e_i, e_j}$ and $|\mathcal{E}(\nabla)|(x)$ to be minus the smallest eigenvalue of the bundle endomorphism $\mathcal{E}(\nabla)$ at the point $x \in M$.

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Assume additionally that M is simply connected, $\dim M \geq 5$ and $c_1(E) = 0$. We will show that rationally, i.e. after eventually passing to a suitable connected sum multiple of (M, E) , the following are equivalent:

1. M admits a Riemannian metric g and E a Hermitian connection ∇ , such that $s(g) > \|\mathcal{E}(\nabla)\|$ on M . We will then say that (M, E) admits large scalar curvature.
2. Both S and W admit an invertible Dirac operator.
3. The characteristic numbers $\hat{A}(TM)[M]$ and $\text{ch}(E)\hat{A}(TM)[M]$ vanish.

As $\|\mathcal{E}(\nabla)\|$ is always nonnegative, the implication (1) \Rightarrow (2) is immediate from the Weitzenböck decomposition. By the index theorem we have (2) \Rightarrow (3). So we are left with the implication (3) \Rightarrow (1).

Therefore we will first extend the surgery theorem for scalar curvature (cf. [2], [8]) to show that positivity of $s + \mathcal{E}(\nabla)$ can be preserved under surgeries of codimension at least 3.

THEOREM 1. *Let $E \rightarrow M$ be a vectorbundle over the smooth manifold M . Assume that there is a Riemannian metric g on M and a unitary connection ∇ on E with $s(g) > \|\mathcal{E}(\nabla)\|$. If the manifold M' is produced from M by surgery in codimension more than 2 and such that the vector bundle E extends over the trace of the surgery giving a vector bundle E' over M' then there are a Riemannian metric g' on M' and a unitary connection ∇' on E' with $s(g') > \|\mathcal{E}(\nabla')\|$.*

Now we look at simply connected spin manifolds M of dimension $\dim M \geq 5$ endowed with a complex vectorbundle E with vanishing first Chern class. Then E — and the spinor bundle $W = S \otimes E$ — are trivial over embedded 2-spheres. As in [2] we obtain that any cobordism can be replaced by a sequence of surgeries of codimension ≥ 3 . Hence we can decide from the cobordism class of (M, E) in $\Omega_n^{spin}(BSU(k))$, whether it admits large scalar curvature. We have

THEOREM 2. *Let $E \rightarrow M$ be a $SU(r)$ -vectorbundle over the smooth simply connected spin manifold M of dimension ≥ 5 . Then the following are equivalent:*

1. For some q the q -fold connected sum $(M, E) \# \dots \# (M, E)$ carries a metric g and a connection ∇ with $s(g) > \|\mathcal{E}(\nabla)\|$.
2. $\hat{A}(TM)[M] = 0$ and $\text{ch}(E)\hat{A}(TM)[M] = 0$.

3. Proof of Theorem 1. Consider surgery on an embedded sphere $S^k \cong S \subset M^{k+l}$, $n = k + l$, with trivial normal bundle and such that the restriction to S of the vector bundle E is trivial. M' is then obtained by cutting out a tubular neighbourhood $f : S^k \times D^l \hookrightarrow M$ of S and glueing back $D^{k+1} \times S^{l-1}$ along the boundary $S^k \times S^{l-1}$. In the end M' will be described as a submanifold of $Z := M \times [0, \delta] \cup_f D^{k+1} \times D^l$.

Let $S^k \times D^l$ carry the metric and the connection induced via f from M . We can extend these data to all of $D^{k+1} \times D^l$, such that in the vicinity of the boundary $S^k \times D^l$ they are compatible to a product structure of a collar neighbourhood. The metric and connection on Z are then obtained by glueing this handle $D^{k+1} \times D^l$ with the product metric and connection on $M \times [0, \delta]$.

