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REVIEW OF GEOMETRY AND ANALYSIS*

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In this article, we shall discuss what the author considers to be important in geometry and related subjects.

Since the time of the Greek mathematicians, geometry has always been in the center of science. Scientists cannot resist explaining natural phenomena in terms of the language of geometry. Indeed, it is reasonable to consider geometric objects as parts of nature. Practically all elegant theorems in geometry have found applications in classical or modern physics. In order to understand the future of geometry, it is perhaps useful to review what was known in the past. Clearly what I consider to be important may not be viewed to be so by others. Also, we should always keep in mind that what is fashionable now may not be so tomorrow.

A theory can be judged to be successful only if its consequences help us understand the basic structure or the beauty of geometry.

While we shall divide the subject into several categories, the division is artificial, as the development of each section depends on other sections heavily.

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I. Submanifolds. Many very important questions in classical geometry remain unanswered. Descriptions of surfaces in Euclidean three space have been important in the subjects of computer graphics and data compression. They also play an important role in modern filmmaking.

Indeed, the theory of surfaces in three space is a central subject in geometry. Many difficult questions remain unanswered. Starting from the time of Gauss [63], geometers have always been interested in the interplay between the intrinsic internal metric structure of surfaces and their extrinsic geometry in the ambient space.

A. Isometric embedding of surfaces. An important well-known question is to characterize those intrinsic metrics on a surface which can be realized as embeddings into three space. Minkowski made the first major progress on this problem by proving that any convex polyhedron can be so realized [152]. For smooth surfaces with positive curvature, this problem is called the Weyl problem, since Weyl found the first significant estimate for the problem [221]. H. Lewy solved the problem in the real analytic category [124], and Pogorelov [169] and Nirenberg [162] solved it in the smooth category. There is also recent work by Y.Y. Li and P. Guan [85].

An important reason that the Weyl problem can be solved is that its solutions must be unique–a theorem due to Cohn-Vossen [46] and Pogorelov [168]. Uniqueness here means that any isometric embedding of the surface is related to any other by a

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rigid motion in three space. Such global rigidity is difficult to prove for nonconvex surfaces. A.D. Alexandrov studied the next easiest case, namely, those surfaces for which the integral of the positive part of the curvature is 4π [1]. He could only prove the rigidity for real analytic metrics. Nirenberg made a beautiful attempt to settle the smooth case [163]. Nirenberg's argument would succeed if one can prove that there is at most one closed asymptotic curve in the surface. It is still an outstanding question to understand the behavior of the asymptotic curves on a surface. They are of fundamental importance for the global isometric embedding problem for surfaces with negative curvature, as they are the characteristic curves for the isometric embedding problem. The isometric problem for surfaces of negative curvature is a very interesting nonlinear hyperbolic problem. As such, it is very difficult to prove global existence theorems for such surfaces. In fact, the celebrated Hilbert-Efimov theorem says that a complete surface with strongly negative curvature cannot be isometrically embedded into three-dimensional Euclidean space [55].

A significant existence theorem was proved by Hong [98], who assumes the curvature decays in the right way. It is a challenging problem to give a transparent and quantitative proof of Efimov's theorem. In other words, take a geodesic disk of radius r whose curvature is not greater than -1 and equal to -1 at the center. What is the largest r so that it can be embedded into $\mathbb{R}^3$ and the second fundamental form is bounded by a given constant?

Global rigidity is false for a general compact surface even when the metric is real analytic. However, an important outstanding problem in classical geometry is whether there exists a continuous family of isometric embeddings of a compact surfaces in $\mathbb{R}^3$ which does not arise from rigid motions.

R. Connelly [50] and D. Bleecker [15] gave beautiful counterexamples to this rigidity conjecture among polyhedra. It is unlikely that those methods can be improved to give smooth examples. There should be (extrinsic) invariants for isometric motions of closed surfaces. Hopefully there are only a finite number of such invariants, so that the space of isometric motions is finite-dimensional. Bleecker's examples show that the volume enclosed by such surfaces is not an invariant. One should be able to generalize these polyhedral examples to piecewise-smooth examples and understand whether the motions are due to the motions of the edges.

Also, Cohn-Vossen developed a theory of infinitesimal rigidity for closed surfaces [47]. The equation involved is linear and thus is easier to handle. But it is only elliptic if the curvature is positive. Hence for surfaces whose curvature changes sign, it is a difficult problem to understand the equation.

We know that there do exist non-trivial first order isometric deformations for some closed surfaces. The question remains how to characterize such surfaces. These are natural uniqueness questions for mixed-type equations on global surfaces.

Of course, we can also ask all the above questions for open surfaces. Cartan proved that every real analytic surface can be locally isometrically embedded into $\mathbb{R}^3$ [35]. The question for a smooth metric is much more difficult. It is an important theorem of C.S. Lin that smooth surfaces with non-negative curvature can be locally isometrically embedded into $\mathbb{R}^3$ [140]. He also settled the problem when the gradient of the curvature is not zero [141]. It remains an open problem whether such an assumption can be dropped.

One can formulate a boundary-value problem for isometric embeddings of compact surfaces with boundary in two different ways. One is the Neumann problem, which is

to require the mean curvature of the boundary to be a given function H. (Clearly we require $H^2 \geq K$.) Of course, one may also require the image of the curve to be a subset of some surface. The other is the Dirichlet problem, which is to require the image of the boundary to be a given Jordan curve. In the first problem, if the curvature is positive, the mean curvature can be bounded and regularity can be guaranteed. In the latter problem, an important necessary condition on the given curve is that the length of its second fundamental form must dominate the geodesic curvature of the original boundary. Hong has made important progress on this problem [97].

For closed surfaces with no boundary in $\mathbb{R}^3$, there are many constraints on their intrinsic metrics. The first important constraint is that the curvature must be positive at one point and that $\int_\Sigma K^+ \geq 4\pi$, where $K^+ = \max(K, 0)$.

Nirenberg [163] proved that if $\int_\Sigma K^+ = 4\pi$, each component of the set $\{K < 0\}$ is bounded by two closed convex planar curves. This fact gives a constraint on metrics that can be isometrically immersed into $\mathbb{R}^3$. Is there any constraint on the topology of the sets $\{K > 0\}$ and $\{K < 0\}$ in general?

If Σ is embedded into $\mathbb{R}^3$, it bounds a domain Ω. It is interesting to relate the spectrum of Σ to the spectrum of Ω. A simple argument shows that the volume of Ω is bounded by $\frac{\sqrt{2}}{3}\mathrm{Area}(\Sigma)\lambda_1^{-\frac{1}{2}}$, where λ_1 is the first eigenvalue of Σ. It follows from this statement and the work of Korevaar [116] that $\dfrac{\mu_i(\Omega)^{3/2}}{\lambda_i(\Sigma)\sqrt{\lambda_1(\Sigma)}} \geq C_g > 0$, where $\mu_i(\Omega)$ is the i-th Dirichlet eigenvalue of Ω and $\lambda_i(\Sigma)$ is the i-th eigenvalue of Σ. Here C_g is a constant depending only on the genus of Σ. What is the optimal C_g and is there a surface that realizes such an optimal constant? Does the set $\{\mu_i(\Omega)\}$ determine $\{\lambda_i(\Sigma)\}$ and vice versa? If a surface Σ can be isometrically embedded into $\mathbb{R}^3$, the spectrum $\{\lambda_i(\Sigma)\}$ is probably constrained. What is the constraint? It is known that asymptotic behavior of $\{\lambda_i(\Sigma)\}$ is closely related to the dynamics of the geodesic flow of Σ. Is the dynamics of geodesic flow constrained? Since Σ cannot have curvature everywhere nonpositive, should the geodesic flow of a closed embedded surface Σ be non-ergodic? If it is not ergodic, can one describe the invariant regions of the geodesic flow in terms of extrinsic geometry? What is the entropy of the flow? Is it possible to describe the geodesics in terms of the coordinates of $\mathbb{R}^3$?

B. Different geometries. One can study properties of surfaces that are invariant under a bigger group than the orthogonal group. There are the special linear group, the conformal group and the projective group each acting on $\mathbb{R}^3$. We can ask for properties of a surface Σ that are invariant under these groups. They are called affine, conformal and projective geometries respectively. Many interesting questions remain unanswered for these geometries.

For affine geometry, important invariants are the concepts of the affine metric (which is the second fundamental form divided by the fourth root of the Gaussian curvature) and the affine normal. When the affine normals of the surface focus to a point, the surface is called affine sphere. Such surfaces were studied extensively by Calabi [23, 24], Calabi-Nirenberg [26], and Cheng-Yau [39, 40]. From these works, we know that for any convex cone, there is a complete affine sphere asymptotic to the cone. It would be very interesting to construct such an affine sphere efficiently. Can one find a representation in terms of pseudoholomorphic functions (see [218])? Are there cases that such spheres can be written in closed form? (The surface $xyz = 1$ is asymptotic to the coordinate cone.) This is especially interesting when the cone is a polyhedral cone. Can we compute the affine metric in these cases? Of course, one asks

the same question for higher dimensions, and it is related to the linear optimization problem [110].

Up to now, most of the progress assumes convexity of the surfaces. What happens if the surface has negative curvature? What are those affine spheres whose induced metrics are complete with negative curvature?

There is a concept of an affine maximal surface, which is defined by a fourth-order elliptic equation (see e.g. [25]). One can therefore fix the boundary and the normal direction along the boundary for the surface. If we fix the boundary to be a triangle, is there an efficient way to solve the boundary value problem for the affine maximal surface? If we fix a polyhedron in $\mathbb{R}^3$, we can attempt to form a C^1-surface that passes through all the edges of the polyhedron so that in each face, the surface is affine maximal. Among all such C^1-surfaces, we can try to find one with maximum total affine area. Recently, Trudinger and Wang [214] solved the Bernstein problem for affine maximal surfaces. Hopefully, the future will bring more estimates for affine maximal surfaces.

As we discuss in Cheng-Yau [40], the affine metric of the affine sphere in $\mathbb{R}^{n+1}$ after a Legendre transform becomes a projective invariant metric on a convex domain in $\mathbb{R}^n$. It is therefore interesting to relate projective-invariant properties of surfaces in $\mathbb{R}^n$ to affine properties of hypersurfaces of the affine sphere in $\mathbb{R}^{n+1}$.

It is known that if a closed surface in $\mathbb{R}^3$ is infinitesimally (metrically) rigid, its image under a projective transformation is also infinitesimally rigid. Is the statement true for the global rigidity of a closed surface? Can one formulate the rigidity problem in terms of projective or affine geometry?

There are many interesting classical questions for curves in complex surfaces in $\mathbb{CP}^3$. We can also ask them for surfaces in $\mathbb{R}^3$ making use of projective geometry. For example, for an affine sphere asymptotic to a polyhedral cone of n sides in $\mathbb{R}^3$, among all closed curves on the affine sphere which are cut out by rational functions on $\mathbb{R}^3$, which are of lowest possible degree? Of course, one can try to find some way to count such curves also. Which affine spheres can be defined by algebraic polynomials? For a closed surface in $\mathbb{R}^3$ defined by algebraic polynomials, can one classify those which admit geodesics defined also by algebraic polynomials?

Algebraic surfaces in $\mathbb{R}^3$ are clearly important surfaces. However, we know virtually nothing about them. There is some information about the number of their components. But their geometry is clearly special. The number of umbilical points, the number of closed geodesics, the number of closed asymptotic lines, the topology of the regions of positive curvature, and the eigenfunctions of the Laplacian are all interesting invariants related to the polynomials that define the surface. For example, one would like to estimate them in terms of the way that these surfaces intersect straight lines. On the one hand, we have techniques and information that we learn from algebraic geometry. We can complexify the surfaces and obtain information related to the topology and the incidence relations among them. On the other hand, we want to understand the classical geometry of these surfaces using classical methods–partial differential equations or connections from real projective geometry. It will be very important to bridge these two approaches. When we complexify the surfaces, in most cases, we can find a complete Kähler-Einstein metric with negative scalar curvature [207]. Since the metric is invariant under an anti-holomorphic involution, we have a canonical metric on the surface. (Canonical in the sense that if there is a non-singular bijective algebraic map from one algebraic surface to another one, it is an isometry

of the canonical metrics.) Some theorems on real algebraic varieties can be proved using the canonical metrics. How do we relate this canonical metric to the Euclidean metric induced on the surface?

In complex algebraic geometry, it is very important to understand algebraic curves in an algebraic manifold. What is the role of real algebraic curves in a real algebraic surface? What is the topology and geometry of these curves? How do we bound the number of components and the geodesic curvatures of the curves in terms of the degree of polynomials defining them?

Potentially, every closed surface can be approximated by a real algebraic surface of large degree. What is the minimal degree needed to approximate a given surface? In other words, given $\varepsilon > 0$, what is the minimal degree of an algebraic surface whose Hausdorff distance is less than ε from a given surface? Can one estimate it in terms of a suitable calculable geometric quantity of the surface (e.g. the L^2 norm of the mean curvature of the surface)? If we minimize the L^2 norm of the mean curvature of surfaces with degree less than a given integer, what kind of algebraic surfaces do we get?

Real algebraic surfaces provide a rich class of natural singularities. It is very interesting to study the behavior of the principal curvatures near the singularities. The behavior of the eigenfunctions of the Laplacian near such points should also be very interesting.

When a surface is defined by a real algebraic set, there is a (singular) foliation attached to it. The study of such a continuous family should be very helpful in understanding the algebraic surface.

Conversely if we have an algebraic one-form which defines a (possibly singular) foliation, it will be interesting to find the closed leaves and bound the number of such leaves. When are the leaves algebraic?

C. Minimal surfaces. There are several important surfaces studied in the literature. Most of these are obtained by variational methods.

The first major class is minimal surfaces. These are surfaces with mean curvature equal to zero. When we are given a closed curve in $\mathbb{R}^3$, we look for surfaces bounded by this curve which minimize area. We can restrict the topology of these surfaces to be within certain classes. For example, we can consider all surfaces of genus g. Then the minimum of the area is a number A_g which may depend on the genus. In general, $A_g \geq A_{g+1} \geq \cdots$ [149]. When the boundary curve is smooth, it is a celebrated theorem of Hardt-Simon that for some g_0, $A_{g_0} = A_{g_0+1} = \cdots$ [90]. There is no effective estimate of g_0 and it will be challenging to find such an estimate.

It is not known whether a smooth simple closed curve in $\mathbb{R}^3$ can bound an infinite number of minimal surfaces. (If the curve is real analytic, this cannot happen (Tomi [212]).) Is there a general algorithm to find all minimal surfaces bounded by a given closed curve (or a set of closed curves)? It is still a difficult problem to calculate unstable minimal surfaces bounded by given curves. (There is also no efficient way to calculate area-minimizing surfaces.) It is not clear how big the genus of a stable or unstable minimal surface must be if it is bounded by a given set of smooth curves. Can it be arbitrarily large?

Many fundamental questions remain unsolved regarding the quantitative behavior of minimal surfaces. One well-known question is that of the best constant for the isoperimetric inequality, i.e. what is the best upper bound of $\frac{4\pi A}{L^2}$, where A is the area and L is the length of the boundary. Naturally one conjectures it to be 1. This

is known if the boundary is a Jordan curve (see Li-Schoen-Yau [132]). This question is related to finding the best constant for the Sobolev inequality for minimal surfaces. This is the L^1 version. It is also interesting to find an L^p version.

It is a very interesting question to find a lower estimate of the first Neumann eigenvalue and an upper estimate of the first Dirichlet eigenvalue of the minimal surface.

While the existence theorem for a single closed curve has been solved, the question for multiple boundary components is largely unknown. Also, when the boundary is singular, very little is known.

Given a one-dimensional skeleton of a simplicial complex in $\mathbb{R}^3$ which is homeomorphic to S^2, we can minimize the area of the image of maps from S^2 into $\mathbb{R}^3$ passing through this one-dimensional skeleton while counting the area of the image with multiplicity one. What type of singularities can there be? For example, for a one-dimensional set which is homeomorphic to the skeleton of the tetrahedron, can one find a minimal set homeomorphic to the cone over this skeleton?

The topological classification of complete proper minimal surfaces in $\mathbb{R}^3$ is close to being complete thanks to the efforts of Meeks, Hoffman, Jorge, Karcher, Rosenberg, and others (see e.g. [58, 96, 148]). However, it is still not clear how to classify the associated conformal structure together with the quadratic differential associated to the second fundamental form (although there has been progress in the recent work of Collin-Kusner-Meeks-Rosenberg [49]). Besides the problem of classification, many interesting questions about the analysis on these surfaces remain unanswered.

For example, we can ask questions about harmonic functions defined on complete minimal surfaces in $\mathbb{R}^3$. If the harmonic function is positive, is it asymptotic to a constant at each end of the surface? If a harmonic function U has at most polynomial growth, does there exist a constant α so that

$$0 < \overline{\lim_{|x|\to\infty}} \frac{U(x)}{|x|^\alpha} < \infty\,?$$

Is the set of such α's infinite and is it asymptotic to the integers? For each α, there should be a finite-dimensional space of such harmonic functions. The space of all polynomial-growth harmonic functions should span the space of all polynomial-growth functions for a suitable weighted Sobolev norm.

