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PROJECTIVITY VIA THE DUAL KÄHLER CONE - HUYBRECHTS' CRITERION*

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Abstract. In this note we give an elementary proof for a remarkable criterion due to Daniel Huybrechts for a Kähler surface to be projective. Our proof is based on Kodaira's celebrated work on surfaces.

Introduction. It was Kunihiko Kodaira who first noticed how one can distinguish projective manifolds among Kähler manifolds ([Kd1], [Kd2], [CK]). His celebrated criteria can be restated in terms of cones (cf. (1.4)) as follows:

KODAIRA'S CRITERION I ([KD1, MAIN THEOREM]). *A compact Kähler manifold is projective if and only if the Kähler cone contains an integral point.* □

KODAIRA'S CRITERION II ([CK], [KD2, THEOREM (3.1)]). *A compact (Kähler) surface is projective if and only if the positive cone contains an integral point.* □

On the other hand, one of the current tendencies in higher dimensional algebraic geometry going back to Kleiman is to study projective varieties via the duality between the ample cone and the so-called Kleiman-Mori cone, the cone of effective curves [KMM].

Quite recently, Daniel Huybrechts [Hu1] took this idea into his study of hyperkähler manifolds and found as a byproduct the following remarkable criterion to distinguish projective surfaces via the dual Kähler cone:

HUYBRECHTS' CRITERION ([HU1, REMARK 3.12 (III)]). *A compact Kähler surface is projective if and only if the dual Kähler cone contains an inner integral point. (For the precise definitions, see (1.4) in Section 1.)*

However his original proof relies on powerful but highly advanced techniques in complex analysis called Demailly's singular Morse theory and he himself asked in the same paper whether or not it is possible to prove this criterion in a more elementary way.

The first purpose of this short note is to answer his question by giving a proof based on Kodaira's Criteria I and II and more or less familiar results on surfaces found now in standard books, [Bea], [BPV]. Our proof also depends on the notion of algebraic dimension due to Kodaira [Kd2, Page 125], while it is almost free from the classification of surfaces, which is, needless to say, one of the other monumental works again due to Kodaira.

Although it is clear that the dual Kähler cone contains both the positive cone and the Kähler cone (1.5), it might not be so apparent whether or not Huybrechts' criterion is really a generalisation of Kodaira's criteria I and II, or more concretely, whether there really exists a surface whose dual Kähler cone contains an inner integral point of negative self-intersection. This was asked by Catanese [Ca].

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However such surfaces indeed exist; we shall argue by looking at the dual Kähler cone of K3 surfaces slightly more closely. One reason why we focus on K3 surfaces is because all K3 surfaces are Kähler [Su] but some of them are not projective as was noticed again by Kodaira [Kd3, Page 778]. (See also Remark (2.4) in Section 3.) Our result in this direction is as follows:

PROPOSITION. *Every K3 surface S of maximal Picard number* 20 *admits an inner integral point of negative self-intersection in its dual Kähler cone.*

Note that there exist countably many K3 surfaces of Picard number 20 [SI]. This Proposition also answers Catanese's question cited above. See also Remark at the end of Section 4 for a slight generalisation.

Of course, it is very interesting to ask whether or not Huybrechts' criterion also holds in higher dimensions, say, in dimension three. We will come back to this question in the forthcoming paper [OP].

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1. Preliminaries.

(1.0). Throughout this note, the term surface means a compact, connected complex manifold of dimension two. We do not assume the minimality. Let S be a surface. The transcendental degree of the meromorphic function field of S over $\mathbb{C}$ is called the algebraic dimension and is denoted by $a(S)$ [Kd2, Page 125]. It is well known that $a(S) \in \{0, 1, 2\}$ and S is projective if and only if $a(S) = 2$ by [Kd2, Theorem (3.1)].

(1.1). A Hermitian metric g on S is called Kähler if the associated positive real $(1, 1)$ form ω_g is $d-$closed. We call ω_g a Kähler form if g is a Kähler metric. A surface is called Kähler if it admits at least one Kähler metric. Note that every projective surface is Kähler [Kd1, Page 28] but the converse is not true in general.

Let S be a Kähler surface.