Let $\varrho \leq \mathcal{R}$ be sufficiently small constants (e.g. less than the injectivity radius of Z) and denote by $d(\cdot, S)$ the distance from S . Define $N_r := \{x \in M \mid d(x, S) \leq r\}$ and $Y_\rho = \partial N_\rho$. If $\rho \leq \mathcal{R}$ then the exponential map provides diffeomorphisms $D^{n-k} \times S^k \cong \rho D\nu(S, M) \rightarrow N_\rho$ and $S^{n-k-1} \times S^k \cong \rho S\nu(S, M) \rightarrow Y_\rho$. Pick a decreasing real function $\phi(\rho)$ defined for $\rho \geq \varrho$, vanishing for $\rho \geq \mathcal{R}$ and such that all derivatives of its inverse function $\chi = \phi^{-1}$ vanish at $\phi(\varrho)$. Let $\delta := \phi(\varrho)$ and

$\psi(x) := \phi(d(x, S))$, $x \in M$. The result of the surgery is

$$M' = \{(m, t) \mid \phi(d(m, S)) = t\} \cup_f \{x \in D^{k+1} \times D^{n-k} \mid d(x, S^k \times D^{n-k}) = \varrho\}.$$

We will show that one can find ϱ and ϕ such that on M' we have $s - \mathcal{E}$ positive.

The calculations in 3.1 are much the same as in [2] and merely included for the reader's convenience.

3.1. Scalar Curvature of M' . M' is glued together from the graph X of ψ on $M \setminus N_\varrho$ and a handle. We express the scalar curvature of $X \subset M \times \mathbb{R}$ at (m, t) in terms of the second fundamental form T of the submanifolds $\psi^{-1}(t) \subset M$ at m . This is a straightforward calculation based on the Gauß equation.

Denote the derivation in direction of the \mathbb{R} -factor by ∂_t and the gradient of ψ by $\partial\psi$. Let $r := -\partial\psi/|\partial\psi| = \partial\psi/\phi'$ and $\hat{n} := (-\partial\psi, \partial_t)/\sqrt{1+|\partial\psi|^2} = (-\phi'r, \partial_t)/\sqrt{1+\phi'^2}$ be the normal unit vectors to $\psi^{-1}(t)$ and X respectively. For a vector $v \in T_m M$ define $\bar{v} := (v, v(\psi)\partial_t) \in T_{(m, \psi(m))} X$.

At a point $(m, t) \in X$ choose an orthonormal basis v_1, \dots, v_{n-1} of the orthogonal complement of the gradient $\partial\psi$ in $T_m M$. We work in the orthonormal basis

$$\left(\bar{v}_1, \dots, \bar{v}_{n-1}, \frac{\bar{\partial\psi}}{|\bar{\partial\psi}|} = -\frac{(r, \phi' \partial_t)}{\sqrt{1+\phi'^2}} \right)$$

of $T_{(m,t)} X$.

First we compare the second fundamental form T of the submanifolds $\psi^{-1}(t) \subset M$ at m with the second fundamental form \bar{T} of $X \subset M \times \mathbb{R}$ at (m, t) . For v, w perpendicular to $\partial\psi$ we obtain

$$\begin{aligned} \bar{T}(\bar{v}, \bar{w}) &= \langle \nabla_{\bar{v}} \bar{w} \mid \hat{n} \rangle = \langle \nabla_v w \mid (-\phi' r) \rangle / \sqrt{1+\phi'^2} \\ &= T(v, w) \frac{-\phi'}{\sqrt{1+\phi'^2}} \\ \bar{T}(\bar{v}, \frac{\bar{\partial\psi}}{|\bar{\partial\psi}|}) &= -\langle \nabla_{\bar{v}}(r, \phi' \partial_t) \mid (-\phi' r, \partial_t) \rangle / (1+\phi'^2) \\ &= (\phi' \langle \nabla_v r \mid r \rangle - v(\phi')) / (1+\phi'^2) \\ &= (\phi' v(|r|^2)/2 - v(\phi')) / (1+\phi'^2) = 0 \\ \bar{T}(\frac{\bar{\partial\psi}}{|\bar{\partial\psi}|}, \frac{\bar{\partial\psi}}{|\bar{\partial\psi}|}) &= \langle \nabla_{(r, \phi' \partial_t)}(r, \phi' \partial_t) \mid (-\phi' r, \partial_t) \rangle / (1+\phi'^2)^{3/2} \\ &= (\langle \nabla_r r \mid -\phi' r \rangle + r(\phi')) / (1+\phi'^2)^{3/2} \\ &= \frac{\phi''}{(1+\phi'^2)^{3/2}} \end{aligned}$$

