

LINES ON NON-DEGENERATE SURFACES*

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Abstract. On an affine variety X defined by homogeneous polynomials, every line in the tangent cone of X is a subvariety of X . However there are many other germs of analytic varieties which are not of cone type but contain “lines” passing through the origin. In this paper, we give a method to determine the existence and the “number” of such lines on non-degenerate surface singularities.

1. Introduction. Let (X, O) be a germ of analytic varieties embedded in (\mathbb{C}^n, O) with a singularity at O . By abuse of language, we say that L is a *line* in (X, O) if (L, O) is a smooth curve germ in (X, O) and $L \setminus \{O\}$ is contained in the regular part of X .

In [3, 5], lines on hypersurfaces with simple singularities are classified by using the classification machinery. All the hypersurfaces of dimension 2 and 3 with simple or simple elliptic singularities passing through x -axis are equivalent to (under the coordinate transformation preserving the x -axis) some surfaces defined by explicit equations. It turns out that the A, D, E singularities split in this classification. This says that different smooth curves on the same surface might have different properties.

Let $\pi : \tilde{X} \rightarrow (X, O)$ be a resolution of a surface (X, O) with an isolated singularity at the origin O and let $\{E_1, \dots, E_r\}$ be the exceptional divisors of π . For an exceptional divisor E_i , let \mathcal{L}_{E_i} denote the set of lines on (X, O) whose strict transform intersect E_i transversally. It is known that \mathcal{L}_{E_i} is non-empty if and only if there exist a function germ h in the maximal ideal \mathfrak{m} such that the multiplicity of π^*h along E_i is one and conversely any line in X is contained in some \mathcal{L}_{E_i} ([1, 2]). We call E_i a *normally smooth divisor* if $\mathcal{L}_{E_i} \neq \emptyset$. Geometrically this implies that $d\pi(v) \neq 0$ for any tangent vector $v \in T_P\tilde{X}$ as long as $P \in E_i \setminus \bigcup_{j \neq i} E_j$ and v is not tangent to E_i . If E_i is normally smooth, any germ of a curve intersecting $E_i \setminus \bigcup_{j \neq i} E_j$ transversely defines a line in X . Any two lines in the same \mathcal{L}_{E_i} can be connected by an analytic family of lines in (X, O) .

For a given resolution $\pi : \tilde{X} \rightarrow X$, we consider the integer $\rho(\pi) := \#\{E_i; \mathcal{L}_{E_i} \neq \emptyset\}$. This number depends on the resolution. Put $\rho(X, O)$ to be the minimal value of $\rho(\pi)$. Obviously $\rho(\pi) = \rho(X, O)$ if $\pi : \tilde{X} \rightarrow X$ is a minimal resolution. We call $\rho(\pi)$ the *line index of the resolution* $\pi : \tilde{X} \rightarrow X$ and we call $\rho(X, O)$ the *line index of* (X, O) .

In this paper, we study $\rho(\pi)$ where π is a toric resolution of a non-degenerate surface singularity. Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be a surface defined by $f(z_1, z_2, z_3) = 0$ with isolated singularity at the origin. We assume that f is non-degenerate in the sense of the Newton boundary ([7]). Let Σ^* be a regular simplicial cone subdivision of the dual Newton diagram $\Gamma^*(f)$ and let $\pi : X_{\Sigma^*} \rightarrow (X, 0)$ be the associated toric resolution. We denote $\rho(\pi)$ by $\rho(\Sigma^*)$ for simplicity. To each vertex $P = {}^t(p_1, p_2, p_3)$ of Σ^* , there corresponds an exceptional divisor $E(P)$ of π , which may have several components. The multiplicity of π^*z_i along $E(P)$ is equal to p_i ([9]). Thus by the result of Gonzalez-Sprinberg and Lejeune-Jalabert ([1]), $E(P)$ is normally smooth if

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and only if $\min(p_1, p_2, p_3) = 1$. We observe that $\rho(\Sigma^*)$ is independent of the choice of Σ^* under certain conditions (see Proposition 6). This allows us to use the canonical toric resolution to determine $\rho(\Sigma^*)$. Note that a toric resolution is not necessarily minimal. So, in general, $\rho(\Sigma^*)$ may be bigger than $\rho(X, O)$ (see Example 28). However to have the equality $\rho(\Sigma^*) = \rho(X, O)$, it is enough that $\pi : X_{\Sigma^*} \rightarrow X$ is line-equivalent to the minimal resolution (see § 2 for the definition). The purpose of this paper is to give a method to compute $\rho(\Sigma^*)$.

2. Line-admissible blowing-ups. Let (X, O) be a germ of a surface with an isolated singularity at O . Suppose that we have a good resolution $\pi_1 : X_1 \rightarrow X$ and let E_1, \dots, E_r be the exceptional divisors of π_1 . Take a divisor E_{i_0} and a point Q on E_{i_0} and let $\pi_Q : \tilde{X}_1 \rightarrow X_1$ be the blowing-up at Q and let E_Q be the exceptional divisor of π_Q . The following statements are obvious.

PROPOSITION 1. *Take a function $h \in \mathfrak{m}$ and let m_i be the multiplicity of π_1^*h along E_i . Then the multiplicity m_Q of the pull-back $\pi_Q^*(\pi_1^*h)$ along E_Q is the sum of m_i for all i such that $Q \in E_i$. In particular, $m_Q \geq 1$, and $m_Q = 1$ if and only if $m_{i_0} = 1$ and $Q \in E_{i_0} \setminus \bigcup_{i \neq i_0} E_i$.*

COROLLARY 2. *Under the situation of Proposition 1, E_Q is a normally smooth divisor of the composition $\pi_1 \circ \pi_Q : \tilde{X}_1 \rightarrow X$ if and only if E_{i_0} is a normally smooth divisor of $\pi_1 : X_1 \rightarrow X$ and Q is contained in $E_{i_0} \setminus \bigcup_{j \neq i_0} E_j$.*

We call $\pi_Q : \tilde{X}_1 \rightarrow X_1$ a *line-admissible* blowing-up if either the center Q is at the intersection of two exceptional divisor or the supporting divisor is not normally smooth. Suppose that we have another good resolution $\pi_2 : X_2 \rightarrow X$. We say that $\pi_2 : X_2 \rightarrow X$ is *line-equivalent* to $\pi_1 : X_1 \rightarrow X$ if there exist a finite chain of resolutions $\pi'_i : Y_i \rightarrow X, i = 1, \dots, s$ such that (1) $Y_1 = X_1$ and $\pi'_1 = \pi_1$ and $Y_s = X_2$ and $\pi'_s = \pi_2$ and (2) any consecutive resolutions factor by either $\sigma_i : Y_i \rightarrow Y_{i+1}$ or $\sigma'_i : Y_{i+1} \rightarrow Y_i$, where σ_i and σ'_i are line-admissible blowing-ups.

An immediate consequence of the definition and Corollary 2 is:

COROLLARY 3. *Assume that $\pi_i : X_i \rightarrow X, i = 1, 2$ are line-equivalent. Then $\rho(\pi_1) = \rho(\pi_2)$.*

3. Toric resolution and the computation of $\rho(\Sigma^*)$.

3.1. Non-degenerate surfaces. We begin with recalling the toric resolutions of surface singularities since this also helps us to fix some notations. We use the notations of [9]. Let (X, O) be the germ of a surface in (\mathbb{C}^3, O) defined by a function $f : (\mathbb{C}^3, O) \rightarrow (\mathbb{C}, O)$. Hereafter we always assume that X has an isolated singularity at O . Let $\sum_{\nu} a_{\nu} z^{\nu}$ be the Taylor expansion of f . The *Newton polyhedron* $\Gamma_+(f)$ is by definition the convex hull of $\bigcup_{\{\nu; a_{\nu} \neq 0\}} \{\nu + \mathbb{R}^3\}$. The *Newton boundary* $\Gamma(f)$ is by definition the union of the compact faces of $\Gamma_+(f)$.

Let $N := \text{Hom}_{\mathbb{Z}}(\mathbb{Z}^3, \mathbb{Z})$ be the set of covectors. We identify N with \mathbb{Z}^3 and we denote the elements of N by column vectors. Let N_+ be the set of covectors $P = {}^t(p_1, p_2, p_3) \in N$ with $p_i \geq 0, i = 1, 2, 3$. Put $E_1 := {}^t(1, 0, 0), E_2 := {}^t(0, 1, 0)$ and $E_3 := {}^t(0, 0, 1)$. P is called *strictly positive covector* if $p_j > 0$ for all j . We denote the minimal value of the linear function P on $\Gamma_+(f)$ by $d(P; f)$. Put $\Delta(P; f) = \{z \in \Gamma_+(f) \mid P(z) = d(P; f)\}$. The *face function* of f with respect to P is by definition $f_P(z) = f_{\Delta(P; f)} := \sum_{\nu \in \Delta(P; f)} a_{\nu} z^{\nu}$. Two covectors $P, P' \in N_+$ are equivalent if

and only if $\Delta(P; f) = \Delta(P'; f)$. The *dual Newton diagram* $\Gamma^*(f)$ of X is a conical polyhedral subdivision of N_+ given by the above equivalent classes.

A surface X is called *non-degenerate* (with respect to the local coordinate z) if for any strictly positive covector $P \in N_+$, $X^*(P) := \{z \in \mathbb{C}^{*3} \mid f_P(z) = 0\}$ is a reduced non-singular surface in the complex torus \mathbb{C}^{*3} . The notion of non-degeneracy can be extended to complete intersection varieties (cf. [6, 9]).