(1.2). By definition, any Kähler metric g on S determines a de Rham cohomology class $[\omega_g]$. This class lies in the real $(1, 1)$ part $H^{1,1}(S, \mathbb{R})$ of the Hodge decomposition of $H^2(S, \mathbb{C})$. We often abbreviate $H^{1,1}(S, \mathbb{R})$ by $H^{1,1}$. We call an element $\eta \in H^{1,1}$ a Kähler class if it is represented by a Kähler form, that is, in the case where there exists a Kähler metric g such that $\eta = [\omega_g]$.

(1.3). The real vector space $H^{1,1}$ carries a natural symmetric bilinear form $(*.*)$ induced by the cup product on the integral cohomology group $H^2(S,\mathbb{Z})$. It is well known that $(*.*)$ on $H^{1,1}$ is non-degenerate and is of signature $(1, h^{1,1}(S)-1)$ by the Hodge index Theorem. We also regard the finite dimensional real vector space $H^{1,1}$ as a linear topological space using some norm $|*|$. Therefore we can speak of the closure $\overline{A}$ and the interior A° of $A \subset H^{1,1}$. For $a \in H^{1,1}$ and for a positive real number $\epsilon > 0$, we set

$$B_\epsilon(a) := \{x \in H^{1,1} | |x - a| \leq \epsilon\}.$$

Furthermore, let $U_\epsilon(a)$ be the interior of $B_\epsilon(a)$ and $\partial B_\epsilon(a)$ its boundary.

(1.4).

(1) The positive cone $\mathcal{C}^+(S)$ is the connected component of $\{x \in H^{1,1} | (x.x) > 0\}$ which contains the Kähler classes.

(2) The Kähler cone $\mathcal{K}(S)$ of S is the subset of $H^{1,1}$ consisting of the Kähler classes of S. By definition, $\mathcal{K}(S)$ is a convex cone of $H^{1,1}$. It is also well known that $\mathcal{K}(S)$ is an open subset of $H^{1,1}$.

(3) The dual cone $\mathcal{K}^*(S)$ of the Kähler cone $\mathcal{K}(S)$ is the set of elements $x \in H^{1,1}$ such that $(x.\eta) > 0$ for any $\eta \in \mathcal{K}(S)$.

(4) An element x of $\mathcal{K}^*(S)$ is called integral if $x \in \mathcal{K}^*(S) \cap \iota^* H^2(S,\mathbb{Z})$, where $\iota : \mathbb{Z} \to \mathbb{R}$ is a natural inclusion of sheaves. An integral element is nothing but an element of $\mathcal{K}^*(S) \cap \mathrm{NS}(S)$ (1.8).

(5) An element x of $\mathcal{K}^*(S)$ is called an inner point if $x \in \mathcal{K}^*(S)^\circ$, that is, there exists a positive real number $\epsilon > 0$ such that $U_\epsilon(x) \subset \mathcal{K}^*(S)$.

The following inclusions are clear by the Hodge Index Theorem:

LEMMA 1.5. $\mathcal{K}(S) \subset \mathcal{C}^+(S) \subset \mathcal{K}^*(S)$. □

LEMMA 1.6. *Let $(H, |*|)$ be a finite dimensional, real normed vector space equipped with a real valued, non-degenerate bilinear form $(*.*)$. Let $K \subset H$ be a non-empty convex cone such that $0 \notin K$. Set $K^* \subset H$ to be the dual of K with respect to $(*.*)$. Let $x \in H$. Then x is an inner point of K^* if and only if there exists a positive real number $r > 0$ such that $(x, \eta) \geq r|\eta|$ for all $\eta \in \overline{K}$. In particular, $K^* \cup \{0\} = \overline{(K^*)^\circ}$.*

Proof. This follows from the compactness of the space $B_\epsilon(x) \times (\overline{K} \cap \partial B_1(0))$. □

The following direct consequence will be applied in our proof:

COROLLARY 1.7. *Let $x \in H^{1,1}$. Then x is an inner point of $\mathcal{K}^*(S)$ if and only if there exists a positive real number $r > 0$ such that $(x.\eta) \geq r|\eta|$ for all $\eta \in \overline{\mathcal{K}(S)}$. Moreover, $\mathcal{K}^*(S) \cup \{0\} = \overline{\mathcal{K}^*(S)^\circ}$.* □

(1.8). The group $H^{1,1} \cap \iota^* H^2(S,\mathbb{Z})$ is called the Néron-Severi group of S and is denoted by $\mathrm{NS}(S)$. The rank of $\mathrm{NS}(S)$ is called the Picard number of S and is written by $\rho(S)$. By the Lefschetz $(1,1)$ Theorem, each element of $\mathrm{NS}(S)$ is represented by the first Chern class of some line bundle [Kd1, Theorem 1]. However, contrary to the projective case, the natural map from the group of Cartier divisors to the Picard group is not surjective in general. Therefore, we CAN NOT say that each element of $\mathrm{NS}(S)$ is represented by a divisor in the Kähler category.