The spectrum of the Laplacian on a complete, properly embedded minimal surface should not differ much from that on $\mathbb{R}^2$. Moreover, we want to know how to describe the continuous spectrum and the approximate eigenfunctions.

Perhaps it is interesting to study the spectrum of the operator $-\Delta + \|x\|^2$, as it is discrete. It should be related to the critical points of $\|x\|^2$ and the classical paths joining these critical points. (Classical paths are paths that are critical with respect to the Lagrangian $\int |\nabla u|^2 + \int \|x\|^2 u^2$.)

Using the construction of conjugate minimal surfaces, it is easy to construct nontrivial continuous families of minimal surfaces. However, only isolated members of such families are embedded. Some classical examples of embedded minimal surfaces, such as the Riemann staircases and Scherck towers, do come in families. Invariants derived from these families can be used as moduli for more general constant mean curvature surfaces (see e.g. the recent work of Grosse-Brauckmann, Kusner and Sullivan [83] on classifying constant mean curvature pairs of pants). The work of Kapouleas [108, 109] has been fundamental in the moduli theories of both minimal and constant

mean curvature surfaces in $\mathbb{R}^3$.

Most complete properly immersed minimal surfaces are asymptotically flat. It will be quite interesting to understand the dynamics of the geodesic flow on these surfaces: When geodesics emanate from infinity, how are they scattered?

It is known [160] that there are complete minimal surfaces properly immersed into the ball. What is the geometry of these surfaces? Can they be embedded? Since the curvature must tend to minus infinity, it is important to find the precise asymptotic behavior of these surfaces near their ends. Are their spectrums discrete?

D. Closed extremal surfaces. The simplest closed extremal surfaces in $\mathbb{R}^3$ are those which extremize area while fixing the enclosed volume. Those surfaces have constant mean curvature. Wente solved the classical problem that an immersed torus with constant mean curvature does exist [220]. While many more examples have been constructed, classification of closed surfaces with constant mean curvature is still far from complete.

The lines of curvature are planar for the Wente surfaces. It will be interesting to understand the combinatorial structure of the lines of self-intersection, the topology of the regions of positive curvature on the surfaces, and their asymptotic lines.

Another important class of surfaces are those which are extremal with respect to the functional $\int H^2$. Leon Simon [189] proved the existence of a torus achieving a global minimum and made important steps toward proving existence for surfaces of higher genus. It is of course very interesting to compute the possible values of $\int H^2$ for such a surface. It will settle the conjecture of Willmore that the global minimum of $\int H^2$ for a torus is given by $2\pi^2$ [222]. There is also a piecewise-linear version of the Willmore problem (see Hsu-Kusner-Sullivan [101]), but it is not yet clear how to guarantee the existence of minimizers even in this case.

In a three-manifold, we can extremize the functional $\sqrt{A}\left(1 - \frac{1}{16\pi}\int \|H\|^2\right)$, where A is the area of the surface. Up to a normalization, this quantity is called the Hawking mass of the surface [92].

E. Motion of surfaces. There are many ways that surfaces move in $\mathbb{R}^3$. We have already mentioned the very interesting question of how to describe motions of surfaces which preserve their intrinsic metric. What is a good way to study these motions (modulo the action of Euclidean motions)? If we move the boundary of the surfaces, is the motion of the compact surface determined up to finite parameters? This is a difficult problem if the curvature of the surface can be negative.

A more general motion is the motion of surfaces that allow the metric to change according to the second fundamental form. In other words, if $X(t)$ is the family of embeddings of the surfaces, we request that $\frac{d}{dt}\langle dX(t), dX(t)\rangle$ be determined by a symmetric tensor determined by the second fundamental form at $X(t)$. What kind of singular behavior of such a family of surfaces do we expect? This question is much more well posed for convex surfaces because of the Weyl theorem. What is the condition to preserve the convexity of surfaces under such a motion?

Much more well known motions of surfaces are those equating the velocity $\frac{dX}{dt}$ with some scalar multiple of the normal of the surface. The scalar can be the curvature, mean curvature or inverse of the mean curvature with suitable sign attached. A beautiful theory has been developed by Hamilton, Huisken and others (e.g. [87], [102]). Complete understanding of the singularities has not been accomplished (except the work of Huisken and Sinestrari [104] for surfaces of positive mean curvature). What

happens to the flow after a singularity develops?

During the motion of these surfaces, it will be very interesting to watch how basic geometric quantities move. These include the behavior of the second fundamental form, the geodesics, the umbilical points, the spectrum of the surfaces, etc.

Singularities that are created by natural motions are perhaps the most natural singularities that occur in nature. Perhaps for simple motions, there is a certain "resolution of singularities" theorem to help to understand how singularities develop. A typical way is to look at the "graph" of the equation and the singularity is obtained by projection or intersection with planes. Can one generalize this kind of construction to understand the motion of surfaces in three space?

Besides motions defined by the previous methods, there are wave motions of surfaces in three space. When we watch water drops, surface waves, and vibrating membranes, we see beautiful geometric pictures. How can we expect to describe these pictures even though we poorly understand the equations governing their formation?

For the vibrating membrane, it is well-known that the wave motion is well approximated by the eigenfunction expansion of the membrane. How does one explain such an approximation? There are two equations related to a vibrating membrane. One is that $\frac{dX^2}{dt^2} = -HN$, where H is the mean curvature and N is the normal of the surface. One can also study time-like minimal hypersurfaces in the flat Minkowski spacetime. In both cases, we know very little about the global time behavior of the hypersurfaces. For the linear wave equation, there are obvious waves periodic in time. It is not clear what this means for the above nonlinear equations.

F. Representation of surfaces in Euclidean space. Minkowski gave the first systematic way to represent a surface in $\mathbb{R}^3$. He successfully treated the case of a convex polyhedron [152]. In general, the Minkowski program is to map the surface Σ to the sphere S^2 via the translation of the normal vector to the original (the Gauss map). If the surface Σ is strictly convex, the Gauss map is one to one. Hence all the information of the surface can be presented on S^2 via the Gauss map. The Gauss curvature can, in particular, be written as a function on S^2. The famous Minkowski problem is to reproduce the surface Σ once we know the curvature on S^2. The surface Σ is smooth if the given curvature function is smooth. It is also unique up to translation. These statements were proved by Pogorelov [169] and Nirenberg [162].

There are several important questions which remain to be answered. How do we solve the Minkowski problem effectively by numerical means? When we discretize the sphere, it is clear that the discretization should be adaptive to the distribution of the value of the curvature function. Where the value of the curvature function is large, there should be more nodes in a neighborhood of those points. What is the best way to choose these points? In many applications of classical geometry, we need to integrate over Σ. Is it possible to give an efficient discretization of Σ via the discretization of S^2 so that the integral of any smooth function is best represented by values of the function at these points?

The Minkowski problem is well studied when surface is convex with no boundary. It is difficult, however, to drop either the assumption of convexity or that of no boundary.

Given a closed curve τ in $\mathbb{R}^3$ and a positive function K defined on S^2, when can we find a convex surface Σ bounded by τ so that the Gauss curvature of Σ is given by K via the Gauss map of Σ? There are quite a few compatibility conditions on

K and τ. First, τ has to bound some convex surface $\widetilde{\Sigma}$. Second, the integral of $\frac{1}{K}$ over D must be greater than the area of the area-minimizing surface bounded by τ. Third, the difference between 2π and the area of D must be bounded by the integral of the length of the second fundamental form of τ. It would be interesting to know whether these are the only compatibility conditions needed to solve the Minkowski problem with Dirichlet boundary data. One can of course formulate a similar problem by requiring $\partial\Sigma$ to be on a given closed surface.

The Minkowski problem is difficult even for closed surfaces if the curvature is allowed to change sign. It is clear that the part of the surface where the curvature is equal to zero creates ambiguity. Perhaps one should assume real analyticity of K to start out. The Gauss map is no longer one to one. As a result, the Minkowski data is a set-valued map. Generically, the value of the curvature function is a finite set with some order. The order is obtained according to the value of the support function defined by the point on S^2. Can this data determine the closed surface if everything is real analytic in a suitably defined sense?

One possible procedure to represent a surface is to dissect the surface into many pieces along some sets of curves $\{\tau_i\}$. Space curves τ_i can be parametrized by their curvature and torsion. The pieces bounded by τ_i can be parametrized by their Minkowski data. Sometimes it may be natural to choose τ_i to be defined by the zero locus of the curvature or mean curvature. Perhaps the techniques of Guan-Spruck [84] will be useful in this regard.

Let p be a point in $\mathbb{R}^3$. Consider the set of all lines l that pass through p. They intersect a given surface at some points and then are reflected according to geometric optics. The reflection can be continued for a number of times. In this way, we get a set-valued map from the unit sphere with center at p to sets on the surface.

This map gives rise to a density on the unit sphere by pulling back the area density of the surface. What can one say about this density? How does it depend on the choice of the point p?

One should be able to answer this question when the surface is closed and convex and p is in the interior of the surface. Given a density on S^2, can one realize it by a closed surface?

Similarly if we have a plane L disjoint from the surface, it receives light rays issued from p and reflected by the surface. One gets a density on L. It is interesting to see to what extent this density can be used to determine the surface. By moving the locations of p or L, one should be able to determine the surface if it is convex. What happens if the surface is not convex?

G. Minimal surfaces in three-manifolds. Minimal surfaces and extremal surfaces for $\int H^2$ are the most natural special surfaces in three-manifolds. It should not be unreasonable to classify these surfaces in S^3. It will also be very interesting to estimate geometric invariants of these surfaces. Many years ago, the author conjectured that the first eigenvalue of an embedded minimal surface in S^3 is equal to two. While the question is still open, progress has been made by Choi and Wang [43]. Should the zeta functions of minimal hypersurfaces in S^n behave nicely? Can one find arithmetic properties of these zeta functions? Are these analogous to the usual functional equations? The determinant of the Laplacian should have special values.

It will be interesting to see whether there is a non-trivial continuous family of closed minimal surfaces in S^3. If such a family does not exist, there is a finite number of minimal surfaces in S^3 for each genus. How does one classify them and what are

their areas?

Besides its own beauty, the study of minimal surfaces in S^n is related to the study of isolated singularities of minimal submanifolds in $\mathbb{R}^{n+1}$. The cone over a minimal submanifold in S^n is a minimal submanifold in $\mathbb{R}^{n+1}$ with an isolated singularity. Therefore there is a close relation between minimal submanifolds in $\mathbb{R}^{n+1}$ and minimal submanifolds in S^n. The eigenfunctions of a minimal surface in S^n are related to homogeneous harmonic functions on the corresponding minimal submanifold in $\mathbb{R}^{n+1}$. The degree of homogeneity α is related to the eigenvalue of the minimal surface in S^n. In fact the eigenvalues are given by $\alpha^2 + k\alpha$, where k is the dimension of the submanifold in $\mathbb{R}^{n+1}$. There are natural questions related to this correspondence. The dimension of the space of harmonic functions on minimal submanifolds in $\mathbb{R}^n$ was estimated by Peter Li [131] and Colding-Minicozzi [48]. In particular this gives an estimate of the multiplicity of the eigenvalues of minimal submanifolds in S^n in terms of area. When the minimal submanifold is linear, α is an integer. This follows from the removable singularity theorem for harmonic functions. Can one find a reasonable constraint on α in terms of the area of the minimal surface? When α is large, it is asymptotic to a positive integer with an error; how do we estimate this error? The multiplicity of a minimal submanifold is probably largest when the submanifold is the geodesic sphere. Can one prove this statement?

In [133] Peter Li and the author introduced the concept of conformal area of conformal structures on a Riemann surface. It is closely related to the eigenvalues of the Riemann surface. In general, for a given conformal structure on a closed Riemann surface, we can associate to each conformal metric the number $\lambda_i A$, where λ_i is the i-th eigenvalue of the metric and A is the area. By the theorem of N. Korevaar [116], there is an upper bound for such a number and it would be very interesting to find an extremal metric that achieves such a maximum. Many minimal surfaces in the sphere give rise to such extremal metrics. Can one give a precise relation?

For a closed minimal surface in S^n which does not lie in any S^{n-1}, is the cn-th eigenvalue of the surface ≥ 2, where c depends only on the genus of the surface? Is it possible to estimate c? Should it be independent of the genus? In particular, a hyperplane in $\mathbb{R}^{n+1}$ that passes through the origin should cut the surface into at most cn components. This would confirm that nontrivial minimal surfaces in S^3 are split into 2 components by such a hyperplane.

H. Submanifolds in higher dimensional space. Since the fundamental work of John Nash on the isometric embedding of manifolds M^n into $\mathbb{R}^N$ [161], very little progress has been made in understanding such embeddings. The codimension of M^n is too high to talk about any meaningful rigidity question. (The large codimension helps to prove existence of isometric embeddings by topological methods.) In order for the isometric embeddings to be visible, one should find a class of these embeddings whose deformations we can describe completely. As we know, the minimum dimension for a manifold M^n to be isometrically embedded is $\frac{n(n+1)}{2}$. However, for $n \geq 3$, no meaningful theory of rigidity or isometric deformation theory is known for M^n in $\mathbb{R}^{\frac{n(n+1)}{2}}$. (For M^n in $\mathbb{R}^{2n-1}$, there are many more known theorems for rigidity.)

For sufficiently large N, we can minimize the quantity $\int H^2$ among all isometric embeddings of a given manifold M^n into $\mathbb{R}^N$. What are the critical points of this functional?

In general, a complete noncompact manifold M^n may not be isometrically embeddable into $\mathbb{R}^N$ with bounded mean curvature. It is presumably possible if the Ricci

curvature is bounded from below. What is the optimal condition for the manifold to be embeddable with bounded mean curvature? By the theorem of Michael-Simon [151], a suitable isoperimetric inequality must hold for these manifolds.

A more tightly constrained problem is to embed a complete noncompact manifold as a minimal submanifold in $\mathbb{R}^N$. For surfaces, the Weierstrass representation can be used to characterize these metrics. For dimension greater than two, the only known constraints are that the metric is real analytic, its Ricci curvature is non-positive, and the isoperimetric inequality and certain inequalities on the heat kernel [38] hold. (For example, the trace of the heat kernel is pointwise bounded by $\frac{C_n}{t^{n/2}}$, where n is the dimension of the minimal submanifold.) The space of metrics that can be realized on some minimal submanifold must be a thin set; how can we describe it? Is every noncompact manifold diffeomorphic to a complete minimal submanifold in Euclidean space? Is every compact manifold diffeomorphic to a minimal submanifold in the sphere? For dimension greater than 3, very few concrete examples are known for minimal submanifolds. Even minimal graphs have not been classified.

A minimal graph has the property that it is a leaf of a foliation of the Euclidean space by minimal submanifolds. Conversely can one classify foliations of $\mathbb{R}^n$ whose leaves are all minimal submanifolds? Such leaves have to be area-minimizing. Is a codimension-one minimal foliation without singularities necessarily a graph? There are many minimal foliations with certain singularities. For example, one of the leaves can be a minimal cone. It will be interesting to classify minimal codimension-one foliations with isolated singularities.

A rich class of minimal submanifolds with higher codimension comes from complex submanifolds. In practice, all higher-codimension minimal submanifolds are constructed either by relating them to complex subvarieties or by reducing the problem to a lower dimension by a compact group acting on the Euclidean space. For the latter method, it is usually related to calculating geodesics for a singular metric. Minimal surfaces for such a singular metric should be studied.

Minimal submanifolds in Euclidean space are closely related to minimal submanifolds in the sphere. Minimal hypersurfaces in spheres S^n are difficult to construct when $n \geq 4$. Are there only a finite number of families of nonsingular minimal hypersurfaces in S^n? It is not known whether there is a nontrivial family of nonsingular embedded minimal hypersurfaces in S^n. If the cones over such hypersurfaces are stable, then this assertion is true. Schoen [179] observed that in this case, the minimal hypersurface admits a conformal metric with positive scalar curvature, which gives rise to a strong constraint on the geometry and topology of minimal hypersurfaces in S^n. One should be able to classify these hypersurfaces.

The volumes of minimal hypersurfaces in S^n are important invariants. What values are possible? The author conjectured that for embedded minimal hypersurfaces, the first eigenvalue is equal to $n-1$. This should be very much related to an estimate of the volume of the submanifold. It follows from the work of Cheng-Li-Yau [38] that one can construct an upper estimate of the trace of the heat kernel of a minimal submanifold M^k in terms of $c_k \operatorname{Vol}(M^k) t^{-k/2}$. Since the coordinate functions of S^n give $n+1$ eigenfunctions of M^k with eigenvalue k, one can then prove that $\operatorname{Vol}(M^k)$ has a lower estimate in terms of $c_k^{-1} \left(\frac{2}{e}\right)^{k/2} (n+1)$ if M^k is not a subset of any subsphere of S^n.