3.2. Canonical subdivisions. We assume that X is defined by $f(z_1, z_2, z_3) = 0$ and f is non-degenerate. Let $\Gamma^*(f)_2^+$ be the union of the two-dimensional cones $\text{Cone}(P, Q)$ of $\Gamma^*(f)$ such that the interior points are strictly positive. Let Σ^* be a regular simplicial subdivision of the dual Newton diagram $\Gamma^*(f)$ and let $\pi : X_{\Sigma^*} \rightarrow X$ be the associated toric modification. Let $\mathcal{V}(\Sigma^*)$ be the set of strictly positive vertices P 's of Σ^* such that $\dim \Delta(P; f) \geq 1$. The exceptional divisors correspond bijectively to $\mathcal{V}(\Sigma^*)$ and for each $P \in \mathcal{V}(\Sigma^*)$ we denote the corresponding divisor by $E(P)$. Note that $E(P)$ need not to be irreducible but it is a disjoint union of rational spheres if $\dim \Delta(P; f) = 1$. The number of connected components is given by $r(P) + 1$, where $r(P)$ is the number of integral points on the interior of $\Delta(P; f)$ ([9, III§6]). The structure of this resolution $\pi : X_{\Sigma^*} \rightarrow X$ depends only on the restriction of Σ^* to $\Gamma^*(f)_2^+$. This follows from the following observation:

PROPOSITION 4. *Assume that Σ_1^* is a regular subdivision of Σ^* such that $\mathcal{V}(\Sigma_1^*) = \mathcal{V}(\Sigma^*)$. Then the canonical morphism $\psi : X_{\Sigma_1^*} \rightarrow X_{\Sigma^*}$, which is induced by the morphism of the ambient toric varieties, is an isomorphism.*

For any two dimensional cone $\sigma = \text{Cone}(P, Q) \in \Gamma^*(f)$, there exists a canonical regular subdivision of σ which is described as follows. Denote by $d := \det(P, Q)$ the greatest common divisor of the absolute values of the 2×2 minors of the matrix (P, Q) . If $d > 1$, there exists a unique integer $d_1, 1 \leq d_1 < d$ such that $Q_1 := (P + d_1 Q)/d$ is an integral covector. If $d_1 > 1$, repeat the process for $\text{Cone}(P, Q_1)$, until a regular subdivision of $\text{Cone}(P, Q)$ is obtained. Let Q_1, \dots, Q_k be the covectors obtained in this way. Let $d/d_1 = [m_1, \dots, m_\ell]$ be the continuous fraction expansion. Then $\ell = k$ and the self-intersection number of each component of $E(Q_i)$ is $-m_i$ (cf. [9, III]). Note that $\Delta(Q_i; f) = \Delta(P; f) \cap \Delta(Q; f)$. This implies $r(Q_i)$ is independent of $i = 1, \dots, k$ and we denote this number by $r(P, Q)$. Recall that the continuous fraction is defined inductively by $[m_1] = m_1$ and $[m_1, m_2, \dots, m_k] = m_1 - 1/[m_2, \dots, m_k]$.

A regular simplicial cone subdivision of $\Gamma^*(f)$ is called a *canonical regular subdivision* if its restriction to each cone σ in $\Gamma^*(f)_2^+$ is canonical in the above sense, and we denote it by Σ_{can}^* . The associated toric resolution is called the *canonical toric resolution* of X .

Let $Q = {}^t(q_1, q_2, q_3)$ and $P = {}^t(p_1, p_2, p_3)$. Put $Q_0 = Q$ and $Q_{k+1} = P$ and let $Q_j := {}^t(q_{1,j}, q_{2,j}, q_{3,j}), j = 0, \dots, k+1$. The canonical subdivision enjoys the following property:

LEMMA 5. *Assume that $\text{Cone}(P, Q) \in \Gamma^*(f)_2^+$. Fix an $\ell = 1, 2, 3$.*

- 1) *If $q_\ell \leq 1$, then $\{q_{\ell,j}\}_{j=0}^{k+1}$ is monotone increasing in j i.e. $q_{\ell,j+1} \geq q_{\ell,j}$ for $0 \leq j \leq k$.*
- 2) *If $q_\ell \geq 2$, then either $\{q_{\ell,j}\}$ is monotone increasing or monotone decreasing in j or there exists a j_0 ($1 \leq j_0 \leq k$) such that $q_{\ell,j_0} \geq 1$ and*

$$p_\ell = q_{\ell,k+1} \geq \dots \geq q_{\ell,j_0+1} \geq q_{\ell,j_0} \leq q_{\ell,j_0-1} \leq \dots \leq q_{\ell,0} = q_\ell.$$

Proof. We prove the assertion 2). If the assertion does not hold, there exists an index $j, 1 \leq j \leq k$ such that $q_{\ell,j-1} \leq q_{\ell,j} > q_{\ell,j+1}$. This implies that the self

intersection number of each component of $E(Q_j)$ is $-(q_{\ell,j-1} + q_{\ell,j+1})/q_{\ell,j} > -2$, which is a contradiction (cf. [9, II(2.3) and III(6.3)]). The assertion 1) follows from 2) as $Q_j, j = 1, \dots, k$ are strictly positive. \square

Let Σ^* be any regular simplicial cone subdivision of $\Gamma^*(f)$ and let $\pi : \tilde{X} \rightarrow X$ be the corresponding toric modification. We denote the line index of π by $\rho(\Sigma^*)$. Take a two dimensional cone $\sigma = \text{Cone}(P, Q) \in \Gamma^*(f)_2^+$. Let $Q_0 := Q, Q_1, \dots, Q_k, Q_{k+1} := P$ be the canonical subdivision of σ and let $S_0 := Q, S_1, \dots, S_\eta, S_{\eta+1} := P$ be the vertices of Σ^* on this cone. By [9, II(2.3)], $\{Q_0, \dots, Q_{k+1}\} \subset \{S_0, \dots, S_{\eta+1}\}$. We consider the condition:

(\sharp): Σ^* has no vertex in the interior of $\text{Cone}(Q, Q_1)$.

We say that Σ^* satisfies the (\sharp)-condition if it satisfies (\sharp)-condition for any $\text{Cone}(P, Q)$ in $\Gamma^*(f)_2^+$ such that Q is not strictly positive. The inclusion $\mathcal{V}(\Sigma_{\text{can}}^*) \subset \mathcal{V}(\Sigma^*)$ implies that the following statements.

THEOREM 6. *There exists a canonical morphism $\phi : X_{\Sigma^*} \rightarrow X_{\Sigma_{\text{can}}^*}$. Furthermore ϕ is a composition of line-admissible blowing-ups if Σ^* satisfies (\sharp)-condition. In particular, the line index $\rho(\Sigma^*)$ does not depend on the choice of a toric resolution associated with any regular simplicial subdivision satisfying (\sharp)-condition and $\rho(\Sigma^*) = \rho(\Sigma_{\text{can}}^*)$.*

Proof. Take a two dimensional cone $\sigma = \text{Cone}(P, Q) \in \Gamma^*(f)_2^+$ and assume that P is strictly positive. Let $Q_0 := Q, Q_1, \dots, Q_k, Q_{k+1} := P$ be the canonical subdivision of σ and let $S_0 := Q, S_1, \dots, S_\eta, S_{\eta+1} := P$ be the vertices of Σ^* on this cone. Write $S_i = {}^t(s_{1,j}, s_{2,j}, s_{3,j})$. Assume that $Q_{i_0} = S_\nu$ and $Q_{i_0+1} = S_\mu$ and $\mu - \nu > 1$. Take S_j with $\nu < j < \mu$ and put $\alpha_j = \det(Q_{i_0}, S_j)$ and $\beta_j = \det(S_j, Q_{i_0+1})$. Then α_j and β_j are positive integers and $S_j = \alpha_j Q_{i_0+1} + \beta_j Q_{i_0}$. This implies that $s_{1,j} > s_{1,\nu} + s_{1,\mu}$. Suppose that $s_1^{\max} = \max\{s_{1,j}; \nu < j < \mu\}$ and put $\gamma = \min\{\gamma; s_{1,\gamma} = s_1^{\max}\}$. Then by [9, II(2.3)] the intersection number of (each component of) $E(S_\gamma)$ is $-(s_{1,\gamma-1} + s_{1,\gamma+1})/s_{1,\gamma} > -2$. Then the negativity of the intersection number implies that $s_{1,\gamma-1} + s_{1,\gamma+1} = s_{1,\gamma}$. Thus each component of $E(S_\gamma)$ is a rational sphere of the first kind. This implies also that $S_\gamma = S_{\gamma-1} + S_{\gamma+1}$ and $\det(S_{\gamma-1}, S_{\gamma+1}) = 1$. Put $\mathcal{V}' = \mathcal{V}(\Sigma^*) - \{S_\gamma\}$. Then we can extend \mathcal{V}' to get a regular simplicial subdivision $\Sigma^{*'}$ such that its restriction to $\Gamma^*(f)_2^+$ is defined by the vertices \mathcal{V}' . Thus we get a toric resolution $\pi' : X_{\Sigma^{*'}} \rightarrow X$. Changing Σ^* outside of $\Gamma^*(f)_2^+$ if necessary, we may assume by Proposition 4 that Σ^* is a subdivision of $\Sigma^{*'}$. Thus we get a canonical morphism $\psi : X_{\Sigma^*} \rightarrow X_{\Sigma^{*'}}$ which factors π by π' . By the definition, ψ is the composition of blowing-up at $r(S_\gamma) + 1$ intersection points of respective components of $E(S_{\gamma-1})$ and $E(S_{\gamma+1})$ in $X_{\Sigma^{*'}}$. Note that ψ is line-admissible unless Q is not strictly positive and $S_\nu = Q_0$ and $S_\mu = Q_1$. This is the situation where ψ is the blowing up at the intersection of $E(Q_1)$ and $E(Q)$. This does not occur if Σ^* satisfies (\sharp)-condition. Now the assertion follows by the induction on the cardinality of $\mathcal{V}(\Sigma^*) \setminus \mathcal{V}(\Sigma_{\text{can}}^*)$. \square

3.3. Computation of $\rho(\Sigma_{\text{can}}^*)$. Let $\pi : X_{\Sigma^*} \rightarrow X$ be a toric resolution. We assume that Σ^* satisfies the (\sharp)-condition. We define $\mathcal{V}_{\text{ns}}(\Sigma^*) := \{P \in \mathcal{V}(\Sigma^*) \mid P \text{ has } 1 \text{ as a coordinate}\}$. We know that $E(P)$ is a normally smooth divisor if and only if $P \in \mathcal{V}_{\text{ns}}(\Sigma^*)$. Thus for each $\text{Cone}(P, Q) \in \Gamma^*(f)_2^+$, we define $\rho_{PQ} := \#\mathcal{V}_{\text{ns}}(\Sigma^*) \cap \text{Cone}(P, Q)^\circ$, where $\text{Cone}(P, Q)^\circ$ is the interior of $\text{Cone}(P, Q)$. This number is independent of Σ^* by Theorem 6. Recall that $r(P, Q)$ is the number of integral points in the interior of $\Delta(P; f) \cap \Delta(Q; f)$. By the definition we have

$$(3.1) \quad \rho(\Sigma^*) = \#\{P \in \mathcal{V}_{\text{ns}}(\Sigma^*); \dim \Delta(P; f) = 2\} + \sum_{\text{Cone}(P,Q) \in \Gamma^*(f)_2^+} (r(P, Q) + 1)\rho_{PQ}$$

Thus we need only to compute ρ_{PQ} for the calculation of $\rho(\Sigma^*)$. Take a cone $\sigma = \text{Cone}(P, Q)$ in $\Gamma^*(f)_2^+$. The following gives a practical method to compute ρ_{PQ} .

