Asian J. Math.
Vol. 3, No. 3, pp. 557–608, September 1999

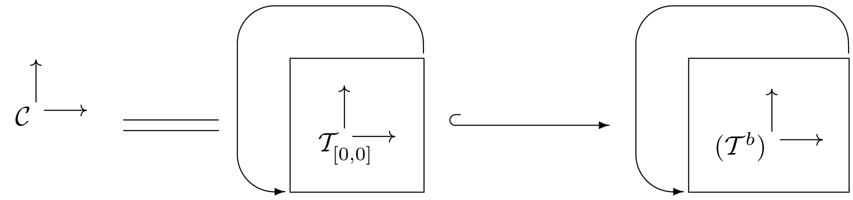
002

K–THEORY FOR TRIANGULATED CATEGORIES III(B): THE THEOREM OF THE HEART*

AMNON NEEMAN†

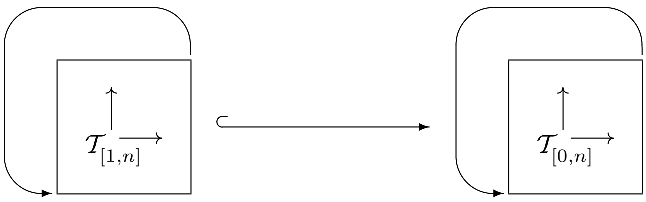
2. A Reduction of the Proof of Theorem I.7.1 (strong case). Let $\mathcal{T}$ be a triangulated category with a t–structure. Suppose $\mathcal{C} = \mathcal{T}_{[0,0]}$ is the heart. We must prove

Theorem I.7.1. *Let $\mathcal{T}$ be a* t*–category. Then the inclusion*



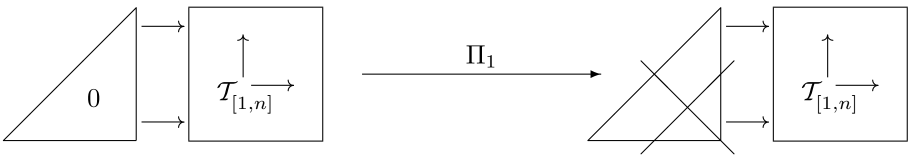
induces a homotopy equivalence.

In Section I.7, we already saw the beginning of the proof. We can reduce ourselves, by very simple arguments, to showing that the inclusion

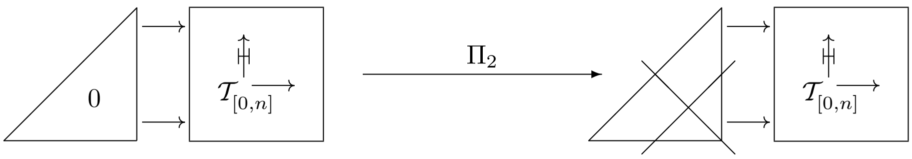


induces a homotopy equivalence for any $n \geq 1$. This much is the trivial stuff; now it is time to get earnest about producing homotopies. From now on, all our simplicial sets carry coherent differentials, and we suppress them throughout in the notation.

By Theorem I.5.1, we know that the natural map



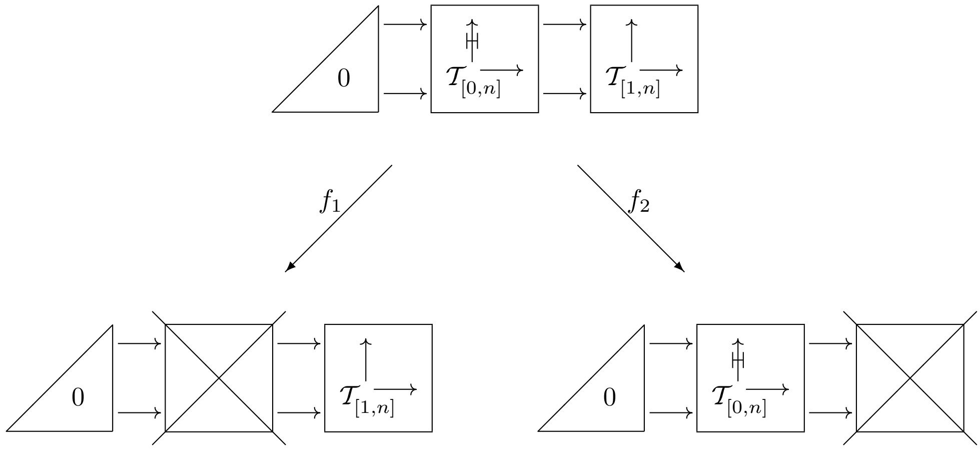
induces a homotopy equivalence. By Theorem 1.2, so does the map



Now we propose to prove:

Lemma 2.1. *In the diagram*

*Received April 30, 1997; accepted for publication April 28, 1998.
†Center for Mathematics and its Applications, Australian National University, Canberra, ACT 0200, Australia (Amnon.Neeman@anu.edu.au).

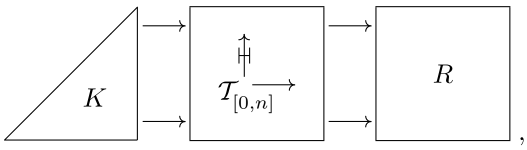


the maps f_1 and f_2 induce homotopy equivalences.

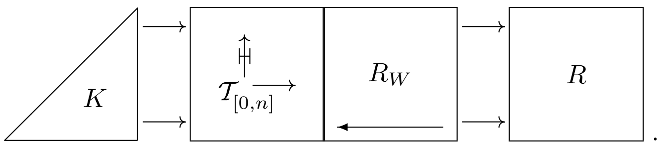
Corollary 2.2. *From the fact that Π_1, Π_2, f_1 and f_2 are all homotopy equivalences, it follows that the inclusion*

induces a homotopy equivalence.

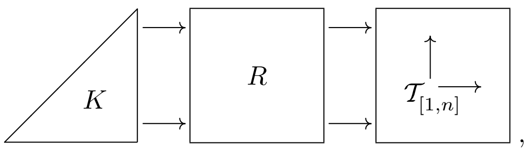
Proof of Lemma 2.1. We need to show the contractibility of the Segal fibers. The Segal fiber of f_1 is the simplicial set



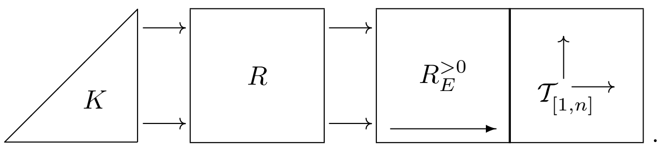
which is contracted by the homotopy



The Segal fiber of f_2 is the simplicial set

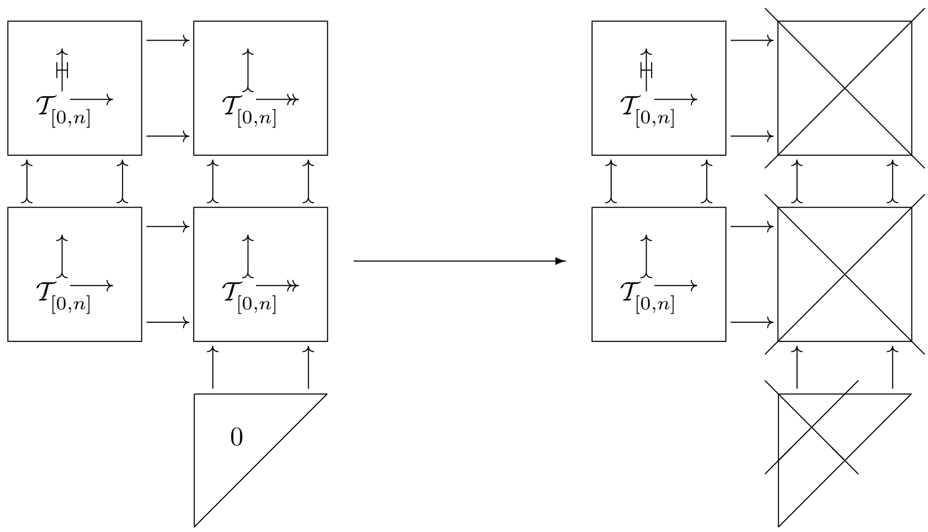


and it is contracted by the homotopy



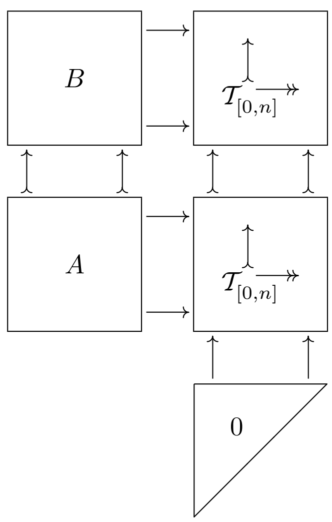
The fact that this truncation homotopy is a contraction, follows because there is no ambiguity in the differentials from the truncation. We can be sure of this because the kernels are present. But this is a point for Group 3 readers, who can see a very full discussion in Section II.1. ∎

Lemma 2.3. *The following map of simplicial sets*

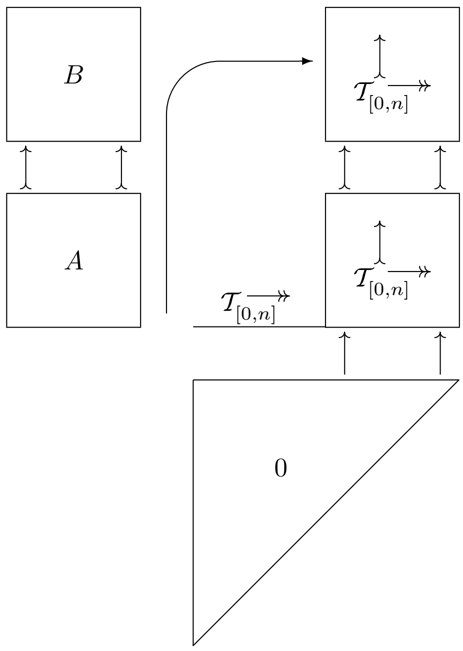


induces a homotopy equivalence.

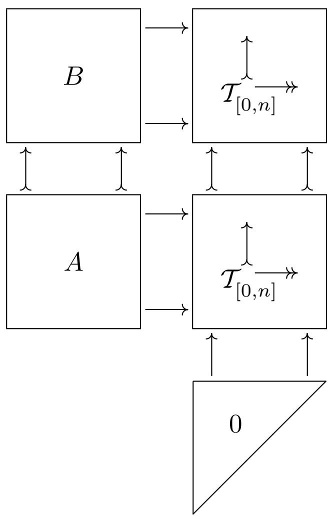
Proof. It suffices to contract the Segal fiber



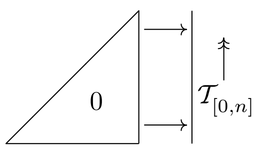
But the homotopy



allows us, up to homotopy, to factor the identity on

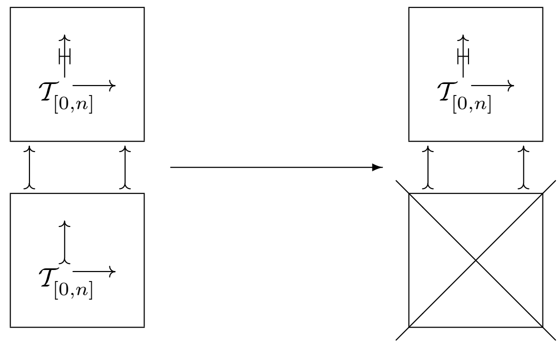


through the contractible set



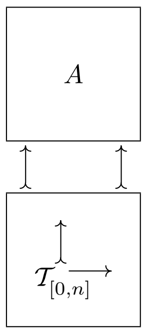
(We leave to the reader the check that the non–trivial homotopy I wrote can be expressed as a deletion of a subdivision of the blueprint homotopy in Theorem 1.1.) ∎

Lemma 2.4. *The natural projection*



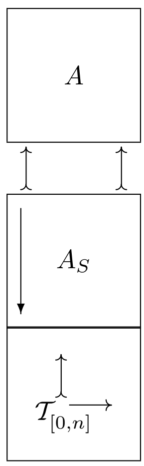
induces a homotopy equivalence.

Proof. We need to contract the Segal fiber



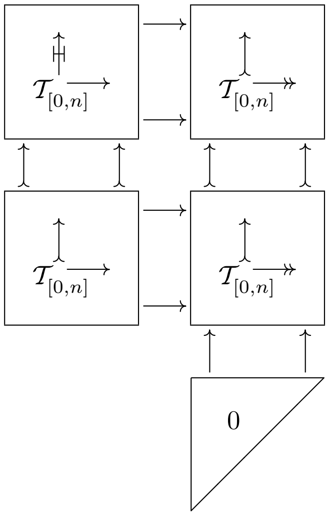
,

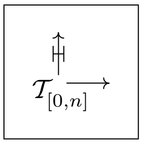
and this is done with the homotopy

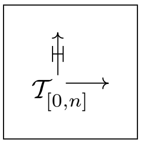


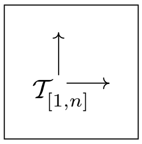
∎

Corollary 2.5. *From Lemma 2.3 and Lemma 2.4, we know that the large simplicial set*

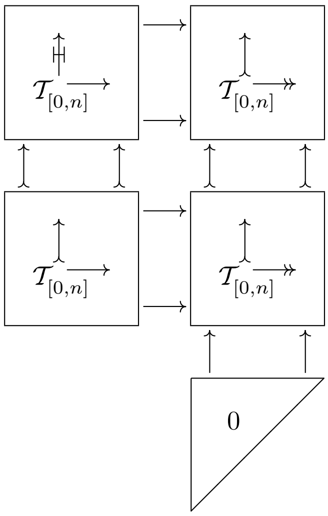


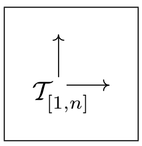
is a model for the smaller simplicial set 

. But we also know, by Corollary 2.2, that 

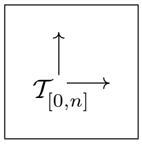
is homotopy equivalent to 

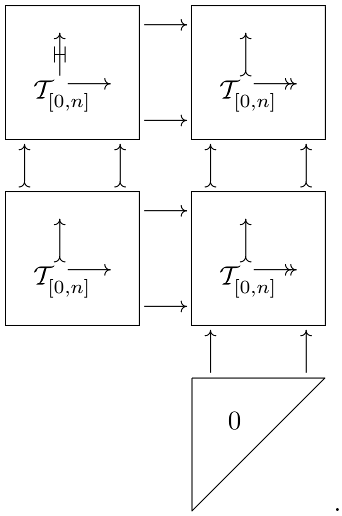
. Therefore



is just a clumsy model for the simplicial set 

.

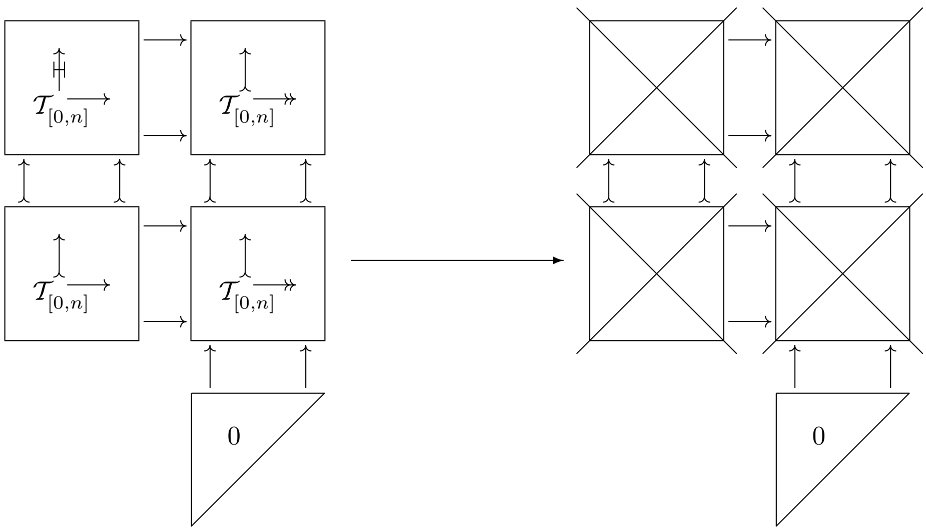
Strategy Session. By Corrolary 2.5, in order to complete the proof of Theorem I.7.1, it sufices to show a natural homtopy equivalence of  with



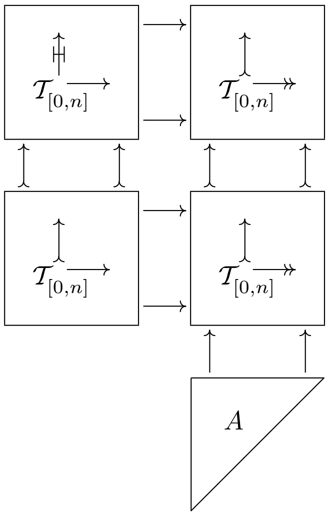
End of Strategy Session.

