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EXCEPTIONAL MODULI PROBLEMS II*

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1. Introduction, Notation. In this note we attempt to formulate moduli problems connected with some arithmetic quotients of each of the four irreducible (real) rank 3 hermitian symmetric tube domains corresponding to the four $\mathbf{R}$-division-algebras $\mathbf{R}, \mathbf{C}, \mathbf{H}$, and $\mathcal{C}$: the reals,complex numbers, Hamilton quaternions, and Cayley division algebra respectively. The ideas in this note were largely inspired by the work of Professors Kunihiko Kodaira and Donald Spencer on the moduli of algebraic varieties [K], [KS].

First some notation. Let $V \cong \mathbf{R}^n$ be an $n-$dimensional real vector space and $\mathcal{P}$ be a self-adjoint homogeneous convex cone such that the tube domain $\mathcal{T} = V + i\mathcal{P}$ is a complex Hermitian symmetric domain whose group G of holomorphic automorphisms is a semi-simple real algebraic Lie group defined over $\mathbf{Q}$. Let $\Gamma \hookrightarrow G(\mathbf{Q})$ be an arithmetic subgroup of $G(\mathbf{Q})$.

In the following discussion, V will most frequently be one of the four Jordan algebras $\mathcal{J}_D$ of 3-by-3 hermitian matrices over $D = \mathbf{R}, \mathbf{C}, \mathbf{H}$, or $\mathcal{C}$, each of which carries a standard, positive involution $a \mapsto \overline{a}$, and $\mathcal{P}_D$ will be the cone of strictly positive elements of $\mathcal{J}_D$. Thus,

$$\mathcal{T}_D = \mathcal{J}_D + i\mathcal{P}_D \subset \mathbf{C}^{n_D}$$

is one of the four rank 3 irreducible hermitian symmetric domains and $n_{\mathbf{R}} = 6, n_{\mathbf{C}} = 9, n_{\mathbf{H}} = 15, n_{\mathcal{C}} = 27$.

Now if $\mathcal{T} = \mathcal{H}_n$ is the Siegel upper half-space and $\Gamma = \Gamma_n$, the Siegel modular group of degree n, then the orbits of Γ_n in $\mathcal{H}_n$ correspond one-to-one to the isomorphism classes of principally polarized Abelian varieties (p.p.A.v's) of dimension n. Moreover [Ba2], there is a Γ_n-invariant complex analytic closed subset $\mathcal{J}_n$ of $\mathcal{H}_n$ and a Zariski-open, Γ_n-invariant subset $\mathcal{J}_n^{\#}$ of it such that the space of orbits of Γ_n in $\mathcal{J}_n^{\#}$ is in one-to-one correspondence with isomorphism classes of canonically polarized Jacobian varieties of curves of genus n, hence, via Torelli's theorem, with the isomorphism classes of non-singular curves of genus n. If $n = 3$, then $\mathcal{J}_3 = \mathcal{H}_{\mathbf{R}} = \mathcal{H}_3$, and this special case is important for the considerations which follow.

Now it can be proved [Ba1] that there is a certain "nice" unicuspidal arithmetic subgroup $\Gamma_{\mathcal{C}}$ of $Hol(\mathcal{T}_{\mathcal{C}})$ for which the associated Eisenstein series have rational Fourier coefficients, thus the quasi-projective variety $\mathcal{T}_{\mathcal{C}}/\Gamma_{\mathcal{C}}$ is defined over $\mathbf{Q}$. However, I do not know of any interpretation of the space of orbits of $\Gamma_{\mathcal{C}}$ in $\mathcal{T}_{\mathcal{C}}$ as the space of moduli of some family of polarized algebraic varieties (possibly with additional structures). The main problem to be discussed here is how to determine such a family if one exists.