THEOREM 7. *Let $P = {}^t(p_1, p_2, p_3)$ be strictly positive and let $Q = {}^t(q_1, q_2, q_3)$ and assume that $d := \det(P, Q) > 1$. Let $Q_i = {}^t(q_{1,i}, q_{2,i}, q_{3,i}), i = 0, \dots, k + 1$ be the vertices defining the canonical subdivision from Q with $Q_0 = Q$ and $Q_{k+1} = P$. Fix an $\ell \in \{1, 2, 3\}$. Then*

1. *For each $1 \leq i \leq k$, there exists positive integers $0 < \alpha_i, \beta_i < d$ such that $Q_i = (\beta_i P + \alpha_i Q)/d$. Putting $\alpha_0 = \beta_{k+1} = d, \alpha_{k+1} = \beta_0 = 0$, they satisfy the inequality:*

$$\alpha_i > \alpha_{i+1}, \quad \beta_i < \beta_{i+1}, \quad i = 0, \dots, k$$

2. *Let $\mathcal{V}_{\text{ns}}^{(\ell)}(P, Q)$ be the set of integral covectors R expressed as $R = (\beta P + \alpha Q)/d$ where α, β are positive integers satisfying*

$$(3.2) \quad \begin{cases} \alpha q_\ell + \beta p_\ell = d, & 0 < \alpha, \beta < d \\ \alpha q_k + \beta p_k \equiv 0 \pmod{d} & (k \neq \ell) \end{cases}$$

and let $\mathcal{V}_{\text{ns}}^{(\ell)}(P, Q; \Sigma_{\text{can}}^)$ be the set of covectors $Q_i, 1 \leq i \leq k$ such that $q_{\ell,i} = 1$. Then $\mathcal{V}_{\text{ns}}^{(\ell)}(P, Q) = \mathcal{V}_{\text{ns}}^{(\ell)}(P, Q; \Sigma_{\text{can}}^*)$. Note that the inequality $\alpha, \beta < d$ follows automatically from the positivity if both p_ℓ and q_ℓ are positive.*

Proof. The first assertion follows by an inductive argument. Write $Q_i = (\beta_i P + \alpha_i Q)/d$ with positive rational numbers α_i, β_i . As $\det(P, Q_i) = \alpha_i$ and $\det(Q_i, Q) = \beta_i$, α_i, β_i are positive integers. By the definition of Q_1 , we can write $Q_1 = (P + \alpha_1 Q)/d$ for some $0 < \alpha_1 < d$. The assertion for Q_1 holds and $\det(P, Q_1) = \alpha_1$. Assume that $Q_j = (\beta_j P + \alpha_j Q)/d$ with $0 < \alpha_j < d$. As $\det(P, Q_j) = \alpha_j$ and $\{Q_j, \dots, Q_{k+1}\}$ is the vertices of the canonical subdivision of $\text{Cone}(P, Q_j)$, there exists $\alpha', 0 < \alpha' < \alpha_i$, such that

$$Q_{j+1} = \frac{1}{\alpha_j} P + \frac{\alpha'}{\alpha_j} Q_j = \frac{1}{\alpha_j} P + \frac{\alpha'}{\alpha_j} \frac{(\beta_j P + \alpha_j Q)}{d} = \left(\frac{1}{\alpha_j} + \frac{\alpha' \beta_j}{\alpha_j d}\right) P + \frac{\alpha'}{d} Q$$

Thus $\alpha_{j+1} = \alpha' < \alpha_j < d$. The inequality $\beta_{j+1} > \beta_j$ can be proved similarly by using the fact that $\{P, Q_k, \dots, Q_1, Q\}$ is the vertices of the canonical subdivision of the cone $\text{Cone}(P, Q)$ from P (cf. [9, II(2.3)]). Now we show the second assertion. The inclusion $\mathcal{V}_{\text{ns}}^{(\ell)}(P, Q; \Sigma_{\text{can}}^*) \subset \mathcal{V}_{\text{ns}}^{(\ell)}(P, Q)$ is obvious. Suppose that $R = (\beta P + \alpha Q)/d \in \mathcal{V}_{\text{ns}}^{(\ell)}(P, Q)$ is not contained in $\mathcal{V}_{\text{ns}}^{(\ell)}(P, Q; \Sigma_{\text{can}}^*)$. Suppose that $R \in \text{Cone}(Q_i, Q_{i+1})^\circ$. Then we can write $R = mQ_i + nQ_{i+1}$ for some positive integers m, n . If $i \geq 1$, this gives a contradiction by comparing the ℓ -th coefficient: $1 = mq_{\ell,i} + nq_{\ell,i+1} \geq m + n$. Suppose that $i = 0$. Write $Q_1 = (P + \alpha_1 Q)/d$ as above. Then $R = mQ + (P + \alpha_1 Q)n/d = nP/d + (md + n\alpha_1)Q/d$. Thus we get $\alpha = md + n\alpha_1 \geq d$ which contradicts to the assumption. \square

REMARK 8. The computation of $\mathcal{V}_{\text{ns}}(P, Q)$ is most difficult for the case $p_\ell, q_\ell > 1$. Assume that $p_\ell, q_\ell > 0$. If we have a solution (α_0, β_0) , the other solutions are reduce

to the following equation. Put $\alpha = \alpha_0 + \alpha', \beta = \beta_0 + \beta'$. Then

$$(3.3) \quad \begin{cases} \alpha' q_\ell + \beta' p_\ell = 0 \\ \alpha' q_k + \beta' p_k \equiv 0 \pmod{d} \quad (k \neq \ell) \end{cases}$$

Let $\Delta := \Delta(P; f) \cap \Delta(Q; f)$. Let $T = {}^t(t_1, t_2, t_3)$ be a covector in $\mathcal{V}_{\text{ns}}^{(\ell)}(P, Q)$ (thus $t_\ell = 1$). Geometrically this implies that $\Delta(T; f) = \Delta$. In particular, $\Gamma_+(f) \subset \{(\nu_1, \nu_2, \nu_3); t_1\nu_1 + t_2\nu_2 + t_3\nu_3 \geq d(T; f)\}$. This gives a practical way to find such a T .

The case $q_\ell = 0$ or 1 , the computation is much easier. See Corollary 11.

The canonical subdivision of $\text{Cone}(P, Q)$ takes sometimes a lot of computations (see Example 9). Theorem 7 gives us a criterion on the existence or non-existence of normally smooth divisors, without computing the whole subdivision $Q_i, i = 1, \dots, k$.

EXAMPLE 9. For simplicity, we write $x = z_1, y = z_2, z = z_3$. Let us consider $f(x, y, z) = x^m + y^n + x^r y^r + z^2$. We assume that $m, n > 2r$. Put $n = n_1 r + n_0, m = m_1 r + m_0$ with $0 \leq m_0, n_0 \leq r - 1$. Then $\Gamma(f)$ has two compact faces whose covectors are $P = {}^t(2(n-r), 2r, nr)/\delta_1$ and $Q = {}^t(2r, 2(m-r), mr)/\delta_2$ where $\delta_1 = \gcd(2(n-r), 2r, nr)$ and $\delta_2 = \gcd(2r, 2(m-r), mr)$ and the corresponding dual Newton diagram is as in Figure 3.1. Note that $d := \det(P, Q)$ is given by $d = 2(mn - mr - nr)/(\delta_1 \delta_2)$. We consider $\mathcal{V}_{\text{ns}}^{(1)}(P, Q)$. First we consider the covector $T_0 = {}^t(1, 1, r)$, which is a weight vector of $x^r y^r + z^2$. As $m, n > 2r$, T_0 must be on $\text{Cone}(P, Q)$. To proceed the further computation, let us assume that n, m, r are odd and $\gcd(m, r) = \gcd(n, r) = 1$. This implies $\delta_1 = \delta_2 = 1$. By Theorem 7, we have

$$\begin{cases} 2\beta(n-r) + 2\alpha r = d \\ 2\beta r + 2\alpha(m-r) \equiv 0 \pmod{d} \\ \beta nr + \alpha mr \equiv 0 \pmod{d} \end{cases}$$

First we have a canonical solution $(\alpha_0, \beta_0) = (n-2r, m-2r)$ which corresponds to the covector $T_0 = {}^t(1, 1, r)$. Thus putting $\alpha = \alpha_0 + a$ and $\beta = \beta_0 + b$, we can reduce the equation as

$$\begin{cases} 2b(n-r) + 2ar = 0 \\ 2br + 2a(m-r) \equiv 0 \pmod{d} \\ bnr + amr \equiv 0 \pmod{d} \end{cases}$$

Taking the positivity of α, β into account, we have the solution

$$\{(\alpha, \beta)\} = \left\{ ((n-2r) + 2j(n-r), (m-2r) - 2jr); 0 \leq j \leq \left\lfloor \frac{m_1 - 2}{2} \right\rfloor \right\}$$

For example, consider the easiest case $m = n$. This has a unique solution $(\alpha, \beta) = (n-2r, n-2r)$ and $\mathcal{V}_{\text{ns}}^{(1)}(P, Q) = \{B\}$ where $B = {}^t(1, 1, r)$. By symmetry, we have $\mathcal{V}_{\text{ns}}^{(2)} = \{B\}$. Note $r(P, Q) = 1$. By writing down the equation described by Theorem 7, we can show $\mathcal{V}_{\text{ns}}^{(3)}(P, Q) = \emptyset$.

Now we look at $\text{Cone}(P, E_1)$ and $\text{Cone}(P, E_3)$. Note that $\det(P, E_1) = r$ and $\det(P, E_3) = 2$. It is easy to see that there are no normally smooth divisor on these cones. Observe that the computation of canonical subdivision of $\text{Cone}(P, Q)$ is not so

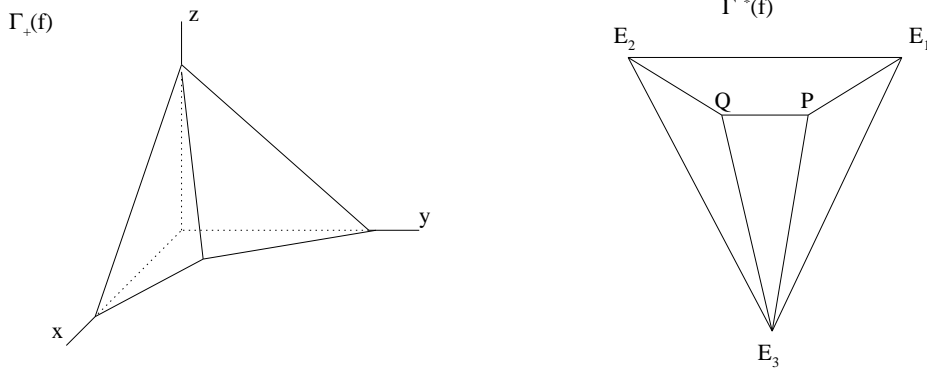


FIG. 3.1. The Newton polyhedron and the dual Newton diagram

easy. For example, if $r = 15, n = 37$, then $B = {}^t(1, 1, 15)$ and first covector B_1 (from Q) is given by $(P + 223Q)/518 = {}^t(13, 19, 240)$ and $518/223 = [3, 2, 2, 12, 2, 2, 3]$ and it takes some computation to complete the subdivision.

