ASIAN J. MATH.
Vol. 4, No. 1, pp. 227–234, March 2000

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SOME NEW OBSERVATION ON INVARIANT THEORY OF PLANE QUARTICS*

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1. Introduction. Let $S(n, m)$ denote the graded ring of projective invariants of an n-ary form (a homogeneous polynomial in n variables) of degree m. We are interested in the case $n = 3$ and $m = 4$. A ternary quartic form $F(x_0, x_1, x_2)$ defines a plane curve of genus 3 if it is nonsingular, and conversely any non-hyperelliptic curve of genus 3 can be realized as such a plane quartic via the canonical embedding, which is unique up to projective transformations. Thus the structure of the ring $S(3, 4)$ is closely related to the moduli of genus 3 curves. (For general background of Invariant Theory, see e.g. [4], [13].)

More than thirty years ago ([5, Appendix]), we calculated the generating function (Poincaré series) of $S(3, 4)$ and made a few guess (or conjecture?) about the structure of the graded ring $S(3, 4)$. More recently, Dixmier [2] has proved the existence of a system of parameters for this ring (suggested in [5]) by exhibiting a system of seven explicit projective invariants.

In this paper, we study some close relationship of the ring $S(3, 4)$ of projective invariants to another invariant theory, i.e. to the invariant theory for the Weyl groups $W(E_7)$ and $W(E_6)$ (cf. [1]). We are led to such a connection from the viewpoint of Mordell-Weil lattices ([8], [9]).

2. Formulation of main results. We consider the case of characterisitic zero. Taking

$$F(x_0, x_1, x_2) = \sum_{i_0+i_1+i_2=4} a_{i_0,i_1,i_2} x_0^{i_0} x_1^{i_1} x_2^{i_2}$$

with variable coefficients $\{a_{i_0,i_1,i_2}\}$, we may regard $S(3, 4)$ as a graded subring of the polynomial ring $\mathbf{C}[a_{i_0,i_1,i_2}]$ graded by the total degree, consisting of those $I = I(F) \in \mathbf{C}[a_{i_0,i_1,i_2}]$ which are invariant under $SL(3)$. Namely, for any $g \in SL(3)$, let $(x'_0, x'_1, x'_2) = (x_0, x_1, x_2)g$ and rewrite $F(x'_0, x'_1, x'_2)$ as a polynomial $F'(x_0, x_1, x_2)$ in x_0, x_1, x_2:

$$F'(x_0, x_1, x_2) = \sum_{i_0+i_1+i_2=4} a'_{i_0,i_1,i_2} x_0^{i_0} x_1^{i_1} x_2^{i_2}.$$

We set $F^g = F'$. Then, by definition, we have

$$I \in S(3, 4) \Longleftrightarrow I(F^g) = I(F) \ (\forall g \in SL(3)).$$

For any ternary quartic form F_0, we call the map $I \to I(F_0)$ the *evaluation map* of $S(3, 4)$ at F_0.

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Actually the **C**-algebra $S(3,4)$ is obtained from the **Q**-algebra $S(3,4)\cap\mathbf{Q}[a_{i_0,i_1,i_2}]$ by the scalar extension of **Q** to **C**. So, in the following, we change the notation so that $S(3,4)$ will denote this **Q**-subalgebra of $\mathbf{Q}[a_{i_0,i_1,i_2}]$

Now we recall the following fact on the normal form of a plane quartic with a given flex (cf. [8, §1]). Take the inhomogeneous coordinates x, t such that $(x_0 : x_1 : x_2) = (1 : x : t)$. The *normal form of type* E_7 is

$$f_\lambda = x^3 + x(p_0 + p_1 t + t^3) + q_0 + q_1 t + q_2 t^2 + q_3 t^3 + q_4 t^4$$

with $\lambda = (p_0, p_1, q_0, \ldots, q_4) \in \mathbf{A}^7$, and the *normal form of type* E_6 is

$$f_\lambda = x^3 + x(p_0 + p_1 t + p_2 t^2) + q_0 + q_1 t + q_2 t^2 + t^4$$

with $\lambda = (p_0, p_1, p_2, q_0, q_1, q_2) \in \mathbf{A}^6$. In either case, let Γ_λ be the plane quartic defined by $f_\lambda = 0$; the flex is given by the point $(x_0 : x_1 : x_2) = (0 : 1 : 0)$. The fact is that every plane quartic with a given flex is isomorphic to Γ_λ for some $\lambda \in \mathbf{A}^7$ or $\mathbf{A}^6$; the distinction depends on whether the given flex is *ordinary* or *special*[1] (i.e. whether the tangent line to the curve at the flex intersects the curve with multiplicity 3 or 4).