2. Approach. There is a quite remarkable coincidence, so far unexplained in satisfactory depth, between these four symmetric tube domains and the four Severi varieties of Zak et al. [LaZ], S_n, $n = 1, 2, 3, 4$, where $\dim(S_n) = 2^n$. These are given explicitly as follows (here $\mathbf{P}^n$ is the projective space of dimension n):

$$S_1 = \mathbf{P}^2 \hookrightarrow \mathbf{P}^5 (\text{Veronese imbedding});$$

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$S_2 = \mathbf{P}^2 \times \mathbf{P}^2 \hookrightarrow \mathbf{P}^8$ (Segre imbedding);
$S_3 = G(2,6) \hookrightarrow \mathbf{P}^{14}$(Plucker imbedding); and
$S_4 = \mathcal{C}\mathbf{P}^2 \hookrightarrow \mathbf{P}^{26}$,

where $G(2,6)$ is the Grassmannian of planes in 6−space, $\mathcal{C}$ stands for the Cayley numbers, and $\mathcal{C}\mathbf{P}^2$ is the Cayley projective plane, realized as the projective variety of the primitive idempotents in the exceptional 27−dimensional Jordan algebra $\mathcal{J}$ of three-by-three hermitian matrices over $\mathcal{C}$, whose projective space is $\mathbf{P}^{26}$, viewed as a 27−dimensional irreducible module of E_6.

Now observe that the *generic* quadric hypersurface section $Q \cap S_1$ of S_1 as imbedded in $\mathbf{P}^5$, where Q is a quadric in $\mathbf{P}^5$, is a non-singular, non-hyperelliptic plane quartic curve C of genus 3 in $\mathbf{P}^2$. Then, as mentioned above, the moduli of such curves are essentially given, via Torelli's theorem, by the orbits of Γ_3 in $\mathcal{H}_3$. By "essentially" we mean that this correspondence holds on the complement of a divisor on the Satake compactification $(\Gamma_3\backslash\mathcal{H}_3)^*$ of the moduli space $\Gamma_3\backslash\mathcal{H}_3$ of normally polarized abelian varieties of dimension 3.

This leads us to consider the four irreducible symmetric Hermitian domains of $\mathbf{R}$−rank 3, of which $\mathcal{H}_3$ is that of lowest dimension, and of which $\mathcal{T}_{\mathcal{C}}$ is that of the highest dimension, 27. The four domains referred to are:

$D_1 = \mathcal{H}_3 = Sp(3,\mathbf{R})/K_1$, where $K_1 = U(3)$, the group of three-by-three complex unitary matrices.

$D_2 = \boldsymbol{H}_3$, the 9-dimensional tube domain in $\mathbf{C}^9 \cong \mathbf{M}_3(\mathbf{C})$, the three-by-three complex matrices $Z = X + iY$, where X and Y are complex Hermitian three-by-three matrices, and $\boldsymbol{H}_3$ consists of those for which Y is positive definite, and can be written as $U(3,3)/K_2$, where $K_2 = U(3) \times U(3)$.

$D_3 = \boldsymbol{Q}_3$ is the tube domain in $\mathbf{C}^{15}$ analogous to $\boldsymbol{H}_3$ with quaternion Hermitian 3×3 matrices in place of complex Hermitian matrices, and $\boldsymbol{Q}_3$ can be written as $SO^*(12)/K_3$, with $K_3 = U(6)$. (The notation $SO^*(2n)$ is from Helgason: "Differential Geometry and Symmetric Spaces", First Ed., p. 354.)

$D_4 = \mathcal{T}_{\mathcal{C}}$ is the tube domain $\{Z = X+iY | X, Y \in 27-$ dimensional real exceptional Jordan algebra $\mathcal{J}$ of 3×3 Hermitian matrices over $\mathcal{C}$ such that Y is positive definite (written $Y > 0$)$\}$. We may write $D_4 = E_{7(-25)}/K_4$, where $K_4 = E_{6(-78)} \times C'$, C' being the unit circle.

We note in each of the above cases that *there is a natural action of K_n on the ambient manifold of S_n*, if we identify that ambient manifold with the projective space of the complexification of the appropriate Jordan algebra of Hermitian matrices over $\mathbf{R}$,$\mathbf{C}$, $\mathbf{H}$, or $\mathcal{C}$, thus:

$\mathbf{P}^5 = \mathbf{P}(\boldsymbol{\Sigma}_{\mathbf{3C}})$, $\boldsymbol{\Sigma}_3$ being the 3×3 symmetric real matrices, where $A \in U(3)$ operates by

$$A : M(\in \Sigma_3) \mapsto^t AMA;$$

$\mathbf{P}^8 = \mathbf{P}(\mathbf{H}_{3\mathbf{C}})$, $\mathbf{H}_3$ being the complex hermitian 3×3 matrices:$\mathbf{H}_{3\mathbf{C}} = \{Z = X + iY | X, Y : 3 \times 3 \text{ hermitian}\} \cong M_3(\mathbf{C})$, and if $Z \in M_3(\mathbf{C})$ and $A, B \in U(3)$, then $(A,B) : Z \mapsto AZB$, where in the identification between $\mathbf{H}_{3\mathbf{C}}$ and $M_3(\mathbf{C})$, one must be careful about the "real" and "imaginary" parts of $Z \in M_3(\mathbf{C}) \cong \mathbf{H}_{3\mathbf{C}}$.

$\mathbf{P}^{14} = \mathbf{P}(\mathbf{Q}_{3\mathbf{C}})$, $\mathbf{Q}_3$ being the quaternion Hermitian 3×3 matrices, $dim_{\mathbf{R}}\boldsymbol{Q}_3 = 15$. (For the explicit action of $U(6)$, see Helgason [He],p.350.)

$\mathbf{P}^{26} = \mathbf{P}(\mathcal{J}_{3\mathbf{C}})$, $\mathcal{J}_3$ being the 3×3 Cayley Hermitian matrices, viewed as the minimal dimensional (> 0) irreducible module of $E_{6\mathbf{C}}$.

3. First step. Certain 9-dimensional domains. Motivated by these considerations, we try the simplest first step. Namely, we examine the configurations of algebraic varieties which arise when, for a generic quadric hypersurface $Q \subset \mathbf{P}^8$, we construct the non-singular 3-fold

$$F = F_Q = Q \cap (\mathbf{P}^2 \times \mathbf{P}^2) \subset \mathbf{P}^8.$$

Here Q is a generic quadric hypersurface in $\mathbf{P}^8$. Let H be a generic hyperplane in $\mathbf{P}^8$. There are (at least) two different ways of seeing that $F^H = F \cap H$ is a K-3 surface. On the one hand, one sees by using the adjunction formula that the three-fold F is a Fano 3-fold because its hyperplane section is the negative of its canonical divisor, and from this, again by the adjunction formula, one sees that the canonical sheaf of F^H is trivial, while F^H itself is simply connected by Lefschetz' theorem. Moreover, the generic hyperplane section of F^H is easily seen to be a canonical curve of genus 7 in P^6; thus, F^H is a K-3 surface of genus 7 and one can calculate that its degree is $2.7 - 2 = 12$.

On the other hand, F^H is generically a double cover of $\mathbf{P}^2$ in two independent ways, via the projections pr_1 and pr_2 of it onto the two factors of $\mathbf{P}^2 \times \mathbf{P}^2$, and the branch locus of each projection is a sextic curve in $\mathbf{P}^2$. The number of moduli $m(F^H)$ is 18, as is not hard to show because the family of all K-3 surfaces has 20 moduli, and since the two projections are independent, one has generically $pic(F^H) = 2$. We now explain more in detail the realization of F as a double cover of P^2 with a sextic branch curve. Explicitly, F is fibered into conics over $\mathbf{P}^2$ as follows:

$$\pi = \pi_Q : Q \cap (\mathbf{P}^2 \times \mathbf{P}^2) \longrightarrow \mathbf{P}^2,$$

where π_Q is the restriction of pr_2 to $F = F_Q$. For $s \in \mathbf{P}^2$, $\pi^{-1}(s) = Q \cap (\mathbf{P}^2 \times \{s\})$ which is a plane conic in the coordinates of the first factor by virtue of the nature of the Segre imbedding $\mathbf{P}^2 \times \mathbf{P}^2 \longrightarrow \mathbf{P}^8$. Let Δ_π be the locus of $s \in \mathbf{P}^2$ such that $\pi^{-1}(s) = F_s$ is the union of two lines. Let $r = [r_0 : r_1 : r_2]$ resp. $s = [s_0 : s_1 : s_2]$ be the coordinates of the first resp. second factor $\mathbf{P}^2$, and $t_{ij}, i, j = 0, 1, 2$, be the coordinates in $\mathbf{P}^8$. Suppose the quadric hypersurface Q in $\mathbf{P}^8$ is given by $A(t) = 0$, where A is the quadratic form