2. Kähler cones of K3 surfaces and complex tori.

THEOREM 2.1 [BEA, PAGE 123, THEOREM 2]. *Let S be a K3 surface, that is, a (Kähler) surface such that $K_S = 0$ in $Pic(S)$ and that $\pi_1(S) = \{1\}$. Then the Kähler* *cone* $\mathcal{K}(S)$ *coincides with the Kähler chamber* $\widetilde{\mathcal{K}(S)}$ *of* $\mathcal{C}^+(S)$ *defined by* $(x.[C]) > 0$ *for all non-singular rational curves* C *in* S*, that is,*

$$\mathcal{K}(S) = \widetilde{\mathcal{K}(S)} := \{x \in \mathcal{C}^+(S) | (x.[C]) > 0 \text{ for all } C \simeq \mathbb{P}^1 \text{ in } S\ \}. \ \square$$

Remark. It is clear that $\mathcal{K}(S) \subset \widetilde{\mathcal{K}(S)}$. However, the other inclusion $\widetilde{\mathcal{K}(S)} \subset \mathcal{K}(S)$ is non-trivial and the proof given in [Bea] is based on the surjectivity of the period mapping. (See also [Hu2, Corollary 3.4] for another proof.)

THEOREM 2.2. *Let* S *be a complex torus of dimension 2. Then* $\mathcal{K}(S) = \mathcal{C}^+(S)$.

Remark. This result should be known. However, the authors could not find any references. We shall give two proofs: The first one is based on an argument of algebraic approximation and the second simpler one is due to Daniel Huybrechts [Hu3].

1st proof. Since any $(1,1)$-class can be represented by a form with constant coefficients, $\mathcal{K}(S)$ and $\mathcal{C}^+(S)$ do not depend on the complex structures of S. Therefore, it is sufficient to check the equality for $S = E_{\sqrt{-1}} \times E_{\sqrt{-1}}$, where $E_{\sqrt{-1}}$ is the elliptic curve of period $\sqrt{-1}$. Then $\rho(S) = 4$ by [SM] and we have $H^{1,1} = \mathrm{NS}(S) \otimes \mathbb{R}$. In particular, the rational points are dense in $H^{1,1}$ and the Kähler cone is then nothing but the ample cone. Note also that S contains no effective curves of negative self-intersection. Then, the result follows from the usual Nakai's criterion for ampleness (see also [CP]) plus the Hodge index Theorem. $\square$

2nd proof. Again we use the fact that any $(1,1)$-class can be represented by a form with constant coefficients. Suppose $\mathcal{K}(S) \neq \mathcal{C}^+(S)$. Since $\mathcal{K}(S) \subset \mathcal{C}^+(S)$, we find a constant $(1,1)$-form φ such that $[\varphi] \in \mathcal{C}^+(S) \cap \partial\mathcal{K}(S)$. Then φ is semipositive but not positive. Therefore $([\varphi]^2) = \int_S \varphi \wedge \varphi = \int_S 0 = 0$, a contradiction. $\square$

In order to prove Huybrechts' criterion, we also need to know the structure of the Néron-Severi groups of K3 surfaces and complex tori of algebraic dimension zero.

PROPOSITION (2.3). *Let* S *be a K3 surface. Assume that* $a(S) = 0$*. Then,*

(1) *Pic*(S) *and NS*(S) *are torsion free and are isomorphic under* c_1*. Moreover NS*$(S) \otimes \mathbb{R}$ *is negative definite with respect to* $(*.*)$.