What we will actually show is

Lemma 2.6. **(Key Lemma).** *The natural projection*

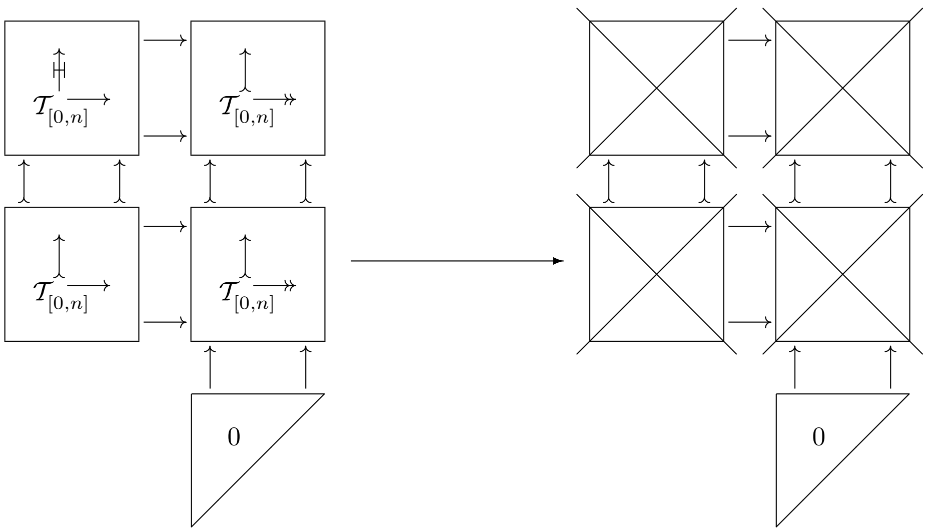


induces a homotopy equivalence. More precisely, the Segal fibers

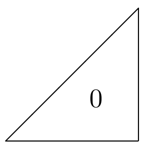


are contractible.

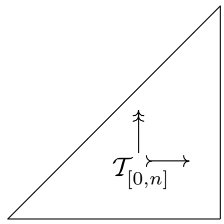
Remark 2.7. I owe the reader an apology for the awful notation. When we write the map



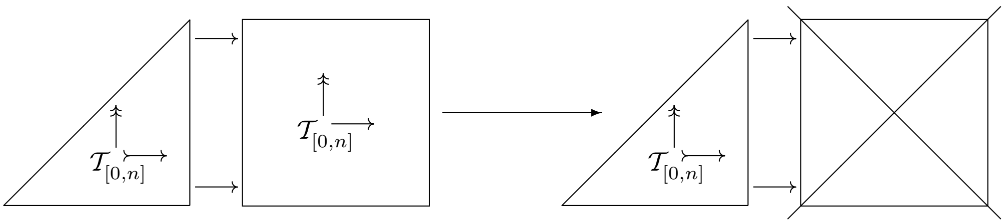
then the simplicial set on the right becomes



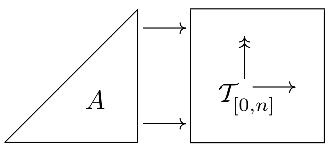
which is a dreadful shorthand for anything. What we mean is, of course, a simplicial set which would be better denoted



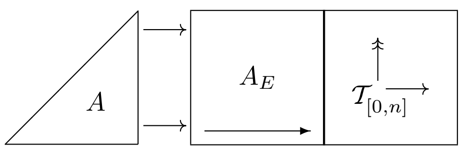
This simplicial set should be very familiar to anyone who knows Waldhausen's K–theory. It is the obvious analogue of the S construction. It is trivial to show that the map



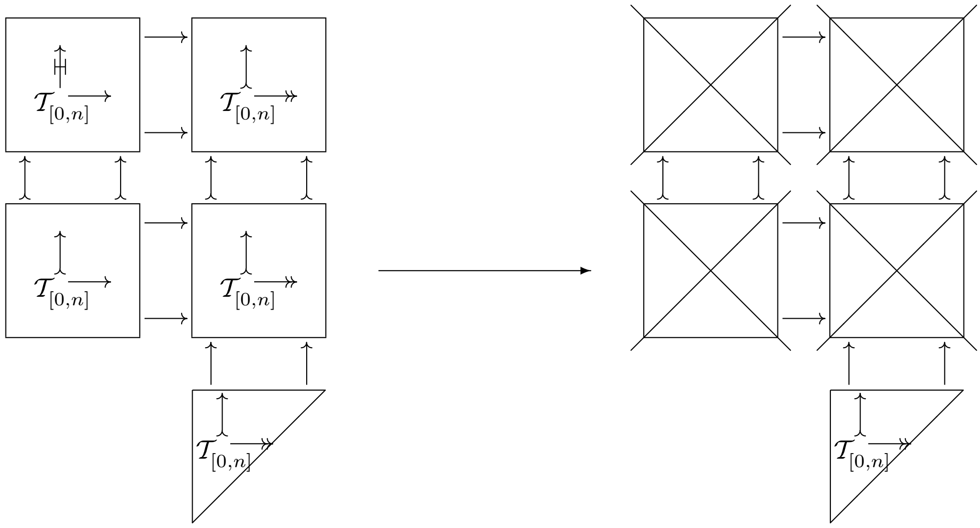
induces a homotopy equivalence. The Segal fiber is just



and it is contracted by the homotopy



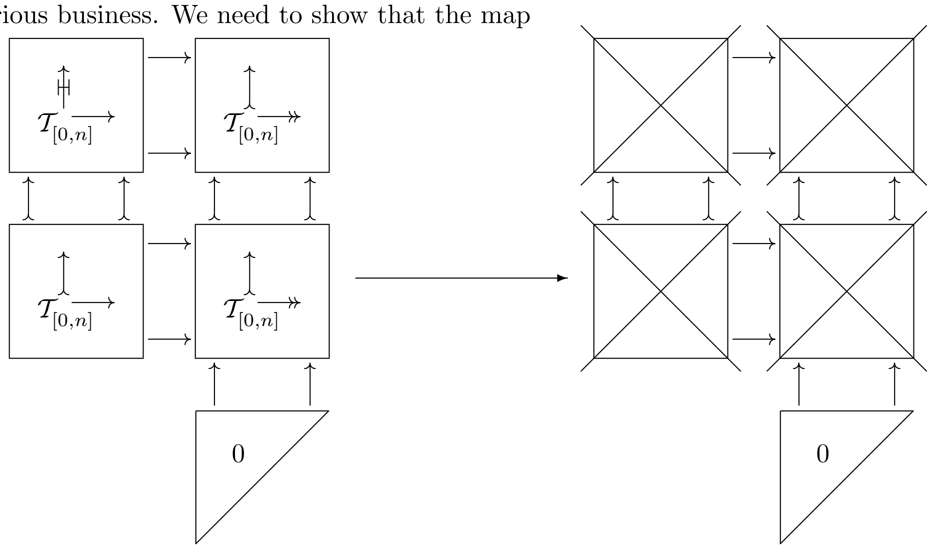
The reason the author has until now labeled triangles with a 0 is twofold. Firstly, when they are part of larger diagrams, the restrictions on the objects and morphisms in the triangle are often forced by the rest of the diagram; it is redundant to give them explicitly. But more importantly, it is easier typographically to incorporate a large label in a rectangle. For a triangle to admit a large label it has to be very large, and our labels for simplicial sets are already quite cumbersome. But, in a more honest notation, Lemma 2.6 says that the following map



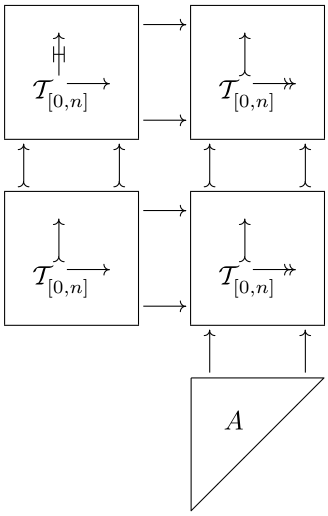
induces a homotopy equivalence.

The next section will consist of a sequence of steps leading us to the proof of the key lemma, Lemma 2.6.

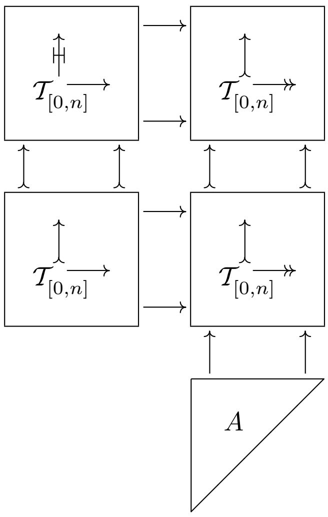
3. Proof of the Key Lemma, Lemma 2.6. It is now time to get down to serious business. We need to show that the map



induces a homotopy equivalence, and more specifically, we will demonstrate it by establishing the contractibility of

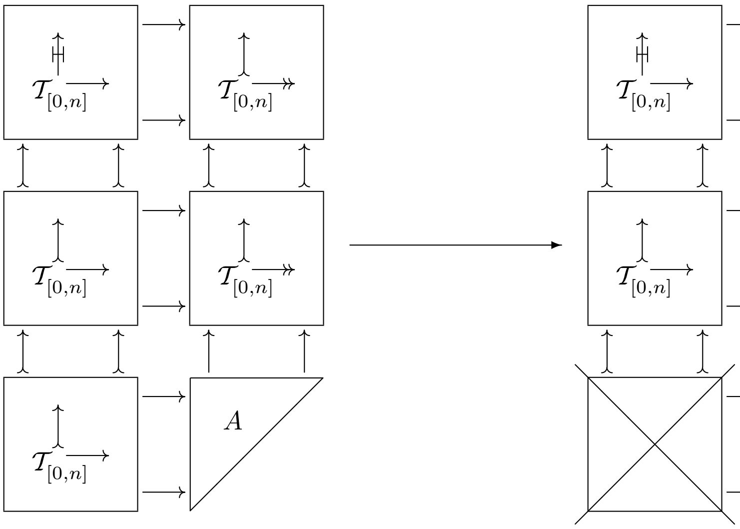


This section, that is the rest of this article [aside from appendices], will concern itself with studying the homotopy type of the fiber



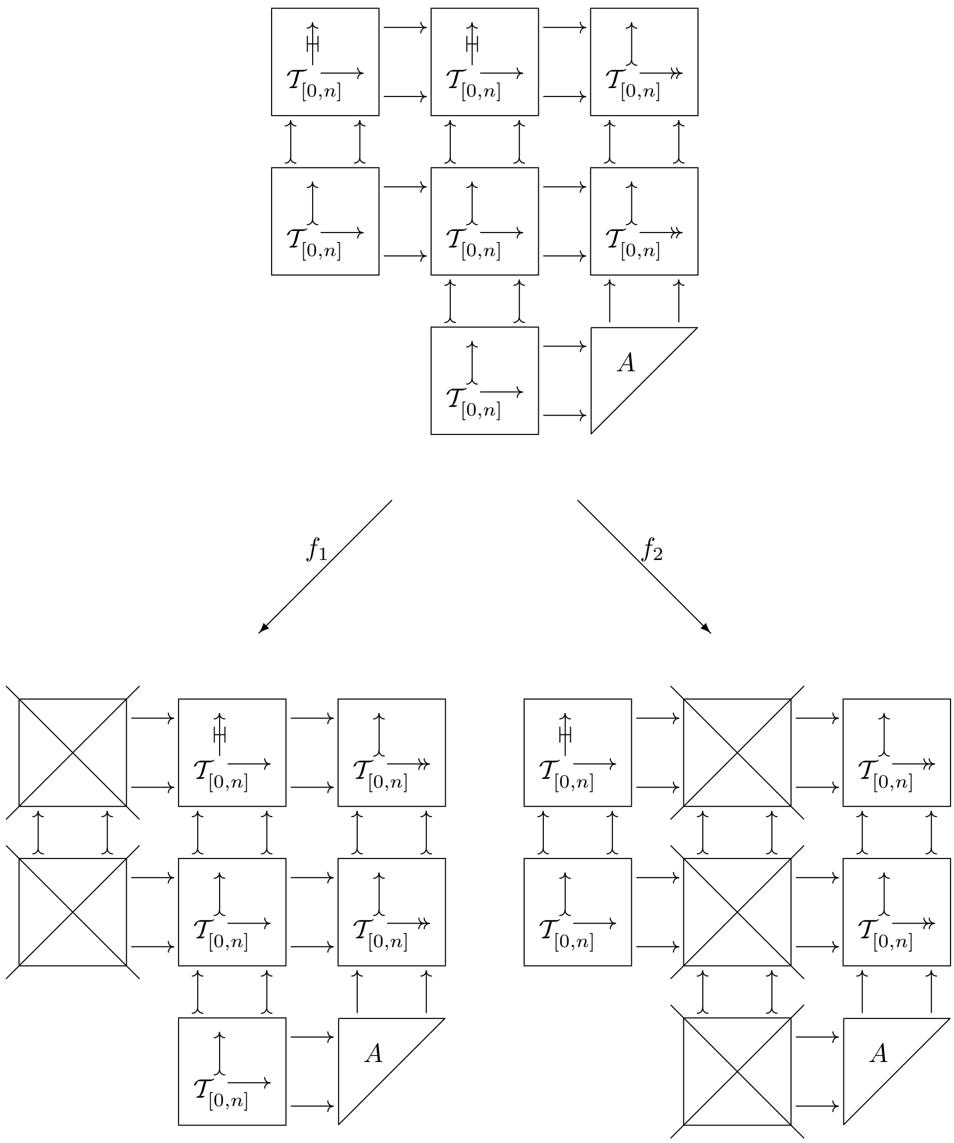
and by a series of slightly convoluted arguments we will establish its contractibility.

Lemma 3.1. *The projection*

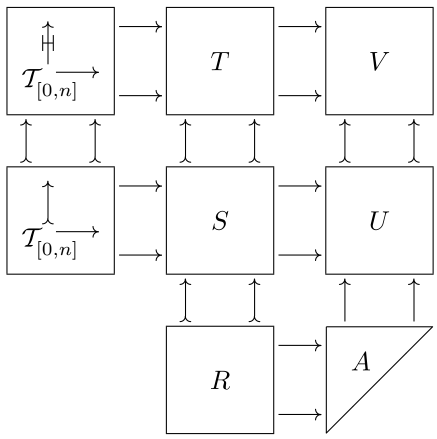


induces a homotopy equivalence.