The following lemma describes the covectors corresponding to the non-compact faces.

LEMMA 10. *Assume that $X = \{f(z_1, z_2, z_3) = 0\}$ and assume that f is non-degenerate and $\Gamma(f)$ has at least one compact two dimensional face for simplicity. Suppose that $z_2 = z_3 = 0$ is a line in X . (So f is not convenient.) Then there is a unique covector $Q = {}^t(q_1, q_2, q_3) \in \text{Vertex}(\Gamma^*(f))$ such that $q_1 = 0$. Furthermore Q takes the form ${}^t(0, 1, q_3)$ or ${}^t(0, q_2, 1)$.*

There exists a unique covector $P = {}^t(p_1, p_2, p_3)$ which corresponds to a compact divisor and adjacent to Q in $\Gamma^(f)_2^+$. Then we have $\det(P, Q) = p_1$.*

Proof. As X has an isolated singularity, f must contain a monomial of type $z_1^a z_2$ or $z_1^a z_3$. Suppose that $B := (a, 1, 0) \in \Gamma(f)$. Let $C = (b, 0, c)$ be the vertex of $\Gamma(f) \cap \{z_2 = 0\}$ adjacent to B by an edge. It is clear that the non-compact face Ξ which has \overline{BC} as a face and is unbounded to the direction of the z_1 -axis has covector $Q = {}^t(0, c, 1)$. One can see that there exists no other non-compact face which is unbounded to the z_1 -axis direction and bounded to z_2, z_3 -direction. Let Δ be the compact face which has \overline{BC} as a boundary and let $P = {}^t(p_1, p_2, p_3)$ be the corresponding covector. As $\Delta(P; f)$ contains B, C , we need to have $p_1 a + p_2 = b p_1 + c p_3$. Now the last assertion follows from $\det(P, Q) = \gcd(p_1, p_2 - c p_3) = \gcd(p_1, p_1(b - a)) = p_1$. \square

The following corollary describes explicitly $\mathcal{V}_{\text{ns}}^{(1)}(P, Q)$ in the case $q_1 = 0$ or 1.

COROLLARY 11. *With the assumptions of Theorem 7, we have the following.*

- 1) *Assume $q_1 = 0$. Then $\mathcal{V}_{\text{ns}}^{(1)}(P, Q) \neq \emptyset$ if and only if $d := \det(P, Q) > 1$ and $d = p_1$. In this cases, $\mathcal{V}_{\text{ns}}^{(1)}(P, Q) = \{Q_1\}$. If $Q \neq E_2, E_3$, then $\{y = z = 0\} \subset X$ and $d = \det(P, Q) = p_1$.*
- 2) *Assume $q_1 = 1$. Then $\mathcal{V}_{\text{ns}}^{(1)}(P, Q) \neq \emptyset$ if and only if $d > p_1$. In this case, we have $Q_i = (iP + (d - ip_1)Q)/d$ for $i = 1, \dots, [d/p_1]$ and $\mathcal{V}_{\text{ns}}^{(1)}(P, Q) = \{Q_i; i = 1, \dots, [d/p_1]\}$.*

Proof. Assume that $Q' = (\beta P + \alpha Q)/d \in \mathcal{V}_{\text{ns}}^{(1)}(P, Q)$ with $0 < \alpha, \beta < d$.

1) If $q_1 = 0$, we have $\gcd(q_2, q_3) = 1$. As $d = \gcd(p_1q_2, p_1q_3, p_2q_3 - p_3q_2) = \gcd(p_1, p_2q_3 - p_3q_2)$, d divides p_1 . Thus $Q' \in \mathcal{V}_{\text{ns}}^{(1)}(P, Q)$ if and only if $d = p_1$ and $\beta = 1$. In this case, $Q' = Q_1$ and $\mathcal{V}_{\text{ns}}^{(1)}(P, Q) = \{Q_1\}$. Assume that $Q \neq E_2, E_3$. By the definition of $\Gamma^*(f)_2^\perp$, $\Delta(Q; f)$ is a non-compact face with dimension 2. In particular, $\{y = z = 0\} \subset X$. By Lemma 10, we have $d = p_1$.

2) Suppose that $q_1 = 1$. Then $\beta p_1 + \alpha = d$. This implies $d > p_1$. Put $d = rp_1 + d'$ with $0 \leq d' < p_1$ and $r = [d/p_1]$. Then by the above equality, we have $(\alpha, \beta) = (d - jp_1, j)$, $j = 1, \dots, [d/p_1]$. Put $Q'_j := (jP + (d - jp_1)Q)/d$. By the definition, d divides the minors of (P, Q) which are $p_1q_2 - p_2, p_1q_3 - p_3, p_2q_3 - p_3q_2$. Thus $\beta p_j + \alpha q_j = \beta p_j + (d - \beta p_1)q_j \equiv \beta(p_j - p_1q_j) \equiv 0 \pmod{d}$ for $j = 2, 3$. Thus Q'_j is an integral covector for $\beta = 1, \dots, r$. It is clear that $Q'_1 = Q_1$. Assume that $Q'_r = Q_\iota$ for some ι . By the monotonicity of the coefficients (Lemma 5), we have $Q_j \in \mathcal{V}_{\text{ns}}^{(1)}(P, Q)$ for $j \leq \iota$. Thus $\iota = r$ and $Q'_j = Q_j$ for $j \leq r$. \square

REMARK 12. In the case of non-convenient surface with $q_1 = 0$, the divisor $E(Q_1)$ corresponds to the deformations of the line $z_2 = z_3 = 0$. In fact, $E(Q)$ is a non-compact divisor which is the strict transform of z_1 -axis and $E(Q)$ intersects transversely with $E(Q_1)$.

For $R \in \mathcal{V}_{\text{ns}}^{(\ell)}$, write $R = (\beta P + \alpha Q)/d$. We call β/d the P -coefficient of R .

COROLLARY 13. *With the assumptions of Theorem 7, suppose that $q_1 > 1$. Let $\bar{Q} = (\bar{\beta}P + \bar{\alpha}Q)/d \in \mathcal{V}_{\text{ns}}^{(\ell)}$ and $\underline{Q} = (\underline{\beta}P + \underline{\alpha}Q)/d \in \mathcal{V}_{\text{ns}}^{(\ell)}$ be the covectors with maximal and minimal P -coefficients in $\mathcal{V}_{\text{ns}}^{(\ell)}$. Then*

$$(3.4) \quad \rho_{PQ}^{(\ell)} = 1 + |\det(\bar{Q}, \underline{Q})| = 1 + \frac{|\bar{\beta}\underline{\alpha} - \bar{\alpha}\underline{\beta}|}{d}$$

Proof. Denote by $d' := |\det(\bar{Q}, \underline{Q})|$. Suppose that $\underline{Q} = Q_i$ and $\bar{Q} = Q_{i+j}$. Then $\mathcal{V}_{\text{ns}}^{(\ell)} = \{Q_i, \dots, Q_{i+j}\}$ by Lemma 5 and $\rho_{PQ}^{(\ell)} = j + 1$. By the assumption, we have $Q_{i+1} = (Q_{i+j} + (d' - 1)Q_i)/d'$. As the continuous fraction $d'/(d' - 1)$ is given by $[2, \dots, 2]$ ($(d' - 1)$ copies of 2), we get $j - 1 = d' - 1$ and the assertion follows immediately. \square

4. Applications.

4.1. Weighted homogeneous surfaces. In this section we study lines on weighted homogeneous surface singularities, which are classified as follows ([12, 9]):

$$\begin{aligned} X_{\text{I}} : & \quad h_{\text{I}} = x^a + y^b + z^c = 0, \\ X_{\text{II}} : & \quad h_{\text{II}} = x^a y + y^b + z^c = 0, \\ X_{\text{III}} : & \quad h_{\text{III}} = x^a y + x y^b + z^c = 0, \\ X_{\text{IV}} : & \quad h_{\text{IV}} = x^a y + y^b z + z^c = 0, \\ X_{\text{V}} : & \quad h_{\text{V}} = x^a y + y^b z + z^c x = 0, \\ X_{\text{VI}} : & \quad h_{\text{VI}} = xy + z^c = 0, \\ X_{\text{VII}} : & \quad h_{\text{VII}} = x^a z + y^b z + z^c + t x^{c_1} y^{c_2} = 0, \quad t \neq 0 \\ X_{\text{VIII}} : & \quad h_{\text{VIII}} = x^a y + x y^b + x z^c + t y^{c_1} z^{c_2} = 0, \quad t \neq 0. \end{aligned}$$

The surface X_{I} is called a Pham-Brieskorn surface. This type of surfaces have been studied in the previous paper [4]. The surface X_{VI} is an A_{c-1} type singularity. There are exact $c - 1$ families of lines on this surface (see [1, 2, 4, 5]). On surface X_{VII} and X_{VIII} , the term $y^{c_1} z^{c_2}$ must be on the supporting plane of the previous three

monomials. Thus a, b, c are not arbitrary. The Newton boundaries of the surfaces other than $X_{\text{VI}}, X_{\text{VII}}$ and X_{VIII} are triangles. Note that for a weighted homogeneous surface, the Newton boundary has only one compact 2-dimensional face. Let $P = {}^t(p_1, p_2, p_3)$ be the corresponding covector. The formula (3.1) in §2 reduces to

$$(4.1) \quad \rho(\Sigma_{\text{can}}^*) = \varepsilon + \sum_{\text{Cone}(P, Q) \in \Gamma^*(f)_2^+} (r(P, Q) + 1) \rho_{PQ}(\Sigma_{\text{can}}^*).$$

where $\varepsilon = 1$ if $P \in \mathcal{V}_{\text{ns}}(\Sigma_{\text{can}}^*)$ and $\varepsilon = 0$ otherwise.

For each type of surfaces, one can calculate $\rho_{PQ}(\Sigma_{\text{can}}^*)$ for each $\text{Cone}(P, Q)$ in the dual Newton diagram by using the method described in the previous sections.