It is obvious that the evaluation map $I \to I(f_\lambda)$ gives a ring homomorphism

$$\phi : S(3,4) \longrightarrow \mathbf{Q}[\lambda] = \mathbf{Q}[p_i, q_j]$$

for either type of f_λ. Let us call it the *evaluation map of type* E_7 *or* E_6, and denote it by ϕ_7 or ϕ_6 when we need to specify the cases.

The main purpose of this paper is to establish less obvious relationship between the invariant theory of a plane quartic and the invariant theory of the Weyl group $W(E_r)$ $(r = 6, 7)$. To formulate the results, note first that the ring of invariants of $W(E_r)$, say $R(E_r)$, can be naturally identified with $\mathbf{Q}[\lambda]$ given above (see [1], [6], [7]), which is a graded polynomial ring with the weights of p_i or q_j assigned as follows:

for E_7 case: $wt(p_i) = 12 - 4i,\ wt(q_j) = 18 - 4j$.

for E_6 case: $wt(p_i) = 8 - 3i,\ wt(q_j) = 12 - 3j$.

On the other hand, let

$$S = S(3,4) = \oplus_m S_m$$

where S_m is the homogeneous part of degree m of S. It is known that $S_m \neq 0$ only if m is a multiple of 3 (cf. §3).

Theorem 1. *(i) The evaluation map of type* E_7

$$\phi_7 : S(3,4) \longrightarrow R(E_7) = \mathbf{Q}[p_0, p_1, q_0, q_1, q_2, q_3, q_4]$$

is a graded homomorphism from $S(3,4)$ *to* $R(E_7)$ *with weight ratio* $3:14$ *in the sense that* ϕ *sends* S_{3d} *to* $R(E_7)_{14d}$ *for all* d.

(ii) The evaluation map of type E_6

$$\phi_6 : S(3,4) \longrightarrow R(E_6) = \mathbf{Q}[p_0, p_1, p_2, q_0, q_1, q_2]$$

[1] see the comments at the end of the paper.

is a graded homomorphism from $S(3,4)$ to $R(E_6)$ with weight ratio $3:8$ in a similar sense.

THEOREM 2. *Let $D \in S(3,4)$ denote the discriminant of a plane quartic: its characteristic property is that $D \in S_{27}$ and $D(f) \neq 0$ if and only if $f = 0$ is smooth. Then the image $\phi(D)$ under the evaluation map ϕ of type E_r (r=7,6) is equal, up to a constant, to the "discriminant" δ of $R(E_r)$ which is defined as the square of the basic anti-invariant of $W(E_r)$; the weight of δ is 126 or 72 for $r = 7$ or 6.*

THEOREM 3. *(i) For $r = 7$, the evaluation map ϕ_7 is injective.*

(ii)For $r = 6$, the evaluation map ϕ_6 has a nontrivial kernel which contains a projective invariant J of degree 60.[2]

For a graded integral domain R, $F(R)$ will denote the field of fractions of R, and $F(R)_{(0)}$ will denote the subfield of homogeneous fractions (i.e. the fractions a/b with $a, b \in R$ of the same weight).

For $S = S(3,4)$, $F(S)_{(0)}$ can be considered as the function field of the moduli space $\mathcal{M}_3$ of curves of genus 3.

THEOREM 4. *Let $P = \mathbf{Q}[I_1, \ldots, I_6, I_9]$ be the polynomial subring of $S = S(3,4)$ generated by the Dixmier's system $\{I_d\ (d = 1, \ldots, 6, 9)\}$, I_d being a suitable projective invariant of degree $3d$. Then we have the algebraic extensions*

$$F(P)_{(0)} \subset F(S)_{(0)} \subset F(\mathbf{Q}[\lambda])_{(0)}$$

with the extension degree

$$[F(S)_{(0)} : F(P)_{(0)}] = 50, \quad [F(\mathbf{Q}[\lambda])_{(0)} : F(S)_{(0)}] = 24.$$

REMARK. (1) Note that both

$$F(P)_{(0)} = \mathbf{Q}(I_d/I_1^d\ (d = 1, \ldots, 6, 9))$$

and

$$F(\mathbf{Q}[\lambda])_{(0)} = \mathbf{Q}(p_0/q_4^6, p_1/q_4^4, q_0/q_4^9, q_1/q_4^7, q_2/q_4^5, q_3/q_4^3)$$

are rational fields (i.e. purely transcendental extensions) over $\mathbf{Q}$. The famous rationality question of the moduli space $\mathcal{M}_3$ of curves of genus 3 is equivalent to asking whether $F(S)_{(0)}$ is a rational field or not. This was answered by Katsylo [3] by a representation-theoretic method. Our approach might be of some use to this question, from a more geometric point of view.