Is the set of the values of volumes of minimal submanifolds M^k discrete? There can be a continuous family of minimal submanifolds with the same volume. A natural

example is obtained by taking the inverse image of the Hopf fibration over a continuous family of minimal submanifolds in $\mathbb{CP}^n$ or $\mathbb{HP}^n$. It is not clear whether in the sphere there is a continuous family of even-dimensional minimal submanifolds M^{2k} with $k > 1$.

Apparently, it is easier to handle the Cauchy-Riemann equations, as they are given by a first-order elliptic system. For such a system, the Atiyah-Singer theorem can help to compute the dimension of the solution space. Unfortunately minimal submanifolds are defined by a second-order elliptic system and it is difficult to understand the deformation theory. (Given a Jacobi field on a minimal submanifold, can we find a deformation by a family of minimal submanifolds along the field?) There may be a class of area-minimizing submanifolds are closely related to some first-order system.

Usually we require the ambient manifold to have special holonomy group in order to find this special class of manifolds. For Kähler manifolds (whose holonomy group is $U(n)$) the idea goes back to Wirtinger's inequality. The idea is to find a closed k-form ω whose pointwise L^∞ norm is one. Then any k-dimensional submanifold M such that $\omega|_M$ is the volume form must be volume-minimizing in its homology class. (This follows by Stokes' Theorem.) Usually for manifolds with special holonomy group, some special closed form can be constructed from the holonomy group. Those forms are in fact parallel and hence have constant norm. When the ambient manifold is Euclidean space, Harvey and Lawson [91] called the corresponding kind of minimal submanifolds calibrated. It is still difficult to construct such submanifolds besides those coming from complex subvarieties. An important example is called a special Lagrangian submanifold. A submanifold L^n of a Kähler manifold M^{2n} is called Lagrangian if the Kähler form restrict to L is trivial. If the holonomy group of M^{2n} is $SU(n)$ (Calabi-Yau manifolds), there is a holomorphic n-form Ω. We can require $\operatorname{Im}\Omega = 0$ on L and $\operatorname{Re}\Omega|_L$ to be the volume form of L. Such submanifolds were called special Lagrangian manifolds by Harvey-Lawson. They were rediscovered independently in string theory a few years ago in the work of Becker-Becker-Strominger [10]. There they want to find supersymmetric cycles in Calabi-Yau 3-folds, i.e. those 3-cycles which preserve half the supersymmetries.

The deformation theory of special Lagrangian submanifolds is studied by McLean [147], who proves that the Jacobi fields of these submanifolds can be identified with harmonic one-forms. It just happens that these classify flat $U(1)$ connections on L. In the paper of Strominger-Yau-Zaslow [195], we looked at the moduli space of the pair consisting of L and a $U(1)$ connection over L. The moduli space has a natural complex structure and a nice "semi-flat" L^2 metric. This moduli space has complex dimension equal to $b_1(L)$.

When L is a three-dimensional torus, the moduli space is then complex three-dimensional and has a holomorphic three-form. Based on reasons motivated by physics, we conjectured that this complex manifold is in fact another Calabi-Yau manifold which is the "mirror" of the original one. In particular, the Hodge diagram of these two complex manifolds are dual to each other and the calculation of number of rational curves can be deduced from the periods of its mirror.

Some further conjectures on the "quantum" monodromy group were made based on their interpretations of the mirror. For example, we conjectured that the semi-flat metric mentioned above is supposed to be correctable to a nonsingular Ricci-flat Kähler metric by including some contributions of holomorphic disks whose boundaries lie on the special Lagrangian torus. Hitchin [94], Gross-Wilson [82], Gross [80, 81], Barannikov-Kontsevich [8], and Fukaya-Oh [59] have made progress on this conjecture.

It is expected that similar constructions can be done on other manifolds with special holonomy group (e.g. G_2 or Spin(7)), which in turn may be of great interest to string theorists.

Unfortunately, there are not too many ways to construct special Lagrangian submanifolds. They can be found by looking at the fixed-point sets of antiholomorphic involutions or by looking at complex Lagrangian submanifolds. Also, Schoen and Wolfson [181] have developed an approach based on the volume-minimizing property of special Lagrangian submanifolds. It would be nice to find a way similar to twistor theory to construct these area-minimizing submanifolds.

Given a bundle on a compact manifold such that both the bundle and the manifold have special holonomy groups, we can use the structure of the holonomy groups to require the curvature of a connection on the bundle to be special. (For example, if the bundle is holomorphic with trivial first Chern class and the connection is Hermitian, we can require the trace of the curvature to be zero.) A sequence of these special connections need not converge; it may blow up along some minimal subvarieties. In general, this may be an uncountable union of open subvarieties; however, Tian has a recent preprint [205] stating that the blow-up set is an integral, closed minimizing current. In the case of Hermitian-Yang-Mills this implies that the blow-up set is a global holomorphic subvariety. In general, this procedure may give a way to construct minimal subvarieties by bundle theory.

Another set of examples of minimal submanifolds are isoparametric submanifolds in spheres. They are defined by a set of functions which satisfy an overdetermined system of equations. If the codimension is one, their principal curvatures are constant and they provide an important class of minimal hypersurfaces with constant scalar curvature. For codimension greater than one, one definition requires the normal bundle to be (geometrically) flat and the principal curvatures to be constant. All compact symmetric spaces can be realized in this way.

II. Intrinsic geometry. The fact that a nondegenerate quadratic form defined on the tangent bundle of a manifold can give so much global information about the manifold is rather fascinating. Up to now, the major results on either positive-definite quadratic forms or Lorentzian quadratic forms come from either direct geometric intuition or the physics of spacetime. There are virtually no results when the signature of the quadratic form is different from these two cases. When on a complex manifold, it may be interesting to develop some theory of holomorphic quadratic forms. None of these theories have achieved much success partially because we do not understand the invariantly defined differential operators associated to them. Both the Laplacian and the wave operator have much more mature histories. Indeed, we understand positive-definite metrics better than Lorentzian metrics partially because the theory of the Laplacian has been developed for a whole century while the theory of the wave equation has seen less development in terms of the precise quantitative behavior of solutions.

When we discuss these quadratic forms, the first important questions are to create quantities that behave well under allowable coordinate transformations. (Sometimes, the manifold has a special structure which allows only a certain type of coordinate transformation.) Unless we are comparing two different structures, differentiating the metric once does not provide an invariant. The most important invariants appear when we differentiate the metric twice. Those parts of the second derivatives invariant under coordinate transformations form the curvature tensor. The local information

given by curvature governs the global structure of the manifold to a large extent, at least if we make the natural assumption that every geodesic can be continued indefinitely. Can this last assumption be weakened somewhat? For example, if we assume only that for all points, the set of unit tangent directions at that point which give rise to incomplete geodesics is a closed set with measure zero, (or a subvariety with given dimension), can we carry out most of the global theorems? Perhaps we can add the assumption that the curvature of the manifold (or part of the curvature tensor) is bounded along each incomplete geodesic and ask for the structure of the completion of the metric space.

A. Constraints on the full curvature tensor. There are more components of the full curvature tensor than of the metric tensor. Making assumptions on the full curvature tensor is therefore an overdetermined condition. Nonetheless, many geometrically intuitive questions can be asked.

A very popular question studied since the time of Rauch [170], Klingenberg [113], and Berger [12] concerns the structure of manifolds with positive curvature. Comparison theorems due to Toponogov [213] are important tools in this field. The following basic question remains unanswered: For dimension large enough, are there topologically any non-locally-symmetric manifolds with positive curvature?

For lower dimensions, there are many examples of non-locally-symmetric examples created by double coset space constructions. They are in general detected by torsions of homology. It would be interesting to know whether the real homology of these manifolds is the same as that of the locally symmetric examples. In particular, it is interesting to know whether the total sum of Betti numbers is dominated by the torus of the same dimension. Gromov did give a bound depending only on the dimension [76].

It is well known that there is a very delicate distinction between metrics with non-negative curvature and those of positive curvature. The famous Hopf problem asks whether $S^2 \times S^2$ admits a metric with positive curvature. Perhaps one should ask about more general phenomena. If a manifold M admits a locally free action of a torus T^k, is it true that any metric with nonnegative curvature must admit a point p where the sectional curvature equals zero on a subset K of the Grassmanian $G(2, T_p(M))$ of all two-planes in the tangent space, with $\dim(K) \geq \dim(G(2, \mathbb{R}^k))$?

A related problem is the work of Gromoll-Meyer on an attempt to construct metrics with positive curvature on exotic spheres [75]. They construct metrics with nonnegative curvature where the sectional curvature vanishes on a thin set.

It was observed by D. Moore and M. Micallef [150] that the celebrated existence theorem of Sacks-Uhlenbeck [177] can be used to study homotopy groups of simply connected manifolds with positive isotropic curvature. (This means that after complexification, the curvature is positive on null planes.) This in particular implies the famous pinching theorems of Klingenberg [113], which in turn depend on the triangle comparison theorems. It would be interesting to see how much more information one can obtain from such variational arguments, including possibly the use of vector bundles.

It is embarrassing that we still do not know whether a manifold with positive curvature operator must be a sphere. By using his Ricci flow, Hamilton does classify four-manifolds with metrics of positive isotropic curvature and shows that a four-manifold with positive curvature operator is a sphere [88].

Given a function on the Grassmannnian of two-planes of the tangent bundle of a

manifold, when will it be the curvature function given by a Riemannian metric?

B. The Ricci tensor. The Ricci tensor is obtained by taking the trace of the curvature tensor. It is a tensor of the same type as the metric tensor. Also, it is obtained by the first variation of the total scalar curvature. This remarkable fact was used to give a variational approach to general relativity. The famous Einstein equation is to construct from the Ricci tensor a divergence-free tensor which is equated to the matter tensor. There have been attempts to generalize the variation of total scalar curvature to a variation of the L^2 norm of curvature tensor. The resulting equation is higher order and has been difficult to understand geometrically. But it does generalize the Einstein equation and perhaps will support a rich theory eventually.

When the Ricci tensor is a constant multiple of the metric tensor, the manifold is called an Einstein manifold. This is perhaps the most natural and beautiful class of manifolds in geometry. The most fundamental question in geometry is to determine which manifolds admit Einstein metrics. If they do exist, how many are there? These are meaningful and difficult questions.

When the dimension of the manifold is greater than five, no obstruction to existence is known. Perhaps every manifold in these dimensions admits an Einstein metric. It is difficult to tell whether, for a given compact manifold, the moduli space of all Einstein metrics has an infinite number of connected components. Continuous families of Einstein metrics exist, and known examples are related to metrics with special holonomy group. The most notable ones come from Kähler-Einstein metrics.

For dimension not greater than four, Einstein manifolds are much more rigid. There are conditions (see Hitchin [93]) such as $\chi(M) \geq \frac{3}{2}|\tau(M)|$, where $\chi(M)$ and $\tau(M)$ are the Euler number and signature of the manifold respectively. Perhaps there is a general structure theorem that every four-manifold is obtained by connecting (1) Einstein manifolds, (2) surface bundles over surfaces (with possible controllable singularities similar to Seifert fibrations), and (3) circle bundles over three manifolds, all connected along three-manifolds which are circle bundles over spheres or tori. This may be considered as a generalization of Thurston's program in dimension four.

One hopes that in dimensions three and four, Hamilton's equation can be used to demonstrate both Thurston's hyperbolization conjecture and the above generalization (see e.g. [89]). The key problem is to understand the singularities which develop as solutions to Hamilton's equation evolve.

The construction of solutions to the Einstein equation is a difficult task. Physicists first used the method of symmetry (group actions) to reduce the dimension. This has been a very important tool. Unfortunately most such solutions are only local and usually have singularities. When the metric is positive definite, McKenzie Wang, Ziller, and others have carried out systematic research and have found many important Einstein manifolds (see e.g. [219]). Recently Böhm has found examples of inhomogeneous Einstein metrics on spheres of dimension five to nine [18].

For four-dimensional stationary Lorentzian metrics with axial symmetry, Geroch [67] introduced the Bäcklund transformations, which take one solution to another. These transformations are highly nontrivial. Unfortunately most of the theory is local. It is highly desirable to understand which metrics constructed by these transformations are complete. It will also be very interesting to see how Bäcklund transformations work for positive definite metrics.

In fact, in the seventies, Hawking and others [68] proposed a way (called Wick rotation) to analytically continue a Lorentzian vacuum solution to a positive definite

Einstein metric. It is rather spectacular to see how a singularity of a solution to the Einstein equation can be "cured" by the Wick rotation. In particular, the Schwarzschild solution becomes a beautiful nonsingular Ricci-flat metric on $S^2 \times \mathbb{R}^2$ which does not have a Kähler structure. Unfortunately the Wick rotation is not really a well-defined procedure, as it seems to depend on a clever choice of local coordinates. As a result, it has only been successful for a limited number of examples. A systematic study of the effect of Wick rotation on Einstein equations would certainly be worthwhile for both physics and geometry.

Penrose's twistor program and the idea of symplectic reduction have also been effective in understanding hyperkähler Ricci-flat metrics [95]. These methods suffer, however, the same problem of global understanding of the metric.

Up to now, the most effective way of constructing Einstein metrics is based on Kähler geometry. In such a geometry, the metric is given by $\sum g_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta$, and the Ricci tensor is given by $R_{i\bar{j}} = -\frac{\partial^2 \log\det(g_{\alpha\bar{\beta}})}{\partial z^i \partial \bar{z}^j}$.

The simplicity of the Ricci tensor led Calabi to believe that it is much easier to construct Einstein metrics in Kähler geometry. If we deform a given metric $\overset{\circ}{g}_{\alpha\bar{\beta}}$ by $\frac{\partial^2 \varphi}{\partial z_\alpha \partial \bar{z}_\beta}$ where φ is a scalar function, then the Einstein equation $R_{\alpha\bar{\beta}} = c g_{\alpha\bar{\beta}}$ is simply given by

$$\begin{aligned} &-\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log\det\left(\overset{\circ}{g}_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi}{\partial z^\alpha \partial \bar{z}^\beta}\right) \\ &= c\left[\overset{\circ}{g}_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}\right]. \end{aligned}$$

If there is a volume form $V\, dz^1 \wedge \cdots dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n$ so that

$$-\frac{\partial^2 \log V}{\partial z^\alpha \partial \bar{z}^\beta} = c\overset{\circ}{g}_{\alpha\bar{\beta}},$$

we can rewrite the equation to be

$$\frac{\partial^2}{\partial z_i \partial \bar{z}_j}\left\{\log\left[\det\left(\overset{\circ}{g}_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi}{\partial z^\alpha \partial \bar{z}^\beta}\right) V^{-1}\right] + c\varphi\right\} = 0.$$

On a compact manifold, we are forced to conclude that

$$\det\left(\overset{\circ}{g}_{\alpha\bar{\beta}} + \frac{\partial^2 \varphi}{\partial z^\alpha \partial \bar{z}^\beta}\right) = Ae^{-c\varphi} V,$$

where A is a constant which can simply be absorbed into V. The choice of the volume form V becomes a very important part of constructing Kähler-Einstein metrics, especially for noncompact manifolds (see e.g. [41, 207]).

From the part of view of partial differential equations, it is clear that $c < 0$ is the easiest case of the above equation. In this case, a necessary and sufficient condition for M to admit a Kähler-Einstein metric with negative scalar curvature is that the canonical line bundle K of the manifold M is ample [6, 225]. When $c = 0$, the hypothesis is that there exists a volume form V so that $\frac{\partial^2 \log V}{\partial z^i \partial \bar{z}^j} = 0$. In this case, every Kähler class admits a unique Kähler metric with zero Ricci curvature [225]. The space of Kähler metrics with zero Ricci curvature is then parametrized by the

moduli space of complex structures with flat canonical line bundle and the Kähler cone for each such complex structure.

Two very important consequences of the existence of a Kähler-Einstein metric are certain relations between the Chern numbers and the stability of the tangent bundle. The stability of a holomorphic bundle is defined by Mumford in terms of some polarization of the complex manifold M^n [159]. (A polarization is given by a Kähler class $[\omega]$.) For any vector bundle V of rank r, the degree of V with respect to ω is defined to be $\deg(V) = c_1(V) \wedge \omega^{n-1}$, and the slope of V is defined to be $\frac{\deg(V)}{r}$. The bundle V is said to be Mumford-stable if the slope of any coherent subsheaf of V is less then the slope of V. It is not trivial to check whether a given bundle is stable. Observing that curvature decreases when passing to subbundles, Kobayashi [114] and Lübke [143] showed that the tangent bundle of a manifold admitting a Kähler-Einstein metric ω is stable with respect to the polarization defined by ω. When the scalar curvature is not zero, therefore, the polarization is either $c_1(M)$ or $-c_1(M)$. In such cases, it is rather interesting to determine whether the tangent bundle is stable with respect to other polarizations.