LEMMA 14. *Assume that $\text{Cone}(P, E_i)$ be a cone in $\Gamma^*(f)_2^+$. Then $\det(P, E_i)$ is given by $\delta_i := \gcd(p_j, p_k)$ with $\{i, j, k\} = \{1, 2, 3\}$. Assume that $\delta_i > 1$.*

- 1) $\mathcal{V}_{\text{ns}}^{(i)}(P, E_i) \neq \emptyset$ if and only if $\delta_i > p_i$ and $\rho_{PE_i}^{(i)} = \left\lfloor \frac{\delta_i}{p_i} \right\rfloor$.
- 2) $\mathcal{V}_{\text{ns}}^{(j)}(P, E_i) \neq \emptyset$ if and only if $p_j | p_k$. In this case, $\rho_{PE_i}^{(j)} = 1$.
- 3)

$$\rho_{PE_i} = \begin{cases} 0, & \text{if } \left\lfloor \frac{\delta_i}{p_i} \right\rfloor = 0 \text{ and } \delta_i < \min\{p_j, p_k\} \\ \max\{1, \left\lfloor \frac{\delta_i}{p_i} \right\rfloor\}, & \text{otherwise} \end{cases}$$

Proof. This follows from Corollary 11. \square

LEMMA 15. *Let $\text{Cone}(P, Q)$ be a cone in $\Gamma^*(f)_2^+$ with $Q = {}^t(0, c, 1)$. Suppose that $\det(P, Q) = p_1 > 1$. Then*

$$\rho_{PQ} = \begin{cases} \max\{1, \left\lfloor \frac{p_1}{p_2} \right\rfloor, \left\lfloor \frac{p_1}{p_3} \right\rfloor\}, & c = 1 \\ \rho_{PQ}^{(2)} + \max\{1, \left\lfloor \frac{p_1}{p_3} \right\rfloor\} - \varepsilon, & c > 1 \end{cases}$$

where $\varepsilon = 1$ if either $Q_1 \in \mathcal{V}_{\text{ns}}^{(2)}(P, Q)$ or $Q_{j_1} \in \mathcal{V}_{\text{ns}}^{(2)}(P, Q)$ with $j_1 := \left\lfloor \frac{p_1}{p_3} \right\rfloor \geq 1$ and $\varepsilon = 0$ otherwise.

Proof. Let Q_1, \dots, Q_k be the primitive covectors in $\text{Cone}(P, Q)$ inserted by the canonical subdivision from Q . If $c = 1$, the assertion is immediate from Corollary 11, as $q_{1,1} = 1$. We assume that $c > 1$. If $\left\lfloor \frac{p_1}{p_3} \right\rfloor = 0$, the assertion is obvious. Assume that $\left\lfloor \frac{p_1}{p_3} \right\rfloor \geq 1$. By Corollary 11, Q_j is given by $(jP + (p_1 - jp_3)Q)/p_1$ for $1 \leq j \leq j_1$. Thus $q_{2,j} = c - j(cp_3 - p_2)/p_1$. If $cp_3 - p_2 < 0$, $q_{2,j}$ is monotone increasing by Lemma 5 and we see that $\mathcal{V}_{\text{ns}}^{(2)}(P, Q) = \emptyset$ and the assertion follows immediately. Assume that $cp_3 - p_2 \geq 0$. Then $q_{2,j}$ is monotone decreasing for $0 \leq j \leq j_1$. Thus $\mathcal{V}_{\text{ns}}^{(2)}(P, Q) \cap \mathcal{V}_{\text{ns}}^{(3)}(P, Q) \neq \emptyset$ if and only if $q_{2,j_1} = 1$. If this is the case, Q_{j_1} is the unique covector in common. Thus the assertion follows from these observations. \square

4.2. Normally smooth divisors on X_{II} . By using Lemmas 14 and 15, we can compute the number $\rho(\Sigma_{\text{can}}^*)$. We show this by considering the surface X_{II} . One can do the same consideration for the other types of surfaces. Let $X_{\text{II}} : h_{\text{II}}(x, y, z) = x^a y + y^b + z^c = 0$. Put $\hat{a} := \gcd(a, b - 1)$, $e := \gcd(b, c)$ and $d := \gcd(c(b - 1), ac, ab) = e \gcd(a, c(b - 1)/e)$. The dual Newton diagram $\Gamma^*(h_{\text{II}})_2^+$ consists of three cones: $\text{Cone}(P, Q)$, $\text{Cone}(P, E_1)$ and $\text{Cone}(P, E_3)$ where $P := {}^t(c(b - 1)/d, ac/d, ab/d)$ and $Q := {}^t(0, c, 1)$.

The following three propositions are special cases of Lemmas 14 and 15.

PROPOSITION 16. Cone (P, E_1) is regular if and only if a divides $c(b-1)/e$. Assume that $a \nmid (c(b-1)/e)$. Then

- 1) $\mathcal{V}_{\text{ns}}^{(1)}(P, E_1) \neq \emptyset$ if and only if $ae > (b-1)c$. And in this case $\rho_{PE_1}^{(1)} = \left\lceil \frac{ae}{(b-1)c} \right\rceil$.
- 2) $\mathcal{V}_{\text{ns}}^{(2)}(P, E_1) \neq \emptyset$ if and only if $c|b$.
- 3) $\mathcal{V}_{\text{ns}}^{(3)}(P, E_1) \neq \emptyset$ if and only if $b|c$.
- 4) $\rho_{PE_1} = \max\{\rho_{PE_1}^{(2)}, \rho_{PE_1}^{(3)}, \left\lceil \frac{ae}{(b-1)c} \right\rceil\}$. □

PROPOSITION 17. As $\det(P, E_3) = \hat{c}a/d$, Cone (P, E_3) is regular if and only if $d = \hat{c}a$. Assume that $\hat{c}a > d$. Then

- 1) $\mathcal{V}_{\text{ns}}^{(1)}(P, E_3) \neq \emptyset$ if and only if $(b-1)|a$.
- 2) $\mathcal{V}_{\text{ns}}^{(2)}(P, E_3) \neq \emptyset$ if and only if $a|(b-1)$.
- 3) $\mathcal{V}_{\text{ns}}^{(3)}(P, E_3) \neq \emptyset$ if and only if $\hat{c}a > ab$ and $\rho_{PE_3}^{(3)} = \left\lceil \frac{\hat{c}a}{ab} \right\rceil$.
- 4) $\rho_{PE_3} = \max\{\rho_{PE_3}^{(1)}, \rho_{PE_3}^{(2)}, \left\lceil \frac{\hat{c}a}{ab} \right\rceil\}$.

Recall that $\rho_{P, E_i}^{(j)} \leq 1$ for $i = 1, 3$ and $j \neq i$ by Lemma 5.

PROPOSITION 18. Cone (P, Q) is regular if and only if $(b-1)c$ divides ae , or equivalently $(b-1)|a$ and $c|b\frac{a}{b-1}$. Assume that Cone (P, Q) is not regular. Then we have

- 1) $\mathcal{V}_{\text{ns}}^{(1)}(P, Q) = \{Q_1\}$.
- 2) $\mathcal{V}_{\text{ns}}^{(3)}(P, Q) \neq \emptyset$ if and only if $c(b-1) > ab$. And in this case $\rho_{PQ}^{(3)} = \left\lceil \frac{c(b-1)}{ab} \right\rceil$.
- 3) $\mathcal{V}_{\text{ns}}^{(2)}(P, Q) \neq \emptyset$ if and only if there exist positive integers α and β such that

$$(4.2) \quad a\beta + d\alpha = b - 1,$$

$$(4.3) \quad ab\beta + d\alpha \equiv 0 \pmod{c(b-1)}.$$

The second condition can be replaced by $a\beta + 1 \equiv 0 \pmod{c}$.

Proof. The last assertion follows from by (4.2) as $ab\beta + d\alpha = (b-1)(a\beta + 1)$. □

The non-trivial computation is required only for $\mathcal{V}_{\text{ns}}^{(2)}(P, Q)$ which we will explain more in detail. Write $b = eb_1$ and $c = ec_1$.

COROLLARY 19. I. For $\mathcal{V}_{\text{ns}}^{(2)}(P, Q) \neq \emptyset$, it is necessary that

$$(4.4) \quad \gcd(a, c) = 1, \quad b > a, c$$

In this case, we have $d = e\hat{a}$ and $\mathcal{V}_{\text{ns}}^{(2)}(P, Q)$ is the set of covectors $T = (\alpha Q + \beta P)/d$ which satisfies

$$(4.5) \quad a\beta + e\hat{a}\alpha = b - 1$$

$$(4.6) \quad 0 < \alpha, \beta$$

$$(4.7) \quad b - e\hat{a}\alpha \equiv 0 \pmod{c}$$

II. Furthermore $\mathcal{V}_{\text{ns}}^{(2)}(P, Q)$ is non-empty if $[b/c] \geq a + \hat{a}$.

Proof. From the congruence $a\beta + 1 \equiv 0 \pmod{c}$, it is clear that $\gcd(a, c) = 1$. Thus $d = e \gcd(a, c_1(b-1)) = e\hat{a}$. The equality (4.7) results from

$$a\beta + 1 = b - d\alpha = e(b_1 - \hat{a}\alpha) \equiv 0 \pmod{c}$$

Thus $b > a\beta \geq a$ and $b > c$. The last congruence equation is equivalent to $b_1 - \hat{a}\alpha \equiv 0$ modulo c_1 .

Assume that $[b/c] - a - \hat{a} \geq 0$. As $\gcd(\hat{a}, b_1) = 1$, there exists positive integer α_0 , $0 < \alpha_0 < c_1$, such that $b_1 - \hat{a}\alpha_0 \equiv 0$ modulo c_1 . Put $b_1 - \alpha_0\hat{a} = j_0c_1$. We see that $j_0 = b_1/c_1 - \alpha_0\hat{a}/c_1 > [b/c] - \hat{a}$. Take α which satisfies the congruence $a\beta + 1 \equiv 0$ modulo c . Then α takes the form $\alpha = \alpha_0 + jc_1$ with $j \in \mathbf{N}$ and thus $b_1 - \hat{a}\alpha = (j_0 - j\hat{a})c_1$. For the positivity of β , we need to have $0 \leq j < j_0/\hat{a}$. The integrity of T implies

$$e(b_1 - \hat{a}\alpha) - 1 = ec_1(j_0 - j\hat{a}) - 1 \equiv 0 \pmod{a}$$

As j can move $0 \leq j < j_0/\hat{a}$ and $j_0 > [b/c] - \hat{a} \geq a$ or $j_0/\hat{a} > a/\hat{a}$, this congruence equation has a positive solution j_1 , $0 \leq j_1 \leq j_0/\hat{a}$. Then put $\beta = (ec_1(j_0 - j_1\hat{a}) - 1)/a$ for such a solution j_1 . This gives a covector $T = (\alpha Q + \beta P) \in \mathcal{V}_{\text{ns}}^{(2)}(P, Q)$. \square

EXAMPLE 20. Consider $X_{\text{II}} : x^9y + y^b + z^8 = 0$ with $b = 22 + 36k$. Then $e = 2, \hat{a} = 3$ and the equation is

$$9\beta + 6\alpha = 21 + 36k, \quad 9\beta + 1 \equiv 0 \pmod{8}$$

In this case, $[b/c] - a - \hat{a} = (22 + 36k)/8 - 12 \geq 0$ if $k \geq 37/18$. For $k \geq 3$ (in fact, for $k \geq 2$), we have a solution $(\alpha, \beta) = (6k - 7, 7)$. In this case, $P = {}^t(28 + 48k, 12, 33 + 54k)$ and $Q = {}^t(0, 8, 1)$ and $T := (\alpha Q + \beta P)/(28 + 48k) = {}^t(7, 1, 8)$. We leave the computation of the other covectors in $\mathcal{V}_{\text{ns}}^{(2)}(P, Q)$ to the reader.