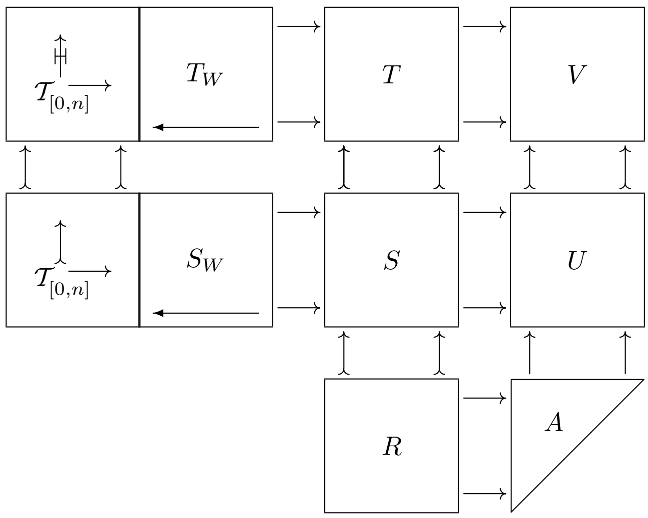
Proof. We treat the slightly more complicated diagram of two projections



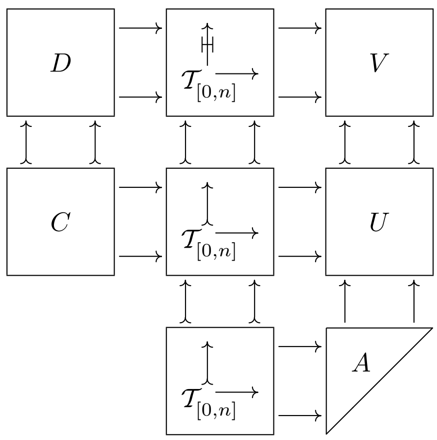
It clearly suffices to prove that f_1 and f_2 induce homotopy equivalences. For f_1 the Segal fiber is



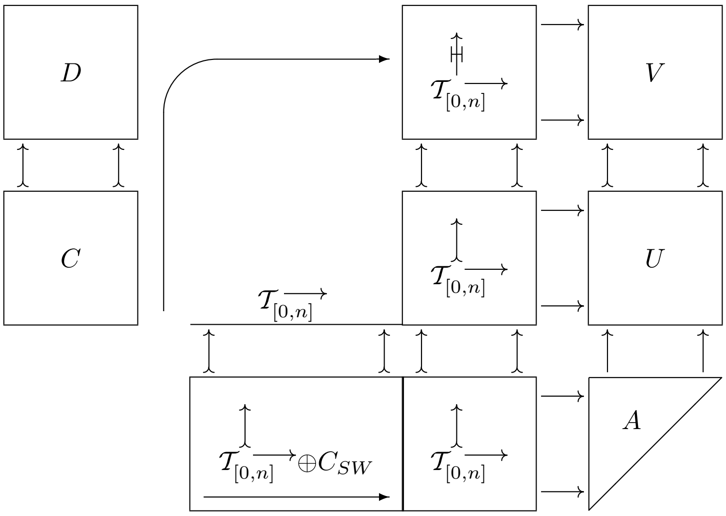
which is contracted by the homotopy



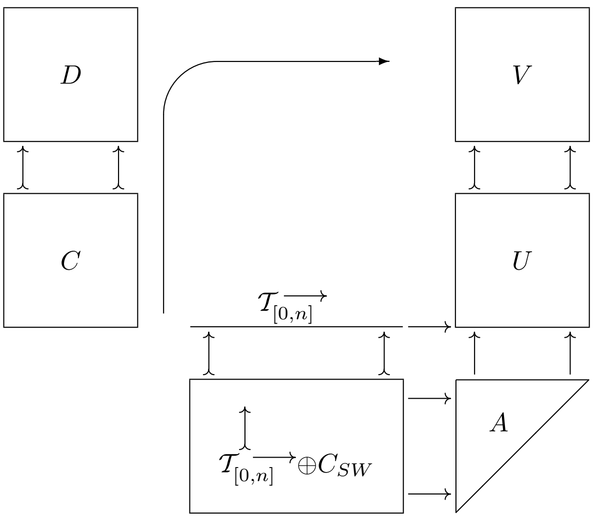
The Segal fiber of the map f_2 is the simplicial set



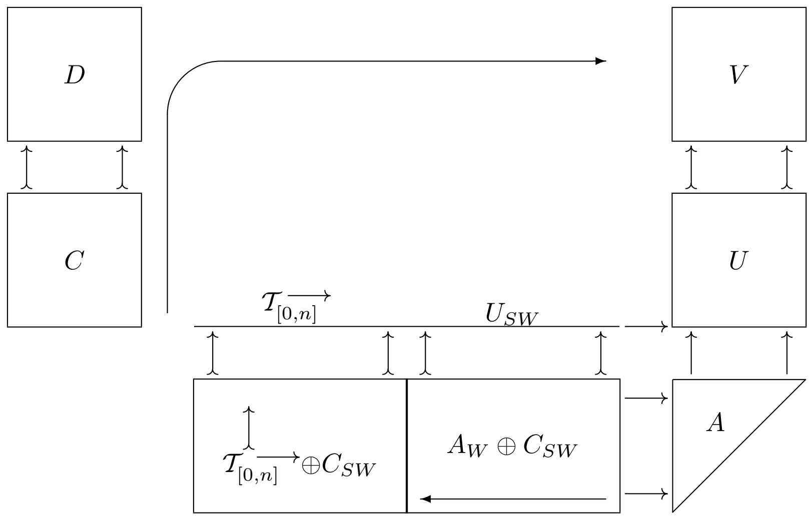
and the homotopy



establishes that the identity on the Segal fiber is homotopic to a map we will denote

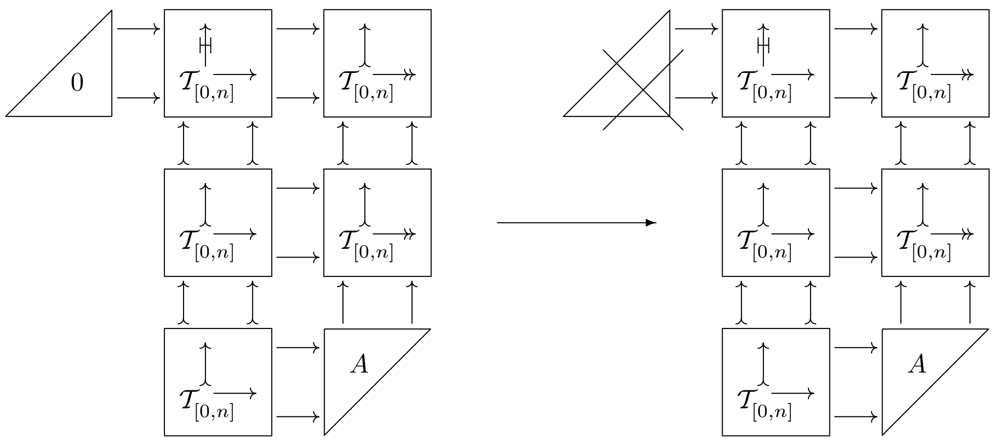


and this map is in turn homotopic to the null map, by the homotopy

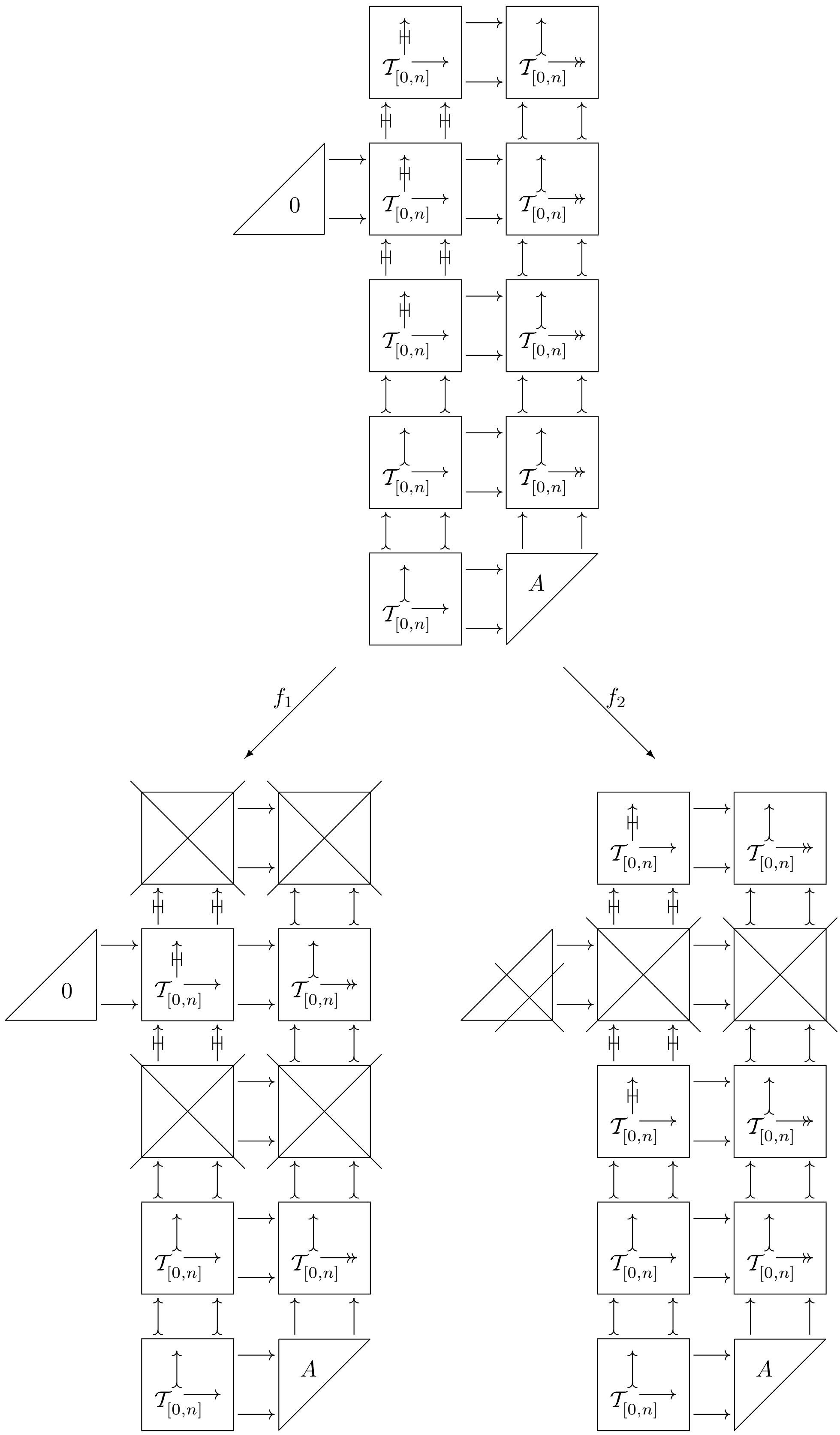


This completes the proof of Lemma 3.1. □

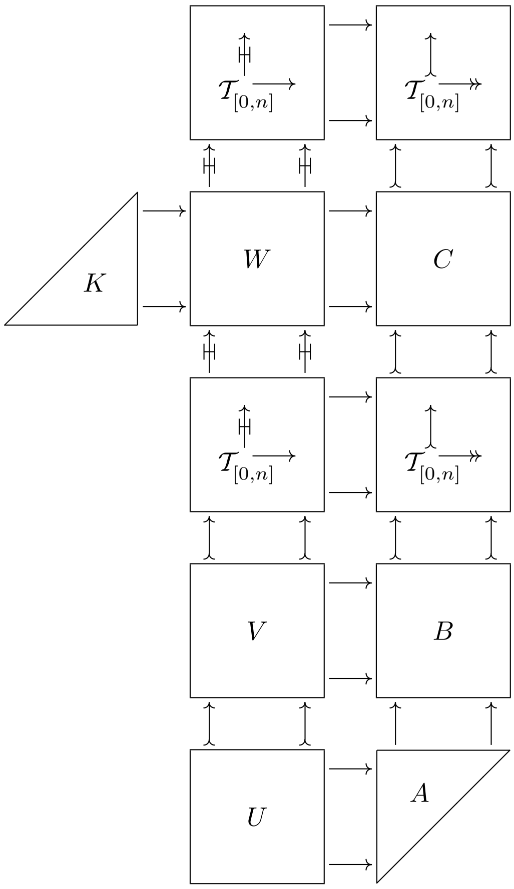
Lemma 3.2. *The following map is a homotopy equivalence*



Proof. The proof is essentially the same as the proof of Lemma 1.2. One considers the more elaborate diagram of two projections

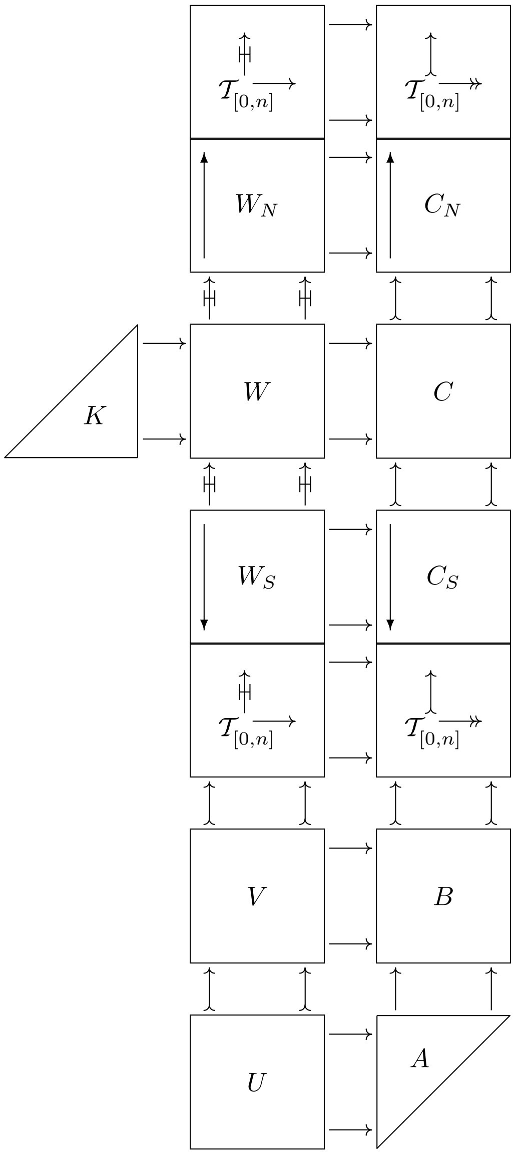

$\mathcal{T}_{[0,n]}$
0
A
f_1
f_2

It clearly suffices to prove that f_1 and f_2 are homotopy equivalences. The Segal fiber of f_1 is the set