(2) The explicit form of the invariants I_d in the Dixmier's system is not necessary to prove Theorem 4, but we shall give it in [11] for a possible use in future.

3. Proof of Theorems. We keep the notation introduced in the above.

First recall that, for any homogeneous invariant $I \in S = S(3,4)$ of degree m ($I \in S_m$), we have

$$I(F^g) = \det(g)^w I(F)\ (\forall g \in GL(3))$$

[2] see the comments at the end of the paper

for some integer w, which is determined by $4m = 3w$ (by comparing the degree in generic coefficients of g). Thus, if $I \neq 0$, $m = 3d$ and $w = 4d$ for some integer d.

Proof of Theorem 1. The key point is the weighted homogeneity of f_λ. For the normal form of type E_7, f_λ is a weighted homogeneous polynomial of total weight 18, if we fix $wt(x) = 6$ and $wt(t) = 4$. Namely we have

$$f_{\lambda'}(u^6 x, u^4 t) = u^{18} f_\lambda(x,t) \quad (\forall u \in \mathbf{G}_m)$$

with $\lambda' = (u^{12}p_0, u^8 p_1, \ldots, u^6 q_3, u^2 q_4)$.

Let g be the diagonal matrix $g = [1, u^6, u^4] \in GL(3)$; note $\det(g) = u^{10}$. Then we have from the above

$$(f_{\lambda'})^g(x,t) = u^{18} f_\lambda(x,t).$$

For any $I \in S_{3d}$, we have then

$$(u^{10})^{4d} I(f_{\lambda'}) = (u^{18})^{3d} I(f_\lambda)$$

which implies

$$I(f_{\lambda'}) = u^{14d} I(f_\lambda) \quad (\forall u \in \mathbf{G}_m).$$

This proves that $\phi_7(I) = I(f_\lambda)$ has weight $14d$ for any $I \in S_{3d}$. Thus part (i) of Theorem 1 is shown.

For the normal form of type E_6, f_λ is a weighted homogeneous polynomial of total weight 12 by taking $wt(x) = 4$ and $wt(t) = 3$. The same argument as above shows part (ii) of Theorem 1.

Proof of Theorem 2. Since the discriminant D of a plane quartic has degree 27 ($D \in S_{27}$), $\phi_7(D)$ has weight $9 \cdot 14 = 126$ and $\phi_6(D)$ has weight $9 \cdot 8 = 72$ by Theorem 1. Hence $\phi(D) \in \mathbf{Q}[\lambda]$ has the same weight as the discriminant δ of $R(E_r)$ (= the number of the roots in E_r) for $r = 7, 6$.

To prove $\phi(D) = \delta$ (up to a constant), the simplest would be to assume the knowledge of singularity theory. From this standpoint, note first that the plane quartic Γ_λ is smooth at the points at infinity (i.e. on $x_0 = 0$). Thus it will be smooth if and only if the affine curve $f_\lambda = 0$ is smooth. By Jacobian criterion, the latter condition is equivalent to the smoothness of the affine surface $S'_\lambda : y^2 = f_\lambda$ (since char $\neq 2$).

Now the singularity theory tells us that the family $y^2 = f_\lambda$ parametrized by $\lambda \in \mathbf{A}^7$ is a so-called semi-universal deformation of the E_r-singularity $y^2 = x^3 + xt^3$ ($r = 7$) or $y^2 = x^3 + t^4$ ($r = 6$) and that S'_λ is smooth if and only if $\delta(\lambda) \neq 0$.

Therefore we have $\phi(D) \neq 0 \Leftrightarrow \delta(\lambda) \neq 0$, proving the assertion.