$$A(t) = \sum a_{ijkl} t_{ij} t_{kl}$$

so that with $t_{ij} = r_i s_j$ we have

$$F_s : \sum b_{ik} r_i r_k = 0,$$

with $b_{ik} = \sum_{j,l} a_{ijkl} s_j s_l$. The conic degenerates to two lines if and only if the discriminant $|b_{ik}| = det(b_{ik}) = 0$, and $|b_{ik}|$ is a homogeneous cubic polynomial in $\{b_{ik}\}$, hence (for fixed a_{ijkl}) is a homogeneous sextic polynomial in s_0, s_1, s_2. Therefore, Δ_π is a sextic plane curve in $\mathbf{P}^2$. It then follows from known formulae [Is, sec.14.5] that the third Betti number $b_3(F)$ is equal to 18. "In general", Δ_π is non-singular (for generic choice of a_{ijkl}). Therefore, by [Be, Théorème 2.1 (with $n = 1$)] the level 3 Hodge structure of F is of the form

$$H^3(F) = H^{2,1}(F) \oplus H^{1,2}(F),$$

so that the intermediate Jacobian $J(F) = H^{2,1}(F)/H^3(F, \mathbf{Z})$ is a normally polarized Abelian variety of dimension $(1/2)b_3(F) = 9$. In fact, the family of varieties:

$\{F_Q | Q$ a quadric hypersurface in $\mathbf{P}^8\}$

does not seem as well suited to the investigation of moduli problems as the family $\{F^H = F_Q \cap H | Q$ a quadric hypersurface and H, a hyperplane in $\mathbf{P}^8\}$. For example, there is no apparent natural relationship between F_Q and any particular one of its hyperplane sections F^H , whereas the latter seem more naturally related to the moduli problems we wish to study, as we shall see.

So let $Q' = Q \cap H$ be a generic irreducible quadric variety of codimension 2 in $\mathbf{P}^8$. As before, $S_2 = \mathbf{P}^2 \times \mathbf{P}^2 \hookrightarrow \mathbf{P}^8$ is a Severi variety and we let

$$F' = F_{Q'} = Q' \cap S_2 \subset \mathbf{P}^8$$

and define $\pi' = \pi_{Q'} : F' \longrightarrow \mathbf{P}^2$, where π' is the restriction of pr_2 to F'. This exhibits F' as a double covering of $\mathbf{P}^2$, with a branch curve $C : \Delta_{\pi'} = 0$ which is the zero-locus of the discriminant of the quadratic equation whose roots are the coordinates of the two points of F' over a given point s of $\mathbf{P}^2$. Direct calculation shows that $\Delta_{\pi'}$ is homogeneous of degree two in both the coefficients of Q and in the coefficients of H, after factoring out a quadratic polynomial depending on the homogeneous coordinate system, which is nowhere vanishing on a given Zariski-open subset of $\mathbf{P}^2$. Hence, $\Delta_{\pi'}$ is homogeneous of degree 6 in the homogeneous coordinates of s, and therefore the branch curve C is a sextic curve. Generically C is non-singular and the desingularization of F' is a K-3-surface.

We know the following [V:§§2.7,2.8; Be:§6.23; Di] the following:

Proposition 3.1. *A sufficiently general homogeneous sextic polynomial*

$$\Sigma(s) = \Sigma(s_0, s_1, s_2)$$

can be expressed as a symmetric determinant

$$det((b_{ik})(s)), b_{ik}(s) = b_{ki}(s),$$

where $b_{ik}(s)$ are quadratic forms in $s = (s_0, s_1, s_2)$. Moreover, given a sufficiently general sextic $\Sigma(s)$, one may reconstruct uniquely the Fano 3-fold as a fibering by conics with the curve $\Delta_\pi : \Sigma(s) = 0$ as the base locus of its singular fibers. Further, one may reconstruct the (possibly singular) K-3 surface F' as a double cover of $\mathbf{P}^2$ having the given sextic branch curve $\Delta_{\pi'}$ in $\mathbf{P}^2$.