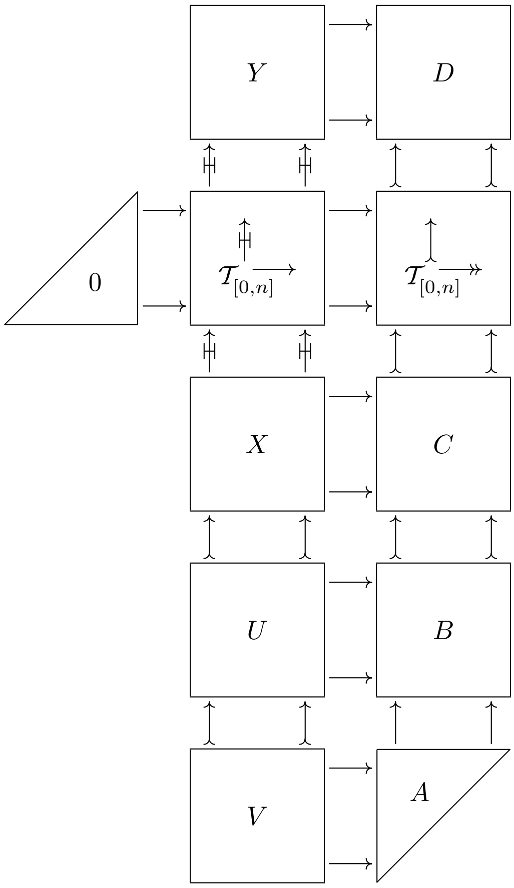


,

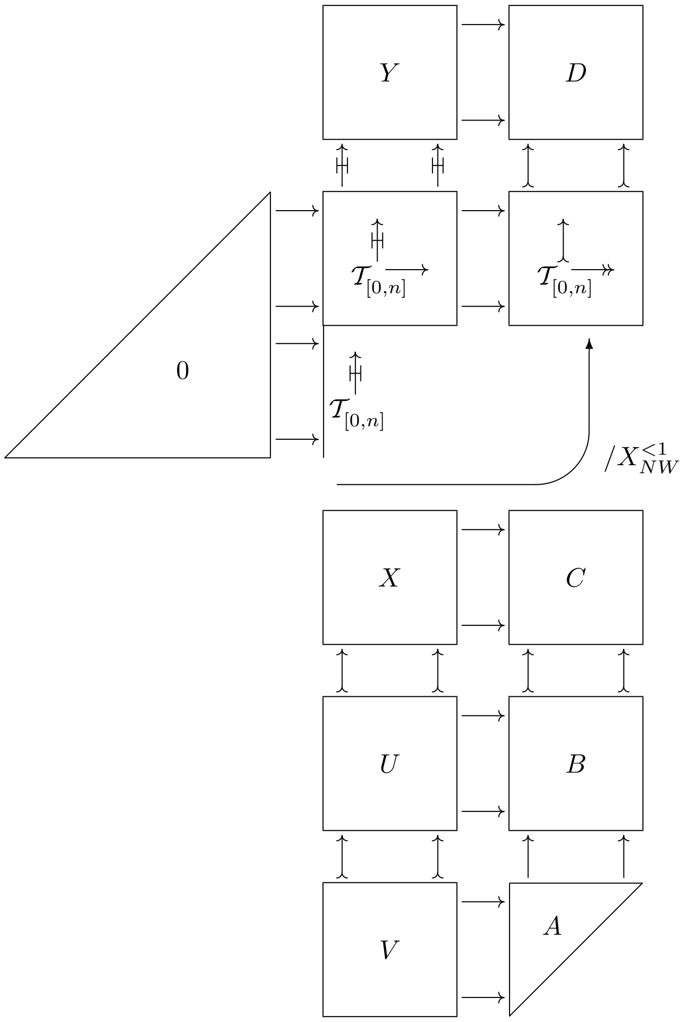
which may be contracted using the homotopy



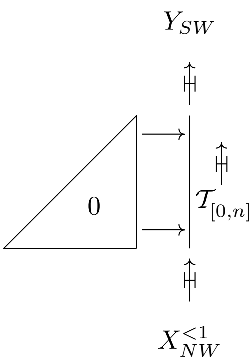
The fiber of the map f_2 is the simplicial set



on which the homotopy

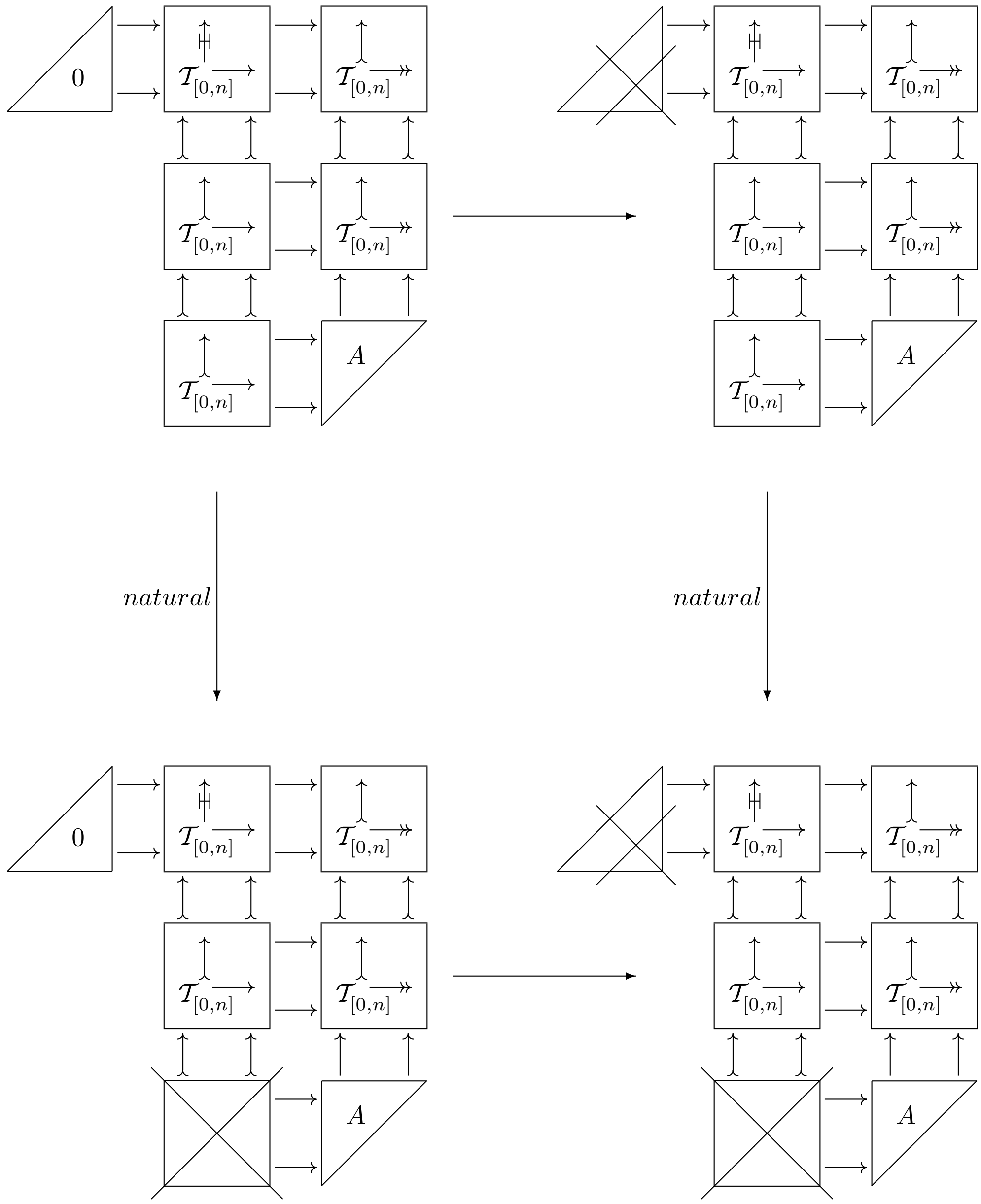


allows us to factor the identity, up to homotopy, through the contractible simplicial set



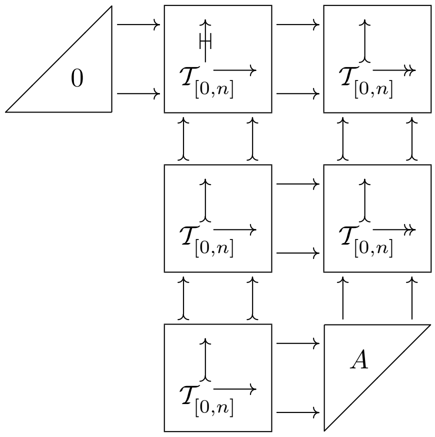
□

Conclusion 3.3. There is a commutative square of maps of simplicial sets

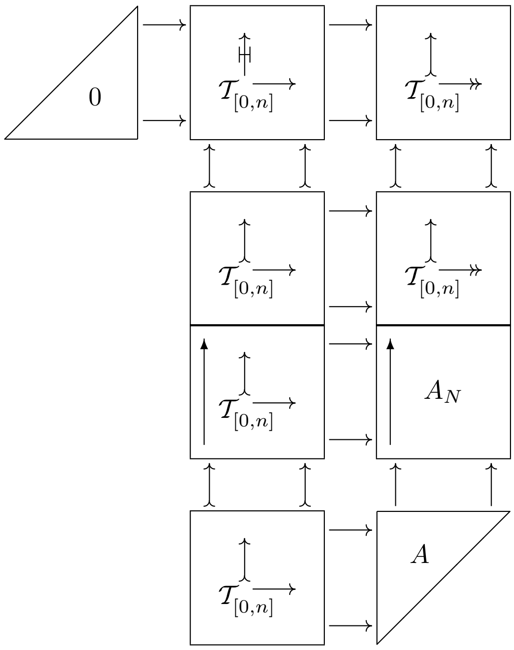


and the sum total of Lemma 3.1 and Lemma 3.2, is that the composites from top left to bottom right (which are of course equal) give a homotopy equivalence.

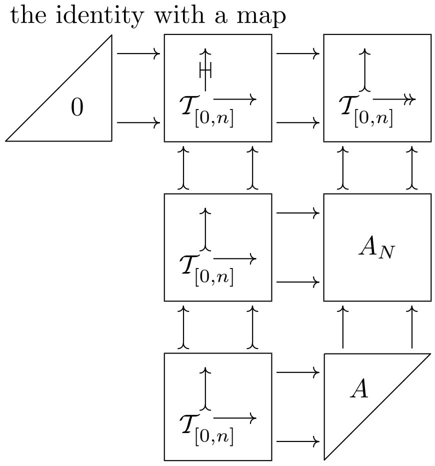
Now, the next point is that both of the natural vertical maps in the diagram of Conclusion 3.3 are homotopic to other maps, which factor through smaller simplicial sets. Precisely, there is on



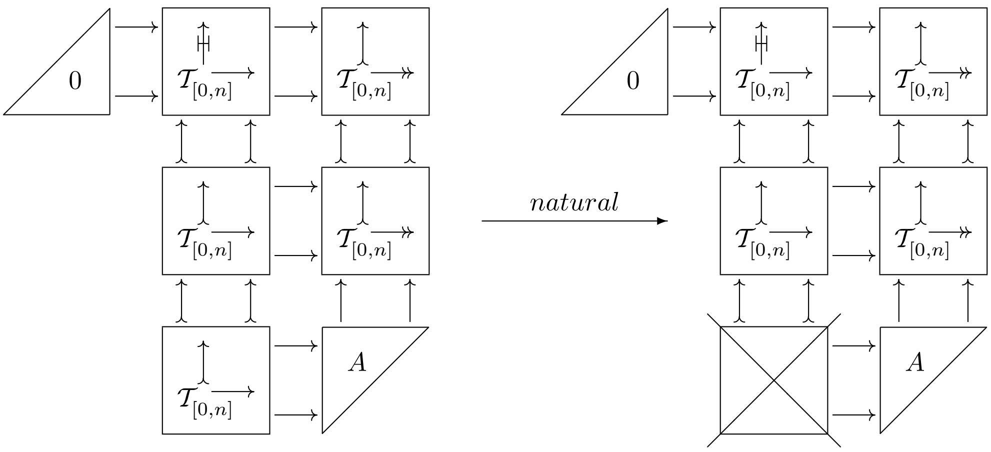
a homotopy



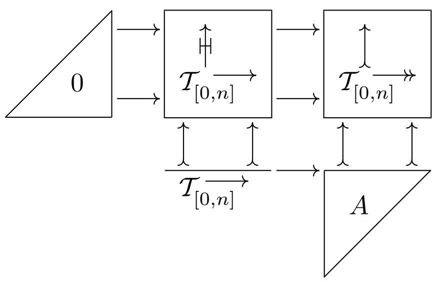
this homotopy connects the identity with a map



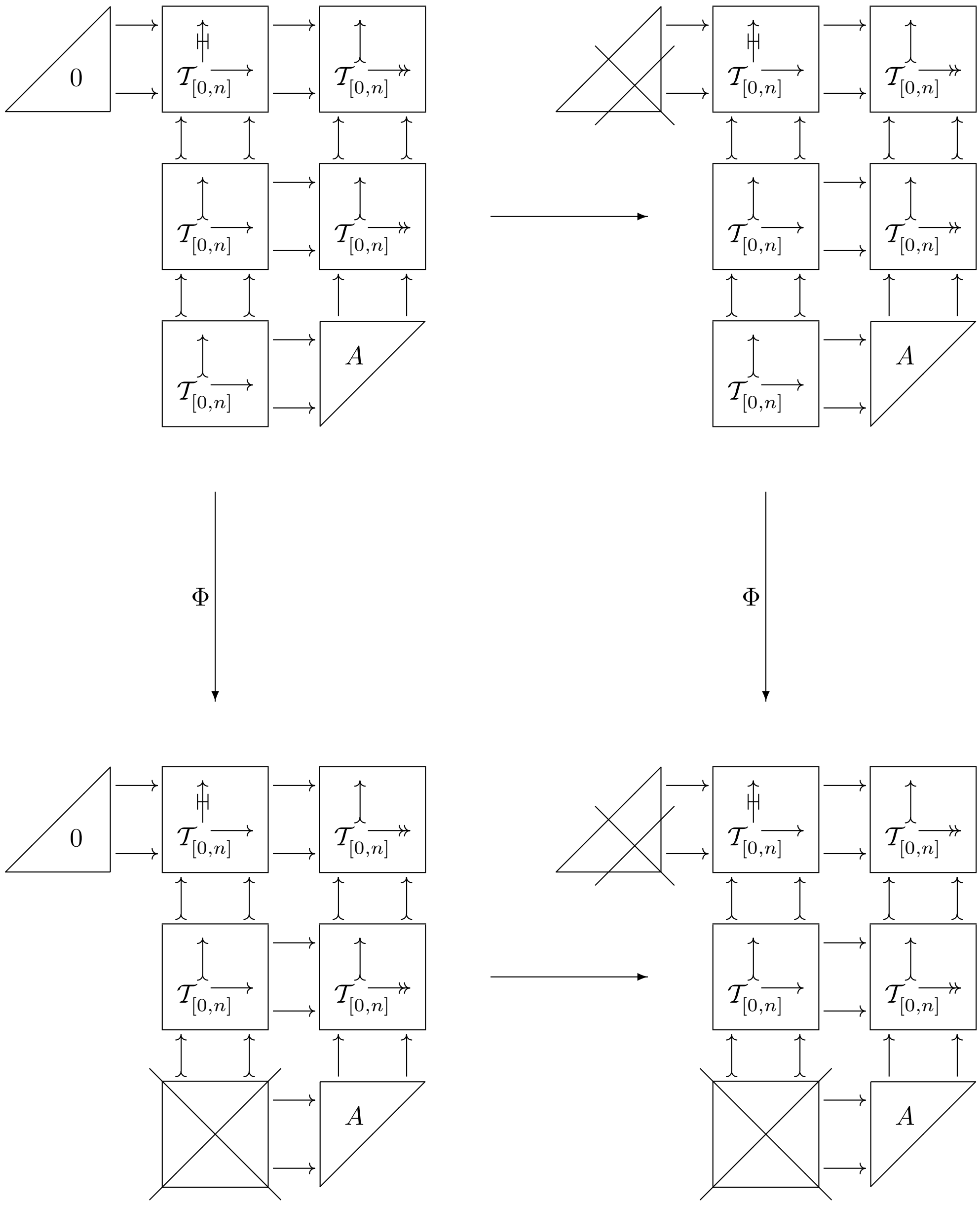
and the composite of this map with the natural projection