We give here an alternative proof based on the theory of Mordell-Weil lattices (MWL) (cf. [6], [7], esp. [8, Th.5]). We consider the elliptic curve

$$E = E_\lambda : y^2 = f_\lambda = x^3 + \cdots$$

defined over $K = k(t)$, k being the algebraic closure of $\mathbf{Q}(p_i, q_j)$. To fix the idea, suppose f_λ is of type E_7 and λ is generic over $\mathbf{Q}$ (i.e. p_i, q_j are algebraically independent over $\mathbf{Q}$). Then the structure of the Mordell-Weil lattice $E(K)$ is isomorphic to E_7^*, the dual lattice of the root lattice E_7, with the narrow Mordell-Weil lattice $E(K)^0$ being isomorphic to E_7. Corresponding to the 56 minimal vectors of norm 3/2 in E_7^*, there are 56 $k(t)$-rational points $P = (x, y)$ of the form:

$$x = at + b, \quad y = ct^2 + dt + e$$

([6], Lemma 9.1). A nice fact is that the map $P \mapsto c$ extends to a group homomorphism $sp : E(K) \to k$ (the specialization map at $t = \infty$, up to a constant), which is injective for λ generic.

We can choose $\{P_1, \dots, P_7\} \subset E(K)$ such that $\langle P_i, P_j \rangle = \delta_{ij} + 1/2$ (see [8], [10]); they generate a subgroup of index 3 in $E(K)$. Then $c_i = sp(P_i) \in k$ $(i = 1, \dots, 7)$ are algebraically independent over $\mathbf{Q}$, and the Weyl group $W(E_7)$ acts on the polynomial ring $\mathbf{Q}[c_1, \dots, c_7]$ in such a way that the ring of invariants is equal to $\mathbf{Q}[p_0, p_1, q_0, \dots, q_4]$. Moreover the coefficients $a, b, \dots, e$ defining $P = (x, y)$ belong to $\mathbf{Q}[c_1, \dots, c_7]$ for all P.

The basic anti-invariant in $\mathbf{Q}[c_i]$ is the product of 63 linear forms:

$$c_i - c_j (i < j), c_i - v, v - c_i - c_j - c_k (i < j < k)$$

where $v = (\sum_i c_i)/3$, which are the image of half of the 126 roots in $E(K)^0 \simeq E_7$. The discriminant $\delta(\lambda)$ is the square of this anti-invariant up to a constant.

Now we consider specializing the generic parameter λ to any $\lambda' \in \mathbf{A}^7$. If the MWL does not degenerate under this specialization, we have the 126 roots in $E_{\lambda'}(K)^0 \simeq E_7$. Recall that a root in E_7 corresponds to a rational point $Q = (x, y)$ of the form

$$x = t^2/u^2 + at + b, \quad y = t^3/u^3 + ct^2 + dt + e$$

with $u = sp(Q) \neq 0$. Therefore none of the 63 linear forms above corresponding to the roots vanish under the specialization, and we have $\delta(\lambda') \neq 0$. In other words, $\delta(\lambda') = 0$ implies the degeneration of MWL (this is the MWL-analogue of "vanishing cycles" in the singularity theory).

Further note that the degeneration of MWL occurs if and only if the affine surface $S_{\lambda'}$ acquires singularities, since both conditions are equivalent to the existence of a reducible fibre in the associated elliptic fibration at $t \neq \infty$.

Thus we have the implication $\delta(\lambda') = 0 \Rightarrow D(\lambda') = 0$. Comparing the degree, we conclude that $\delta = \phi(D)$ up to a constant.

The case of E_6 can be treated in a similar way. □

Remark. It is also possible to directly verify $\phi(D) = \delta$ (up to a constant) by means of computer algebra (cf. [11]).

Proof of Theorem 3. The injectivity of the homomorphism ϕ_7 is clear, because a generic plane quartic can be put in the normal form $\Gamma_\lambda : f_\lambda = 0$ (over a field of rationality of the curve and a flex) ([8, §1]).

To prove the second part, we use the notation in the above proof of Theorem 2. For each of the 56 $k(t)$-rational points $P = (x, y) \in E_\lambda$, we have the identity in t:

$$(ct^2 + dt + e)^2 = f_\lambda(at + b, t).$$

This means that the line $L : x = at + b$ in $\mathbf{P}^2$ is a bitangent to the plane quartic $\Gamma_\lambda : f_\lambda = 0$, i.e. we have $L \cdot \Gamma_\lambda = 2A + 2B$ for the two points $A, B \in \Gamma_\lambda$, which are determined by the equation $ct^2 + dt + e = 0$. In this way, we get all the 28 bitangents to Γ_λ, since $\pm P = (x, \pm y)$ give the same bitangent.