4.3. The minimality of the canonical toric resolutions. We study when the canonical toric resolution of a weighted homogeneous surface is minimal. Though the canonical toric resolution is not always minimal (see Example 28), we can expect that the minimality hold for almost all classes of non-degenerate surfaces. By [9, III(6.3)], for each weighted homogeneous surface the resolution graph associated with the canonical toric resolution is star-shaped. Hence, when the resolution graph has at least three arms, the canonical resolution is minimal.

We have the following general statement which is very helpful to see if a given toric modification is minimal.

LEMMA 21. *Let $X := f^{-1}(0)$ be a non-degenerate surface. Suppose that $P \in \Gamma^*(f)$ is the strictly positive covector corresponding to a compact face Δ of the Newton boundary $\Gamma(f)$.*

- 1) *Let $\Delta_1, \dots, \Delta_\ell$ be the boundary edges of Δ . The exceptional divisor $E(P)$ is rational if and only if*

$$-\frac{6\text{Vol}(\text{Cone } \Delta)}{d(P; f)} + \sum_{i=1}^{\ell} (r(\Delta_i) + 1) = 2$$

where Cone Δ is the cone over Δ with vertex O and $r(\Delta_i)$ is the number of integral points in the interior of Δ_i .

- 2) *The canonical toric resolution $\pi : \tilde{X} \rightarrow (X, 0)$ is not minimal if and only if there exists a compact face Δ of $\Gamma(f)$ such that $E(P)$ is rational, $E(P)^2 = -1$ and $E(P)$ intersects at most two other exceptional divisors where P is the covector corresponding to Δ .*

Proof. The first statement is a conclusion of [9, III(6.4)]. The assertion 2) follows from the Castelnuovo-Enriques criterion and [9, III §4(A) and §6]. \square

THEOREM 22. *Let X be one of the surfaces of type $X_{\text{II}}, X_{\text{III}}, X_{\text{IV}}, X_{\text{V}}, X_{\text{VI}}$ or X_{VIII} . We assume that $a, b, c > 1$ in 4.1. Then the canonical toric resolution of X is minimal. In particular, $\rho(X, 0) = \rho(\Sigma_{\text{can}}^*)$.*

Proof. We first check when the central exceptional divisor $E(P)$ is rational by using Lemma 21 (see also [9, III(6.9)]). If this is the case, we compute the number of arms from $E(P)$. If this number is less than 3, we show that $E(P)^2 \leq -2$. Recall that the number of arms in the resolution graph is the sum of $r(P, Q) + 1$ for non-regular cones $\text{Cone}(P, Q) \in \Gamma^*(f)_2^+$.

(II). Let $X = X_{\text{II}} : x^a y + y^b + z^c = 0$. Put $e = \gcd(b, c)$, $\hat{a} = \gcd(a, b - 1)$. Then $P = {}^t(c(b - 1), ac, ab)/d$ with $d = e \gcd(a, c(b - 1)/e)$. Note that $r(P, Q) + 1 = 1$, $r(P, E_1) + 1 = e$ and $r(P, E_3) + 1 = \hat{a}$. By loc. cit. $E(P)$ is rational if and only if 1) $e = \gcd(c, a/\hat{a}) = 1$ or 2) $\hat{a} = \gcd(a, c/e) = 1$. If 1) holds, then $d = \hat{a}$. We have $\det(P, Q) = c(b - 1)/\hat{a} > 1$, $\det(P, E_3) = c > 1$ and $\det(P, E_1) = a/\hat{a}$. If $\hat{a} = a$, $\text{Cone}(P, E_3)$ gives $\hat{a} = a$ arms. Hence, in any case the resolution graph of X_{II} has at least three arms centered at $E(P)$.

In case 2), we have $\det(P, Q) = c(b - 1)/e > 1$, $\det(P, E_1) = a > 1$ and $\det(P, E_3) = c/e$. If $e < c$, we have at least three arms in the resolution graph. Suppose that $e = c$. Then the number of arms at $E(P)$ is $e + 1 \geq 3$, unless $b = 2$ and $e = c = 2$. In this case, the resolution graph has two similar arms and $E(P)$ is normally smooth with $E(P)^2 \leq -2$.

Outline of other cases:

(III) Let $X_{\text{III}} : x^a y + xy^b + z^c = 0$. Then $P = {}^t(c(b - 1), c(a - 1), ab - 1)/d$ with $d = e \gcd(c, (ab - 1)/e)$ and $e = \gcd(a - 1, b - 1)$. The dual Newton diagram $\Gamma^*(f)_2^+$ has 3 arms $\text{Cone}(P, E_3)$, $\text{Cone}(P, Q)$, $\text{Cone}(P, R)$ where $Q = {}^t(0, c, 1)$ and $R = {}^t(c, 0, 1)$. The central divisor $E(P)$ is rational if and only if $\gcd(c, (ab - 1)/e) = 1$. If $E(P)$ is rational, then $d = e$ and $\det(P, Q) = c(b - 1)/e > 1$, $\det(P, R) = c(a - 1)/e > 1$, and $\det(P, E_3) = c > 1$. Hence, the resolution graph has at least three arms.

(IV) Let $X_{\text{IV}} : x^a y + y^b z + z^c = 0$. Then $P := {}^t(bc - c + 1, a(c - 1), ab)/d$ with $d = e \gcd(a, (bc - c + 1)/e)$ and $e := \gcd(b, c - 1)$. The dual Newton diagram $\Gamma^*(f)_2^+$ has 3 arms $\text{Cone}(P, E_1)$, $\text{Cone}(P, Q)$, $\text{Cone}(P, S)$ where $Q = {}^t(0, c, 1)$ and $S = {}^t(1, 0, a)$. The divisor $E(P)$ is rational if and only if $\gcd(a, (bc - c + 1)/e) = 1$ which is equivalent to $d = e$. We have $\det(P, E_1) = a > 1$, $\det(P, S) = a(c - 1)/e > 1$ and $\det(P, Q) = (bc - c + 1)/e$. As $\text{Cone}(P, E_1)$ has e -copies of arms, $E(P)$ has at least three arms.

(V) Let $X_{\text{V}} : x^a y + y^b z + z^c x = 0$. Then $P := {}^t(bc - c + 1, ca - a + 1, ab - b + 1)/d$ with $d = \gcd(bc - c + 1, ca - a + 1, ab - b + 1)$. The dual Newton diagram $\Gamma^*(f)_2^+$ has 3 arms $\text{Cone}(P, Q)$, $\text{Cone}(P, S)$, $\text{Cone}(P, T)$ where $Q = {}^t(0, c, 1)$, $S = {}^t(1, 0, a)$ and $T := {}^t(b, 1, 0)$. The divisor $E(P)$ is rational if and only if $d = 1$. In this case, we have $\det(P, Q) = bc - c + 1 > 1$, $\det(P, S) = ca - a + 1 > 1$ and $\det(P, T) = ab - b + 1 > 1$. Thus $E(P)$ has three arms.

(VII) Let $X_{\text{VII}} : x^a z + y^b z + z^c + tx^{c_1} y^{c_2} = 0$. Then $P = {}^t(b(c - 1), a(c - 1), ab)/\delta$ with $\delta = \gcd(b(c - 1), a(c - 1), ab)$. The dual Newton diagram $\Gamma^*(f)_2^+$ has 4 arms $\text{Cone}(P, Q)$, $\text{Cone}(P, S)$, $\text{Cone}(P, E_1)$, $\text{Cone}(P, E_2)$ where $Q = {}^t(0, 1, c_2)$ and $S = {}^t(1, 0, c_1)$. By the weighted homogeneity, we have the equality $b(c - 1)c_1 + a(c - 1)c_2 = abc$ which implies that $(c - 1) | ab$. Hence $\delta = (c - 1) \gcd(a, b, ab/(c - 1))$. By loc. cit., $E(P)$ is rational if and only if either (i) $\gcd(a, b) = \gcd(a, c - 1) = 1$, or (ii)

$\gcd(a, b) = \gcd(b, c - 1) = 1$. By symmetry, we may assume that the first case (i). Then $\delta = c - 1$, $\det(P, Q) = b > 1$, $\det(P, S) = a > 1$, $\det(P, E_1) = a > 1$. Thus the resolution graph has at least three arms.

(VIII) Let $X_{\text{VIII}} : x^a y + xy^b + xz^c + ty^{c_1} z^{c_2} = 0$. Then $P = {}^t(c(b-1), c(a-1), b(a-1))/\delta$ with $\delta = \gcd(c(b-1), c(a-1), b(a-1))$. By the weighted homogeneity, we must have $c(a-1)c_1 + b(a-1)c_2 = c(ab-1)$ which implies that $(a-1)|c(ab-1)$ and $cc_1 + bc_2 = bc + c(b-1)/a - 1$. Thus $\delta = (a-1)\gcd(b, c, c(b-1)/(a-1))$. The dual Newton diagram $\Gamma^*(f)_2^+$ has 4 arms Cone (P, E_3) , Cone (P, Q) , Cone (P, S) and Cone (P, T) where $Q = {}^t(0, c, 1)$, $S = {}^t(c_2, 0, 1)$ and $T = {}^t(c_1, 1, 0)$. The divisor $E(P)$ is rational if and only if $(b-1) = k(a-1)$ for some $k \in \mathbf{N}$ and $\gcd(b, c) = 1$. Then $d = a - 1$ and $\det(P, Q) = ck > 1$, $\det(P, S) = c > 1$, $\det(P, T) = b > 1$ and $\det(P, E_3) = c$. Thus the $E(P)$ has at least 3 arms. \square

4.4. Normally smooth divisors on $T_{p,q,r}$ -surfaces. Let $T_{p,q,r} : x^p + y^q + z^r + xyz = 0$ with $1/p + 1/q + 1/r < 1$.