The Gauss formula then yields for the sectional curvature \bar{K} of the submanifold X :

$$\begin{aligned} \bar{K}(\bar{v}, \bar{w}) &= K(v, w) + \frac{\phi'^2}{1+\phi'^2} (T(v)T(w) - T(v, w)^2) \\ \bar{K}(\bar{v}, \frac{\bar{\partial\psi}}{|\bar{\partial\psi}|}) &= K^{M \times \mathbb{R}}(v, \frac{\bar{\partial\psi}}{|\bar{\partial\psi}|}) - \frac{\phi' \phi''}{(1+\phi'^2)^2} T(v) \\ &= \frac{1}{1+\phi'^2} K(v, r) - \frac{\phi' \phi''}{(1+\phi'^2)^2} T(v) \end{aligned}$$

Taking sums over the basis above we end up with

$$(3.1) \quad \begin{aligned} \bar{s} &= s + \frac{\phi'^2}{1 + \phi'^2} \sum_{i,j} (T(v_i)T(v_j) - T(v_i, v_j)^2) \\ &\quad - \frac{\phi'^2}{1 + \phi'^2} 2 \sum_i K(v_i, r) - \frac{\phi' \phi''}{(1 + \phi'^2)^2} 2 \sum_i T(v_i). \end{aligned}$$

We will need the asymptotic behaviour when approaching S of the functions on $N_{\mathcal{R}} \setminus N_{\varrho}$ defined by the sums in (3.1):

LEMMA 3.2. *As $\rho = d(x, S) \rightarrow 0$ the asymptotic behaviour of the functions $\mathcal{A} := \sum_{i,j} (T(v_i)T(v_j) - T(v_i, v_j)^2) = (\text{Tr } T)^2 - \text{Tr } T^2$, $\mathcal{B} := -2 \sum_i T(v_i) = 2 \text{Tr } T$ and $\mathcal{C} := 2 \sum_i K(v_i, r) = 2 \text{Ric}(r)$ is*

$$\mathcal{A}(x) = a_2 \rho^{-2} + a_1(x) \rho^{-1} + a_0(x), \quad \mathcal{B}(x) = b_1 \rho^{-1} + b_0(x),$$

with bounded functions $a_1(x)$, $a_0(x)$ and $b_0(x)$ and positive constants a_2 and b_1 . \mathcal{C} also extends to a bounded function on $N_{\mathcal{R}}$.

In fact since the codimension l of the submanifold S is ≥ 3 we have $a_2 = (l - 1)(l - 2)/2 > 0$ and $b_1 = l - 1 > 0$.

Proof. Consider the diffeomorphism $S \times \mathbb{R}^l = \nu(S, M) \rightarrow N_{\mathcal{R}}$ given by the exponential map i.e. mapping $(p, v) \mapsto \exp_p v$. For unit speed curves $p(t)$ in S and A_t in $SO(l)$ define vectorfields $h = \frac{d}{dt} \exp_{p(t)} v$, $u = \frac{d}{dt} \exp_p A_t v$, $\tilde{r} = \frac{d}{dt} \exp_p tv$. Then for small $\rho = |v|$ expand $|u| = \rho + b_u \rho^2$ and $|\tilde{r}| = \rho + b_r \rho^2$ with smooth functions b_u , b_r . We compute

$$T\left(\frac{u}{|u|}\right) = \frac{1}{|u|^2 |\tilde{r}|} \langle \nabla_u u \mid \tilde{r} \rangle = -\frac{1}{2|u|^2 |\tilde{r}|} \tilde{r}(|u|^2)$$

because u and \tilde{r} commute and are mutually perpendicular. Since $r = \tilde{r}/|\tilde{r}| = \frac{\partial}{\partial \rho}$, we infer from the asymptotics of $|u|$, that this is

$$\frac{-1}{2|u|^2} \frac{\partial}{\partial \rho} (\rho + b_u \rho^2)^2 = -\frac{1}{\rho} + O(1).$$

A similar computation shows that $T(\frac{h}{|h|})$ and $T(\frac{u}{|u|}, \frac{h}{|h|})$ are bounded. The Lemma then follows from polarisation. \square