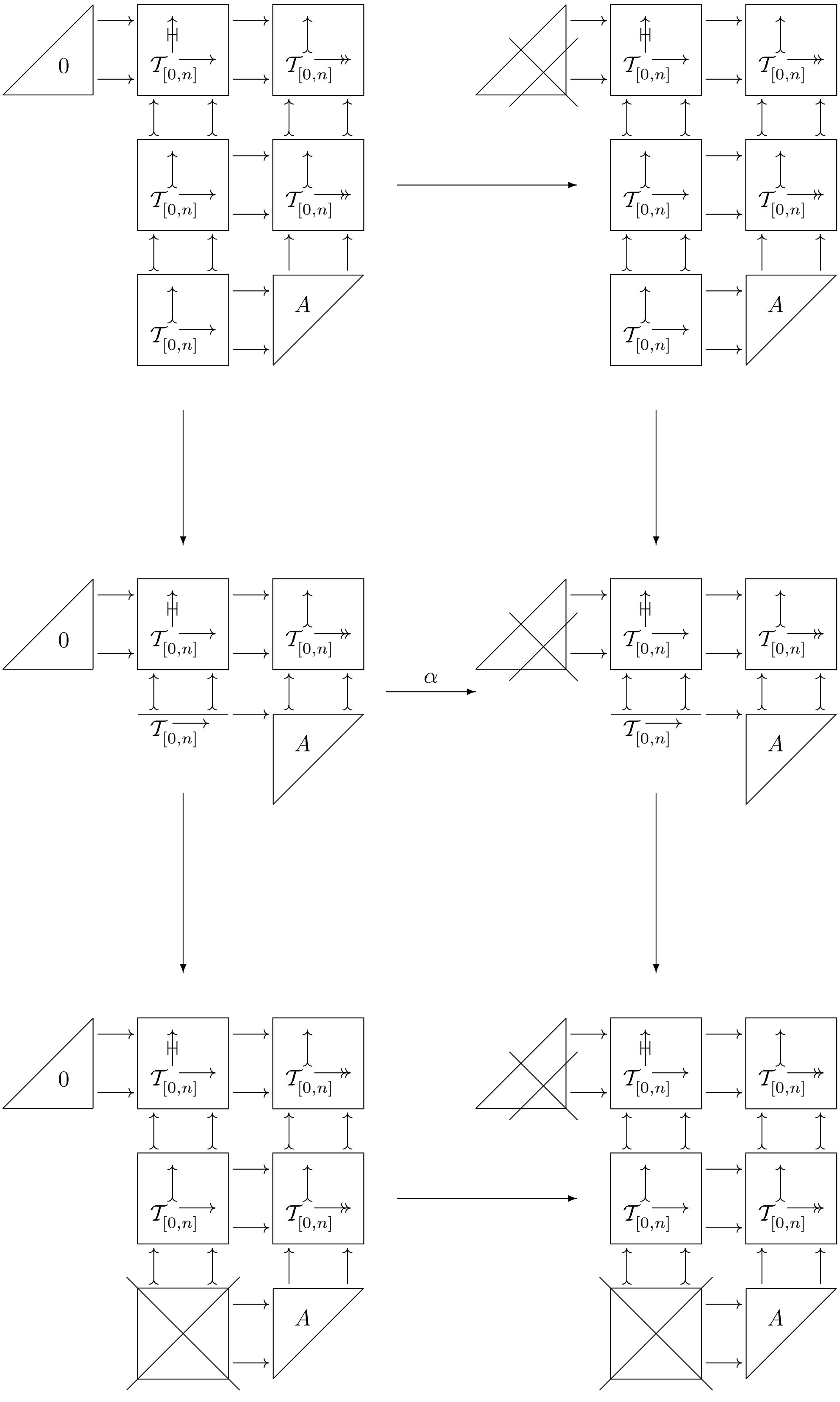
is a map we will call Φ. Finally and most importantly, Φ factors through the simplicial set



The diagram of Conclusion 3.3 can now be replaced by a homotopic diagram

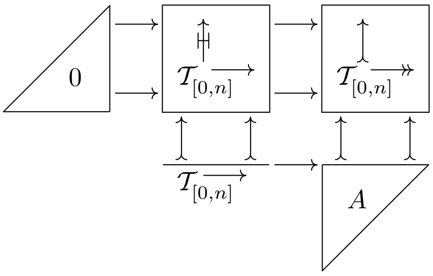


and it is still true that the composite from top left to bottom right is a homotopy equivalence. But the vertical maps factor, to give a larger commutative diagram

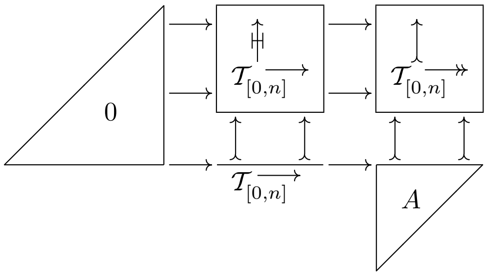

0
$\mathcal{T}_{[0,n]}$
A
α

The map from top left to bottom right, which we know to be a homotopy equivalence, factors through α. The proof will therefore be complete if we show that α is null homotopic.

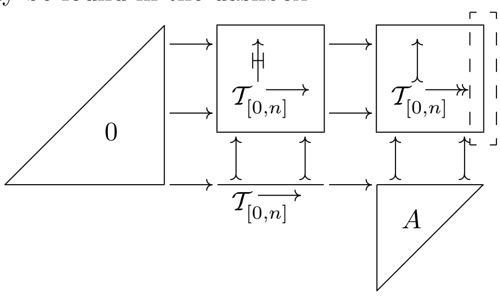
The first observation to make is that the simplicial set



is equal to

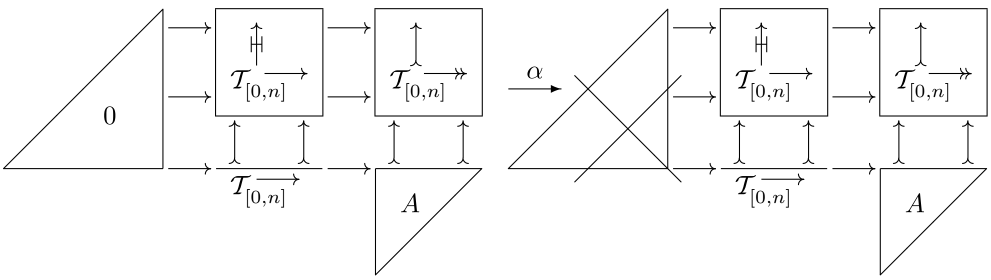


The point is that the kernels we seem to be adding are already part of the structure. Their suspensions may be found in the dashbox

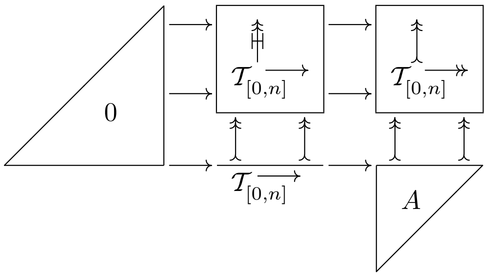


Next, we prove

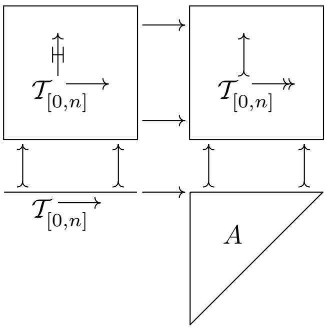
Lemma 3.4. *The map α below*



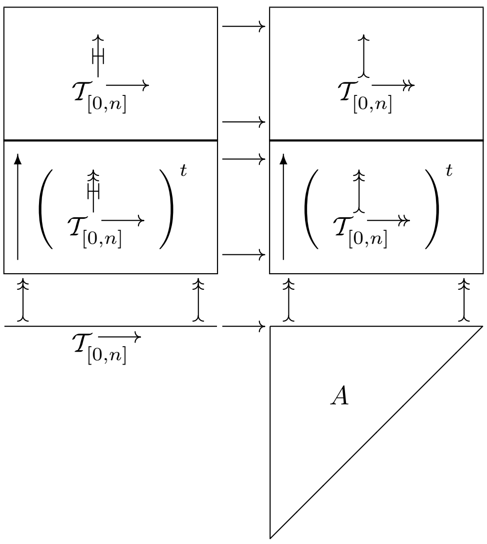
factors, up to homotopy, through the simplicial set



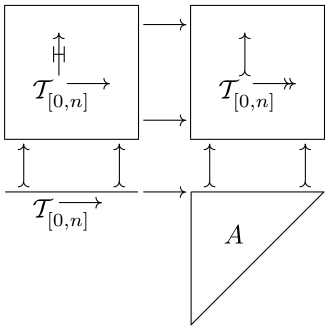
Proof. The point is that the simplicial set



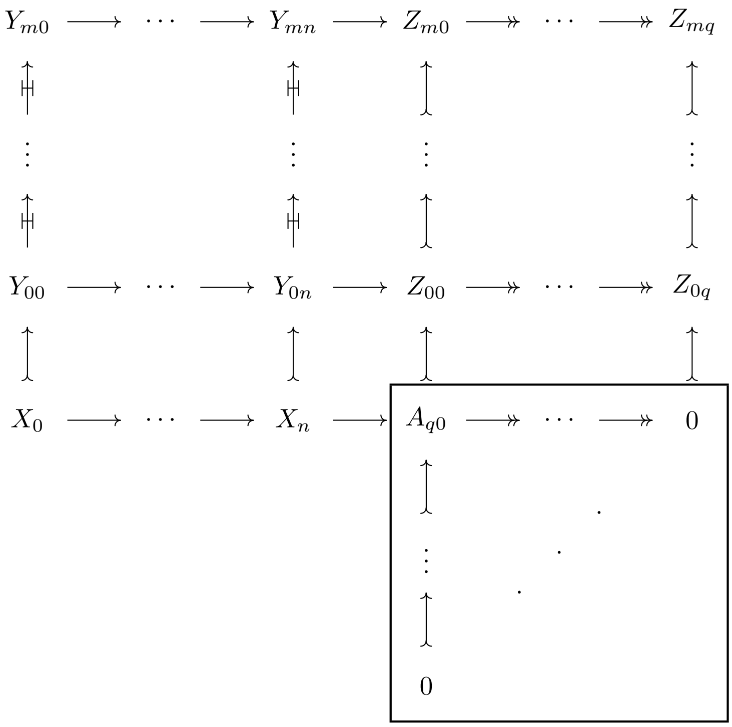
admits a truncation homotopy



This is one of the occasions where it might be clearer if we actually wrote the homotopy in terms of what it does to a typical cell. Given a cell of the simplicial set



that is a diagram

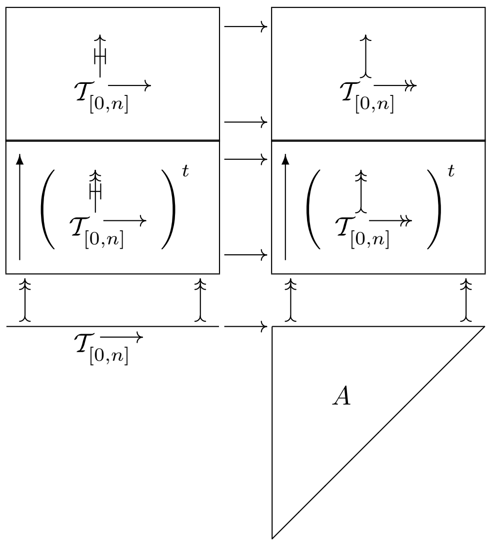


then the homotopy carries it to cells of the form

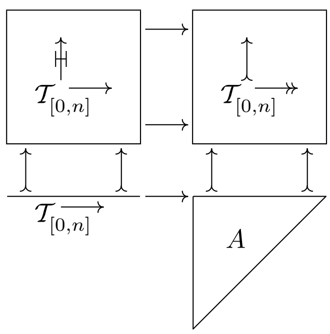
$$
\begin{array}{ccccccccccc}
\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. & Y_{m0} & \longrightarrow & \cdots & \longrightarrow & Y_{mn} & \longrightarrow & Z_{m0} & \twoheadrightarrow \ \cdots \ \twoheadrightarrow & Z_{mq} & \\
& \uparrow & & & & \uparrow & & \uparrow & & \uparrow & \\
& \vdots & & & & \vdots & & \vdots & & \vdots & \left.\right\} \begin{pmatrix} (n-i+1) \\ terms \end{pmatrix} \\
& \uparrow & & & & \uparrow & & \uparrow & & \uparrow & \\
& Y_{i0} & \longrightarrow & \cdots & \longrightarrow & Y_{in} & \longrightarrow & Z_{i0} & \twoheadrightarrow \ \cdots \ \twoheadrightarrow & Z_{iq} & \\
& \uparrow & & & & \uparrow & & \uparrow & & \uparrow & \\
& Y^t_{i0} & \longrightarrow & \cdots & \longrightarrow & Y^t_{in} & \longrightarrow & Z^t_{i0} & \twoheadrightarrow \ \cdots \ \twoheadrightarrow & Z^t_{iq} & \\
& \uparrow & & & & \uparrow & & \uparrow & & \uparrow & \\
& \vdots & & & & \vdots & & \vdots & & \vdots & \left.\right\} \begin{pmatrix} (i+1) \\ terms \end{pmatrix} \\
& \uparrow & & & & \uparrow & & \uparrow & & \uparrow & \\
& Y^t_{00} & \longrightarrow & \cdots & \longrightarrow & Y^t_{0n} & \longrightarrow & Z^t_{00} & \twoheadrightarrow \ \cdots \ \twoheadrightarrow & Z^t_{0q} & \\
& \uparrow & & & & \uparrow & & \uparrow & & \uparrow & \\
& X_0 & \longrightarrow & \cdots & \longrightarrow & X_n & \longrightarrow & A_{q0} & \twoheadrightarrow \ \cdots \ \twoheadrightarrow & 0 & \\
& & & & & & & \uparrow & & & \\
& & & & & & & \vdots & \ddots & & \\
& & & & & & & \uparrow & & & \\
& & & & & & & 0 & & &
\end{array}
$$

where the map $A_{qj} \longrightarrow Z^t_{ij}$ is constructed to be epi. In other words, $Z^t_{iq} = Z^{<n}_{iq}$ because the map $0 \longrightarrow Z^t_{iq}$ is assumed epi, and the other objects and morphisms in the diagram are truncations forced by this. Thus Z^t_{ij} is just the kernel $Ker\Big(Z_{ij} \longrightarrow Z^{\geq n}_{iq}\Big)$, and Y^t_{ij} is just the kernel $Ker\left(Y_{ij} \longrightarrow Z^{\geq n}_{iq}\right)$.