(2) S *contains at most 19 distinct smooth rational curves and contains no other curves.*

Proof of (1). The first part of the assertion is well known. Using $a(S) = 0$ and the Riemann-Roch Theorem, we readily see that $L^2 < 0$ for all $L \in \mathrm{Pic}(S) - \{0\}$. Since $(*.*)$ is defined over $\mathbb{Z}$, this implies the result. $\square$

Proof of (2). Let C be an irreducible curve on S. Then $C \simeq \mathbb{P}^1$, because $0 > C^2 = (K_S + C.C) = 2p_a(C) - 2$ by (1) and the adjunction formula.

CLAIM 1. *Let* $C_1, ..., C_m$ *be* m *distinct irreducible curves on* S*. Then* $[C_1], ..., [C_m]$ *are linearly independent in NS*$(S) \otimes \mathbb{R}$.

Proof. Since the classes $[C_i]$ defined over $\mathbb{Z}$, it is enough to show that if $\sum_{i\in I} p_i[C_i] = \sum_{j\in J} q_j[C_j]$, where $I \cap J = \emptyset$, $p_i \in \mathbb{Z}_{\geq 0}$ and $q_j \in \mathbb{Z}_{\geq 0}$ then $p_i = q_j = 0$. By (1), we have

$$0 \geq (\sum_{i\in I} p_i[C_i]. \sum_{i\in I} p_i[C_i]) = (\sum_{i\in I} p_i[C_i]. \sum_{j\in J} q_j[C_j]) \geq 0.$$

Therefore, $(\sum_{i\in I} p_i[C_i]. \sum_{i\in I} p_i[C_i]) = 0$. Then again by (1), we have

$$[\sum_{i\in I} p_i C_i] = \sum_{i\in I} p_i[C_i] = 0$$

in $\mathrm{NS}(S)$ and $\sum_{i \in I} p_i C_i = 0$ in $\mathrm{Pic}(S)$. This is possible only in the case where $p_i = 0$ for all $i \in I$. Similarly, $q_j = 0$ for all $j \in J$. □

Claim 2. *S contains at most 19 distinct $\mathbb{P}^1$'s.*

Proof. Recall that $(H^{1,1}, (*.*))$ is of dimension 20 and of signature $(1, 19)$. Assume that S contains more than or equal to 20 distinct $\mathbb{P}^1$'s. Let $C_1, ..., C_{20}$ be 20 $\mathbb{P}^1$'s among them. Then, since $\mathbb{R}\langle [C_1], ..., [C_{20}] \rangle \subset \mathrm{NS}(S) \otimes \mathbb{R} \subset H^{1,1}$ and $\dim_{\mathbb{R}} \mathbb{R}\langle [C_1], ..., [C_{20}] \rangle = 20 = \dim_{\mathbb{R}} H^{1,1}$ by Claim 1, we get $\mathrm{NS}(S) \otimes \mathbb{R} = H^{1,1}$. However, $\mathrm{NS}(S) \otimes \mathbb{R}$ is of signature $(0, 20)$ by (1) while $H^{1,1}$ is of signature $(1, 19)$, a contradiction. □

Now we are done. □

Remark (2.4). For each integer m such that $0 \le m \le 19$, there actually exists a K3 surface of $a(S) = 0$ which contains exactly m distinct $\mathbb{P}^1$'s and no other curves. (See also [Og2, Example 4] for a relevant example.)

Construction. This construction is much inspired again by the work of Kodaira [Kd3, Section 5]. By [OZ], based on [SI], there exists a projective K3 surface T which contains 19 $\mathbb{P}^1$'s, say, $C_1, ..., C_{19}$ whose intersection matrix $(C_i.C_j)$ is of type A_{19}. Let $f : \mathcal{X} \to \mathcal{B}$ be the Kuranishi family of T. By [Kd3, Theorem 17], we may identify the base space $\mathcal{B}$ with an open set $\mathcal{U}$ of the period domain $\mathcal{P}$ of the K3 surfaces under some marking $\tau : R^2 f_* \mathbb{Z} \simeq \Lambda_{K3} \times \mathcal{B}$:

$$\mathcal{B} \simeq \mathcal{U} \subset \mathcal{P} := \{[\omega] \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) | (\omega.\omega) = 0,\ (\omega.\overline{\omega}) > 0\}.$$