The scalar curvature of Y_{ρ} is also obtained from the Gauß formula (substitute $\phi'' = 0$ and $\phi' = \infty$ in (3.1)). Hence for small ρ we get:

$$s^{Y_{\rho}} = s + \mathcal{A} - \mathcal{C} = a_2 \rho^{-2} + a_1 \rho^{-1} + a_0 - \mathcal{C}.$$

3.2. The Curvature Endomorphism. The manifold X can also be viewed as obtained from $M \setminus N_{\varrho}$ by blowing up the metric in direction of r . More precisely X is isometric to $(M \setminus N_{\varrho}, \bar{g})$ with

$$\bar{g}(v, w) := g(v, w) + g(v, \partial\psi)g(\partial\psi, w) = g(v, w) + |\partial\psi|^2 g(v, r)g(r, w).$$

Especially the length of r becomes $\sqrt{1 + \phi'^2}$. The transition matrix between the metrics g and \bar{g} gives an isomorphism between the spinor bundles of (M, g) and of

(M, \bar{g}) . The pull back via this isomorphism of the curvature endomorphism of (M, \bar{g}) to the spinor bundle over (M, g) is:

$$\begin{aligned} \bar{\mathcal{E}} &= 4 \sum_{i,j} v_i v_j \otimes R_{v_i, v_j} + \frac{8}{\sqrt{1 + \phi'^2}} \sum_i r v_i \otimes R_{r, v_i} \\ (3.3) \quad &= \mathcal{E} - 8 \left(1 - \frac{1}{\sqrt{1 + \phi'^2}} \right) \sum_i r v_i \otimes R_{r, v_i} \end{aligned}$$

and its smallest eigenvalue is estimated by

$$\begin{aligned} \|\bar{\mathcal{E}}\| &\leq |\mathcal{E}| + 8 \left(1 - \frac{1}{\sqrt{1 + \phi'^2}} \right) \left\| \sum_i r v_i \otimes R_{r, v_i} \right\| \\ (3.4) \quad &\leq |\mathcal{E}| + 8 \frac{\phi'^2}{1 + \phi'^2} \left\| \sum_i r v_i \otimes R_{r, v_i} \right\| \end{aligned}$$

Herein $\mathcal{D} := 8 \left\| \sum_i r v_i \otimes R_{r, v_i} \right\|$ extends to a bounded function on $N_{\mathcal{R}}$.

3.3. Solution of The Differential Inequality. Finally we need to solve the differential estimate $\bar{s} - \|\bar{\mathcal{E}}\| > 0$. From (3.1), (3.4) and Lemma 3.2 we infer that

$$\begin{aligned} \bar{s} - \|\bar{\mathcal{E}}\| &\geq s - |\mathcal{E}| + \frac{\phi'^2}{1 + \phi'^2} (\mathcal{A} - \mathcal{D} - \mathcal{C}) + \frac{\phi' \phi''}{(1 + \phi'^2)^2} \mathcal{B} \\ &= s - |\mathcal{E}| + \frac{\phi'^2}{1 + \phi'^2} (a_2 \rho^{-2} + a_1(x) \rho^{-1} + a_0(x) - \mathcal{D} - \mathcal{C}) \\ &\quad + \frac{\phi' \phi''}{(1 + \phi'^2)^2} (b_1 \rho^{-1} + b_0(x)) \end{aligned}$$

So we have solved the problem on X if we can find a decreasing function ϕ on $[\varrho, \mathcal{R}]$ such that this expression is positive. Furthermore we need that ϕ vanishes identically near \mathcal{R} and that all derivatives of its inverse function $\chi = \phi^{-1}$ vanish at $\phi(\varrho)$ so that X will inherit a product metric and connection near its boundary.