Now it should be emphasized that the sextic curves Δ_π and $\Delta_{\pi'}$ are in general distinct. More generally, we should replace F' by its minimal desingularization, which will then also be a K-3 surface. Thus, one obtains a family of K-3 surfaces $F^H = F'$, generically as double covers of $\mathbf{P}^2$, and realized as minimal desingularizations of the intersections $Q' \cap S_2$. The generic member F' of this family has $m(F') = 18$ moduli, as we have already said. In a written communication [T], sent to me following the Yaroslavl' conference on algebraic geometry in 1992, S. Tregub explained how it should be possible to single out a subfamily of such F^H having 9 moduli. This is important because we should like, if possible, to link this set-up with the second symmetric hermitian space $\boldsymbol{H}_3$ and Hermitian modular functions, for the reasons suggested earlier. The 9-dimensional family described to me by him is the family of K-3 surfaces in our family which may be described as follows: Let $\mathcal{F}$ be the 18-dimensional family of K-3 surfaces F^H described above, let $\mathcal{E}$ be the family of all K-3 surfaces having a fixed-point-free involution, which are therefore the 2-fold unramified covers of Enriques surfaces E. Put $\mathcal{F}_\mathcal{E} = \mathcal{F} \cap \mathcal{E}$. Each of the families $\mathcal{F}$, $\mathcal{E}$, and $\mathcal{F}_\mathcal{E}$

can be characterized by a property of its Hodge structure. Then using this, Tregub suggests how one can show that $\mathcal{F}_{\mathcal{E}}$ has a component $\mathcal{F}_{\mathcal{E},0} = M$, say, such that $dimM = 9$ and that the Enriques surfaces E corresponding to K-3 surfaces from the family M are exactly those which contain a smooth rational curve (cf. [N]). In fact, the family M is generically the family of Reye congruences described by Cossec [Co]. Thus, if F^H is the double cover of an Enriques surface E containing a non-singular rational curve, then $m(F^H) = m(E) = 9$. Such an E is a 2-fold branched covering of a 4-nodal (Cayley) cubic hypersurface $\mathcal{C}$ in $\mathbf{P}^3$ and the branch locus of $\pi : E \longrightarrow \mathcal{C}$ is $C \cup sing(\mathcal{C})$, where, as I. Dolgachev has informed me, C is one of a determined finite number of smooth curves of genus 4. (To be more precise, according to him, there is, in another way, a natural map of degree 24 from the moduli space of nodal Enriques surfaces to M_4, the moduli space of smooth curves of genus 4. He has outlined a proof of this in a longer written communication [D].)

Tregub's first proposed proof of the existence of the component $\mathcal{F}_{\mathcal{E},0} = M$ was based on Hodge theory and the cohomology group $L = H^2(S, \mathbf{Z})$. He improved his first ideas which lacked a demonstration of the ampleness of certain divisors. But it turns out, as Brendan Hassett explained to me using F. Cossec's paper "Reye Congruences"[Co], that there is a more geometric, direct way of seeing this based on Cossec's description of the K-3 double cover S of a generic nodal Enriques surface as a double cover of P^2. In this description, the branch curve C of S as a double cover of P^2 has the form $C = E_1 \cup E_2$, where each E_j is a plane cubic such that E_1 and E_2 intersect transversally and C has a totally tangent conic K such that $K \cap E_1 \cap E_2 = \emptyset$. Using this description it is easy to see, by counting constants, that the number of moduli of this family is in fact 9 (*vide infra*).