(1) Suppose that p, q, r are pairwise coprime and $p < q < r$. The diagram $\Gamma^*(f)_2^+$ has three strictly positive vertices $P := {}^t(rq - r - q, r, q)$, $Q := {}^t(r, pr - p - r, p)$, and $R := {}^t(q, p, pq - q - p)$. The cones Cone (P, E_1) , Cone (Q, E_2) and Cone (R, E_3) are regular. Put $\delta := pqr - pr - qr - pq$. Then $\det(P, Q) = \det(Q, R) = \det(P, R) = \delta$.

PROPOSITION 23. *Under the above assumption, we have*

$$\rho(X_{p,q,r}, O) = \rho_{QR}^{(1)} + \rho_{QR}^{(2)} + \rho_{QR}^{(3)} + \rho_{PR}^{(2)} + \rho_{PR}^{(3)} + \rho_{PQ}^{(3)} - 2 - \epsilon,$$

where $\epsilon = 1$ if $p = 3$, and $\epsilon = 0$ if $p \neq 3$.

Proof. This is a summary of the following three lemmas. \square

LEMMA 24.

- 1) $\mathcal{V}_{\text{ns}}^{(1)}(Q, R) = \{P_k = {}^t(1, k, p - k - 1) \mid p/q < k < (rp - r - p)/r\}$.
- 2) $\mathcal{V}_{\text{ns}}^{(2)}(Q, R) = \{P'_k = {}^t(k, 1, pk - k - 1) \mid r/(pr - p - r) < k < q/p\}$.
- 3) $\mathcal{V}_{\text{ns}}^{(3)}(Q, R) = \{P''_k = {}^t(k, pk - k - 1, 1) \mid q/(pq - p - q) < k < r/p\}$.
- 4) $\mathcal{V}_{\text{ns}}^{(1)}(Q, R) \cap \mathcal{V}_{\text{ns}}^{(2)}(Q, R) \cap \mathcal{V}_{\text{ns}}^{(3)}(Q, R) \neq \emptyset$ if and only if $p = 3$.
- 5) $\rho_{QR} = \rho_{QR}^{(1)} + \rho_{QR}^{(2)} + \rho_{QR}^{(3)} - 1 - \epsilon$, where $\epsilon = 1$ if $p = 3$, and $\epsilon = 0$ if $p \neq 3$.

Proof. We mainly use Theorem 7. Let $P' := (\beta Q + \alpha R)/\delta = {}^t(p_1, p_2, p_3)$. The equation is

$$\begin{cases} \beta r + \alpha q = p_1 \delta \\ \beta(pr - p - r) + \alpha p = p_2 \delta \\ \beta p + \alpha(pq - p - q) = p_3 \delta \end{cases} \quad \text{this implies} \quad \begin{cases} \alpha = (pr - p - r)p_1 - rp_2 \\ \beta = qp_2 - pp_1 \\ p_2 + p_3 = (p - 1)p_1 \end{cases}$$

Hence, we have the following conclusions.

- 1) $p_1 = 1$ if and only if there exists an integer $p_2 > 0$ such that $\alpha > 0$ and $\beta > 0$. This is equivalent to $p/q < p_2 < (pr - p - r)/r$. And in this case $P' = (1, p_2, p - 1 - p_2)$.
- 2) $p_2 = 1$ if and only if there exists an integer $p_1 > 0$ such that $r/(pr - p - r) < p_1 < q/p$. And in this case $P' = (p_1, 1, (p - 1)p_1 - 1)$.
- 3) $p_3 = 1$ if and only if there exists an integer $p_1 > 0$ such that $q/(pq - p - q) < p_1 < r/p$. And in this case $P' = {}^t(p_1, pp_1 - p_1 - 1, 1)$.
- 4) is obvious now.

5) One can see this by comparing the three sets $\mathcal{V}_{\text{ns}}^{(i)}(Q, R)$. In case $p = 2$, we have $\mathcal{V}_{\text{ns}}^{(1)}(Q, R) = \emptyset$ and $\mathcal{V}_{\text{ns}}^{(2)}(Q, R) \cap \mathcal{V}_{\text{ns}}^{(3)}(Q, R) = \{{}^t(2, 1, 1)\}$. Hence, $\rho_{QR} = \rho_{QR}^{(2)} + \rho_{QR}^{(3)} - 1$.

In case $p = 3$, we have $\mathcal{V}_{\text{ns}}^{(i)}(Q, R) \cap \mathcal{V}_{\text{ns}}^{(j)}(Q, R) = \mathcal{V}_{\text{ns}}^{(1)}(Q, R) \cap \mathcal{V}_{\text{ns}}^{(2)}(Q, R) \cap \mathcal{V}_{\text{ns}}^{(3)}(Q, R) = \{^t(1, 1, 1)\}$ for $i \neq j$. Hence, $\rho_{QP} = \rho_{QR}^{(1)} + \rho_{QR}^{(2)} + \rho_{QR}^{(3)} - 2$.

In case $p > 3$, we have $\mathcal{V}_{\text{ns}}^{(1)}(Q, R) \cap \mathcal{V}_{\text{ns}}^{(2)}(Q, R) = \{^t(1, 1, p-2)\}$ and $\mathcal{V}_{\text{ns}}^{(1)}(Q, R) \cap \mathcal{V}_{\text{ns}}^{(3)}(Q, R) = \mathcal{V}_{\text{ns}}^{(2)}(Q, R) \cap \mathcal{V}_{\text{ns}}^{(3)}(Q, R) = \emptyset$. Hence, $\rho_{QP} = \rho_{QR}^{(1)} + \rho_{QR}^{(2)} + \rho_{QR}^{(3)} - 1$. \square

Similarly, one can prove the following two lemmas.

LEMMA 25.

- 1) $\mathcal{V}_{\text{ns}}^{(1)}(P, R) = \emptyset$.
- 2) $\mathcal{V}_{\text{ns}}^{(2)}(P, R) = \{Q'_\ell = {}^t(q - \ell - 1, 1, \ell) \mid q/r < \ell < (pq - p - q)/p\}$.
- 3) $\mathcal{V}_{\text{ns}}^{(3)}(P, R) = \{Q''_\ell = {}^t(q\ell - \ell - 1, \ell, 1) \mid p/(pq - p - q) < \ell < r/q\}$.
- 4) Let $Q' = {}^t(q_1, q_2, q_3) = (\beta P + \alpha R)/\delta$. Then $(q-1)q_2 = q_1 + q_3$.
- 5) $\rho_{PR} = \rho_{PR}^{(2)} + \rho_{PR}^{(3)} - 1$.

LEMMA 26.

- 1) $\mathcal{V}_{\text{ns}}^{(1)}(P, Q) = \mathcal{V}_{\text{ns}}^{(2)}(P, Q) = \emptyset$.
- 2) $\mathcal{V}_{\text{ns}}^{(3)}(P, Q) = \{R'_\ell = {}^t(r - \ell - 1, \ell, 1) \mid r/q < \ell < (pr - p - r)/p\}$ and $\rho_{PQ} = \rho_{PQ}^{(3)}$.

EXAMPLE 27. (1) Let $p = 2, q = 3$ and $r \geq 7$. By the canonical subdivisions of the three cones, we see that $\rho_{QR} = \lceil \frac{r-6}{2} \rceil \geq 1$, $\rho_{PR} = \lceil \frac{r-6}{3} \rceil \geq 1$, and $\rho_{PQ} = \lceil \frac{r-3}{6} \rceil$. (2) Let $p = 3, q = 4$ and $r > 4$. By the canonical subdivisions of the three cones, we see that $\rho_{QR} = \lceil \frac{r}{3} \rceil \geq 1$, $\rho_{PR} = \lceil \frac{r}{4} \rceil \geq 1$ and $\rho_{PQ} = \lceil \frac{2r}{3} \rceil - \lceil \frac{r}{4} \rceil - 1$.

(2) Another case. Let $f(x, y, z) = x^n + y^n + z^n + xyz$ ($n \geq 4$). The dual Newton diagram has three covectors $P_i, i = 1, 2, 3$ corresponding to the three compact faces. They are given by ${}^t(n-2, 1, 1), {}^t(1, n-2, 1), {}^t(1, 1, n-2)$. And for $i \neq j$, $\det(P_i, P_j) = n-3$. Let B_1, \dots, B_k be the vertices of the canonical subdivision of $\text{Cone}(P_1, P_2)$ from P_1 . Then $B_1 = (P_2 + (n-4)P_1)/(n-3) = {}^t(n-3, 2, 1)$. Thus $(n-3)/(n-4) = [2, \dots, 2]$ with $(n-4)$ -copies of 2. This implies $k = n-4$ and $B_j = {}^t(n-2-j, 1+j, 1), j = 1, \dots, n-4$. In fact, by Lemma 5 the third coordinate of B_j is always 1 as both of P_1, P_2 have 1 as the third coordinate. Hence $\rho_{P_1 P_2} = n-4$. The branch $\text{Cone}(P_i, E_i)$ is regular. Thus $\rho(V, O) = \rho(\Sigma_{\text{can}}^*) = 3n-9$ and every exceptional divisor is normally smooth.

5. Remarks.

5.1. Example of the inequality $\rho(\Sigma_{\text{can}}^*) > \rho(X, O)$. Let us consider A_{2c-1} -singularity, $X = \{x^2 + y^2 + z^{2c} = 0\}$. The resolution graph has two arms and the central divisor $E(P)$ is a rational curve with $E(P)^2 = -1$. Thus we have to blow-down the central divisor once (Example (6.7.1) in [9, III]). However in this example, the central exceptional divisor is not normally smooth, i.e., the extra blowing-up is line-admissible. So $\rho(\Sigma_{\text{can}}^*) = \rho(X, O)$. The following gives an example of $\rho(\Sigma_{\text{can}}^*) > \rho(X, O)$.

EXAMPLE 28. Let X be defined by $h = xy + y^{bc} + z^c$ with $b, c \geq 2$. This is an A_{c-1} -singularity and a special case of X_{II} with $P := {}^t(bc-1, 1, b)$ and $Q := {}^t(0, c, 1)$.