It turns out that the concept of stability makes sense even when ω is not closed. Since the first Chern form is defined up to $\partial\bar{\partial}f$, where f is a globally defined function, $\int_M c_1(V) \wedge \omega^{n-1}$ is well defined as long as $\partial\bar{\partial}(\omega^{n-1}) = 0$. Hermitian metrics ω with such a property were studied by Gauduchon [61, 62], who proved that the equation $\partial\bar{\partial}(\omega^{n-1}) = 0$ can always be solved by a conformal deformation of a given ω.

Right after the author's work on the existence of Kähler-Einstein metrics in the mid-seventies, the author and others made attempts to find similar canonical metrics on holomorphic vector bundles. The natural concept is a Hermitian Yang-Mills connection. Consider Hermitian connections on a vector bundle V. Contract the two base indices of the curvature F, so that $\mathrm{tr}(F)$ becomes an endomorphism of V. We require $\mathrm{tr}(F) = cI_V$, where c is a constant and I_V is the identity endomorphism.

There is another concept, Gieseker stability, that is equally natural from the view of geometric invariant theory. A holomorphic bundle V is Gieseker-stable with respect to a positive line bundle L if and only if for any nontrivial coherent subsheaf S of V,

$$\begin{aligned}&\frac{1}{\operatorname{rank} S}\sum_i (-1)^i \dim H^i(M,\, S\otimes L^k)\\ &< \frac{1}{\operatorname{rank} V}\sum_i (-1)^i \dim H^i(M,\, V\otimes L^k),\end{aligned}$$

for k large enough.

By using the Riemann-Roch formula, one knows that $\sum_i (-1)^i \dim H^i(M,\, S\otimes L^k)$ is given by a polynomial in k called the Hilbert polynomial of S with respect to L. Gieseker [69] and Maruyama [145] proved that the space of Gieseker-stable vector bundles over a projective surface form a quasiprojective variety.

To explain how Gieseker stability arises, we note that there is a large integer k_0 depending only on V, so that for $k \geq k_0$, $H^i(M,\, V\otimes L^k) = 0$ for $i > 0$, and $V \otimes L^k$ is generated by global sections. Consider the family of holomorphic bundles V with fixed Hilbert polynomial such that $\wedge^r V$ is isomorphic to a fixed line bundle H. (Here $r = \operatorname{rank} V$.) Let $W = H^0(M,\, H\otimes L^{rk})$.

Then by the Riemann-Roch theorem, $\dim H^0(M,\, V\otimes L^k)$ is constant and $H^0(M,\, V\otimes L^k)$ can be identified with a fixed vector space S. We then have a natural

homomorphism

$$\bigwedge^r S \longrightarrow W$$

which determines the holomorphic structure of V. The group $SL(S)$ acts on $\mathrm{Hom}\,(\wedge^r S, W)$ and the quotient space, suitably defined, is the moduli space of all such V.

Gieseker proved that Gieseker stability is equivalent to the stability of such an action of $SL(S)$ in the sense of geometric invariant theory.

In his thesis my former student Conan Leung [121] considered the space U of unitary connections on the bundle to reinterpret Gieseker's work. Let D_A be such a connection on the bundle V, and let B and C be tangent vectors of U at D_A. Also, let ω be the Kähler form on the manifold and $k > 0$ be an integer. Then we define a two-form on U by

$$\Omega_k(D_A)(B,C) = \int_M \mathrm{tr}\left[B \wedge e^{\left(k\omega I_V + \frac{i}{2\pi} R_A\right)} \wedge C\right]_{\mathrm{sym}} \mathrm{Td}(M)$$

Here $[\quad]_{\mathrm{sym}}$ indicates the graded symmetric product of the forms inside, R_A is the curvature two-form of D_A, ω is the Kähler form, I_V is the identity endomorphism, and $\mathrm{Td}(M)$ is the Todd class.

When k is large enough, $k\omega$ dominates R_A and Ω_k becomes non-degenerate symplectic form. The gauge group G acts symplectically on U with respect to these symplectic forms.

The moment map can be computed to be

$$\begin{aligned} &\mu_k \colon U \longrightarrow \mathcal{G}^* \\ &\mu_k(D_A) = \left[e^{k\omega I_V + \frac{i}{2\pi} R_A} \mathrm{Td}(M)\right]^{2n}, \end{aligned}$$

where $\mathcal{G}$, the Lie algebra of G, can be identified as the space of endomorphism-valued top forms on M.

In general $\mu_k^{-1}(0)$ may be empty. Hence one chooses a constant multiple of ω^n and the moment map equation is given by

$$\begin{aligned} &\left[e^{\frac{i}{2\pi} R_A + k\omega I_V} \mathrm{Td}(M)\right]^{2n} \\ &= \frac{1}{\mathrm{rk}(V)} \chi(M,\, V \otimes L^k) \frac{\omega^n}{n!} I_V. \end{aligned}$$

This is Leung's equation. He observed that when $k \to \infty$, it reduces to the equation of Donaldson-Uhlenbeck-Yau [52, 215]. He demonstrated that if V is irreducible, the existence of uniformly-bounded-curvature solutions to the above equations for sufficiently large k is essentially equivalent to Gieseker stability.

The introduction of the Todd class may not be so essential for the analysis, although it does give rise to the Hilbert polynomial, which in turns leads us to the concept of Gieseker stability. If we replace the Todd class by some other class, e.g. the $\hat{\mathrm{A}}$ class, we may obtain other concepts of stability.

If the bundle V is the tangent bundle of the manifold M and if the background Kähler metric g, as a metric on V, is the one used to define the Donaldson-Uhlenbeck-Yau equation, we obtain the Kähler-Einstein equation. One can therefore interpret

the equation of Donaldson-Uhlenbeck-Yau as the (semi-) linearized version of the Kähler-Einstein equation, which is itself fully nonlinear. Since bundle stability is the criterion for existence for the former equation, it is clear to me that a certain nonlinear stability of manifolds must be involved for the existence of Kähler-Einstein metrics. Hence in the mid-eighties, I proposed that Gieseker-Mumford stability of an algebraic manifold with positive-definite first Chern class should be the criterion for the existence of a Kähler-Einstein metric (see [227]). This view has been picked up recently by Tian [204]. While he considered a slightly different concept of stability, it is most likely that my original conjecture is correct.

If we let the Hermitian metric on the manifold be g and the (tangent) bundle metric to be h, then g need not be Kähler. However, according to my work with Jun Li [129] (see Buchdahl for the case of two dimensions [21]), the Hermitian Yang-Mills equation $\mathrm{tr}(F(h)) = \lambda I$ makes sense even when g is not Kähler. The concept of stability of a bundle with respect to the metric g also makes sense as long as $(\partial\overline{\partial}\omega_g) \wedge \omega_g^{n-1} = 0$, where ω_g is the $(1,1)$-form associated to g. If the bundle is stable with respect to ω_g, it can still be proved that there is a Hermitian metric h which satisfies the Hermitian Yang-Mills equation. By a conformal change of h, we can obtain $\tilde{h}$ which satisfies the condition $(\partial\overline{\partial}\omega_h)\wedge\omega_h^{n-1} = 0$ (by [61]). Hence if the bundle is the tangent bundle, then we have a map from the space of Hermitian metrics g with $(\partial\overline{\partial}\omega_g)\wedge\omega_g^{n-1} = 0$ to itself. It is tempting to believe that the existence of a fixed point of this map is relevant to the question of the existence of a Kähler-Einstein metric. A Kähler-Einstein metric is clearly a fixed point. It is interesting to know whether there are other, non-Kähler fixed points. The iteration of the above map on the space of Hermitian metrics may converge to a fixed point under reasonable conditions on the stability of the manifold. The arguments of Uhlenbeck-Yau [215] perhaps can be strengthened to provide the needed estimate. This is a worthwhile project: It may not only provide the solution to the existence problem for Kähler-Einstein metrics, but may also give canonical Hermitian metrics on non-Kähler manifolds.

While there are only a limited member of non-Kähler surfaces, it is clear that many more non-Kähler manifolds exist in higher dimensions. Furthermore, as we explain in more detail below, canonical Hermitian metrics on non-Kähler manifolds will be useful in string theory since such manifolds, according to an idea of Reid [172], may be used to connect the moduli space of all Calabi-Yau threefolds.

The idea of using such bundle metrics to find canonical Hermitian metrics on non-Kähler surfaces was already used by Jun Li, F.Y. Zheng and myself [130] in giving the first comprehensive proof of the theorem of Bogomolov [17] that class VII_0 surfaces with no holomorphic curves are the Inoue surfaces [105]. A very important observation here is that the non-existence of holomorphic curves actually helps to establish the stability of the tangent bundle in the sense Jun Li and I described in [129]. It remains an unsolved problem to classify those VII_0 surfaces which contain a finite number of holomorphic curves. It is tempting to think that a similar argument can be applied to the cotangent bundle with poles along those curves.

For manifolds with many subvarieties, the proof of the stability of the tangent bundle may be a nontrivial task. This is especially true for the concept of stability we introduced in [129].

Roček, in his paper in Mirror Symmetry I [173], proposed a class of non-Kähler manifolds which admit $(2,2)$ supersymmetry. The condition he imposed is rather strong and it has been difficult to produce fruitful examples. One condition is that the tangent bundle of the manifold M must split into two bundles $V_1 \oplus V_2$ with $\det V_1$

isomorphic to $\det^{-1} V_2$. Furthermore, V_1 and V_2 are required to define foliations of M whose leaves are Kähler, i.e. M admits a global Hermitian metric which is Kähler when restricted to those leaves. It is an interesting question to see how restrictive such a class of manifolds is.

When the manifold admits Kähler-Einstein metrics with negative scalar curvature, I was able to characterize those manifolds uniformized by the ball and other Hermitian symmetric domains [224, 228]. The former characterization depends solely on the Chern numbers and the latter depends on the existence of nontrivial holomorphic sections of certain bundles. The Chern-number characterization has been widely used. However, a very interesting question remains unanswered. Namely, how do we characterize geometrically those rank-one Hermitian locally symmetric manifolds that are defined by arithmetic groups? One can probably use the many Hecke correspondences, but these conditions are not easy to check.

It would be a very interesting and difficult task to find a topological characterization of those manifolds that can be covered by symmetric domains. There is of course already a difficult theorem of Thurston for three-dimensional manifolds. Naturally we can make the problem easier by assuming the manifold is Einstein. Is there a suitable concept of stability in Riemannian geometry to help to characterize locally symmetric manifolds? Only recently, Besson et al [13], LeBrun [119], Gursky-LeBrun [86], and C. Leung [123] were able to make some progress on the characterization of those Einstein manifolds that are rank-one locally symmetric spaces. While the first work is based on the concept of the center of gravity, the latter works depend on the Seiberg-Witten invariants and apply only to four-dimensional manifolds. As we mentioned earlier, the major problem is the lack of powerful existence theorems for Einstein metrics.

For an existence theorem rather close to the existence of Einstein metric, Taubes [198] proved a remarkable theorem that after blowing up a four-dimensional manifold at suitable number of points, there is a metric with self-dual curvature (i.e. a metric whose curvature form is invariant under the star operator). It is obtained by a beautiful singular-perturbation argument. However, the moduli space of such metrics is difficult to control, and the topological implications of their existence are unclear. It would be very nice if one can apply similar arguments to produce Einstein metrics. It should be remarked that Taubes' theorem leads to many complex threefolds which are twistor spaces of the four-manifolds with Taubes' metrics. What would be a good canonical Hermitian metric on such threefolds, which are not Kähler?

Let us now discuss several points about Einstein metrics which are also Kähler. For such metrics, the case of negative scalar curvature is much better understood, even for noncompact Kähler manifolds. However, many questions, especially the relation to Arakelov geometry, remain to be pursued.

For complex one dimension, such metrics are simply Poincaré metrics. However, it is a difficult problem to find an explicit form of the metric in terms of the homogeneous polynomials which define the algebraic curve as a subset of projective space. This problem amounts to finding an explicit uniformization of such a curve by the upper half plane. We can use the Weierstrass $\wp$ function to uniformize elliptic curves to $\mathbb{C}$. For certain noncompact curves, one can use the Picard-Fuchs equation to help find the uniformization through hypergeometric series. In his thesis [53], C. Doran studied this kind of question much more extensively by looking at the fiber space over a curve whose generic fiber is an elliptic curve or a K3 surface.

Many years ago, the author conjectured that such metrics on algebraic manifolds can be approximated by the induced metrics of projective embeddings into projective spaces by high powers of the canonical line bundle. This was proved by Tian [201] in his Harvard thesis using the $\overline{\partial}$-localization method of Siu-Yau [191]. Zelditch [231] improved the estimates in a recent paper using asymptotic spectral analysis. This kind of embedding has applications to Arekelov geometry, as was studied by S.W. Zhang [232].

A related object is the complete Kähler-Einstein metric on a quasiprojective manifold $M \setminus D$, studied first by Cheng-Yau [41] and later by Tian-Yau [207] and others. (Here M is algebraic, D is a normal-crossing divisor, and $K_M \otimes [D] > 0$.) While the principal term of the metric near D is known, it should be possible to find an asymptotic expansion of the metric along D.

One of the uses of Kähler-Einstein metrics is to provide analytic tools to study the underlying complex manifold. Given a projective manifold M, its universal cover is a rather transcendental object. However, one would like to capture the algebraic meaning of it. A natural question that the author asked more than ten years ago is to find a meromorphic map from the universal cover of M to an open subset of another algebraic manifold N so that the covering transformations of M can be extended to birational transformations of N. While the statement may be too good to be true for all projective manifolds, it should not be too far off. An important question is how to produce holomorphic functions of slowest growth. For example, if we want to realize the universal cover of M as a bounded domain, we need to produce bounded holomorphic functions. R. Schoen and the author do have a way of producing bounded harmonic functions on a manifold which covers a compact manifold [187]. (For manifolds which have strongly negative curvature, this problem was first studied by Sullivan [196] and Anderson [2].) On the other hand, holomorphic functions are more constrained and no method has been produced to handle their existence.

In the theory of Kähler-Einstein metrics with negative scalar curvature, the Kähler class is canonical and is the negative of the first Chern class. In this case, the metric is determined by the complex structure. Therefore, any invariants of the Kähler-Einstein metric give intrinsic invariants of the complex structure. For example, such a metric provides Laplacian operators acting on various natural bundles of the manifold. The zeta functions associated to the eigenvalues of these Laplacians should have interesting properties related to the complex structure. However, besides the holomorphic torsion introduced by Ray-Singer [171], little is known about such zeta functions. It would be interesting to use information about the eigenvalues to give a compactification of the moduli space of such Kähler-Einstein manifolds. Can one use methods of differential geometry to prove that there are only a finite number of components of the family of projective complex structures over such manifolds? It is an interesting result of Catanese-LeBrun [36] and Kotschick [117] that there do exist examples of two Kähler-Einstein manifolds diffeomorphic to each other but with opposite signs of scalar curvature. One of the major accomplishments of Seiberg-Witten theory is the proof that this is not possible for Kähler surfaces [57].

In order to understand the problem of compactifying the moduli space, a natural approach is to study its Weil-Petersson metric. In general, the Weil-Petersson metric is not complete and has unbounded curvature. However, its Ricci tensor may be negative-definite outside a compact set and then its behavior should reflect the behavior of the canonical Kähler-Einstein metric on the moduli space. The Weil-Petersson metric for moduli space of Kähler-Einstein metrics with negative scalar curvature is

quite different from the one on the moduli space of Kähler-Einstein metrics of zero Ricci curvature. In the former case, we probably should expect that the Ricci curvature has an upper bound; and in the latter case, the Ricci curvature should have a lower bound. It should be interesting to prove that they have finite volume. C.-L. Wang [217] has understood the condition for the Weil-Petersson metric on the moduli space of Ricci-flat Kähler manifolds to be complete near the singular points of the moduli space.

Can one compactify a complete Kähler manifold whose volume is finite and whose Ricci curvature is bounded from below? Many years ago, I initiated the program of compactifying complete manifolds with finite volume using geometric means. (See Siu-Yau [191].) I suggested to Mok-Zhong and my former student F.Y. Zheng to work on this program. While it has not been completed, progress has been made by Mok-Zhong [154], Mok [153], Yeung [230], and unpublished work of Zheng and myself.