4. A Certain Construction. One starts from a quartic Cayley symmetroid $X = H(W)$ defined as a quartic surface in P^3 by an equation $det(\lambda_{ij}) = 0, \lambda_{ij} = \lambda_{ji}$, each λ_{ij} being a linear form in four variables. The following facts may be verified from F. Cossec's paper "Reye Congruences". Generically $X = H(W)$ has 10 ordinary double points or nodes of the type given in local affine coordinates by $x^2 + y^2 + z^2 = 0$. Denote these nodes by $P_1, \ldots, P_{10}$ and let $\tilde{S}(W) = \tilde{X}$ be the minimal desingularization of X described by Cossec, with $\pi : \tilde{X} \longrightarrow X$ the canonical projection. Then $E_i = \pi^{-1}(P_i)$ is a nodal rational curve with $E_i^2 = -2, i = 1, \ldots 10$. Let $L = \pi^* \mathcal{O}_X(+1)$, the invertible sheaf pulled back from the sheaf associated to the hyperplane section on X. Then $L^2 = 4$ since X is a quartic surface. The following discussion follows the written communication [T] from S. Tregub. Let $L_j = L - E_j$, which is the proper transform of L under the quadratic transform π_j blowing up the node at $P_j, j = 1, \ldots, 10$. Thus L_j is an irreducible curve on the K-3 surface $\tilde{X}$. Now $L.E_j = 0$ because $E_j = \pi^* P_j$ and $P_j.(hyperplane) = 0$, and $E_i.E_j = 0$ for $i \neq j$ because E_i and E_j are disjoint. Hence we have the relations $L_i^2 = 2, L_i.L_j = 4$ for all $i \neq j$. Therefore the linear system $|L_j| = |L - E_j|$ defines a finite morphism (it has no fixed components, hence no base points; cf.[S-D], Theorem 3.1) $\varphi_j : S \overset{2:1}{\to} \mathbf{P}^2$, which corresponds to the projection of X from its singular point P_j onto $\mathbf{P}^2$. Then it is geometrically clear that $\varphi_i \times \varphi_j : S \longrightarrow \mathbf{P}^2 \times \mathbf{P}^2$ is a finite, birational morphism, and the image, contained in P^8 by way of the Veronese imbedding of $\mathbf{P}^2 \times \mathbf{P}^2$ is of the form

$$(\mathbf{P}^2 \times \mathbf{P}^2) \cap \mathbf{Q} \cap H \subset \mathbf{P}^8$$

for a hyperplane H and quadric hypersurface $\mathbf{Q}$ in $\mathbf{P}^8$. To get H, note that by the Riemann-Roch formula for K-3 surfaces, we have $dim|L_i + L_j| = 7$, while

$dim|\mathcal{O}_{\mathbf{P}^2\times\mathbf{P}^2}(1,1)| = 8$, which implies that the image of S is contained in a unique hyperplane (of a form of bi-degree $(1,1)$ on $\mathbf{P}^2\times\mathbf{P}^2$). To obtain a quadric hypersurface $\mathbf{Q}$ such that the above inclusion holds, one proceeds as follows: It is well known and easy to prove via the Künneth formula that $h^0(\mathbf{P}^2\times\mathbf{P}^2, \mathcal{O}(2)) = 36$. On the other hand, $h^0(\tilde{X}, \mathcal{O}(2L_1+2L_2)) = 26$ because $dim|2L_1+2L_2| = 24+1 = 25$, which is the projective dimension. Hence the family of quadrics containing (the image of) $\tilde{X}$ has (projective) dimension greater than 9. But the family of degenerate quadrics, splitting into two hyperplanes, has dimension 8. Therefore, there exists an irreducible quadric containing $\tilde{X}$. Thus, generically, the non-singular K-3 surface $\tilde{X}(W)$ can be realized in the form

$$(\mathbf{P}^2\times\mathbf{P}^2)\cap\mathbf{Q}\cap H\subset\mathbf{P}^8.$$

Now the family of branch curves C of the form $C = E_1\cup E_2$, having a totally tangent conic K, and where E_1 and E_2 are transversal smooth cubic plane curves, and such that $E_1\cap E_2\cap K = \emptyset$, has 9 moduli as we now prove. Since the choice of conic is arbitrary, we can fix it once for all, call it K, and consider the family of all plane cubics to which K is totally tangent. By elementary elimination, one can see that the family of such cubics for the fixed K is six-dimensional. Since one may apply any element of the orthogonal group of K to the plane coordinates, and hence to the two cubics, the number of moduli for the pairs of cubics totally tangent to K is $6+6-3 = 9$ (3 being the dimension of the complex orthogonal group of K).

5. Comparison of two 9-dimensional domains. The 9-dimensional moduli space of such nodal Enriques surfaces is a Zariski-open set on an arithmetic quotient of a symmetric tube domain of type IV [N]. Also, a generic smooth curve C of genus 4 is a space sextic realized as $\mathbf{Q}\cap\mathcal{C}$, where the quadric hypersurface $\mathbf{Q}$ is determined up to an element of the finite group of projective automorphisms of $\mathcal{C}$, and the nodal Enriques surface can be recovered from the genus 4 curve $C\subset\mathcal{C}$.