Consider the product

$$J = \prod_{\nu=1}^{28} (d_\nu^2 - 4c_\nu e_\nu)$$

which is an element of $\mathbf{Q}[c_1, \ldots, c_7]$ of weight $28 \cdot 10 = 280$. Since the Weyl group $W(E_7)$ acts (transitively) on the 56 minimal vectors, J is an invariant. Hence

$$J \in \mathbf{Q}[c_1, \ldots, c_7]^{W(E_7)} = \mathbf{Q}[p_0, p_1, q_0, \ldots, q_4].$$

LEMMA 5. *For the normal form of type E_7, the vanishing of the invariant J is equivalent to the existence of a special flex.*

Proof Assume $J(\lambda') = 0$ for $\lambda' \in \mathbf{A}^7$. Then some factor in the product must be 0, so the two points of contact of a bitangent coincide. In other words, we have

$$L \cdot \Gamma_{\lambda'} = 4A$$

for this bitangent L. Then this point A is a special flex of $\Gamma_{\lambda'}$ with flex tangent L. The converse is clear. □

By the lemma, the vanishing of J has an invariant meaning in the sense of projective geometry. Hence J is a projective invariant, or more precisely, we have $J = \phi_7(I)$ for a unique projective invariant $I \in S(3, 4)$. In view of Theorem 1, I has degree 60. Finally I belongs to the kernel of the map ϕ_6, since the normal form of type E_6 has a special flex by definition.

This complete the proof of Theorem 3. □

Presumably the invariant I constructed above should be the generator of $\mathrm{Ker}(\phi_6)$, but it is not yet proven.

Proof of Theorem 4. By [2], P is a polynomial subring of S such that S is integral over P. Then by [5, Lemma 1], $F(S)$ is an algebraic extension of $F(P)$ of degree $N(1)$ if $N(T)$ denotes the numerator of the generating function of S. In our case ([5, Appendix]), the generating function is equal to

$$\frac{N(T)}{\prod_{d=1}^{6}(1 - T^d) \cdot (1 - T^9)} \quad (T = t^3)$$

with

$N(T) = 1 + T^3 + T^4 + T^5 + 2T^6 + 3T^7 + 2T^8 + 3T^9 + 4T^{10} + 3T^{11} + 4T^{12} + 4T^{13} + 3T^{14} + 4T^{15} + 3T^{16} + 2T^{17} + 3T^{18} + 2T^{19} + T^{20} + T^{21} + T^{22} + T^{25}$.

Hence we have $N(1) = 50$, which shows $[F(S) : F(P)] = 50$. It follows easily that we have $[F(S)_{(0)} : F(P)_{(0)}] = 50$.

On the other hand, we view $F(S)$ as a subfield of $\mathbf{Q}(p_0, p_1, q_0, \ldots, q_4)$ via the injective map ϕ_7 (Theorem 3). Suppose Γ is a generic plane quartic. Then it has

24 flexes, say ξ_ν, which are all ordinary flexes. For each choice of the flex ξ_ν, Γ is isomorphic to Γ_λ for some $\lambda = (p_i, q_j) \in \mathbf{A}^7$, with ξ_ν mapped to $(0,1,0) \in \Gamma_\lambda$; moreover $\lambda = \lambda^{(\nu)}$ is uniquely determined by the condition $q_4 = 1$ for the given pair (Γ, ξ_ν) (see [8, §1]). Thus the 24 values of $\lambda^{(\nu)}$ corresponding to the 24 flexes are mutually conjugate over $F(S)_{(0)}$. It follows that $[F(\mathbf{Q}[\lambda])_{(0)}, F(S)_0] = 24$. □

Acknowledgement. The author owes the following valuable remarks to the referee:

(1) about the terminology. *A point of undulation* is a more standard word for a special flex used in this paper and [8]. See Salmon's book [12], no. 50, p. 37 and no. 247, p. 218.

(2) about a characterization of undulation. Theorem 3 (ii) is classically known, and is a special case of the following fact. Salmon describes a projective invariant of degree $6(m-3)(3m-2)$ for a plane curve of degree m whose vanishing expresses the condition that the curve has a point of undulation ([12], no. 400, p. 362).

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