Since $\det(P, E_1) = \det(P, E_3) = 1$ and $\det(P, Q) = bc-1$, we make the canonical subdivision of $\text{Cone}(P, Q)$. The first covector T_1 from P is given by

$$T_1 = (Q + (bc - c - 1)P)/(bc - 1) = {}^t(bc - c - 1, 1, b - 1)$$

We have the continuous fraction expansion $(bc-1)/(bc-c-1) = [2, \dots, 2, 3, 2, \dots, 2]$ where the number of 2 in the first 2-series (respectively in the second 2-series) is

$(b-2)$ (resp. $c-2$). Thus we have $c+b-3$ covectors T_1, \dots, T_{b+c-3} . The exceptional divisor $E(P)$ is rational with $E(P)^2 = -1$ and $E(T_j)$ with self intersection number $E(T_j)^2 = -2$ for $j \neq b-1$ and -3 for $j = b-1$ (see Theorem (6.3), Chapter III, [9]). In fact first $b-2$ covectors are given by

$$\begin{aligned} Q_j &= {}^t(cb - jc - 1, 1, b - j), \quad j = 1, \dots, b - 1 \\ Q_{b-1+j} &= {}^t(c - j - 1, j + 1, 1), \quad j = 1, \dots, c - 2 \end{aligned}$$

and we see that they are normally minimal. To get a minimal reslution, we need to blow down $b-1$ divisors $E(P), E(T_1), \dots, E(T_{b-2})$ in this order. Then the self-intersection number of $E(T_{b-1})$ changes to -2 and we get A_{c-1} graph. In this example, we have $\rho(X, O) = c - 1$ and $\rho(\Sigma_{\text{can}}^*) = b + c - 2$.

5.2. Parametrization of lines. The normally smooth divisors on a surface X correspond to the lines on X . By using a toric resolution, one can give the exact parameterizations of the lines on X . This was done already for the Pham-Brieskorn surfaces in [4].

PROPOSITION 29. *Suppose that we have a line L in a non-degenerate surface $X : f(x, y, z) = 0$ and assume that L is parametrized as*

$$x(t) = \alpha t^a + \alpha_1 t^{a+1} \dots, \quad y(t) = \beta t^b + \beta_1 t^{b+1} + \dots, \quad z(t) = \gamma t^c + \gamma_1 t^{c+1} + \dots$$

with $\alpha, \beta, \gamma \neq 0$ and $\min(a, b, c) = 1$. Let $P = {}^t(a, b, c)$. Then the pull back of L intersects $E(P)$ transversally and $f_P(\alpha, \beta, \gamma) = 0$. Conversely any curve in $\mathcal{L}_{E(P)}$ has such a parametrization.

EXAMPLE 30. (1) Let X be defined by $h = x^a y + y^b - z^b = 0$ with $a = a_1(b-1)$ and $a_1 > 1$. This is a special case of X_{II} . We use the notations in §4.2. Note that $P = {}^t(1, a_1, a_1)$, $Q = {}^t(0, b, 1)$, $\det(P, Q) = \det(P, E_3) = 1$ and $\det(P, E_1) = a_1$. By canonical subdivision of Cone (P, E_1) we have $R_i := {}^t(1, i, i)$ with $i = 0, 1, \dots, i_1 = a_1$, where $R_0 := E_1$ and $R_{i_1} := P$. Hence $\rho_{PE_1} = a_1 - 1$. Since $r(P, E_1) + 1 = b$, each $E(R_i)$ has b components. By [9, III(6.3)], $E(P)^2 = -b < -1$. Hence π is minimal and $\rho(X, 0) = b(a_1 - 1) + 1$. The restriction of π on the toric chart associated with $\sigma_i := \text{Cone}(R_i, R_{i-1}, E_2)$ is given by

$$\pi_{\sigma_i} : \quad x = uv, \quad y = u^i v^{i-1} w, \quad z = u^i v^{i-1}.$$

and the pull-back of h is given by

$$h \circ \pi_{\sigma_i} = u^{ib} v^{(i-1)b} \left(u^{(a_1-i)(b-1)} v^{(a_1-i+1)(b-1)} w + w^b - 1 \right)$$

The divisor $E(R_i)$ is defined by $u = 0$ and $w^b - 1 = 0$, hence $E(R_i)$ has b components. On this toric chart, the resolution \tilde{X} of X is defined by

$$\tilde{h}_i(u, v, w) := u^{(a_1-i)(b-1)} v^{(a_1-i+1)(b-1)} w + w^b - 1 = 0$$

and in a neighborhood of $q \in E(R_i)$ we take u, v to be the local coordinates of \tilde{X} . Let $q = (0, s)$ in this coordinates. We consider the lines C_s defined by $t \mapsto (t, s)$. The image of C_s by π_{σ_i} is given by

$$\pi_{\sigma_i}(C_s) : \quad x = st, \quad y = s^{i-1} w_k(t, s)t, \quad z = s^{i-1} t^i,$$

where $w_k(t, s)$ is the solution of $\tilde{h}_i(t, s, w) = 0$ with $w_k(0) = \exp(2\pi ki/b)$. As a special case, take $i = 1$. Then C_s is a normal line on $E(Q_1)$. When we moves $s \rightarrow 0$, this line approaches to $E(E_1)$ and $w_k(t) \equiv \exp(2k\pi i/b)$ and the image is the obvious line $t \mapsto (x, y, z) = (0, w_k t, t)$.

(2) Let $X = T_{2,3,7} : x^2 + y^3 + z^7 + xyz = 0$. We have three covectors

$$P = {}^t(11, 7, 3), \quad Q = {}^t(7, 5, 2), \quad R = {}^t(3, 2, 1)$$

and we do not need any other covector. Consider the toric chart $\sigma := (Q, R, E_3)$ with coordinates (u, v, w) . Then the line $u = 1, v = t$ produces a line parametrized as $t \mapsto (t^3, t^2, -2t + 128t^2 + \dots)$.

5.3. Obvious lines on surfaces. We consider a surface $X = \{f(x, y, z) = 0\}$ where f has a non-degenerate Newton boundary. There are surfaces having obvious lines which can be read off from the polynomial defining the surface.

(1) Assume that $f(x, y, z)$ is not convenient and assume for example $\{y = z = 0\} \subset X$. Then as we have seen in Lemma 10, there is a unique non-compact face, different from the coordinate planes, which has the covector of the type $Q = {}^t(0, c, 1)$ or ${}^t(0, 1, c)$ and a unique covector P such that $\text{Cone}(P, Q)$ is in $\Gamma^*(f)_2^+$ and P corresponds to a compact face. Let Q_1, \dots, Q_k be the covectors defining the canonical regular subdivision from Q . Then Q_1 is a normally smooth divisor and \mathcal{L}_{Q_1} contains the canonical line $\{y = z = 0\}$.

(2) Assume that $h(x, y) := f(x, y, 0)$ (the section of f with $z = 0$) is a non-monomial homogeneous polynomial of degree d . Then we can factor $h(x, y) = cx^a y^b \prod_{i=1}^k (y - \alpha_i x)$. Thus X has the lines $z = 0, y = \alpha_i x$ for $i = 1, \dots, k$. Combinatorially this says the following. There exists a compact face Δ such that $\Delta \supset \Delta(h)$. The corresponding covector takes the form $P = {}^t(p, p, r)$ with $\gcd(p, r) = 1$. Then the first covector Q_1 from E_3 in the canonical regular subdivision of $\text{Cone}(P, E_3)$ takes the form $Q_1 = {}^t(1, 1, s)$ with $s = 1 + [r/p]$. So we can see that $Q_1 \in \mathcal{V}_{\text{ns}}(P, E_3)$. A typical example is $T_{n,n,n} : x^n + y^n + z^n - xyz = 0$. Another example is (1) of Example 30.

(3) Assume that the monomial x^A in f such that $(A, 0, 0) \in \Gamma(f)$. We say that x^A is negligibly truncatable if $f_t(x, y, z) = (f(x, y, z) - f(x, 0, 0)) + tf(x, 0, 0)$ defines a μ -constant family for $0 \leq t \leq 1$ (cf. [11]). Assume for example, the monomials $x^a y$ and $x^b z^c$ are on the non-compact face of $\Gamma(f_0)$. Let $Q' := {}^t(c/d, c(A-a)/d, (A-b)/d)$ with $d = \gcd(c, A-b)$. The covector Q' corresponds to the negligible compact face of f_1 containing $(a, 1, 0), (b, 0, c)$ and $(A, 0, 0)$. Then there is a normally smooth divisor on $\text{Cone}(Q, E_3)$. In fact, $\det(Q', E_3) = c/d$. If $c = d$, Q gives normally smooth divisor. If $c > d$, the first covector Q'_1 of the canonical regular subdivision of $\text{Cone}(Q', E_3)$ is normally smooth. An example is given by $f(x, y, z) = x^2 y + y^2 + z^5 + x^5$. Then x^5 is negligibly truncatable.

(4) Assume that $\Gamma(f)$ has a compact face whose covector P has 1 in its coefficients. Then $E(P)$ is a normally smooth divisor. This is the case, for example, if $P = {}^t(1, 1, 1)$ and $f_P(x, y, z)$ has a two-dimensional support. We can see easily that $E(P)$ is isomorphic to the projective curve $f_P(x, y, z) = 0$ in \mathbb{P}^2 . The tangent cone of X at O is given by the cone of $f_P = 0$.

5.4. Normally smooth divisors on complete intersections. In this paper we mainly considered normally smooth divisors on two dimensional hypersurface singularities. However every assertion can be generalized to non-degenerate complete intersections. We give an example. Consider the surface given by $X = \{f_1(x, y, z, w) =$

$f_2(x, y, z, w) = 0\}$ where f_1 and f_2 has the same Newton boundary. Assume that f_1, f_2 are Pham-Brieskorn polynomials of the same type, with generic coefficients:

$$f_i = a_i x^{p_1} + b_i y^{p_2} + c_i z^{p_3} + d_i w^{p_4}, i = 1, 2$$

We assume that $p_1, \dots, p_4 \geq 2$ and mutually coprime. Then the dual Newton diagram $\Gamma^*(f_1, f_2)$ is the same with $\Gamma^*(f_i)$ and $\Gamma^*(f)_2^+$ is star-shaped with the center $P = {}^t(p_2 p_3 p_4, p_1 p_3 p_4, p_1 p_2 p_4, p_1 p_2 p_3)$ and four arms $\text{Cone}(P, E_i), i = 1, \dots, 4$. We consider the Cone (P, E_1) . First $\det(P, E_1) = p_1$. By Lemma 11, $\mathcal{V}_{ns}^{(i)}(P, E_1) = \emptyset$ for $2 \leq i \leq 4$. As for $\mathcal{V}_{ns}^{(1)}(P, E_1) \neq \emptyset$ if and only if $p_2 p_3 p_4 < p_1$ and putting $r = \lfloor p_1 / p_2 p_3 p_4 \rfloor$, $\mathcal{V}_{ns}^{(1)}(P, E_1) = \{Q_j = (jP + (p_1 - j p_2 p_3 p_4)E_1) / p_1; j = 1, \dots, r\}$.

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