Let c_i be the element of the K3 lattice Λ_{K3} which corresponds to the class $[C_i]$ under the marking τ. Define the subset $c_i^{\perp} \subset \mathcal{U}$ by $c_i^{\perp} := \{[\omega] \in \mathcal{U} | (\omega.c_i) = 0\}$. Let $0 \le m \le 19$ and choose a very general point P of the space $c_1^{\perp} \cap ... \cap c_m^{\perp}$. Here we note that this space is of dimension $20 - m > 0$ by (2.3). Then the fiber $\mathcal{X}_P$ is a K3 surface which contains exactly m distinct $\mathbb{P}^1$'s whose intersection matrix is of type A_m and has no other curves. This also implies $a(\mathcal{X}_P) = 0$. □

Proposition (2.5). *Let S be a complex torus of dimension 2. Assume that $a(S) = 0$. Then, $NS(S) \otimes \mathbb{R}$ is negative definite with respect to $(*.*)$.*

Remark. We learned the following proof from Fabrizio Catanese [Ca].

Proof. Write $S := \mathbb{C}^2/\Lambda$, where Λ is a lattice of rank four. It is clear that $\mathrm{NS}(S) \otimes \mathbb{R}$ is negative semi-definite. Suppose, to the contrary, that there exists an element $L \in \mathrm{NS}(S)$ such that $L \ne 0$ but $(L^2) = 0$. Via the Poincaré duality, we regard $L \in \wedge^2 \Lambda = H_2(S, \mathbb{Z})$. Since $\mathrm{rank}\Lambda = 4$ and $L \ne 0$, by considering the alternating matrix corresponding to L, we see that there exists a basis $\langle e_i \rangle_{i=1}^4$ of $\Lambda \otimes \mathbb{Q}$ such that $\mathbb{Q}L$ coincides with either $\mathbb{Q}e_1 \wedge e_2$ or $\mathbb{Q}(e_1 \wedge e_2 + e_3 \wedge e_4)$. Since $\int_S L \wedge L = (L^2) = 0$, where we identify L with the corresponding constant differential form, the latter case is impossible. Hence we have $\mathbb{Q}L = \mathbb{Q}e_1 \wedge e_2$. Replacing L and e_i by their integral multiples, we may assume that $L = e_1 \wedge e_2$ and $e_1, e_2 \in \Lambda$. Set $\Lambda' := \mathbb{Z}\langle e_1, e_2 \rangle \subset \Lambda$. Then, $\mathrm{rank}\Lambda' = 2$. In addition, by a direct calculation using the global coordinates of the universal covering space $\mathbb{C}^2$ and the fact that L is of type $(1, 1)$ and is real, we see that Λ' is a lattice of a linear subspace $V \simeq \mathbb{C}$ of $\mathbb{C}^2$. In particular, V/Λ' is a complex subtorus of S. Set $E := V/\Lambda'$. Then S/E is an elliptic curve and we have a natural surjective map $S \to S/E$. However, this implies $0 = a(S) \ge a(S/E) = 1$, a contradiction. □

3. Proof of Huybrechts' Criterion. In this section, we shall give an elementary proof of Huybrechts' criterion.

Proof of the "only if" part. Any ample class gives a desired point. □

Proof of the "if" part. Let S be a Kähler surface which has an inner integral point of $\mathcal{K}^*(S)$. It is sufficient to show that $a(S) \neq 0, 1$ by (1.0).

LEMMA (3.1). $a(S) \neq 1$.

Proof. Assume to the contrary that $a(S) = 1$ and take the algebraic reduction $f : S \to C$, which, in the surface case, is a surjective morphism to a non-singular curve with connected fibers [Kd2, Theorem 4.1]. Let F be a general fiber of f and set $f(F) = P$. Then $[F] = [f^*(P)] \in \mathrm{NS}(S)$, where P is regarded as a divisor on C. Since P is ample on C, the class $[P]$ is represented by a positive definite real d-closed $(1,1)$-form θ. Set $\Theta := f^*\theta$. Then $[F] = [\Theta]$ and Θ is positive semi-definite at each point $Q \in S$. Therefore for a Kähler form ω and for any $\epsilon > 0$, we have $[\Theta + \epsilon\omega] \in \mathcal{K}(S)$. Thus, $[F] = \lim_{\epsilon\to 0}[\Theta + \epsilon\omega] \in \overline{\mathcal{K}(S)}$. Moreover, since $([F].[\omega]) = \int_F \omega > 0$, we see that $[F] \neq 0$. Let M be an inner integral point of $\mathcal{K}^*(S)$. Then, by (1.7), we have $(M.[F]) > 0$, whence $(M + n[F])^2 = M^2 + 2n(M.[F]) > 0$ for a large integer n. In addition, we have $M + n[F] \in \mathrm{NS}(S)$. However, S is then projective by Kodaira's criterion II, a contradiction. □

The next Lemma reduces our problem to the case of minimal surfaces.