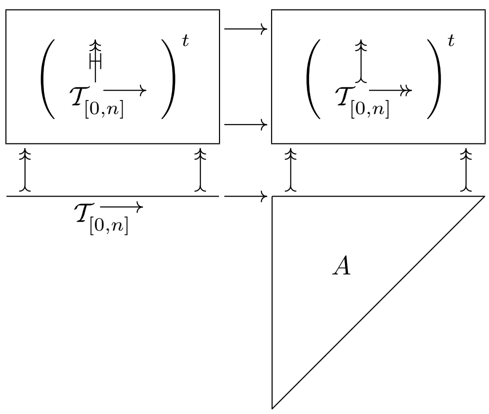
The homotopy



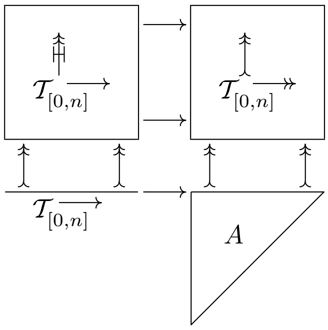
connects the identity on



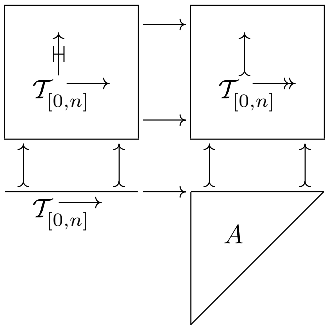
with a map we denote



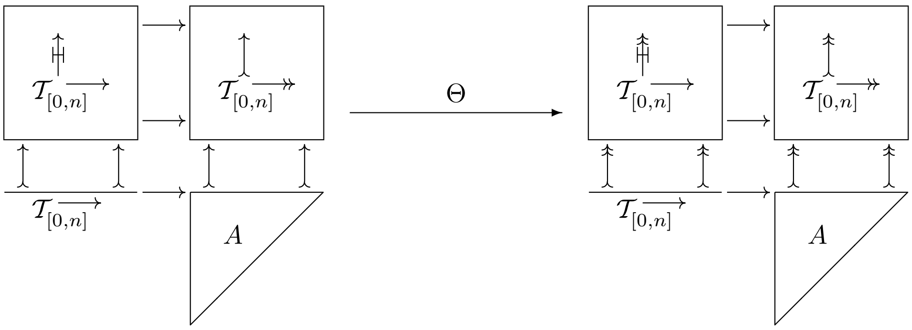
and this last map factors through the simplicial set



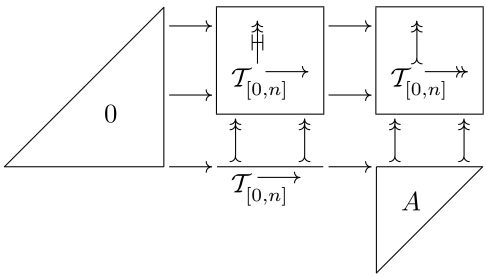
Thus, up to homotopy the identity on



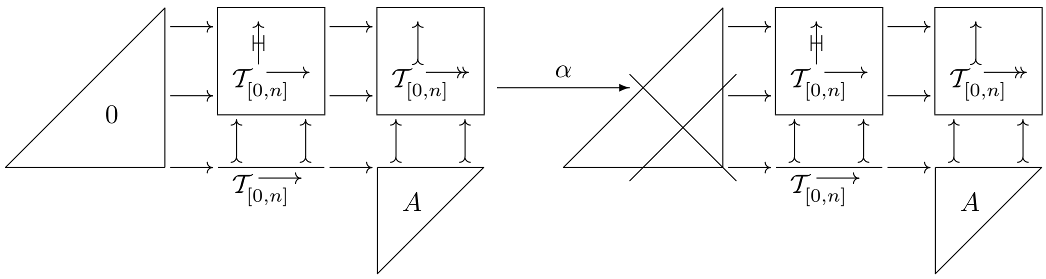
factors through a truncation map



and the reader can easily verify that the composite $\Theta \circ \alpha$ factors through the simplicial set



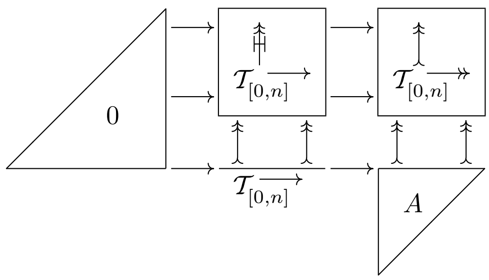
That is, the truncations used to define Θ can be extended to the kernels. We remind the reader that α is the projection



This completes the proof of Lemma 3.4. ☐

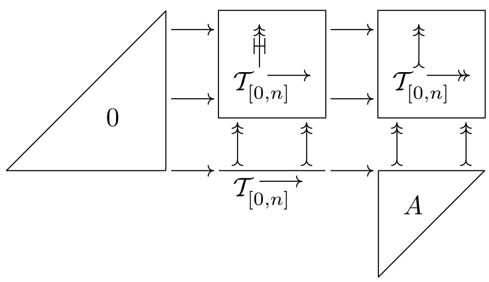
So we will be completely done once we establish:

Lemma 3.5. *The simplicial set*

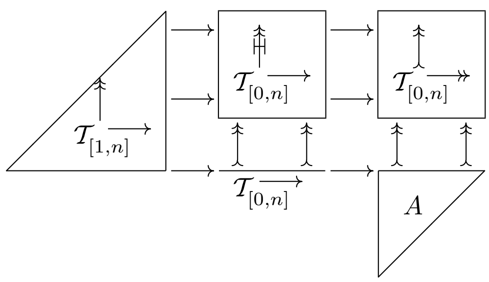


is contractible.

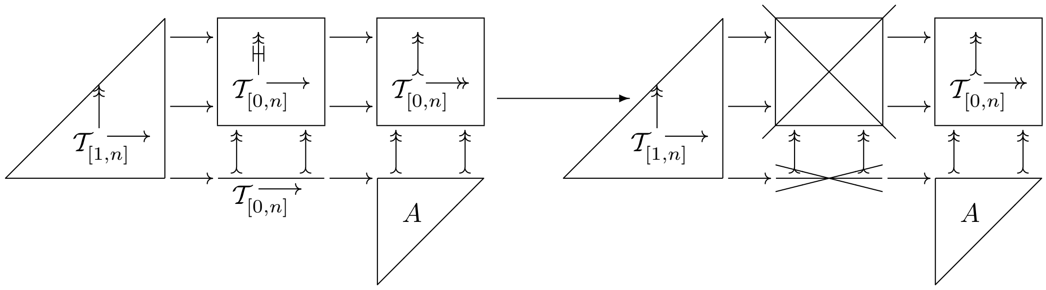
Proof. The simplicial set



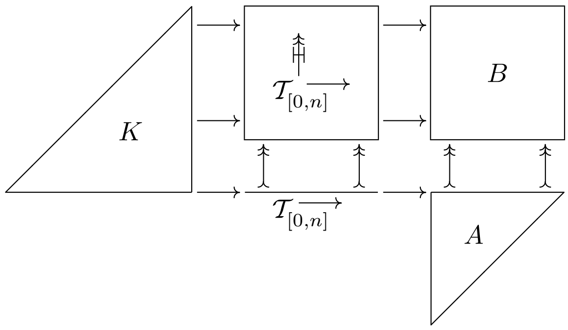
is perhaps better denoted



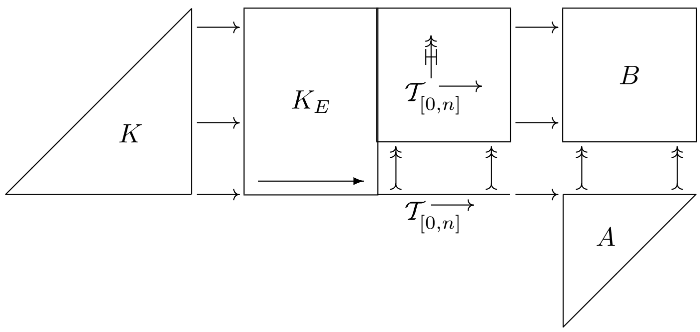
This highlights the fact that the kernels of monoepis are constrained to be in $\mathcal{T}_{[1,n]}$. The projection



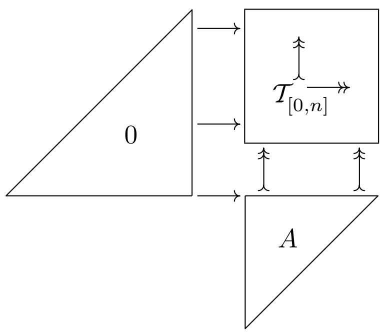
is a homotopy equivalence since the Segal fiber



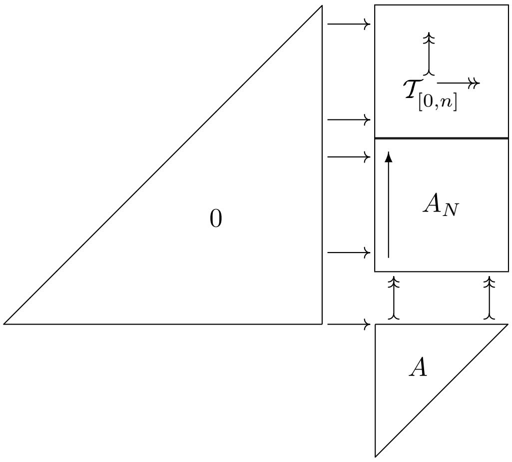
is contracted by the homotopy



And finally, the simplicial set



is contracted by the homotopy



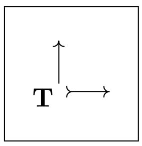
This completes the proof of Lemma 3.5, and hence the proof of Lemma 2.6, and hence the proof of Theorem I.7.1, and hence this article. □

Appendix A. The Comparison with Waldhausen's K–Theory.

By the term *Waldhausen category*, we will understand a category with cofibrations and weak equivalences, satisfying the extension axiom, the gluing lemma and the cylinder axiom. Given a Waldhausen category, it is possible to associate to it a K–theory. This K–theory has reasonable functoriality properties, developed in the excellent foundational article [12].

Let me briefly remind the reader of the definition of Waldhausen's K–theory. One begins with a small Waldhausen category $\mathbf{T}$. Out of it one constructs a simplicial category denoted $wS.\mathbf{T}$. The geometric realisation of this simplicial category is by definition a (delooping of) the Waldhausen K–theory of $\mathbf{T}$.

Instead of giving Waldhausen's precise simplicial category, let me give one which is easily seen to be homotopy equivalent. We can consider a bisimplicial category. The bisimplicial set of objects in this bisimplicial category is what we have been denoting



.

We recall what this means. A (p, q)–object in our bisimplicial category is a diagram of pushout squares in $\mathbf{T}$

$$\begin{array}{ccccc} X_{p0} & \rightarrowtail & \cdots & \rightarrowtail & X_{pq} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ X_{00} & \rightarrowtail & \cdots & \rightarrowtail & X_{0q} \end{array}$$

A morphism of (p, q)–objects is a morphism of diagrams

$$
\begin{array}{ccccc}
X_{p0} & \rightarrowtail & \cdots & \rightarrowtail & X_{pq} \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
X_{00} & \rightarrowtail & \cdots & \rightarrowtail & X_{0q}
\end{array}
\qquad\qquad
\begin{array}{ccccc}
Y_{p0} & \rightarrowtail & \cdots & \rightarrowtail & Y_{pq} \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
Y_{00} & \rightarrowtail & \cdots & \rightarrowtail & Y_{0q}
\end{array}
$$

where, for each i and j, the map $X_{ij} \longrightarrow Y_{ij}$ is a weak equivalence.

This defines a bisimplicial category. Its nerve is a trisimplicial set; a (p, q, r)–simplex is a sequence of r composable morphisms of (p, q)–objects as above. The realisation of this trisimplicial set is a delooping of Waldhausen's K–theory of $\mathbf{T}$. As I said, this is not quite Waldhausen's model, but it is easy and well–known that it is homotopy equivalent to Waldhausen's $wS.\mathbf{T}$.

The first observation is that a Waldhausen category admits a suspension functor. The suspension of an object X is defined to be the quotient, of the mapping cylinder on $1 : X \longrightarrow X$, by the inclusion of the front and back faces. It is a theorem of Waldhausen's, that this functor Σ induces the map -1 in K–theory. We can form the category which is the direct limit of the inclusions

$$\mathbf{T} \xrightarrow{\Sigma} \mathbf{T} \xrightarrow{\Sigma} \mathbf{T} \xrightarrow{\Sigma} \cdots$$

and the category we obtain is still a Waldhausen category. Let us denote it $\Sigma^{-1}\mathbf{T}$. It is easy to see that the inclusion $\mathbf{T} \subset \Sigma^{-1}\mathbf{T}$ induces a homotopy equivalence on K–theory. The categories $\mathbf{T}$ and $\Sigma^{-1}\mathbf{T}$ have isomorphic Waldhausen K–theories. We may henceforth replace $\mathbf{T}$ by $\Sigma^{-1}\mathbf{T}$, without loss of generality. Assume, therefore, that all our Waldhausen categories $\mathbf{T}$ have an invertible suspension functor Σ.

Now recall: for each (p, q), we have a category of (p, q)–objects

$$
\begin{array}{ccccc}
X_{p0} & \rightarrowtail & \cdots & \rightarrowtail & X_{pq} \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
X_{00} & \rightarrowtail & \cdots & \rightarrowtail & X_{0q}
\end{array}
$$

where the morphisms are the weak equivalences. Given any category, it is possible to form the associated groupoid. We formally invert every morphism in the category. In particular, there is a category whose objects are diagrams

$$\begin{array}{ccccc}
X_{p0} & \rightarrowtail & \cdots & \rightarrowtail & X_{pq} \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
X_{00} & \rightarrowtail & \cdots & \rightarrowtail & X_{0q}
\end{array}$$

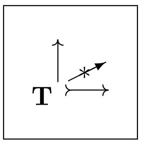
and whose morphisms between objects

$$\begin{array}{ccccc}
X_{p0} & \rightarrowtail & \cdots & \rightarrowtail & X_{pq} \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
X_{00} & \rightarrowtail & \cdots & \rightarrowtail & X_{0q}
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccccc}
Y_{p0} & \rightarrowtail & \cdots & \rightarrowtail & Y_{pq} \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
Y_{00} & \rightarrowtail & \cdots & \rightarrowtail & Y_{0q}
\end{array}$$

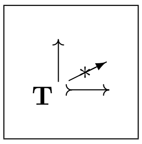
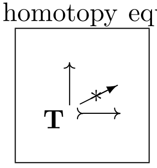
are strings

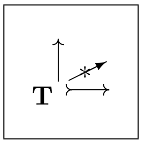
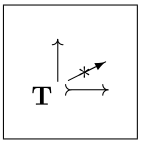
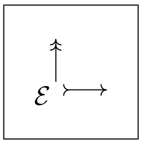
$$\phi_1^{-1}\phi_2\phi_3^{-1}\cdots\phi_{n-1}^{-1}\phi_n$$

where ϕ_i is a weak equivalence of diagrams, and the relations among strings are the ones generated by the relations among weak equivalences. We deduce a new bisimplicial category, in which the category structure is very simple; every morphism is an isomorphism. Taking the nerve of this bisimplicial category, we obtain a trisimplicial set. We call this trisimplicial set



.

Remark A.1. The reader should note that  is *not* in general homotopy equivalent to Waldhausen's $wS.\mathbf{T}$. There is a natural map from $wS.\mathbf{T}$ to  , which is not in general a homotopy equivalence.

However, the theory  has many of the good functoriality properties of Waldhausen's K–theory. By essentially the same proof as Waldhausen's, it satisfies an additivity theorem. The consequences of additivity therefore also hold. In particular, it easily follows that, if $\mathbf{T}$ is the category of bounded chain complexes in a Karoubian exact category $\mathcal{E}$, then the homotopy type of  is the same as  .

Remark A.2. For each $(p, q) \in \mathbb{N} \times \mathbb{N}$, we constructed a category with invertible morphisms, whose objects were diagrams

$$\begin{array}{ccccc} X_{p0} & \rightarrowtail & \cdots & \rightarrowtail & X_{pq} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ X_{00} & \rightarrowtail & \cdots & \rightarrowtail & X_{0q} \end{array}$$

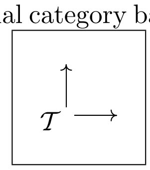
The stupid case, namely $p = 0 = q$, deserves some more thought. We construct a category whose objects are the objects of $\mathbf{T}$, and whose morphisms are chains of weak equivalences and their inverses

$$\phi_1^{-1}\phi_2\phi_3^{-1}\cdots\phi_{n-1}^{-1}\phi_n.$$

We wish to do something slightly different. We wish to consider the category whose objects are the objects of $\mathbf{T}$, and the morphisms are strings

$$\phi_1^{-1}\phi_2\phi_3^{-1}\cdots\phi_{n-1}^{-1}\phi_n$$

as before. But we insist only that the ϕ_i^{-1}'s are inverses of weak equivalences. The maps ϕ_i in the string, which are not inverted, are free to be any morphisms of $\mathbf{T}$. We obtain a category, which we denote $\mathcal{T}$. It is the category obtained from $\mathbf{T}$ by formally inverting the weak equivalences. It is well–known, although the author knows of no reference, that the category $\mathcal{T}$ is a triangulated category.