Since the revolutions of string theory in theoretical physics, the theory of Kähler manifolds with zero Ricci curvature (i.e. Calabi-Yau manifolds) has gone through a vigorous change. The fundamental paper of Candelas, Horowitz, Strominger and Witten [28] studied the Kaluza-Klein model, where one wants to compactify a ten-dimensional spacetime to a four-dimensional spacetime by using compact six-dimensional manifolds with nontrivial parallel spinors. The final analysis shows that the compactification is given by a Calabi-Yau manifold of three complex dimensions. This famous paper immediately called for a great deal of work on constructing such manifolds, especially those with Euler number equal to ± 6 and with nontrivial fundamental group. At the beginning, physicists thought that there are only a couple of Calabi-Yau manifolds with three dimensions. During the first major conference on string theory [226], the author described many ways to construct these manifolds, and the physicists were rather surprised to find out that there should be at least on the order of ten thousand such manifolds. The author proposed to construct a large class of these manifolds by taking complete intersections of hypersurfaces in products of weighted projective spaces. The first important example is the complete intersection of two cubics in $\mathbb{CP}^3 \times \mathbb{CP}^3$ and a bidegree $(1,1)$ hypersurface. This manifold has Euler number equal to -18. I was able to find a group of order three which acts on it with no fixed point. The quotient manifold then has Euler number -6 and non-trivial fundamental group. Tian and the author [206] then found more examples in a similar way. It was observed by B. Greene that all these constructions lead to manifolds with the same topology. Greene and his coauthors even discussed the phenomenological implication of those manifolds [3].

The first general theory of Calabi-Yau manifolds was the study of two-dimensional surfaces due to Piatetski-Shapiro and Shafarevich [167] (Burns-Rapoport [22] for the case of Kähler manifolds). They found that the period map must be injective for the moduli space of K3 surfaces. The question of surjectivity was done much later and was due to Kulikov [118] and Pinkham-Perrson [166]. Both of these papers are deep works and require a great deal of algebraic machinery.

These theorems were drastically simplified by the observations of Todorov [210], that the author's existence theorem for Ricci-flat metric can be applied. The key point is an observation of Hitchin [93] that the metric provides an S^2 family of complex structures. This rational curve of complex structures provides a way to move in the moduli space. (Much more rigorous and detailed treatments were then given by Siu [190].) There were expectations to generalize these methods to higher-dimensional Calabi-Yau manifolds. While this has not been carried out, the famous theorem of

Bogomolov [17] on the unobstructedness of holomorphic symplectic Kähler manifolds was generalized to general Calabi-Yau manifolds by Tian [200] in his thesis and by Todorov [211] independently. This basic theorem played an important role on the later development of Calabi-Yau manifolds. (The analog of the formula for proving unobstructedness is being using by Kontsevich, Fukaya, and others to construct higher products in their attempts to work out the algebraic formulation of mirror symmetry [8, 59].)

String theory demands extensive calculations on the moduli space of Calabi-Yau manifolds. Since the Torelli theorem holds, the period of the top-dimensional holomorphic form determines the local geometry of the moduli space. It was observed by Tian [200] and the physicists that the Kähler potential can be written as $\log \|\Omega\|^2$ where Ω is a local holomorphic family of top-dimensional holomorphic forms. The fact that the holomorphic n-form defines a sub-line bundle of the (flat) bundle of n-dimensional cohomology classes gives a way to calculate the Weil-Petersson geometry with extra data. The quotient of this flat bundle by the line bundle describes the infinitesimal deformation of complex structures and hence gives the tangent bundle of the moduli space.

Two groups studied this kind of geometry (Candelas et al [27] and Strominger [194]). Strominger coined the name special geometry for it (he originally called it Kähler geometry of restricted type and the author suggested changing it to special geometry). Special geometry turns out to play an important role in later calculations of mirror symmetry.

The works of Gepner [66] and Greene-Vafa-Warner [74] show heuristically how to attach a conformal field theory and a path integration to certain Calabi-Yau manifolds. Soon after, Dixon [51] and Lerche-Vafa-Warner [120] made the prediction of mirror symmetry, which asserts that for any Calabi-Yau manifold M, one can associate another Calabi-Yau manifold M' so that by going from M to the mirror M', two three-point correlation functions (one associated to the complex deformations and the other associated to the Kähler deformations) are mapped to one another. The correlation function for complex deformations of M is simply the natural triple product of $H^1(T_M)$ (this works since $\Lambda^3 T$ is trivial). The correlation function for Kähler deformations is much more complicated. Besides the classical topological cup product on $H^1(T_M^*)$, one needs to add corrections due to integration over rational curves! B. Greene and the author called the last triple product the quantum cup product during the first conference on mirror manifolds in 1990 in Berkeley. Vafa called the cohomology arising from such a ring structure quantum cohomology.

For the important example of the quintic in $\mathbb{CP}^4$, Greene-Plesser [73] demonstrated the existence of the mirror based on arguments from conformal field theory. Immediately afterwards, Candelas et al [29] carried out the complete detailed calculation of the correlation functions based on the mirror statements. The identification of the special geometry on both the Kähler and the complex sides plays an important role. The calculation of such an identification is a spectacular piece of work in mathematics. It depends on studying the periods of holomorphic three-forms which satisfy a Picard-Fuchs equation and on understanding the monodromy associated to the degeneration of complex structure. This work of Candelas et al has greatly influenced the development of Calabi-Yau manifolds in the past ten years. In particular, it provides a beautiful formula to calculate the number of rational curves (which needs to be defined suitably) on the quintic. Even the existence of this formula was not expected in mathematics literature. Later developments due to many mathematicians

are all basically reinterpretations of Candelas' formula in various forms.

Candelas' method of calculation was immediately carried out by many groups of mathematicians when the complex deformation space is one-dimensional. When the deformation space is multidimensional, the calculation requires a new method and this was carried out independently by Hosono-Klemm-Thiesen-Yau [99] and by Candelas-de la Ossa-Font-Katz-Morrison [30]. A further generalization was also done by Hosono-Lian-Yau [100]. In the former paper the Frobenius method and the hypergeometric system of Gelfand-Kapranov-Zelevinsky [64, 65] were extensively used. The formal parameter in the Frobenius method was later replaced by the hyperplane class in equivariant geometry. This gives the right interpretation of Candelas' formula in terms of equivariant geometry.

It makes sense to talk about the quantum cohomology ring structure for any Kähler manifolds. For manifolds with positive first Chern class, the associativity of quantum cohomology is sometimes enough to determine the instanton sum. This statement comes from the WDVV equations, which are due to a group of physicists (see [223, 54]). For these manifolds, mathematicians were able to exploit the associativity of the quantum cohomology to calculate the instanton sums. The concept of a Frobenius manifold was developed to understand these calculations, which in turn led to formulas for counting curves in homogeneous manifolds. On the other hand, it took a much longer time to actually prove the associativity of quantum cohomology.

The first proof of this associativity (for semi-positive symplectic manifolds) was due to Ruan-Tian [175]. First of all, one needs to define the meaning of the instanton sum. Ruan [174] defined some special cases for symplectic manifolds when the curve has genus zero. Then Ruan-Tian extended this to a complete set of genus-zero invariants [175] and generalized it to curves of arbitrary genus [176]. The definition is modeled after Donaldson's definition of his gauge invariants for four-dimensional manifolds. A basic ingredient is the compactness argument for pseudoholomorphic curves essentially due to Sacks-Uhlenbeck [177]. It was observed by Gromov [77] that pseudoholomorphic curves can be used to study the rigidity of symplectic manifolds. Ruan-Tian's definition and proof of associativity for quantum cohomology works only for pseudoholomorphic curves with respect to a generic choice of almost complex structure. However integrable complex structures are far away from being generic, and therefore the instanton sum needs to be defined differently if we restrict ourselves to projective manifolds only.

Based on the works of Sacks-Uhlenbeck [177], Gromov [77], Parker-Wolfson [165] and others, Kontsevich [115] defined the concept of the compactification of the moduli space of rational maps from pointed rational curves to a projective manifold. When the projective manifold is a complete intersection in a certain homogeneous space, there is a way to define a certain obstruction bundle over the above compactified space. If the obstruction bundle has the same rank as the moduli space of maps, we can take the Euler number of the bundle. In general, however, one has to use the construction of the virtual cycle first done by Li-Tian [127] to define such a number. For a generic choice of projective hypersurface, the "number" of curves in a fixed topology can be defined in terms of these Euler numbers. For the quintic, we get

$$K_d = \sum_{k|d} n_{d/k} k^{-3},$$

where K_d is the Euler number and $n_{d/k}$ is the expected number of rational curves. This formula, called the covering formula, was discovered by Candelas et al [31] and

rigorously justified by Aspinwall-Morrison [5] and Manin [144]. The number n_i is a projective invariant and should be called differently from the symplectic invariant mentioned above. A natural name should be the Schubert invariant to honor the fundamental work begun by Schubert a century ago.

In many important cases, Li-Tian [128] and Siebert [188] were able to prove these Schubert invariants are the same as the one defined by Ruan-Tian-Gromov. In particular, this demonstrates that the associativity law is valid for these Schubert invariants.

Candelas' formula for the quintic threefold is the following equation of formal power series in T:

$$\frac{5T^3}{6} + \sum_{d>0} K_d e^{dT} = \frac{5}{2}(\frac{f_1}{f_0}\frac{f_2}{f_0} - \frac{f_3}{f_0}),$$

where $T = \frac{f_1}{f_0}$, K_d is the Euler number as above, and for $i = 0, 1, 2, 3$,

$$f_i = \frac{1}{i!}(\frac{d}{dH})^i|_{H=0} \sum_{d\geq 0} e^{d(t+H)} \frac{\prod_{m=1}^{5d}(5H+m)}{\prod_{m=1}^{d}(H+m)^5}.$$

The f_i form a basis for the solution space of $L(f) = 0$, where L is the hypergeometric differential operator

$$L = (\frac{d}{dt})^4 - 5e^t(5\frac{d}{dt} + 1)\cdots(5\frac{d}{dt} + 4).$$

Many people have made serious attempts to prove this formula. Witten [223] defined the concept of a linear sigma model, and Plesser-Morrison [157] made an (unsuccessful) attempt to use this concept to justify Candelas' formula. However, they did demonstrate the importance of the linear sigma model. Soon after, Kontsevich made a serious attempt to apply the Atiyah-Bott localization to prove Candelas' formula [115]. While he succeeded in computing the degree-four invariant for the quintic, his formulation is too complicated to be carried out in general. It is important to note that the above H used in the Frobenius method (see [99]) is interpreted as the equivariant hyperplane class. Following Kontsevich, Givental [70] made another attempt, using ideas of Witten and others and introducing quantum differential equations (these are just equations for determining a flat section of a certain canonically defined connection). However, his claimed proof is far from being complete. Finally, based on the works of Witten, Kontsevich, Li-Tian, and some new ideas on the concept of Euler data, Lian-Liu-Yau [134] gave the first complete proof of Candelas' formula in 1997. Some six months after the publication of [134], two works attempting to complete Givental's program appeared. The first one was due to Procesi et al [14] and the other one to Pandharipande [164]. The first paper did not claim to prove Candelas' formula in its final form, and the second used some ideas of Lian-Liu-Yau.

While the work of Lian-Liu-Yau does not give a construction of the mirror manifold, it does raise many interesting mathematical questions. One should interpret this theory as a theory of characteristic classes or K-theory over a mapping-space sigma model of algebraic manifolds. One advantage of such sigma models is that they allow us to restrict the maps to those from curves of a fixed topology, resulting in a finite-dimensional mapping space.

An important question involved in the theory of Lian-Liu-Yau is the following: Given an algebraic bundle V over an algebraic manifold M and the stable moduli space of maps $\mathcal{M}(g,k)$ from curves of genus g to M with homology class $k \in H_2(M,\mathbb{Z})$, one can form a virtual bundle $\widetilde{V}$ over $\mathcal{M}(g,k)$ by looking at $H^0(C,f^*V) - H^1(C,f^*V)$, where $f : C \to M$ is a map in $\mathcal{M}(g,k)$. Given a theory of characteristic classes, i.e. a map b from the ring of holomorphic vector bundles to homology classes (which can be refined to algebraic cycles), one can then consider $b(\widetilde{V})$ and consider several numbers related to $b(\widetilde{V})$. For example, we can evaluate $b(\widetilde{V})$ over the Li-Tian class [127], which was defined by Li-Tian as a virtual moduli cycle (and subsequently understood by Behrend-Fantechi [11] using a somewhat different method), or we can consider a product of $b(\widetilde{V})$ with the Chern class of the tautological line bundle of $\mathcal{M}(g,k)$ and then evaluate this product over the Li-Tian cycle. The method of Lian-Liu-Yau can be used to compute these numbers for a large class of bundles V and M. This class includes, for example, convex and concave bundles over toric varieties or balloon manifolds. The computation of $b(\widetilde{V})$ can be considered as part of the K-theory over sigma models of algebraic manifolds.

It is important to carry out the computations of Lian-Liu-Yau in the most general possible setting. Equally important is to interpret the geometric meaning of the numbers computed. When b is the Euler class and $H^1(C,f^*V) = 0$, the number is interpreted to be related to the counting of the "number" of curves of genus g. This is how one computes the number of rational curves in a generic quintic in $\mathbb{CP}^4$. In that case, one takes V to be the line bundle $\mathcal{O}(5)$ over $\mathbb{CP}^4$. When V is $\mathcal{O}(-3)$ over $\mathbb{CP}^2$, we are dealing with numbers which arise in "local mirror symmetry," i.e. the "number" of rational curves in $\mathbb{CP}^2$ embedded as a hypersurface in a Calabi-Yau manifold (see the works of Vafa et al, e.g.[112], and the recent work of Chiang-Klemm-Yau-Zaslow [42]). The set of all these characteristic numbers over sigma models is very much related to the hypergeometric series of Gelfand-Kapranov-Zelevinsky [64, 65]. It would be very interesting to understand the internal structure of these numbers as a map from the K-groups of M.

When b is the Euler class, it is a remarkable theorem of Li-Tian that it is the same as counting the number (up to sign) of pseudoholomorphic curves of a generic almost-complex structure compatible with the given symplectic structure. Using the proof of Lian-Liu-Yau, one should be able to extend the methods of Li, Ruan and Tian to show that the coefficients n_d of the generating function are integers. This should have deep interest for both number theorists and combinatoricists. The transformation from the hypergeometric series to the generating function is called the mirror transformation. It is also a remarkable fact that by choosing the right coordinates, the mirror transformation has a good q-expansion whose coefficients are integers (as was computed experimentally by Hosono-Klemm-Thiesen-Yau [99] and publicized by the authors). When the deformation of the mirror manifold is one-dimensional, this integral condition was verified by Lian-Yau [138]. This is a very important fact, as it was used by Lian-Yau to prove divisibility properties of the number of rational curves. For example, it was proved that n_i, the number of rational curves in a quintic, is divisible by 125 in the case i is not divisible by 5. However, such integral properties of the mirror map are not known for the multivariable case and pose a challenging problem. Note that when the Calabi-Yau manifold has one or two dimensions, the mirror map is related to the j-function. In fact, Lian-Yau [135, 136, 137] observed that when the Calabi-Yau manifold is the K3 surface or when the Calabi-Yau manifold contains a pencil of K3 surfaces, the mirror map should be related to the automorphic form

which appears in the moonshine conjecture related to the monster group. In his Harvard thesis, Chuck Doran made remarkable progress on this question, as he studied the Painlevé VI equation and its algebraic solutions extensively [53].

Duality conjectures in the recent progress of string theory have clear implications in number theory as was indicated by work of Moore-Witten [156]. Also G. Moore has questions on the values of the mirror maps on certain special points on the moduli space determined by a variational principle [155]. All these questions imply that a very rich structure of number theory is hidden in the theory of mirror symmetries. Klemm-Lian-Roan-Yau [111] developed a generalization of the Schwarzian equation for the mirror map. It was based on such equations that the divisibility properties of number of rational curves were found.

While the theory of Lian-Liu-Yau is able to tackle many important questions in enumerative geometry, it does not explain the geometric meaning of mirror manifolds. As we mentioned above, the construction of Strominger-Yau-Zaslow [195], however, does provide such a framework. Vafa [216] has recently extend the SYZ conjecture to include vector bundles in the picture. While Gross [80, 81] and Hitchin [93] have made significant progress on the SYZ conjecture, a full understanding of the theory of SYZ is still far away. Key missing ingredients are explicit constructions of special Lagrangian submanifolds in general Calabi-Yau manifolds and holomorphic disks whose boundaries lie on given Lagrangian submanifolds. In any case, the SYZ picture is likely to be correct and it will be very interesting to combine the rigorous treatment of Lian-Liu-Yau with the picture of SYZ. It predicts a construction of a Ricci-flat metric and hopefully can be carried out by understanding the instanton corrections to the semi-flat metric.

Many years ago, Mukai [158] observed that the moduli space of $SU(n)$ bundles over a K3 surface has natural hyperkähler structure. (This can be generalized to other hyperkähler manifolds.) He introduced the concept of the Mukai transform, which is clearly related to the above theories. Hopefully, a complete mathematical theory encompassing all these ideas can be found soon.

Another important problem is to classify all three-dimensional Calabi-Yau manifolds and those four-dimensional ones that are elliptic fiber spaces. A very much related question is the understanding of construction of manifolds with G_2 and Spin(7) holonomy groups.