LEMMA (3.2). *Let $\tau : S \to T$ be the blow down of a (-1)-curve E. Then,*

(1) *S is projective if and only if T is projective.*

(2) *S is Kähler if and only if T is Kähler.*

(3) *Assume that there exists an inner integral point x of $\mathcal{K}^*(S)$. Then there also exists an inner integral point of $\mathcal{K}^*(T)$.*

Proof. The assertions (1) and (2) are well known. (However, it might be worth reminding here that the "only if" part of both (1) and (2) is false in general if dimension is three or higher and the center is not a point. One instructive counterexample is found in [Og1].) Let us show the assertion (3). Recall that $H^2(S, K) = \tau^*H^2(T, K) \oplus K[E] \simeq H^2(T, K) \oplus K[E]$ for $K = \mathbb{Z}$, $\mathbb{R}$. Moreover, this equality and isomorphism are compatible with the cup product and the Hodge decompositions. Let us regard $H^{1,1}(S)$ as a normed space by the product norm of $H^{1,1}(T)$ and $\mathbb{R}[E]$. Set $e := [E]$. Then the inner integral point $x \in \mathcal{K}^*(S)$ is of the form $x = \tau^*y + ae$ where $y \in \mathrm{NS}(T)$ and $a \in \mathbb{Z}$. We show that y is an inner point of $\mathcal{K}^*(T)$. Let $\sigma \in \mathcal{K}(T)$. Then $\tau^*\sigma \neq 0$ and $\tau^*\sigma - \epsilon e \in \mathcal{K}(S)$ for all sufficiently small positive real numbers ϵ. Therefore $\tau^*\sigma \in \overline{\mathcal{K}(S)}$. Since x is an inner point of $\mathcal{K}^*(S)$, there exists $r > 0$ such that $(x.\eta) \geq r|\eta|$ for all $\eta \in \overline{\mathcal{K}(S)}$ by (1.7). In particular, $(x.\tau^*\sigma) \geq r|\tau^*\sigma|$. On the other hand, using $x = \tau^*y + ae$ and applying the projection formula, we calculate $(x.\tau^*\sigma) = (y.\sigma)$. This together with the compatibility of the norms implies $(y.\sigma) \geq r|\sigma|$ for all $\sigma \in \mathcal{K}(T)$, hence for all $\sigma \in \overline{\mathcal{K}(T)}$. □

LEMMA (3.3). *Let S be a minimal Kähler surface. Assume that $a(S) = 0$. Then S is either a K3 surface or a complex torus of dimension 2.*

Proof. This is of course well known, see, for example, [BPV]. We give a proof to convince the reader that no deep result from classification theory is involved. Since $a(S) = 0$, we have $\kappa(S) = 0$ or $-\infty$, where $\kappa(S)$ is the Kodaira dimension of S. Moreover, if $h^0(K_S) = 0$, then by the Serre duality $h^2(\mathcal{O}_S) = 0$ and S is then projective by Kodaira's criterion II (see also [Kd2, Theorem (3.5)]), a contradiction. Therefore $K_S = 0$ in $\mathrm{Pic}(S)$ by the minimality of S. Since S is Kähler, this gives the result. □

By virtue of (3.1), (3.2) and (3.3), in order to conclude the "if" part, it is now sufficient to show the following:

LEMMA (3.4).

(1) *Let S be a K3 surface. Assume that $\mathcal{K}^*(S)$ contains an inner integral point x. Then $a(S) \neq 0$.*

(2) *Let S be a complex torus of dimension 2. Assume that $\mathcal{K}^*(S)$ contains an inner integral point x. Then $a(S) \neq 0$.*

Proof of (1). Assume to the contrary that $a(S) = 0$. Let $C_1, ..., C_m$ $(0 \leq m \leq 19)$ denote the distinct smooth rational curves on S ((2.3)(2)). We argue dividing into two cases:

Case 1. $x \in \mathbb{R}\langle [C_1], ..., [C_m] \rangle$;

Case 2. $x \notin \mathbb{R}\langle [C_1], ..., [C_m] \rangle$.