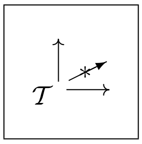
Given the triangulated category $\mathcal{T}$, one can also form a bisimplicial category based on it. The objects form the bisimplicial set we have been denoting  . A

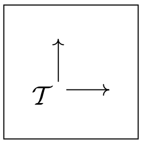
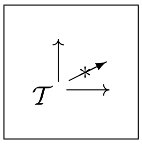
(p, q)–object is a diagram

$$\begin{array}{ccccc} X_{p0} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0q} \end{array}$$

together with a coherent differential $X_{pq} \longrightarrow \Sigma X_{00}$. A morphism of (p, q)–objects is an ismorphism of diagrams

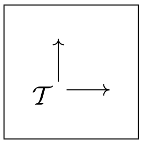
$$\begin{array}{ccccc} X_{p0} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0q} \end{array} \longrightarrow \begin{array}{ccccc} Y_{p0} & \longrightarrow & \cdots & \longrightarrow & Y_{pq} \\ \uparrow & & & & \uparrow \\ \vdots & & & & \vdots \\ \uparrow & & & & \uparrow \\ Y_{00} & \longrightarrow & \cdots & \longrightarrow & Y_{0q} \end{array}$$

The trisimplicial set  is defined to be the nerve of this bisimplicial category.

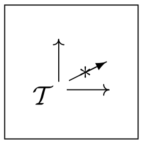
Remark A.3. As always, the precise definition of  raises delicate problems, and the same problems persist with . I do not understand this well, and hence in this conjectural Appendix, I propose to say nothing about the subject. The reader is referred to the introduction of *K–theory for triangulated categories I* for some more detail.

There is one fact that one can prove very easily.

Lemma A.4. *Let $\mathcal{T}$ be any triangulated category. The simplicial sets*

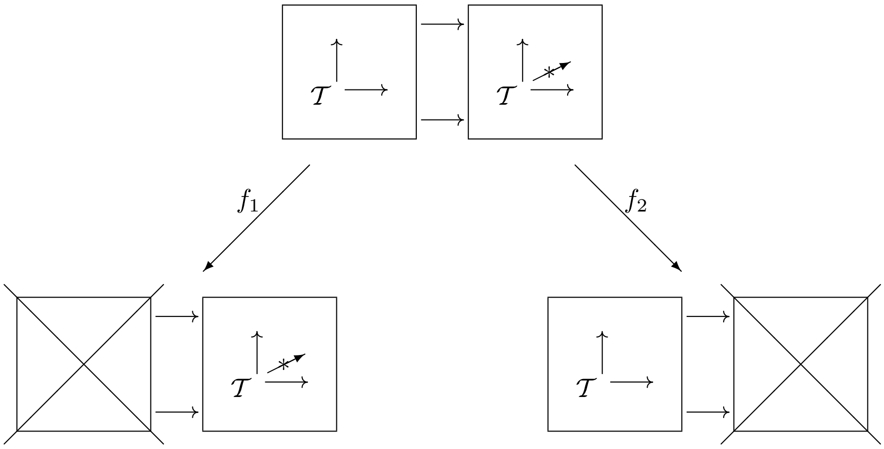


and

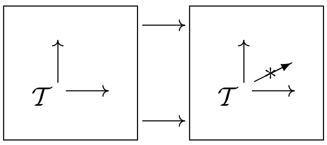


are homotopy equivalent.

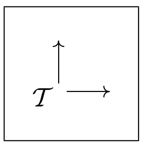
Proof. This is really a special case of Waldhausen's swallowing lemma. We briefly outline this for the reader. In the diagram below, of a quartisimplicial set and two projections



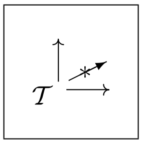
we want to show that the maps f_1 and f_2 are both homotopy equivalences. Perhaps it would be most illuminating if we explained what the simplicial set



might be.

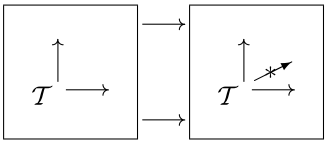
Recall first that a (p,q)–simplex in  is a diagram

$$\begin{array}{ccccc}
X_{p0} & \longrightarrow & \cdots & \longrightarrow & X_{pq} \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0q}
\end{array}$$

together with a coherent differential. A (p,m,n)–simplex in  is, for every integer $0 \le j \le n$, a diagram

$$\begin{array}{ccccc}
Y^j_{p0} & \longrightarrow & \cdots & \longrightarrow & Y^j_{pm} \\
\uparrow & & & & \uparrow \\
\vdots & & & & \vdots \\
\uparrow & & & & \uparrow \\
Y^j_{00} & \longrightarrow & \cdots & \longrightarrow & Y^j_{0m}
\end{array}$$

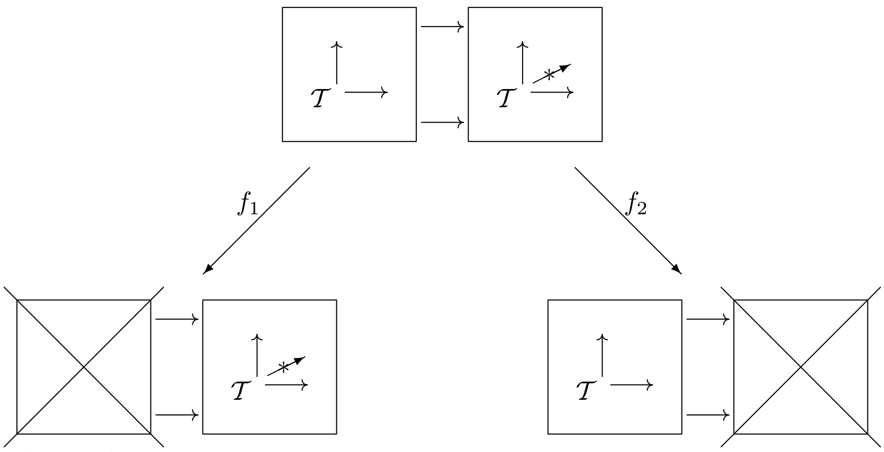
together with coherent differentials, and isomorphisms of diagrams for different j. A (p,q,m,n)–simplex in



is the concatenation of the two. For every $0 \le j \le n$, it is a diagram

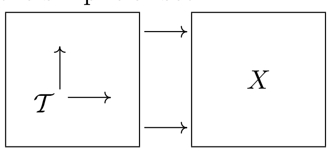
$$\begin{array}{ccccccccccc}
X_{p0} & \longrightarrow & \cdots & \longrightarrow & X_{pq} & \longrightarrow & Y^j_{p0} & \longrightarrow & \cdots & \longrightarrow & Y^j_{pm} \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
\vdots & & & & \vdots & & \vdots & & & & \vdots \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0q} & \longrightarrow & Y^j_{00} & \longrightarrow & \cdots & \longrightarrow & Y^j_{0m}
\end{array}$$

together with coherent differentials. For every pair of different j, we want an isomorphism of the diagrams, restricting to the identity on the X's. We are asserting that, with the obvious projection maps

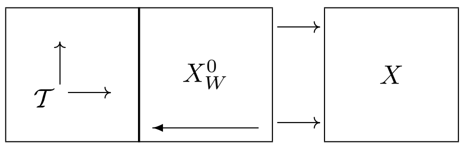


both f_1 and f_2 are homotopy equivalences.

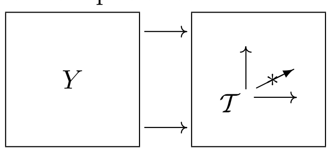
The Segal fiber of f_1 is the simplicial set



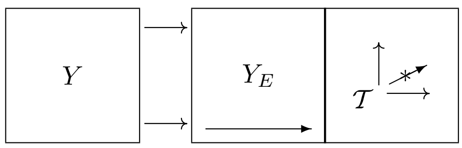
and is contracted by



while the Segal fiber of f_2 is the simplicial set

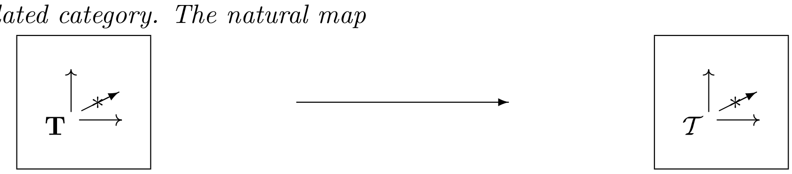


and is contracted by



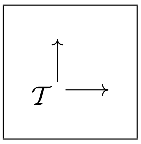
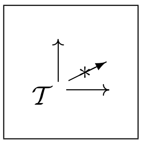
□

Conjecture A.5. *Let* $\mathbf{T}$ *be a Waldhausen category. Let* $\mathcal{T}$ *be the associated triangulated category. The natural map*

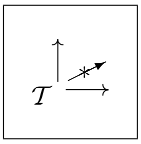
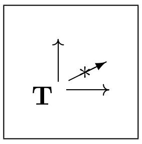


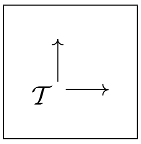
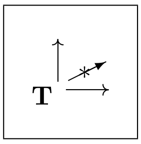
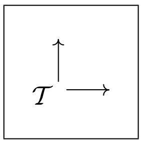
induces a homotopy equivalence.

Remark A.6. Suppose Conjecture A.5 is true. By Lemma A.4,

 and 

have the same homotopy type. By Conjecture A.5,

 and 

also are homotopy equivalent. Combining these facts, we would deduce that triangulated K–theory, given by , has a description in terms of Waldhausen–like theories . This would allow us to deduce, as in Remark A.1, that one can recover the K–theory of an exact category $\mathcal{E}$ from , where $\mathcal{T}$ is the bounded derived category of $\mathcal{E}$.

Remark A.7. I owe the reader a summary of the status of the conjecture. At some point I believed I had a proof. I even submitted a manuscript, *K–theory for triangulated categories IV*, for publication. I retracted the manuscript after finding an error. At the time, I believed I knew how to fix the error. But that was five years ago. I never completed the job, and no longer remember any of the details. I have the original manuscript (with the error), as well as a draft of the corrected manuscript. Both are over 300 pages.

If nothing else, this should explain the strange numbering of the manuscripts. *K–theory for triangulated categories* $3\frac{1}{2}$ and $3\frac{3}{4}$ were written when I still had every intention of going back to complete *K–theory for triangulated categories IV*.

Appendix B. Localisation.

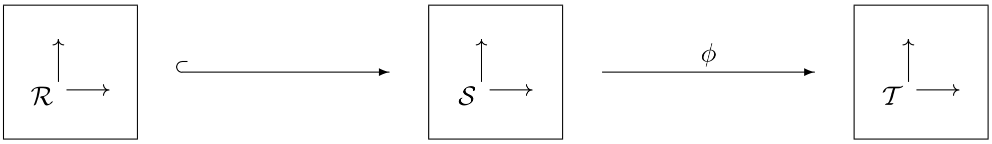
Let $\mathcal{S}$ be a triangulated category, $\mathcal{R} \subset \mathcal{S}$ a thick subcategory. We remind the reader that a full subcategory $\mathcal{R} \subset \mathcal{S}$ is called *thick*, if it is closed under the formation of triangles on its morphisms, and under the formation (in $\mathcal{S}$) of direct summands of its objects.

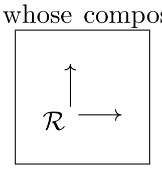
It is a theorem of Verdier that one can form a triangulated category $\mathcal{T} = \mathcal{S}/\mathcal{R}$. There is a triangulated functor

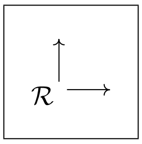
$$\mathcal{S} \longrightarrow \mathcal{T}$$

under which $\mathcal{R}$ maps to zero. Furthermore, this functor is universal among all triangulated $\mathcal{S} \longrightarrow \mathbf{T}$ taking $\mathcal{R} \subset \mathcal{S}$ to zero. If $\mathcal{S}$, and therefore also its subcategory $\mathcal{R}$, are both essentially small, then so is the quotient $\mathcal{T}$. Assume from now on that all our categories are essentially small.

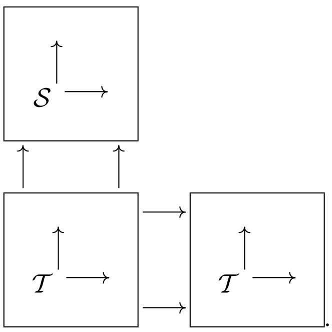
We deduce simplicial maps



whose composite is clearly the null map. There is therefore a canonical map from  to the homotopy fiber of the map ϕ. Our key conjecture in this appendix is

Conjecture B.1. *The natural map from*  *to the homotopy fiber of ϕ is a homotopy equivalence.*

It turns out to be easy to give a simplicial model for the homotopy fiber of ϕ. For the purpose of what we will now do, it is convenient to study the simplicial set



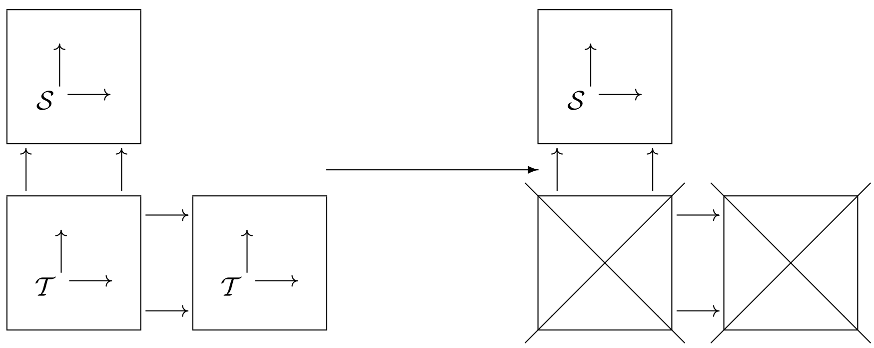
What this means is the following. A simplex is a diagram

$$
\begin{array}{ccccccccccccc}
Y_{n0} & \longrightarrow & \cdots & \longrightarrow & Y_{np} & & & & & & & & \\
\uparrow & & & & \uparrow & & & & & & & & \\
\vdots & & & & \vdots & & & & & & & & \\
\uparrow & & & & \uparrow & & & & & & & & \\
Y_{00} & \longrightarrow & \cdots & \longrightarrow & Y_{0p} & & & & & & & & \\
\uparrow & & & & \uparrow & & & & & & & & \\
X_{m0} & \longrightarrow & \cdots & \longrightarrow & X_{mp} & \longrightarrow & Z_{m0} & \longrightarrow & \cdots & \longrightarrow & Z_{mq} \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
\vdots & & & & \vdots & & \vdots & & & & \vdots \\
\uparrow & & & & \uparrow & & \uparrow & & & & \uparrow \\
X_{00} & \longrightarrow & \cdots & \longrightarrow & X_{0p} & \longrightarrow & Z_{00} & \longrightarrow & \cdots & \longrightarrow & Z_{0q}
\end{array}
$$

where the X's and Z's lie in $\mathcal{T}$, while the Y's lie in $\mathcal{S}$. The diagram is to be understood as having maps and coherent differentials as always. But since there are two categories involved, are the morphisms to be taken in $\mathcal{S}$ or in $\mathcal{T}$?

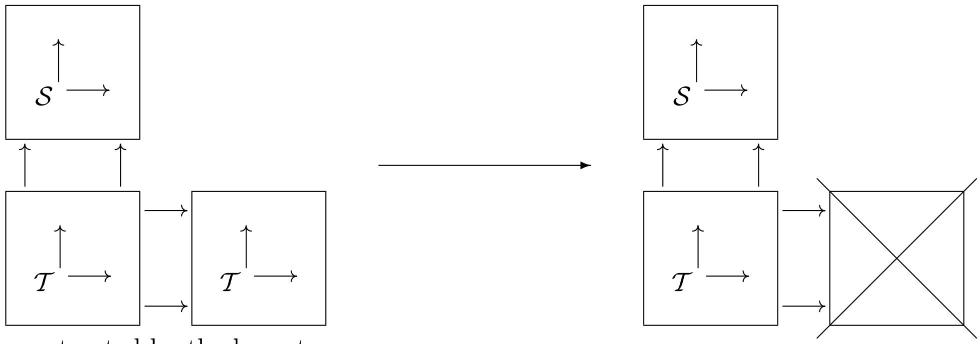
The convention we adopt is that a morphism between two objects in $\mathcal{S}$ must be a morphism in $\mathcal{S}$, while if one of the objects is in $\mathcal{T}$, the morphism lies in $\mathcal{T}$. The commutativity of the diagram, and coherence of the differentials, is to be understood in $\mathcal{S}$ is all the objects lie in $\mathcal{S}$, and in $\mathcal{T}$ otherwise. A morphism in $\mathcal{S}$ gives rise to a morphism in $\mathcal{T}$ via the natural functor $\mathcal{S} \longrightarrow \mathcal{T}$. Next we prove

Lemma B.2. *The map*

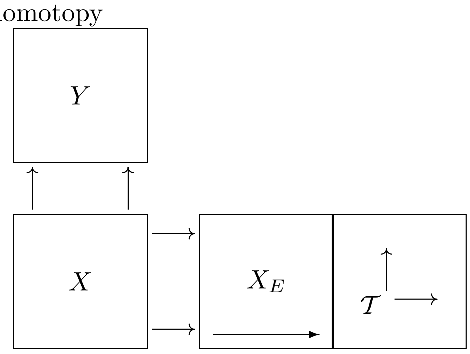


induces a homotopy equivalence.