Eventually after taking an even smaller value of \mathcal{R} , we pick positive constants a, b such that on $N_{\mathcal{R}}$ the estimates $a \rho^{-2} \leq a_2 \rho^{-2} + a_1(x) \rho^{-1} + a_0(x) - \mathcal{D} - \mathcal{C}$ and $b \leq b_1 \rho^{-1} + b_0(x)$ hold. Furthermore let $\epsilon := \min(s - |\mathcal{E}|) > 0$. Then it suffices to solve

$$(3.5) \quad \epsilon + \phi'^4 a \rho^{-2} + \phi' \phi'' b \rho^{-1} > 0$$

Consider the the differential equation $\phi'^4 \rho^{-2} a/2 + \phi' \phi'' \rho^{-1} b = 0$ and its solutions

$$\phi_C(\rho) = \int_{\rho}^{\mathcal{R}} \frac{1}{\sqrt{\frac{a}{b} \log x + C}} dx$$

defined for $\rho \geq \varrho := e^{-Ca/b}$ for some $C \in \mathbb{R}$. For a sufficiently large value of C we can find a decreasing solution of $\epsilon + \phi' \phi'' b \rho^{-1} > 0$ in the interval $[\mathcal{R}/2, \mathcal{R}]$ which vanishes identically near \mathcal{R} and extends ϕ_C smoothly from $[\varrho, \mathcal{R}/2]$ to $[\varrho, \mathcal{R}]$ to ensure the proper boundary condition at $\rho = \mathcal{R}$. At the other boundary (3.5) for the inverse function χ reads $\epsilon \chi^2 \chi'^4 + a - b \chi \chi'' \geq 0$. Let χ_C be the inverse function of ϕ_C on $[0, \phi(\varrho)]$ extended by the constant ϱ to all of \mathbb{R}^+ . Then we have $a/2 - b \chi_C(y) \chi_C''(y) = 0$ for all $y \neq \phi(\varrho)$. But χ_C can clearly be smoothed keeping $a - b \chi(y) \chi''(y) \geq 0$.

for $M_J^r = (\epsilon_{\mu,\nu})_{\mu=1\dots n, \nu=1\dots r}$. Here γ_q is the canonical complex line bundle over the q th factor S^2 in (4.1). Slightly abusing the system of notation above define

$$(4.3) \quad X^{2n+2}(2, (n+1)) := (\mathbb{C}P^{2n+1} \times S^2, (\eta \otimes \gamma) \oplus (\eta^{-1} \otimes \gamma^{-1}))$$

and

$$(4.4) \quad X^{2n+1}(3, (a, b, c)) := (\mathbb{C}P^{2n+1}, \eta^a \oplus \eta^b \oplus \eta^c)$$

for $a, b, c \in \mathbb{Z}$, $-n \leq a, b, c \leq n$, $a + b + c = 0$, if $n \geq 2$. If $n = 1$ we take $a = 2$, $b = c = -1$. In (4.3) and (4.4) η, γ denote the canonical bundles over $\mathbb{C}P^{2n+1}$ and S^2 .

LEMMA 4.5. *The $X^n(r, J)$ above admit large positive scalar curvature with exception of $X^2(2, (2))$ and $X^3(3, ((2, -1, -1)))$.*

Proof. In [3] Hitchin has proved that $(\mathbb{C}P^q, \eta^s)$ admits large scalar curvature if $q \geq 2s$ and that $s - |\mathcal{E}(\eta^s)| = 0$ if $q = |s| = 1$. It is immediate from the definition that $|\mathcal{E}(E \oplus F)| = \max(|\mathcal{E}(E)|, |\mathcal{E}(F)|)$ and $|\mathcal{E}(E \otimes F)| \leq |\mathcal{E}(E)| + |\mathcal{E}(F)|$. Thus we can estimate

$$|\mathcal{E}(E(M_J^r))| \leq \max_i \left(\sum_{q=1}^n \epsilon_{q,i} |\mathcal{E}(\gamma)| \right) < n |\mathcal{E}(\gamma)|$$

because, with the above exceptions, in every row of the matrices M_J^r at least one entry vanishes. Since the scalar curvature of the round S^2 equals $|\mathcal{E}(\gamma)|$, we thus get that the scalar curvature of $S^2 \times \dots \times S^2$ is larger than $|\mathcal{E}(E(M_J^r))|$. The cases involving $\mathbb{C}P^{2n+1}$ are similiar. \square

LEMMA 4.6. *The matrix $(c_{J'}(X^n(s, J)))_{J', (s, J)}$, $s \leq r$, has full rank.*