Case 1. By (2.3), the subspace of $H^{1,1}$

$$[C_1]^\perp \cap ... \cap [C_m]^\perp$$

is of signature $(1, 19 - m)$ (where $\perp$ is taken with respect to $(*.*)$). Therefore

$$[C_1]^\perp \cap ... \cap [C_m]^\perp \cap \mathcal{C}^+(S) \neq \emptyset.$$

Let η be an element of this set. Then by (2.1), $\eta \in \overline{\mathcal{K}(S)}$ and $\eta \neq 0$. On the other hand, by our assumption, we have $(x.\eta) = 0$. This contradicts (1.7).

Case 2. In this case $m \leq 18$. Indeed, otherwise we would have $\mathbb{R}\langle x, [C_1], ..., [C_{19}] \rangle = \mathrm{NS}(S) \otimes \mathbb{R} = H^{1,1}$ and would get the same contradiction as in Claim 2 of (2.3). Therefore the subspace

$$x^\perp \cap [C_1]^\perp \cap ... \cap [C_m]^\perp$$

is of signature $(1, 19 - m - 1)$ and then

$$x^\perp \cap [C_1]^\perp \cap ... \cap [C_m]^\perp \cap \mathcal{C}^+(S) \neq \emptyset.$$

Let η be an element of $x^\perp \cap [C_1]^\perp \cap ... \cap [C_m]^\perp \cap \mathcal{C}^+(S)$. Then by (2.1), $\eta \in \overline{\mathcal{K}(S)}$ and $\eta \neq 0$. On the other hand, by the choice of η, we have $(x.\eta) = 0$, a contradiction. □

Proof of (2). Note that $(H^{1,1}, (*.*))$ is non-degenerate and of signature $(1, 3)$. Assume to the contrary that $a(S) = 0$. Then $x^2 < 0$ and $x \neq 0$ by (2.5). Thus the subspace $x^\perp \subset H^{1,1}$ is of index $(1, 2)$. Combining this with (2.2), we have $x^\perp \cap \mathcal{K}(S) = x^\perp \cap \mathcal{C}^+(S) \neq \emptyset$. Therefore there exists an element $\eta \in \mathcal{K}(S)$ such that $(x.\eta) = 0$. However, this contradicts $x \in \mathcal{K}^*(S)$. □

4. Proof of Proposition. In this section, we shall prove the Proposition in the Introduction.

Proof of Proposition. Since $\rho(S) = 20$, we have $H^{1,1} = \mathrm{NS}(S) \otimes \mathbb{R}$. In particular, S is projective and the rational points are dense in $H^{1,1}$. By the classification of K3 surfaces of $\rho = 20$ due to [SI] or by the classification of Kleiman-Mori's cones of algebraic K3 surfaces due to [Kv], S contains a smooth rational curve C. Then $[C] \in \mathcal{K}^*(S)$ and $([C]^2) = -2 < 0$. Therefore, there exists a neighbourhood U of $[C]$ in $H^{1,1}$ such that $(y^2) < 0$ for all $y \in U$ and that $U \cap \mathcal{K}^*(S)^\circ \neq \emptyset$ by (1.7). By the density of rational points, there then exists an inner rational point x of $\mathcal{K}^*(S)$ such that $(x^2) < 0$. Taking an appropriate integral multiple of this x, we obtain a desired point. □

Remark.

(1) In this example, the Kähler cone of S coincides with the ample cone by $H^{1,1} = \mathrm{NS}(S) \otimes \mathbb{R}$. Therefore, the dual Kähler cone (plus $\{0\}$) also coincides with the Kleiman-Mori cone.

(2) For the same reason, any Kähler surface T such that $H^{1,1} = \mathrm{NS}(T) \otimes \mathbb{R}$ which contains a pseudo-effective curve of negative self-intersection admits an inner integral point of negative self-intersection in its dual Käher cone. In particular, any non-minimal Kähler surface T such that $p_g(T) = 0$ satisfies this property. Actually every surface T of general type with K_T not ample with $p_g(T) = 0$ also shares this property.

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