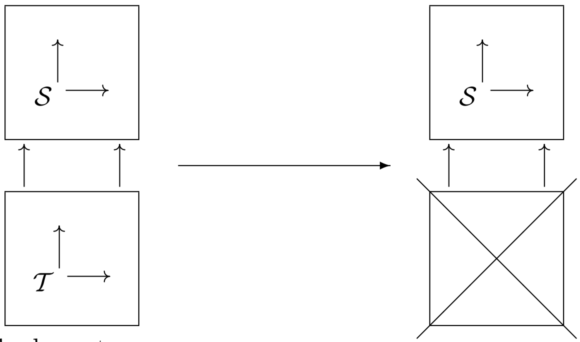
Proof. The Segal fiber of the map



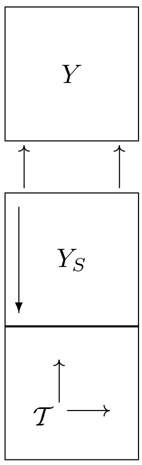
is contracted by the homotopy



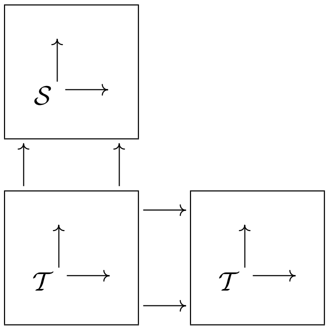
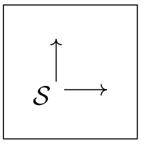
while the Segal fiber of



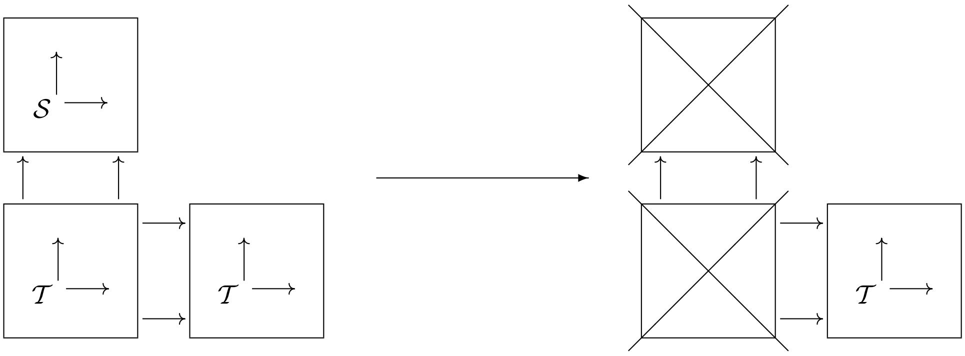
is contracted by the homotopy



□

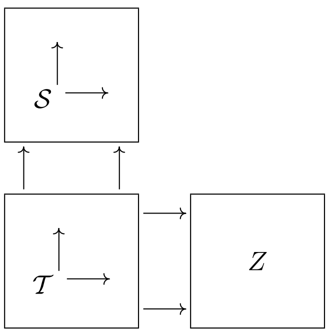
Lemma B.2 establishes that  is just another model for . Now we will prove

Lemma B.3. *The projection*

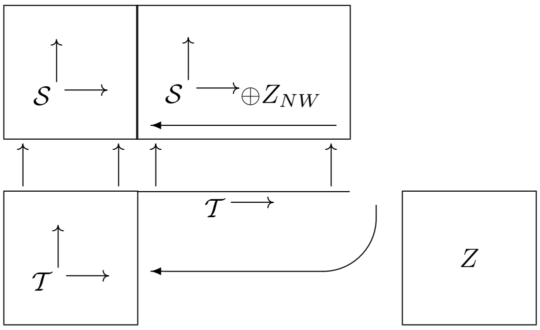


is a quasi–fibration.

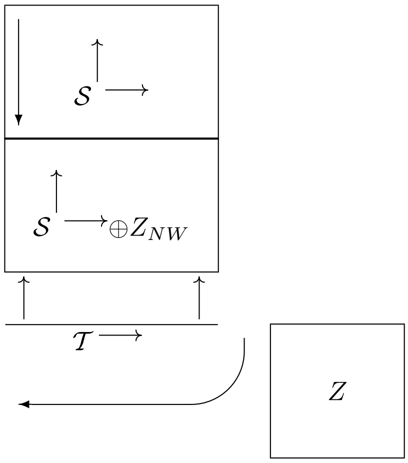
Proof. This is an example of Prototype Quasifibration II.1.2. We briefly remind the reader. The Segal fiber is the simplicial set



On it there are homotopies



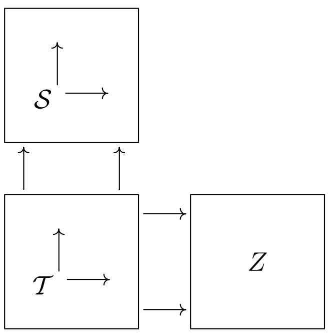
and



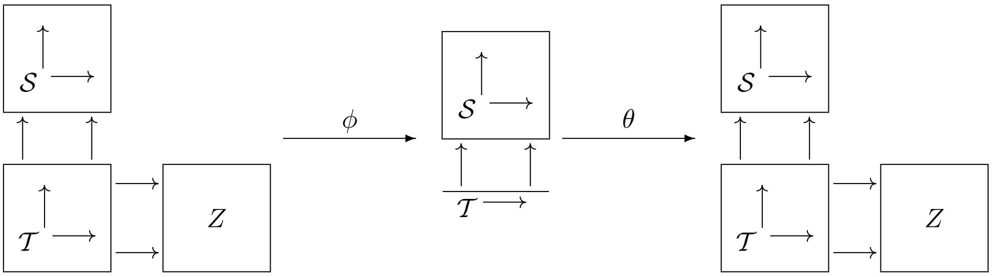
and the reader will naturally object that the object Z_{NW} lies in $\mathcal{T}$, and we have no business adding it to objects of the category $\mathcal{S}$. We need to explain how to fix this problem.

The point is that the functor $\mathcal{S} \longrightarrow \mathcal{T}$ is surjective on objects. This follows from Verdier's construction of the quotient $\mathcal{T} = \mathcal{S}/\mathcal{R}$. We may choose an object in $\mathcal{S}$, whose image under $\mathcal{S} \longrightarrow \mathcal{T}$ is isomorphic to Z_{NW}. Choose and fix such an object, and denote it also by Z_{NW}. Then we can form, in $\mathcal{S}$, direct sums with Z_{NW}, and since all the maps in the $\mathcal{S}$ square invloving Z_{NW} are either 0 or identities, the homotopies above are well–defined.

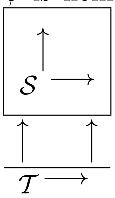
This was the only subtle point. The homotopies above show that the identity on the Segal fiber



factors, up to homotopy, as

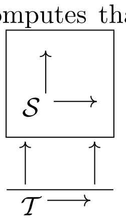


That is, $\theta\phi$ is homotopic to the identity. But $\phi\theta$ is translation in the connected

H–space  with respect to the 0–cell

$$\begin{array}{c} 0 \\ \uparrow \\ Z_{NW} \end{array}$$

and hence $\phi\theta$ is also homotopic to the identity. It follows that ϕ and θ are homotopy equivalences. One easily computes that for any face map ∂, the composite $\phi\partial\theta$ is

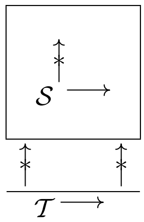
translation in the H–space  with respect to the 0–cell

$$\begin{array}{c} 0 \\ \uparrow \\ Z_{ij} \end{array}$$

and hence all the face maps ∂ induce homotopy equivalences. □

In the light of Lemmas B.2 and B.3, Conjecture B.1 becomes equivalent to the assertion that the natural inclusion

induces a homotopy equivalence. If we try to copy Quillen's proof of the analogous statement for abelian categories, we find ourselves with two conjectural lemmas to prove. To state them, we need to introduce the intermediate simplicial set



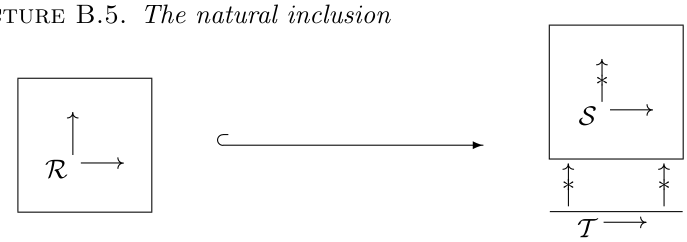
.

The restriction on the vertical maps, indicated by the star in the middle of the arrow, is that they be isomorphisms in $\mathcal{T}$. With this definition, Quillen's proof in the case of abelian categories translates to two conjectures about triangulated categories

Conjecture B.4. *The natural inclusion*

induces a homotopy equivalence.

Conjecture B.5. *The natural inclusion*



induces a homotopy equivalence.

Remark B.6. I did once try very hard to prove these conjectures; I had some ideas, involving descent arguments. But I never managed to produce a proof.

One can also try to adapt Waldhausen's proof of his localisation theorem. But this turns out to be far less promising. Waldhausen's proof is based on a clever application of the additivity theorem. In triangulated K–theory, the additivity theorem probably fails.

Appendix C. Wild Conjecture.

The fact that one should be able to recover the K–theory of an exact category from its derived category is, as was explained in the introduction of *K–theory for triangulated categories I*, a generalisation of Quillen's resolution theorem. We should remind the reader. The resolution theorem asserts the following.

Theorem C.1. *Let $F : \mathcal{E} \longrightarrow \mathcal{F}$ be a fully faithful, exact inclusion of exact categories. Suppose further that every object $y \in \mathcal{F}$ admits a resolution*

$$0 \longrightarrow x_n \longrightarrow x_{n-1} \longrightarrow \cdots \longrightarrow x_1 \longrightarrow x_0 \longrightarrow y \longrightarrow 0,$$

with all the x_i's in $\mathcal{E}$. Then the natural map

$$Q(\mathcal{E}) \longrightarrow Q(\mathcal{F})$$

is a homotopy equivalence.

The hypothesis in Quillen's resolution theorem, namely that every object of $\mathcal{F}$ have a resolution by objects of $\mathcal{E}$, is one of the standard conditions under which the induced map of derived categories

$$D^b(\mathcal{E}) \longrightarrow D^b(\mathcal{F})$$

is an equivalence. The theorem therefore strongly suggests that Quillen's K–theory should depend not on the exact category $\mathcal{E}$, but only on its derived category $D^b(\mathcal{E})$. The theorems one can prove in this direction have been the content of this series.

It is natural to wonder about generalisations of Quillen's devissage theorem. We remind the reader of the statement of the theorem.

Theorem C.2. *Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a fully faithful, exact inclusion of abelian categories. Suppose further that every object $y \in \mathcal{B}$ admits a filtration*

$$0 = x_n \subset x_{n-1} \subset \cdots \subset x_1 \subset x_0 = y,$$

with all the intermediate quotients x_i/x_{i+1} in $\mathcal{A}$. Then the natural map

$$Q(\mathcal{A}) \longrightarrow Q(\mathcal{B})$$

is a homotopy equivalence.

The devissage theorem has always been very puzzling. There is no known generalisation to exact (as opposed to abelian) categories. And since the statement is so similar to the resolution theorem, one has to wonder whether the two have a common generalisation.

Let me try to propose one. In both cases, the theorem asserts that an inclusion $\mathcal{E} \subset \mathcal{F}$ is a homotopy equivalence. Let us, for simplicity, look at resolutions and filtrations of length 1. The condition in the resolution theorem is

C.3.1. *Every object $y \in \mathcal{F}$ admits an exact sequence*

$$0 \longrightarrow x \longrightarrow x' \longrightarrow y \longrightarrow 0$$

with x, x' in $\mathcal{E}$. The condition in the devissage theorem is

C.3.2. *Every object $y \in \mathcal{F}$ admits an exact sequence*

$$0 \longrightarrow x \longrightarrow y \longrightarrow x' \longrightarrow 0$$

with x, x' in $\mathcal{E}$. The point I want to make is that, in the derived category, these become indistinguishable. In other words, if the inclusion $\mathcal{E} \subset \mathcal{F}$ satisfies the hypothesis of devissage, then the natural map

$$D^b(\mathcal{E}) \longrightarrow D^b(\mathcal{F})$$

should satisfy a something analogous to the hypothesis of resolution. But we know that the resolution theorem is a consequence of the fact that the K–theory of $\mathcal{E}$ is really a functor of $D^b(\mathcal{E})$.

This leads one to expect that there should be some construction, which we will call the derived category of a triangulated category. In fact, categories ought to be infinitely differentiable. Given a category $\mathcal{T}$, it should be possible to define its derived category $D^b(\mathcal{T})$, and this category should have a K–theory isomorphic to the K–theory of $\mathcal{T}$. Devissage is presumably the statement that the K–theory of an abelian category depends only on the derived category of its derived category.

I have some suggestions for what the derived category of the derived category might be. But this is pure speculation. I have no theorems. But perhaps the best evidence that some such theorem should hold, is Theorem I.4.8 in this series. If any homological functor of abelian categories induces a map in K–theory, then presumably K–theory depends on rather less than the derived category.

REFERENCES

[1] A. A. Beilinson, J. Bernstein, and P. Deligne, *Analyse et Topologie sur les Éspaces Singuliers*, Astérisque, 100, Soc. Math. France, 1982.
[2] A. Borel and J. P. Serre, *Le théorème de RiemannRoch*, Bull. Soc. Math. France, 86 (1958), pp. 97–136.
[3] C. Giffen and A. Neeman, *K–theory for triangulated categories*, Preprint.
[4] R. Hartshorne, *Residues and Duality*, SLN, 20 (1966), pp. 1–423.
[5] V. A. Hinich and V. V. Schechtman, *Geometry of a category of complexes and algebraic K–theory*, Duke Math. J., 52 (1985), pp. 399–430.
[6] A. Neeman, *Some new axioms for triangulated categories*, J. of Algebra, 139 (1991), pp. 221–255.
[7] A. Neeman, *The Brown representability theorem and phantomless triangulated categories*, J. of Algebra, 151 (1992), pp. 118–155.
[8] D. Quillen, *Higher algebraic K–theory I*, SLN, 341 (1973), pp. 85–147.
[9] R. Thomason and T. Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, In The Grothendieck Festschrift (a collection of papers to honor Grothendieck's 60'th birthday), Birkhäuser, Volume 3, 1990, pp. 247–435.
[10] J. L. Verdier, *Catégories dérivées, état 0. SGA* $4\frac{1}{2}$, SLN, 569 (1977), pp. 262–308.
[11] F. Waldhausen, *Algebraic K–theory of generalized free products I, II*, Ann. of Math, 108 (1978), pp. 135–256.
[12] F. Waldhausen, *Algebraic K-theory of spaces*, SLN, 1126 (1985), pp. 318–419.