We want to connect naturally the Zariski-open subset of an arithmetic quotient of a domain of type IV and dimension 9 with (a Zariski-open subset of) an arithmetic quotient of the irreducible type I domain $\boldsymbol{H}_3$ of dimension 9. If Γ_3 is the usual Hermitian modular group, then $\Gamma_3\backslash\boldsymbol{H}_3$ parametrizes normally polarized Abelian varieties (A,i) of dimension 6 with complex multiplication by i of type $(i,i,i,-i,-i,-i)$ in the tangent space $T_e(A)$ to A at e. Thus we have the relations or finite correspondences:

$$F^H \longrightarrow E \mapsto C\subset\mathcal{C}$$

and we want some finite set of (A,i)'s corresponding to the curve C of genus 4. This brings us to the subject of Janus-type varieties. (Janus: Ancient Roman god of gates and doorways, depicted with two faces looking in opposite directions.)

There are examples of pairs of quite different symmetric domains, say D_1 resp. D_2 on which arithmetic groups Γ_1 resp. Γ_2 operate, such that if $(D/\Gamma)^{\#}$ is a suitable smooth toroidal compactification of D/Γ, then there are normal crossings divisors Δ_1 and Δ_2 on the respective smooth compactifications $(D/\Gamma_1)^{\#}$ and $(D/\Gamma_2)^{\#}$ such that the latter two toroidal compactifications are both isomorphic to a compact projective variety V, and such that

$$(1)\qquad V-\Delta_1 = D_1/\Gamma_1, V-\Delta_2 = D_2/\Gamma_2.$$

Such examples are given by B. Hunt in [H] and by Hunt and S. Weintraub in [H-W]. Moreover, there is a natural geometric interpretation of both terms in (1) and the geometric relation is *not random.*

Supposing the genus $g(C) = 4$, let C^{**} be a 4-fold unramified cyclic covering of C, and C^{*} be the intermediate covering of C of degree 2. There are only finitely many C^{**} for the given C, corresponding to the points of order 4 on $J(C)$. We have $g(C^{**}) = 13$ and $g(C^{*}) = 7$. The Jacobian $J(C^{*})$ is naturally a subvariety of $J(C^{**})$, and B. van Geemen [vG1, vG2] has shown that

$$A(C^{**}) = J(C^{**})/J(C^{*})$$

is a normally polarized Abelian variety of dimension 6 and that the period-four automorphism of C^{**} over C induces complex multiplication of $A(C^{**})$ of type $(i, i, i, -i, -i, -i)$, thus determining an orbit of Γ_3 in $\boldsymbol{H}_3$. At this point, the first question was whether the correspondence $C^{**} \longrightarrow A(C^{**})$ is generically finite. Now, O. Debarre [De] has given a positive answer to this question, and we have

$$F^H \longrightarrow E \overset{[24:1]}{\longrightarrow} C \overset{[1:128]}{\mapsto} C^{**} \overset{\pi}{\longrightarrow} A(C^{**}).$$

One obvious question is to describe the boundary points of the finiteness domain/range of π, as well as the fibers of positive dimension.

But from the point of view of our original motivation, we should like to know what kind of analogous phenomena might take place in the cases $\boldsymbol{Q}_3$ and $\mathcal{T}_e$?? Might we reasonably, for example, take a generic section of the respective Severi variety in its ambient space $\mathbf{P}^N$ by a smooth quadric variety of suitable co-dimension in $\mathbf{P}^N$. If so, then can we find some objects contained in the complex Cayley projective plane whose moduli are connected with an arithmetic quotient of $\mathcal{T}_e$?

6. The 15-dimensional domain. Now we want to offer some discussion of what we think might be true in case of the tube domain $\boldsymbol{Q}_3$ of dimension 15.

Let us then consider what the first two cases had in common and how this might be extended to the case of the domain $\boldsymbol{Q}_3$. In the first case one has the moduli of non-hyperelliptic curves C_3 of genus 3, realized as the intersections of quadric hypersurfaces with the Veronese surface, the first Severi variety, in $\mathbf{P}^5$. One has $dim(C_3) = 1$, the canonical class on C_3 is very ample when C_3 is not hyperelliptic, and each C_3 has its Jacobian variety $J(C_3)$ of dimension 3. In the second case, one considers the family $\mathcal{F}$ of intersections of the second Severi variety $\mathbf{P}^2 \times \mathbf{P}^2 \subset \mathbf{P}^8$ with quadrics Q' of codimension 2, and these intersections are K-3 surfaces, of which we consider the subfamily, having 9 moduli, of those which are double covers of nodal Enriques surfaces. The canonical class of a K-3 surface or of an Enriques surface is trivial. We have seen how there is generically a finite correspondence between these and the isomorphism classes of 6-dimensional p.p.A.v.'s with complex multiplication by $\sqrt{-1}$.