Only recently Joyce [106, 107] was able to construct non-trivial examples of such manifolds. They were obtained by singular perturbation which is similar to the construction of C. Taubes on self-dual $SU(2)$ connections over four-manifolds [197]. While these manifolds clearly play an important role in the recent progress of string theory, their global structure is still hard to be understood. How do we parametrize them? Are they related to Kähler manifolds in a systematic way? How can we understand the moduli space of bundles with special holonomy groups over these manifolds or Calabi-Yau manifolds?

A recent development of string theory demands that a given Calabi-Yau manifold can be deformed to another. Since these manifolds may have different topology, one must go through singular manifolds to achieve such a goal. One is also allowed to identify manifolds which give rise to the same conformal field theory. Aspinwall-Greene-Morrison [4] has studied the deformation of conformal field theory of Calabi-Yau manifolds when these is a "flop" construction which changes the topology of the manifolds. Greene-Morrison-Strominger [72] also discussed how the quantum field

theory changes when the manifold is deformed to acquire conifold points. These theories demonstrate the possibility of good physical theories even when the target space has singularities. This should mean that we can develop a good geometric theory even when the manifolds acquire singularities. This includes a good metric, a good Hodge theory, a good bundle theory, and a good enumerative geometry on such singular manifolds. Such geometries should reflect the quantum field theory mentioned above. In particular, one would like to see new geometric quantities to capture the limit of the "quantum" geometry when a smooth manifold approaches a singular one. Supersymmetric cycles which represent cycles collapsing to the singularities should play an important role in all these discussions.

In the discussion of connecting different Calabi-Yau manifolds, a particularly important process was suggested by M. Reid [172] (some initial ideas date back to Clemens [45]). We can destroy the second cohomology of a Calabi-Yau manifold by blowing down rational curves with negative normal bundle. There are theorems by Clemens [44], Friedman [56] and Tian [203] on how to deform the complex structure of the resulting singular manifold to that of a smooth complex manifold. These manifolds need not be algebraic (although they are birational to such manifolds). By passing through this kind of process, Reid suggests to connect all Calabi-Yau threefolds together. It is a rather tempting conjecture. However, since the manifold obtained by smoothing is not Kähler, a canonical Hermitian metric has to be defined to account for properties similar to those given by the Ricci-flat metric. A Weil-Petersson metric on the moduli space based on such canonical metrics would be important because it should help to identify the mirror map.

A few years ago Zaslow and I [229] demonstrated the relation between counting singular rational curves with nodes in a K3 surface and automorphic forms. Motivated by the formula, Göttsche [71] made the following conjecture for a more general Kähler surface X:

Let C be a sufficiently ample divisor on X, and K be the canonical divisor. Then the number of curves of genus g in $|C|$ passing through $r = -KC + g - \chi(\mathcal{O}_X)$ points is given by the coefficient of $q^{\frac{1}{2}C(C-K)}$ in the following power series in q:

$$B_1^{K^2} B_2^{CK} (DG_2)^r \frac{D^2 G_2}{(\Delta(D^2 G_2))^{\chi(\mathcal{O}_X)/2}},$$

where $D = q\frac{d}{dq}$, G_2 is the Eisenstein series

$$G_2(q) = -\frac{1}{24} + \sum_{k>0} (\sum_{d|k} d) q^k,$$

Δ is the discriminant

$$\Delta(q) = q \prod_{k>0} (1 - q^k)^{24},$$

and the $B_i(q)$ are certain universal power series.

Bryan-Leung [20] made the first step to give rigorous proof of the Yau-Zaslow formula for K3 surfaces when the cohomology class is primitive. It is remarkable that A.-K. Liu [142] was recently able to obtain the formula for general Kähler surfaces. (Some special cases were obtained with T.-J. Li jointly.) Using the idea of a family of Seiberg-Witten invariants, he is also able to study a similar question for algebraic

threefolds which are elliptic fiber spaces. It is rather mysterious that the generating function for counting curves is related to automorphic forms. Perhaps some generalized theory of those forms will be developed in the near future.

Besides the WDVV equation and the theory of mirror symmetry, there is also the theory of Seiberg-Witten equations. Taubes [199] was the first one to demonstrate that the Seiberg-Witten invariants are related to counting the number of pseudo-holomorphic curves in a symplectic manifold. This theorem lays the foundation for the structure of symplectic manifolds in four dimensions. It is basically a trivial corollary of Taubes' theorem that there is only one symplectic structure on $\mathbb{CP}^2$. (It is still not known whether this is true on a homotopy $\mathbb{CP}^2$, where I proved it to be the case if the symplectic structure is Kähler.) The theorem of Taubes on $\mathbb{CP}^2$ was generalized by A.-K. Liu and T.-J. Li to other rational surfaces [126].

Coming back to the classification of Calabi-Yau manifolds, it may be interesting to understand geometric cobordism among such manifolds. When do two Calabi-Yau threefolds bound a seven-dimensional manifold with G_2 holonomy? For G_2-manifolds, one can of course look for a Spin(7) manifold to be the total space.

The recent development of string theory gives rise to the following interesting question. If a manifold M is a metric cone over a compact manifold N such that M has special holonomy group, what conditions does this place on N? This is particularly interesting in the case when M is Calabi-Yau.

For manifolds with positive Ricci curvature, my solution of the Calabi conjecture gave a complete understanding of the topology for compact Kähler manifolds with positive Ricci curvature. (While Tian and I did make substantial progress on noncompact Kähler manifolds with positive Ricci curvature [208], there is still work to be accomplished. For example, it is not clear whether all such manifolds can be compactified.) When the manifold is not Kähler, the solution is quite different. Even for exotic spheres, this has not been answered. A tempting conjecture is the following:

An exotic sphere admits a metric with positive scalar curvature if and only if it bounds a spin manifold. An exotic sphere admits a metric with positive Ricci curvature if and only if it bounds a parallelizable manifold. An exotic sphere admits a metric with positive sectional curvature if and only if it can be written as a vector bundle over a compact manifold.

The first statement is true due to the theorem of Stolz [192] using the surgery result of Schoen-Yau [184] and Gromov-Lawson [78]. Also, the sufficiency parts of the second and third statements are known to be true.

The famous theorem of Cheeger-Gromoll [37] basically reduces the study of manifolds with non-negative Ricci curvature to those with finite fundamental group. Does every finite group appear as the fundamental group of a compact manifold with positive Ricci curvature? Perhaps it is already true for algebraic manifolds with positive first Chern class. In particular, it is not clear whether a connect sum of simply connected manifolds with positive Ricci curvature admits a metric with positive Ricci curvature. More generally, for a compact manifold M with finite fundamental group, if M admits metrics with positive scalar curvature, does it also admit metrics with positive Ricci curvature? Stolz [193] made a proposal to generalize the famous theorem of Lichnerwicz to loop space in the following way: If the Ricci curvature of M is positive and if $\frac{1}{2}p_1(M)$ is zero, the Witten index of M is zero. Note that $\frac{1}{2}p_1(M) = 0$ is interpreted as the loop space of M is spin. However, there is not enough evidence for such a conjecture.

It would be very nice to find a condition to see whether there is an obstruction for manifolds with positive Ricci curvature to admit Einstein metrics with positive scalar curvature. For dimension greater than five, there is presumably no obstruction. Even when the manifold is Kähler, the existence of a Kähler-Einstein metric with positive scalar curvature is still an open question. While there is known obstruction from the fact that the group of automorphisms must be reductive (a theorem of Matsushima [146]) and that the Futaki invariants [60] have to be zero, the author believes that the key obstruction comes from the stability of the polarized complex structure as was defined in geometric invariant theory by Mumford and Giesecker. G. Tian [204] made the first major step toward this conjecture (besides settling the problem for two complex dimensions [202]) by introducing the notion of K-stability.

It is possible that the equation of R. Hamilton can be useful for this problem. His equation does preserve the complex structure and Cao [32] has shown long-time existence of the equation. Basic estimates such as a Harnack inequality were also obtained by Cao [33]. Some soliton solutions were also found by Cao [34]. It remains an important question to classify all such soliton solutions. Questions of uniqueness of Kähler-Einstein metrics and Kähler solitons were treated by Bando-Mabuchi [7] and Tian-Zhu [209] respectively. The asymptotic behavior of Hamilton's equation may be related to the question of stability.

Since Kähler-Einstein metrics with positive scalar curvature do not always exist for Kähler manifolds with positive scalar curvature, it is perhaps interesting to look for slightly more general form of Einstein metrics. The ideas of Conan Leung can lead to somewhat more general metrics (see his recent paper [122]). However Leung has shown that the existence of such a metric implies the vanishing of Futaki's invariants. These may be most general canonical metrics to be studied. It could be useful to study the possible relationship with quantum cohomology.

C. Scalar curvature. While scalar curvature is one of the weakest invariant for a compact manifold with dimension greater than two, it plays an important role in classical general relativity. Scalar curvature basically represents the matter distribution when the Lorentzian metric is restricted to a spacelike hypersurface. (The precise matter distribution includes a contribution from the second fundamental form of the spacelike hypersurface.)

The physical interpretation then demands that positivity of the scalar curvature is much more significant than negativity, as ordinary matter density is supposed to be non-negative. Physical intuition is a very good guide for the development of the theory of manifolds with positive scalar curvature.

The first achievement is the consequence of Dirac operator. Lichnerwicz proved the following vanishing theorem [139]: For spin manifolds, positive scalar curvature implies, by the Atiyah-Singer index theorem, that a certain KO-characteristic number of the manifold must vanish. For quite a long time, this was the only known topological constant. Then the question of the positive mass conjecture was raised in the Stanford conference on differential geometry in 1973. It was immediately realized that a much simpler question is whether there is a metric with positive scalar curvature on the three-dimensional torus. Schoen and I managed to prove that a such a metric does not exist [182]. The proof depends on the existence theorem of incompressible minimal surfaces (which is also due to Sacks-Uhlenbeck independently [177]) and a careful study of the second-variation formula. In particular, we prove that there is no complete metric with positive scalar curvature on a three-dimensional manifold

whose fundamental group admits a subgroup isomorphic to the fundamental group of a surface of genus not less than one [185].

By 1978, we were able to settle the positive mass conjecture completely [183] and we were also able to generalize the theorem on metrics with positive scalar curvature to higher dimensions [184]. The basic principle is to proceed by induction on dimension. We observed that for a stable minimal hypersurface in a manifold with positive scalar curvature, one can (by making use of the first eigenfunction of the second variation operator) conformally deform the metric to a metric with positive scalar curvature. Topological properties of the ambient manifold must be used to guarantee the existence of such minimal hypersurfaces. For example, the first cohomology of the manifold must have enough classes to provide nontrivial intersections.

At the same year, by using ideas from our proof of the positive mass conjecture, we showed that, up to codimension three, one can perform surgery in the category of manifolds with positive scalar curvature [184]. (Subsequently this basic fact was also obtained by Gromov-Lawson using a somewhat different construction [78].) This surgery result enables tools from spin cobordism theory to be applied to manifolds with positive scalar curvature. Stolz [192] proved the following important result: The obstruction from Lichnerwicz's theorem is the only one for the class of simply connected manifolds.

While we had some preliminary results about extending the Lichnerwicz vanishing theorem to manifolds with nontrivial fundamental group, Gromov-Lawson were able to develop an extensive theory about it. They thought that the vanishing of the obstructions from $KO[\pi_1]$ would be good enough for the existence of positive scalar curvature [79]. The recent work of Schick [178] shows that this is not correct, as the obstruction coming from Schoen-Yau on minimal hypersurfaces was not taken into account.

It remains an open question to find a necessary and sufficient condition for a manifold to admit a metric with positive scalar curvature. An interesting problem is to prove that if a manifold M represents a non-trivial homology class in $H^*(K(\pi,1),\mathbb{Q})$ for some group π, then it does not admit a metric with positive scalar curvature. In a course that Schoen and I gave in Berkeley on manifolds with positive scalar curvature in 1981, we settled this problem when $\dim M \leq 4$. In the past ten years, there has been work by A. Connes and others using ideas from operator algebra (see e.g. [9]). However, it has not led to a final solution of the problem. The surgery argument is not significant enough when $\dim M \leq 4$. The result of Schoen-Yau shows that three-manifolds with positive scalar curvature must be connect sums of manifolds with finite fundamental group [185]. If the Poincaré conjecture and the spherical space-form conjecture hold, the converse is also true.

When $\dim M = 4$, the best result is due to A.-K. Liu and T.-J. Li [125]. They gave necessary and sufficient conditions for a symplectic manifold to admit a metric with positive scalar curvature.

Besides questions related to topology, the geometry of manifolds with positive scalar curvature is a fruitful subject, since it is very much related to questions in general relativity. For example, the following statement (of Schoen-Yau [186]) has significance in the theory of black holes:

For a compact three-dimensional manifold M, the first eigenvalue of the operator $-\Delta_M + \frac{1}{2}R_M$ (with Dirichlet condition) is bounded above by $\frac{4\pi^2}{3r^2}$, where r is the smallest radius of any tube around a closed Jordan curve σ in M so that σ is

homotopically trivial inside the tube.

It should be possible to generalize this type of result to higher-dimensional manifolds.

A good formulation of the positive mass conjecture for compact manifolds with boundary is desirable and is not known. The positive mass theorem, and also the solution of the famous Yamabe problem by Schoen [180], have also been applied by Schoen and the author to study conformally flat, positive-scalar-curvature metrics on domains $\Omega \subset S^n$ [187].

Twenty years ago, Penrose made a conjecture in general relativity that the total mass is greater than the square root of the area of the black hole up to a constant. When the spacelike hypersurface has nonnegative scalar curvature, this conjecture has recently been settled by Huisken-Ilmanen [103] and Bray [19]. It remains an open problem to deal with the case when the scalar curvature is not nonnegative.

A rather interesting question which arises in general relativity is the following: For a compact three-dimensional manifold M with nonnegative scalar curvature, the Hawking mass of ∂M should not be greater than a fixed multiple of the diameter of ∂M unless there is a closed stable minimal surface inside M. Of course, there is a generalization of such a statement to manifolds whose scalar curvature is not positive. These statements are related to black-hole formation.

III. Conclusion. It is perhaps appropriate for me to summarize some of the highlights of this report by recording some important questions.

1. Understand the precise nature of the existence and uniqueness of an isometric embedding of an n-dimensional manifold into $\frac{n(n+1)}{2}$-dimensional Euclidean space. When the manifold is compact with no boundary, can there be nontrivial isometric deformations?

There is clearly a distinction between $n = 2$ and $n > 2$, local and global embedding, real analytic and smooth assumptions on metrics. For global embedding with $n = 2$, there was work by Weyl [221], Nirenberg [162] and Pogorelov [169]. For local embedding, there was important work by Lin [140]. The global uniqueness for $n = 2$ was studied by Cohn-Vossen [46]. The uniqueness for the associated linearized operator is also very important and studied by Cohn-Vossen. While the linearized problem is interesting by itself, it is clearly related to the nonlinear problem. Develop a global theory for such linear operators which may neither be elliptic nor hyperbolic. Assume the curvature operator of M^n be positive definite and the embedding is in $R^{\frac{n(n+1)}{2}}$. Does the kernel of the linearized operator consist of only infinitesimal rigid motions? Is there any way to apply the index theorem in this setting?

The isometric embedding problem consisting of finding a Riemannian connection on the abstract normal bundle such that after combining with the Levi-Civita connection on the tangent bundle via the (unknown) second fundamental form, a flat connectionis formed. (Both the Gauss equations and the Codazzi equations have to be used.) Viewed in this way, one notes that the normal bundle is in general topologically decomposable if $n > 2$. This may be one cause of nonuniqueness of the isometric embedding. Perhaps one should decompose the normal bundle into several subbundles and construct the desired Riemannian connections and second fundamental form into several parts according to the decomposition of the normal bundle. Geometrically, we may consider the possibility of structification of the isometric embedding by embedding M^n into $M_1^{n_1} \subset M_2^{n_2} \subset \cdots \subset R^{\frac{n(n+1)}{2}}$ where $n < n_1 < n_2 < \cdots < \frac{n(n+1)}{2}$. If we

add certain structification on the isometric embedding, it may force uniqueness and the existence theorem can be proved easier. For example, one can isometric embed a surface of arbitrary genus by embedding into suitable hyperbolic three manifold first. One can also isometric embed a genus one surface into R^4 through a suitable chosen three dimensional manifold.

One should have an effective way to check whether a metric on M^n can be isometrically embedded into another manifold with dimension $m < \frac{n(n+1)}{2}$. When $n > 2$, the only known uniqueness theorems occur when the codimension is not greater than $n-1$. There is virtually no existence theorem known when $n > 2$.