Proof. We compute the Chern class of the vectorbundle $E(M_J^r)$: Denoting by $x_q = c_1(\gamma_q)$ the generator of the second cohomology group of the q th factor S^2 in (4.1) we obtain from (4.2) that:

$$c_k(E) = \sum_{\mu_1, \dots, \mu_k, \nu_1, \dots, \nu_k} \epsilon_{\mu_1, \nu_1} \cdots \epsilon_{\mu_k, \nu_k} x_{\nu_1} \cdots x_{\nu_k},$$

where the μ_s respectively ν_s in this sum are pairwise distinct. Order the partitions I, J lexicographically. Observing that $x_s^2 = 0$ we get for $r \geq 4$ that

$$(4.7) \quad c_I(X^n(r, J)) = \begin{cases} 0 & \text{if } I > J \\ \neq 0 & \text{if } I = J \end{cases}$$

Thus this part of the matrix is triangular. If $r = 3$, then a straightforward calculations gives that $c_{j_3, j_2}(X^n(3, (j_3, j_2))) = (-1)^{j_3+j_2-1} (j_3 - 1)(j_3 + j_2)j_3!(2j_3 + 2j_2 - 1)!$. If n is even then $j_3 \neq 1$ and (4.7) still holds. For the remainder of the matrix we use the manifolds defined in (4.3) and (4.4). For $r = 2$ we clearly have $c_2^{n+1}((\eta \otimes \gamma) \oplus (\eta^{-1} \otimes \gamma^{-1})) = 2(-1)^{j_2} j_2 \neq 0$. We are left with the case $r = 3$ and n odd. The Chernclasses of $\eta^a \oplus \eta^b \oplus \eta^c$ are given by the elementary symmetric polynomials σ_3, σ_2 in a, b, c . Assume that the polynomial

$$P(a, b, c) := \sum_{j_3, j_2} \alpha_{j_3, j_2} c_{j_3, j_2}(X^n(3, (a, b, c))) = \sum_{j_3, j_2} \alpha_{j_3, j_2} \sigma_3^{j_3} \sigma_2^{j_2}$$

of degree $2n+1$ vanishes for all a, b, c as after (4.4). Then the polynomial $P(a, b, -a-b)$ vanishes for all $a, b \in \mathbb{Z}$ with $-n \leq a, b, a+b \leq n$. Since it is homogeneous it must be divisible by all $(na+sb)$ and $(sa+nb)$, $s = 0 \dots n$ and if $n \geq 2$ it must also contain $(a-b)$ hence have degree at least $2n+2$. Therefore P vanishes on the entire plane $a+b+c=0$. Since it does not contain σ_1 and since there are no algebraic relations between the elementary symmetric polynomials, the coefficients α_{j_3, j_2} are all 0. \square

Let $\mathcal{K}_{n,r} \subset \Omega_{2n}^{spin}(BSU(r))$ be the kernel of those c_{JpI} with nontrivial I . We have shown that the span of the $X^n(r, J)$ as above projects onto $\mathcal{K}_{n,r}$. It is well known that $\bigoplus_n \Omega_n^{spin}$ is polynomially generated by the Kummer surface K and the quaternionic projective spaces $\mathbb{H}P^n$, $n \geq 2$. In view of the direct sum decomposition

$$(4.8) \quad \Omega_{2n}^{spin}(BSU(r)) = \bigoplus_{p=0}^n \mathcal{K}_{p,r} \times \Omega_{2n-2p}^{spin}$$

we infer from Lemma 4.6 that there is a basis of $\Omega_n^{spin}(BSU(r))$ consisting of monomials in K , quaternionic projective spaces and one of the $X^n(r, J)$. Among these only K , $X^2(2, (2))$ and $X^3(3, ((2, -1, -1)))$ do not admit large scalar curvature. Therefore the only monomials not admitting large scalar curvature are of the form $K^{d/4-1} \times X^2(2, (2))$ or $K^{d/4}$ if the dimension d is divisible by 4 and $K^{(d-2)/4-1} \times X^3(3, ((2, -1, -1)))$ if the dimension is $d = 2 \pmod{4}$. These monomials are also detected by the characteristic numbers \hat{A} and $\text{ch } \hat{A}$.

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