Therefore, in the third case let us consider the third Severi variety $G = G(2, 6)$, also denoted $G(1, 5)$ with projective dimensions, which is contained in $\mathbf{P}^{14}$ and consider its intersections with quadrics Q" of codimension 4 in $\mathbf{P}^{14}$, $Q" = Q \cap H \cap H' \cap H"$, where Q is a (generic) quadric hypersurface and $H, H', H"$ are hyperplanes in $\mathbf{P}^{14}$. Generically $G \cap Q" = F$ is a Fano 4-fold and its <u>anti</u>-canonical class is very ample and equivalent to a hyperplane section. According to calculations of Brendan Hassett we have $m(F) \leq 63 = 15 + 4.12$: Hassett obtained the number 63 by computing the dimension of the Hilbert scheme of these Fano fourfolds (as subvarieties of the Grassmanian) and subtracting the dimension of the automorphism group of the Grassmannian; implicitly he uses the fact that each small deformation of the Fano fourfolds actually lies in the Grassmannian (cf. [Bo, Se, Ku2]). Moreover, according to Oliver Kuechle [Ku1,2,3], its <u>non-zero</u> Hodge numbers are determined by

$h(1,1) = h(0,0) = 1, h(1,3) = 15$, and $h(2,2) = 106$. Therefore we would like to see if there might be a 15-dimensional family of such Fano 4-folds F' such that for each F' in the family there is a finite set of naturally corresponding p.p.A.v.'s of dimension 12 admitting complex multiplication by the Hurwitz quaternions. In this case also one may conjecture the need to consider a subfamily of $F's$ having certain automorphisms, just as the K-3 double cover of a (nodal) Enriques surface has a fixed-point-free involution. In general there are technical problems with constructing a moduli space for arbitrary Fano varieties which, in general, may have many automorphisms. At this point, our main efforts, with significant cooperation from Brendan Hassett, are concentrated on trying to construct some more or less reasonable moduli space for the particular Fano 4-folds $F' = G \cap Q$" which we have just been describing. Since $H^2(F', T_{F'}) = 0$ by application of a vanishing theorem of Kodaira-Akizuki-Nakano, where $T_{F'}$ is the holomorphic tangent bundle to F', a local moduli space should exist. One thing that still has to be checked is whether the candidate local moduli space is non-singular, on which Hassett has some ideas related to the analogous case of the Fano variety of lines on a cubic 4-fold. Moreover,the problem would be simpler if we could also prove that $H^0(F', T_{F'}) = 0$, for in this case, the criteria of Kodaira and Spencer [K,Theorem 6.4], [KS] for the existence of moduli for F' are satisfied. Then one can try to construct a global moduli space and see if some part of the conjectured moduli space may be identified with an arithmetic quotient of some symmetric domain. As for the question of whether $H^0(F', T_{F'}) = 0$, there are results of Carrell and Friedman [CF] which imply that for any non-zero holomorphic tangent field X on F' the variety V of zeros of X must be non-empty (since $H^{1,0}(X) = 0$ by [Ku1]) and of complex dimension ≥ 2. While this does not prove that a non-zero holomorphic vector field X cannot exist on F', it implies some restrictions on the existence of such. One might hope from further such restrictions to prove that X must in fact be 0, and thus $H^0(F', T_{F'}) = 0$.

In any event, we do consider it an interesting problem to see what the nature might be of some moduli- or parameter space for a suitable subfamily of these Fano 4-folds, similar to the situation described earlier of a subfamily of the K-3 surfaces

$$F^H = (\mathbf{P}^2 \times \mathbf{P}^2) \cap \mathbf{Q} \cap H \subset \mathbf{P}^8,$$

where we considered the subfamily of those which are double covers of nodal Enriques surfaces. We hope someone can see something interesting in these problems and suggestions.

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