An interesting global existence Theorem for codimensional one can be obtained as corollary of my work with Schoen [183]: Given a metric on a three dimensional manifold which is strongly asymptotically flat (the mass is zero). Assume that for some symmetric tensor h_{ij} which vanishes at infinity quadratically, the following inequality holds

$$\frac{1}{2}\left[R - \sum h_{ij}^2 + \left(\sum h_{ii}\right)^2\right] \geq \left[\sum_i \left(\sum_j h_{ij,j} - \sum_j h_{jj,i}\right)^2\right]^{\frac{1}{2}}.$$

Then the metric can be isometrically embedded into the flat Mirkonski spacetime as a spacelike hypersurface.

2. Understand the spectrum of the Laplacian of a complete manifold. What is the precise condition for a set of discrete numbers in $\mathbb{R}^+$ to be the spectrum of the Laplacian of some manifold? So far, only necessary conditions are know. When the manifold has a special structure, e.g. if it has an Einstein metric or is a minimal submanifold, one expects there to be more symmetry. (The symmetry is perhaps exhibited in the associated zeta function.) For a generic metric, the spectrum should determine the metric. How can we prove such a statement?

Let $\{\alpha_1, \ldots, \alpha_n, \ldots\}$ be a sequence of nonnegative numbers. (Some α_i may repeat itself finite number of times.) There are two well-known important conditions for then to be spectrum.

(a) There is a positive integer n (dimension) so that $\sum_{i=1}^{\infty} \exp(-\alpha_i t)$ has asymptotic series expansion $t^{-n/2} \sum_{i=1}^{\infty} a_i t^i$.

(b) The distribution defined by $\sum_{j=1}^{\infty} \exp(i\sqrt{\lambda_j} t)$ has singular support in a sequence of countable numbers $\{l_i\}$.

Both **(a)** and **(b)** are asymptotic informations of the sequence $\{\alpha_i\}$. The first one are the heat invariants and the second one are the wave invariants. Unless we have more apriori informations of the manifold (e.g. a minimal hypersurface [38]), there is no other constrains that we know for $\{\alpha_i\}$ to be spectrum of a manifold.

Since these are all asymptotic constrains, it will be interesting to see how much informations can they provide for the finite part of the sequence if they are spectrum of a manifold. If $\{\alpha_i\}$ and $\{\beta_i\}$ are two sequences which can be realized as spectrums of two manifolds. If $t^{n/2}\left(\sum_i e^{\alpha_i t} - \sum_i e^{\beta_i t}\right) = 0(t^m)$ for any $m > 0$ when $t \to 0$ and if $\sum_j \left(e^{i\sqrt{\lambda_j} t} - e^{i\sqrt{\beta_j} t}\right)$ has no singularity as a distribution, can we conclude that $\alpha_j = \beta_j$ for all j.

The heat invariants are integrals of local geometric quantities while the wave invariants are more global and are related to the lengths of closed geodesics. Is it possible to construct more geometric invariants by looking at other functions of Laplacian, by studying space of subsequence of eigenvalues or by constructing new sequence out of eigenvalues (e.g. taking differences of eigenvalues).

When we say that we can "hear" a geometric quantity g, we should mean that for all $\varepsilon > 0$, there is a set of integers $\{n_1, \ldots, n_k\}$ so that n_i depends only on $\lambda_1, \ldots, \lambda_{n_{i-1}}$ and ε and there is a function f defind on $R_T^{n_k+1}$ with

$$|f(\lambda_1, \ldots, \lambda_{n_k}, \varepsilon) - g| < \varepsilon.$$

At present, it is not clear how to "hear" any geometric quantities of a manifold. Unless one has an apriori knowledge of the manifold, it is not even clear that we can hear the dimension of the manifold. For a convex domain, Peter Li and the author did work out a way to hear its area. It will be nice to be able to hear the volume of a manifold if we know its Ricci curvature is bounded form below.

It is important to complete the full spectrum of known Einstein manifolds or closed minimal submanifolds. Only a handful of examples were calculated and they are constructed by group theoretical method.

3. Find an explicit method (similar to the Weierstrass representation method) to exhibit a large class of minimal submanifolds in $\mathbb{R}^n$ or S^n. Can all calibrated minimal submanifolds be produced by such an explicit method? It would be especially interesting to produce them in ambient manifolds with special holonomy group. (In the case of a Calabi-Yau manifold, we are looking at special Lagrangian submanifolds.) Calculate the moduli space of these minimal submanifolds.

It may also be interesting to find a class of hypersurfaces with constant mean curvature that are defined by first order elliptic systems. In most classes, we may need to parametrize these calibrated submanifolds with certain geometric structure over them. For example, in case of special Lagrangial submanifold in a Calabi-Yau manifold, we parametrize then together with a unitary flat line bundle. In some other cases, it can be some twisted harmonic spinors. It is an interesting question of finding a good compactification of the moduli space of these submanifolds together with the exterior structure. The behaviour of the Weyl-Peterson metric near singular point should be interesting. Note that sequence of minimal submanifolds always converge to a minimal current in the sense of geometricmeasure theory, the question is how to generalized flat line bundle or twisted harmonic spinor define on such minimal currents. We hope that the compactified moduli space has extra structure such as (possibly singular) algebraic structure. Can they be realized as moduli space of some algebraic geometric objects?

4. Classify compact smooth manifolds which admit Einstein metrics. If $\dim M > 4$, does M always admit an Einstein metric? If so, how can we parametrize them?

Is there any criterion to guarantee that the moduli space of Einstein metrics has finite number of components? For manifolds with odd dimension ≥ 5, there were examples of Wang-Ziller in Journal of Diff. Geom., 1990, where infinite number of components were found. The problem is interesting already for dimension equal to four and six. When we change the topology of M, how does the topology of the moduli space of Einstein metrics change? Most manifolds with nontrivial continuous family of Einstein metrics come from torus bundle over Kähler-Einstein manifolds or special holonomy manifolds. What are others? For dimension 4, can one decom-

pose M^4 into open pieces of the form (a) Einstein manifold (b) circle bundles over a three dimensional manifolds with constant curvature (c) surface bundle over another surface.

The above open pieces are suppose to connected along three manifolds which are circle bundles over either S^2 or T^2. For dimension 3, there is the analogous statement of Thurston's geometrization conjecture.

5. Classify compact Riemannian manifolds with special holonomy group. The major holonomy groups are $SU(n)$, Spin(7) and G_2. Prove that there are only a finite number of deformation types of these manifolds for each dimension.

Most likely, manifolds with holonomy group Spin (7) and G_2 can be constructed through some geometric construction on those with $SU(n)$ holonomy. It will therefore be interesting to find a canonical structure on their module space similar to the special geometry on Calabi-Yau manifold. There are also calibrated submanifolds and bundles with special holonomy group on these manifolds. What will be the precise "mirror symmetry" for these manifolds? For example, can one find a way to "count" those calibrated minimal submanifolds in this kind of geometry. What is the condition for a holonomy class to be represented by some calibrated submanifold? For the case of Lagrangian minimal surface in a Kähler-Einstein surface, this was studied extensively by Schoen-Wolfson.

6. Prove that if M^{2n} admits an almost-complex structure and $n > 2$, then it also admits an integrable complex structure. For $n = 2$, there are well-known obstructions. Any complex surface with even first Betti number can be deformed to an algebraic surface, and any algebraic surface admits a Lefschetz fibration (possibly after blowing up several times). Therefore, it is interesting to find a sufficient condition for a surface bundle (with Morse type singularity) to admit a complex structure. It will also be interesting to find a condition on the homotopy type of a four-manifold to admit a singular surface-bundle structure. An interesting class of manifolds to be studied in this direction is quotients of the complex ball, which we know to be globally rigid. A related question is how to glue complex manifolds together. The Seiberg-Witten invariants and relative versions of pseudoholomorphic curve theory can perhaps help us understand such gluings. Another direction is to find a sufficient condition for a four-manifold which branches over $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$ to admit a complex structure.

By the work of Bogomolov and Li-Yau-Zheng [130], one knows how to classify those Kodaira class VII surfaces with no curves. It remains to classify those with curves. Perhaps the arguments in [124] can be used.

It is an important problem to find a suitable large class of non Kähler complex manifolds with canonical Hermitian metrics. String theory may provide some directions to look for possible equations for such Hermitian metrics. It is especially important to find canonical Hermitian metrics for these complex manifolds obtained by smoothing singularities of Calabi-Yau manifold obtained by blowing down a rational curve with negative normal bundle. One likes to find those with "sypersymmetry" so that mirror symmetry still exists.

7. Find necessary and sufficient conditions for a complex manifold to admit a Kähler structure. Prove that any Kähler manifold can be deformed to an algebraic manifold.

For a Kähler manifold, we have the well-known theorems of Lefschitz and Hodge on it cohomology. The theorem of Deligne-Griffiths-Morgan-Sullivan also demands the rational homotopic type to be formally determined. These are unknown general

necessary conditions beyond the fact that it is a complex manifold (which also gives integrability conditions on Chern numbers based on the index theorem). If a complex manifold satisfies all these conditions on its homotopy type, is it homotopic equivalent to a Kähler manifold? The corresponding question on the diffeomorphic type is far more delicate. For example, the author was able to characterize those Kähler manifold homotopic to algebraic manifolds with constant holomorphic sectional curvature in terms of Chern number equalities.

8. Given a complex vector bundle V over an algebraic manifold whose Chern classes are of (k, k) type, will V admit a holomorphic structure if we add or subtract from it holomorphic bundles in the sense of K theory? Prove the Hodge conjecture that rational (k, k)-classes are Poincaré dual to algebraic cycles.

The only progress of constructing integrable complex structure over a complex vector bundle is due to C. Taubes [198] and comes from existence of antiself-dual connection over a four dimensional manifold. However, it does not tell much about the higher dimensional manifolds. Even in the case of four manifold, one has to restrict to the case when the bundle is constructed topologically from bubbling process. It is very much desirable to have a construction without using singular perturbation from known data. The construction of Uhlenbeck-Yau [215] for Hermitian Yang-Mills connections should be applied to general topological bundles. The construction of integrable complex structure on bundles may encounter obstructions coming from algebraic cycles. A deep understanding of their relationship should be rewarding.

9. What is the structure of the singular set of an elliptic variational problem? In particular, what is the structure of the singular set of an area-minimizing variety? Is such a singular set stable under perturbations of the ambient metric? What are the singular sets for solutions to the mean curvature flow and the Ricci flow?

For hyperbolic system, the most important question is the development of singularity for Einstein equations in general relativity. The nonlinearity of the system has exhibited spectacular rich interaction between physics and geometry.

For a long time, geometers have been interesting in overdetermined systems of differential inequalities. For example, the existence of metrics with positive curvature on amanifold has been center of activities for a long time. (It is not known, for example, that when dimension is large enough, only locally symmetric spaces admit metrics with positive sectional curvature. It is still not known whether any nontrivial product manifold admit metrics with positive sectional curvature.) It is natural to see whether one can develop the concept of weak form of sectional curvature from the point of view of differential equations. The idea is to allow weak convergence of such metrics. Singularity of such metrics should be very interesting to understand. This question of course can be asked for such determinal system as Ricci tensor.

10. Give a full and rigorous geometric explanation of the concept of mirror symmetry for Calabi-Yau manifolds. Does it exist for other geometric structures without special holonomy group? Explain the structure of the generating function of the instanton numbers and the structure of the mirror map which come from mirror symmetry. What is the arithmetic meaning of these functions?

The works of Lian-Liu-Yau called 'Mirror Principle' I, II and III (Asian J. Math., 1997, 1999), do give a beginning of systematic understanding of enumerative geometry motivated by mirror symmetry. The work of Strominger-Yau-Zaslow [195] began a geometric understanding of mirror symmetry. Lian-Yau [136], [137] did begin to study the arithmetic nature of the mirror map.

REFERENCES

[1] A. D. Alexandrov, *On a class of closed surfaces*, Recueil Math. (Moscow), 4 (1938), pp. 69–77.

[2] M. T. Anderson, *The Dirichlet problem at infinity for manifolds of negative curvature*, J. Diff. Geo., 18 (1983), pp. 701–721.

[3] P. S. Aspinwall, B. R. Greene, K. H. Kirklin, and P. J. Miron, *Searching for three-generation Calabi-Yau manifolds*, Nucl. Phys. B, 294 (1987), pp. 193–222.

[4] P. S. Aspinwall, B. R. Greene, and D. Morrison, *Calabi-Yau moduli space, mirror manifolds and spacetime topology change in string theory*, Nuclear Phys. B, 416:2 (1994), pp. 414–480.

[5] P. S. Aspinwall and D. Morrison, *Topological field theory and rational curves*, Comm. Math. Phys., 151 (1993), pp. 245–262.

[6] T. Aubin, *Equations du type Monge-Ampère sur les variétés kählériennes compactes*, Bull. Sc. Math., 102 (1978), pp. 63–95.

[7] S. Bando and T. Mabuchi, *Uniqueness of Einstein Kähler metrics modulo connected group actions*, in Algebraic geometry (Sendai, 1985), Adv. Stud. Pure Math. 10, North-Holland, Amsterdam-New York, 1987, pp. 11-40.

[8] S. Barannikov and M. Kontsevich, *Frobenius manifolds and formality of Lie algebras of polyvector fields*, Internat. Math. Res. Notices, 4 (1998), pp. 201–215, alg-geom/9710032.

[9] P. Baum, A. Connes, and N. Higson, *Classifying space for proper actions and K-theory of group C^*-algebras*, in C^*-algebras: 1943–1993 (San Antonio, TX, 1993), Contemp. Math. 167, Amer. Math. Soc., Providence, RI, 1994, pp. 240–291.

[10] K. Becker, M. Becker, and A. Strominger, *Fivebranes, membranes, and non-perturbative string theory*, Nucl. Phys. B, 456 (1995), pp. 130–152, hep-th/9507158.

[11] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math., 128:1 (1997), pp. 45–88.

[12] M. Berger, *Sur certaines variétés riemanniennes à courbure positive*, C. R. Acad. Sci., Paris, 247 (1958), pp. 1165–1168; *Les variétés riemanniennes à courbure positive*, Bull. Soc. Math., Belg., 10 (1958), pp. 89–104.

[13] G. Besson, G. Courtois, and S. Gallot, *Entropies et rigidités des espaces localement symétriques de courbure strictement négative*, Geom. Funct. Anal., 5:5 (1995), pp. 731–799.

[14] G. Bini, C. De Concini, M. Polino, and C. Procesi, *On the work of Givental relative to mirror symmetry*, math. AG/9805097.

[15] D. D. Bleecker, *Volume increasing isometric deformations of convex polyhedra*, J. Diff. Geom., 43:3 (1996), pp. 505–526.

[16] F. A. Bogomolov, *Hamiltonian Kählerian manifolds*, Soviet Math. Dokl., 19:6 (1979), pp. 1462–1465.

[17] F. A. Bogomolov, *Surfaces of class* VII_0 *and affine geometry*, Math. USSR-Izv., 21:1 (1983), pp. 31–73.

[18] C. Böhm, *Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces*, Invent. Math., 134 (1998), pp. 145–176.

[19] H. Bray, *Proof of the Riemannian Penrose conjecture using the positive mass theorem*, Preprint.

[20] J. Bryan and N. C. Leung, *The enumerative geometry of $K3$ surfaces and modular forms*, alg-geom/9711031, 1997.

[21] N. P. Buchdahl, *Hermitian-Einstein connections and stable vector bundles over compact complex surfaces*, Math. Ann., 280:4 (1988), pp. 625–648.

[22] D. Burns and M. Rapoport, *On the Torelli problem for Kählerian $K3$ surfaces*, Ann. Sci. École Norm. Sup., (4) 8:2 (1975), pp. 235–273.

[23] E. Calabi, *Improper affine hypersurfaces of convex type and a generalization of a theorem of Jörgens*, Michigan Math. J., 5 (1958), pp. 105–126.

[24] E. Calabi, *Complete affine hyperspheres I*, Sympos. Math., 10 (1977), pp. 19–38.

[25] E. Calabi, *Hypersurfaces with maximal affinely invariant area*, Am. J. of Math., 104:1 (1982), pp. 91–126.

[26] E. Calabi and L. Nirenberg, unpublished (see the introduction to [40]).

[27] P. Candelas, P. S. Green, and T. Hübsch, *Rolling among Calabi-Yau vacua*, Nucl. Phys. B, 330 (1990), pp. 49–102.

[28] P. Candelas, G. Horowitz, A. Strominger, and E. Witten, *Vacuum configuration for superstrings*, Nucl. Phys. B, 258 (1985), pp. 46–74.

[29] P. Candelas, M. Lynker, and R. Schimmrigk, *Calabi-Yau manifolds in weighted* $\mathbb{P}^4$, Nucl.

Phys. B, 341 (1990), pp. 383–402.

[30] P. Candelas, X. de la Ossa, A. Font, S. Katz, and D. R. Morrison, *Mirror symmetry for two-parameter models I*, Nucl. Phys. B, 416:2 (1994), pp. 481–538.

[31] P. Candelas, X. de la Ossa, P. Green, and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nucl. Phys. B, 359 (1991), pp. 21–74.

[32] H. D. Cao, *Deformation of Kähler metrics to Kähler-Einstein metrics on compact Kähler manifolds*, Invent. Math., 81:2 (1985), pp. 359–372.

[33] H. D. Cao, *On Harnack's inequalities for the Kähler-Ricci flow*, Invent. Math., 109:2 (1992), pp. 247–263.

[34] H. D. Cao, *Existence of gradient Kähler-Ricci solitons*, in Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994), AK Peters, Wellesley, MA, 1996, pp. 1–16.

[35] E. Cartan, *Sur la possibilité de plonger un espace riemannien donné dans un espace euclidien*, Ann. Soc. Pol. Math., 6 (1927), pp. 1–7.

[36] F. Catanese and C. LeBrun, *On the scalar curvature of Einstein manifolds*, Math. Res. Lett., 4:6 (1997), pp. 843–854.

[37] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Diff. Geom., 6 (1971/1972), pp. 119–128.

[38] S. Y. Cheng, P. Li, and S.-T. Yau, *Heat equations on minimal submanifolds and their applications*, Amer. J. Math., 106 (1984), pp. 1033–1065.

[39] S. Y. Cheng and S.-T. Yau, *On the regularity of the Monge-Ampère equation* $\det((\partial^2 u/\partial x^i \partial x^j)) = F(x,u)$, Comm. Pure Appl. Math., 30 (1977), pp. 41–68.

[40] S. Y. Cheng and S.-T. Yau, *Complete affine hypersurfaces, Part* I. *The completeness of affine metrics*, Comm. Pure Appl. Math., 39 (1986), pp. 839–866.

[41] S. Y. Cheng and S.-T. Yau, *Inequality between Chern numbers of singular Kähler surfaces and characterization of orbit space of discrete group of* $\mathrm{SU}(2,1)$, in Complex differential geometry and nonlinear differential equations (Brunswick, Main, 1984), Contemp. Math. 49, Amer. Math. Soc., Providence, RI, 1986, pp. 31–44.

[42] T.-M. Chiang, A. Klemm, S.-T. Yau, and E. Zaslow, *Local mirror symmetry: calculations and interpretations*, hep-th/9903053.

[43] H. I. Choi and A. Wang, *A first eigenvalue for minimal hypersurfaces*, J. Diff. Geom., 18 (1983), pp. 559–562.

[44] C. H. Clemens, *Degeneration techniques in the study of threefolds*, in Algebraic threefolds (Varenna, 1981), Lecture Notes in Math. 947, Springer, Berlin-New York, 1982, pp. 93–154.

[45] C. H. Clemens, *Homological equivalence modulo algebraic equivalence is not finitely generated*, Publ. Math. IHES, 58 (1983), pp. 19–38.

[46] E. Cohn-Vossen, *Zwei Sätze über die Starrheit der Eiflächen*, in Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematish-Physikalische Klasse, 1927, pp. 125–134.

[47] E. Cohn-Vossen, *Unstarre geschlossene Flächen*, Math. Annalen, 102 (1929/1930), pp. 10–29.

[48] T. H. Colding and W. P. Minicozzi II, *Harmonic functions on manifolds*, Ann. of Math., 146:3 (1997), pp. 725–747.

[49] P. Collin, R. Kusner, W. Meeks, and H. Rosenberg, *Topology, geometry and conformal invariants of properly embedded minimal surfaces*, Preprint.

[50] R. Connelly, *A counterexample to the rigidity conjecture for polyhedra*, Inst. Hautes Études Sci. Publ. Math., 47 (1977), pp. 333–338.

[51] L. J. Dixon, *Some world-sheet properties of superstring compactifications, on orbifolds and otherwise*, in Superstrings, unified theories and cosmology 1987 (Trieste, 1987), ICTP Ser. Theoret. Phys. 4, World Sci. Publishing, Teaneck, NJ, 1988, pp. 67–126.

[52] S. K. Donaldson, *Anti Self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, Proc. London Math. Soc., 50 (1985), pp. 1–26.

[53] C. Doran, *Picard-Fuchs uniformization and geometric isomonodromic deformations: Modularity and variation of the mirror map*, Harvard thesis, 1999.

[54] B. Dubrovin, *Integrable systems in topological field theory*, Nucl. Phys. B, 379 (1992), pp. 627–689.

[55] N. V. Efimov, *Impossibility of a complete surface in* 3*-space whose Gaussian curvature has a negative upper bound*, Soviet Math., 4 (1963), pp. 843–846.

[56] R. Friedman, *Simultaneous resolution of threefold double points*, Math. Ann., 274:4 (1986), pp. 671–689.

[57] R. Friedman and J. W. Morgan, *Algebraic surfaces and Seiberg-Witten invariants*, J. Algebraic Geom., 6:3 (1997), pp. 445–479.

[58] C. Frohman and W. H. Meeks III, *The ordering theorem for the ends of properly embedded minimal surfaces*, Topology, 36:3 (1997), pp. 605–617.

[59] K. Fukaya and Y.-G. Oh, *Zero-loop open strings in the cotangent bundle and Morse homotopy*, Asian J. Math., 1:1 (1997), pp. 96–180.

[60] A. Futaki, *An obstruction to the existence of Einstein-Kähler metrics*, Invent. Math., 73 (1983), pp. 437–443.

[61] P. Gauduchon, *Le théorème de l'excentricité nulle*, C.R.A.S. Paris, Série A, 285 (1977), pp. 387–390.

[62] P. Gauduchon, *Fibrés Hermitiens a endomorphisme de Ricci non négatif*, Bull. Soc. Math. France, 105 (1977), pp. 113–140.

[63] K. F. Gauss, *General Investigations of Curved Surfaces*, Raven Press, New York, 1965.

[64] I. M. Gelfand, A. V. Kapranov, and A. V. Zelevinsky, *Hypergeometric functions and toric varieties*, Func. Anal. Appl., 23 (1989), pp. 94–106; Correction in Func. Anal. Appl., 27 (1993), pp. 295.

[65] I. M. Gelfand, A. V. Kapranov, and A. V. Zelevinsky, *Generalized Euler integrals and A-hypergeometric functions*, Adv. Math., 84:2 (1990), pp. 255–271.

[66] D. Gepner, *Exactly solvable string compactifications on manifolds of* $\mathrm{SU}(N)$ *holonomy*, Phys. Lett. B, 199 (1987), pp. 380–388.

[67] R. Geroch, *A method for generating solutions of Einstein's equations, I, II*, J. Math. Phys., 12 (1971), pp. 918–924; J. Math. Phys., 13 (1972), pp. 394–404.

[68] G. W. Gibbons, S. W. Hawking, and M. J. Perry, *Path integrals and the indefiniteness of the gravitational action*, Nuclear Phys. B, 138:1 (1978), pp. 141–150.

[69] D. Gieseker, *On moduli of vector bundles on an algebraic surface*, Ann. of Math., 106 (1977), pp. 45–60.

[70] A. Givental, *Equivariant Gromov-Witten invariants*, Int. Math. Res. Notices, 1996:13, pp. 613–663.

[71] L. Göttsche, *A conjectural generating function for numbers of curves on surfaces*, alg-geom/9711012.

[72] B. R. Greene, D. R. Morrison, and A. Strominger, *Black hole condensation and the unification of string vacua*, Nuclear Phys. B, 451:1–2 (1995), pp. 109–120.

[73] B. R. Greene and M. R. Plesser, *Duality in Calabi-Yau moduli space*, Nucl. Phys. B, 338 (1990), pp. 15–37.

[74] B. R. Greene, C. Vafa, and N. P. Warner, *Calabi-Yau manifolds and renormalization group flows*, Nucl. Phys. B, 324 (1989), pp. 371–390.

[75] D. Gromoll and W. Meyer, *An exotic sphere with nonnegative sectional curvature*, Ann. of Math., 100:2 (1974), pp. 401–406.

[76] M. Gromov, *Curvature, diameter and Betti numbers*, Comm. Math. Helv., 56 (1981), pp. 179–195.

[77] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math., 82:2 (1985), pp. 307–347.

[78] M. Gromov and H. B. Lawson, *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math., 111:3 (1980), pp. 423–434.

[79] M. Gromov and H. B. Lawson, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Publ. Math. IHES, 58 (1983), pp. 83–196.

[80] M. Gross, *Special Lagrangian fibrations* I: *Topology*, alg-geom/9710006.

[81] M. Gross, *Special Lagrangian fibrations* II: *Geometry*, math. AG/9809072.

[82] M. Gross and P. M. H. Wilson, *Mirror symmetry via 3-tori for a class of Calabi-Yau threefolds*, Math. Ann., 309:3 (1997), pp. 505–531.

[83] K. Grosse-Brauckmann, R. Kusner, and J. Sullivan, *Classification of embedded constant mean curvature surfaces with genus zero and three ends*, PNAS, to appear.

[84] B. Guan and J. Spruck, *Boundary-value problems on* S^n *for surfaces of constant Gauss curvature,* Ann. of Math., 138:3 (1993), pp. 601–624.

[85] P. Guan and Y. Y. Li, *The Weyl problem with nonnegative Gauss curvature*, J. Diff. Geom., 39:2 (1994), pp. 331–342.

[86] M. Gursky and C. LeBrun, *On Einstein manifolds of positive sectional curvature*, math. DG/9807055.

[87] R. S. Hamilton, *Harnack estimate for the mean curvature flow*, J. Diff. Geom., 41:1 (1995), pp. 215–226.

[88] R. S. Hamilton, *Four-Manifolds with positive isotropic curvature*, Comm. Anal. Geom., 5:1 (1997), pp. 1–92.

[89] R. S. Hamilton, *Non-singular solutions of the Ricci flow on three-manifolds*, Comm. Anal. Geom., to appear.

[90] R. Hardt and L. Simon, *Boundary regularity and embedded solutions for the oriented Plateau problem*, Ann. of Math., 110:3 (1979), pp. 439–486.

[91] R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Math., 148 (1982), pp. 47–157.

[92] S. W. Hawking, *Gravitational radiation in an expanding universe*, J. of Math. Phys., 9:4 (1968), pp. 598–604.

[93] N. J. Hitchin, *Compact four-dimensional Einstein manifolds*, J. Diff. Geom., 9 (1974), pp. 435–441.

[94] N. J. Hitchin, *The moduli space of special Lagrangian submanifolds*, dg-ga/9711002.

[95] N. J. Hitchin, A. Karlhede, U. Lindström, and M. Roček, *Hyperkähler metrics and supersymmetry*, Commun. Math. Phys., 108 (1987), pp. 535–589.

[96] D. Hoffman and H. Karcher, *Complete embedded minimal surfaces of finite total curvature*, in Geometry, V, Encyclopedia Math. Sci. 90, Springer, Berlin, 1997, pp. 5–93 & pp. 267–272.

[97] J. X. Hong, *Dirichlet problems for general Monge-Ampère equations*, Math. Z., 209:2 (1992), pp. 289–306.

[98] J. X. Hong, *Realization in $\mathbb{R}^3$ of complete Riemannian manifolds with negative curvature*, Comm. Anal. Geom., 1:3-4 (1993), pp. 487–514.

[99] S. Hosono, A. Klemm, S. Theisen, and S.-T. Yau, *Mirror symmetry, mirror map and applications to Calabi-Yau hypersurfaces*, Comm. Math. Phys., 167:2 (1995), pp. 301–350; *Mirror symmetry, mirror map and applications to complete intersection Calabi-Yau spaces*, Nucl. Phys. B, 433:3 (1995), pp. 501–522.

[100] S. Hosono, B. Lian, and S.-T. Yau, *Maximal degeneracy points of GKZ systems*, J. A. M. S., 10:2 (1997), pp. 427–443.

[101] L. Hsu, R. Kusner, and J. Sullivan, *Minimizing the squared mean curvature integral for surfaces in space forms,* Exper. Math., 1:3 (1992), pp. 191–207.

[102] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Diff. Geom., 31:1 (1990), pp. 285–299.

[103] G. Huisken and T. Ilmanen, J. Diff. Geom, to appear.

[104] G. Huisken and C. Sinestrari, *Mean curvature flow singularities for mean convex surfaces*, April 1997, Preprint.

[105] M. Inoue, *On surfaces of class VII_0*, Invent. Math., 24 (1974), pp. 269–310.

[106] D. D. Joyce, *Compact Riemannian 7-manifolds with holonomy G_2, I, II*, J. Diff. Geom., 43:2 (1996), pp. 291–328 & pp. 329–375.

[107] D. D. Joyce, *Compact 8-manifolds with holonomy* Spin(7), Invent. Math., 123:3 (1996), pp. 507–552.

[108] N. Kapouleas, *Complete constant mean curvature surfaces in Euclidean three-space*, Ann. of Math., 131:2 (1990), pp. 239–330.

[109] N. Kapouleas, *Complete embedded minimal surfaces of finite total curvature*, J. Diff. Geom., 47:1 (1997), pp. 95–169.

[110] N. Karmarkar, *Riemannian geometry underlying interior-point methods for linear programming*, in Mathematical developments arising from linear programming (Brunswick, ME, 1988), Contemp. Math. 114, 1990, pp. 51–75.

[111] A. Klemm, B. H. Lian, S. S. Roan, and S.-T. Yau, *A note on ODEs and mirror symmetry*, in Functional Analysis on the eve of the 21st Century, Vol. II, Progr. Math., 132 (1993), pp. 301–323.

[112] A. Klemm, P. Mayr, and C. Vafa, *BPS states of exceptional non-critical strings*, hep-th/9607139.

[113] W. Klingenberg, *Über Riemannische Mannigfaltigkeiten mit positiver Krümmung*, Comment. Math. Helv., 35 (1961), pp. 47–54.

[114] S. Kobayashi, *Differential Geometry of Complex Vector Bundles*, Princeton University Press, 1987.

[115] M. Kontsevich, *Enumeration of rational curves via torus actions*, in The moduli space of curves, R. Dijkgraaf, C. Faber and G. van der Geer, ed., Birkhäuser, 1995, pp. 335–368.

[116] N. Korevaar, *Upper bounds for eigenvalues of conformal metrics*, J. Diff. Geom., 37 (1993), pp. 73–93.

[117] D. Kotschick, *Einstein metrics and smooth structures*, Geom. Topol. (electronic), 2 (1998), pp. 1–10.

[118] V. S. Kulikov, *Degenerations of* K3 *surfaces and Enriques surfaces*, Math. USSR-Izv., 11:5 (1978), pp. 957–989.

[119] C. LeBrun, *Einstein metrics and Mostow rigidity*, Math. Res. Lett., 2:1 (1995), pp. 1–8.

[120] W. Lerche, C. Vafa, and N. P. Warner, *Chiral rings in $N = 2$ superconformal theories*, Nucl. Phys. B, 324 (1989), pp. 427–474.

[121] N. C. Leung, *Einstein type metrics and stability on vector bundles*, J. Diff. Geom., 45 (1997), pp. 514–546.

[122] N. C. Leung, *Bando Futaki invariants and Kähler Einstein metric*, Comm. in Analysis and Geom., 6:4 (1998), pp. 799–808.

[123] N. C. Leung, *Uniformization of four manifolds*, dg-ga/9705001.

[124] H. Lewy, *On the existence of a closed convex surface realizing a given Riemannian metric*, Proceedings of the National Academy of Sciences, U.S.A., 24:2 (1938), pp. 104–106.

[125] A.-K. Liu and T.-J. Li, *General wall crossing formula*, Math. Res. Lett., 2:6 (1995), pp. 797–810.

[126] A.-K. Liu and T.-J. Li, *Symplectic structure on ruled surfaces and a generalized adjunction formula*, Math. Res. Lett., 2:4 (1995), pp. 453–471.

[127] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. Amer. Math. Soc., 11:1 (1998), pp. 119–174.

[128] J. Li and G. Tian, *Comparison of the algebraic and the symplectic Gromov-Witten invariants*, alg-geom/9712035.

[129] J. Li and S.-T. Yau, *Hermitian-Yang-Mills connection on non-Kähler manifolds*, in Mathematical aspects of string theory, (San Diego, 1986), Adv. Ser. Math. Phys. 1, World Sci. Publ., Singapore, 1986, pp. 560–573.

[130] J. Li, S.-T. Yau, and F. Zheng, *A simple proof of Bogomolov's theorem on class* VII_0 *surfaces with* $b_2 = 0$, Illinois J. Math., 34:2 (1990), pp. 217–220.

[131] P. Li, *Harmonic sections of polynomial growth*, Math. Research Letters, 4 (1997), pp. 35–44.

[132] P. Li, R. Schoen, and S.-T. Yau, *On the isoperimetric inequality for minimal surfaces*, Ann. Scuola Norm., 9 (1984), pp. 237–244.

[133] P. Li and S.-T. Yau, *A new conformal invariant and its applications to the Willmore Conjecture and the first eigenvalue of compact surfaces*, Invent. Math., 69 (1982), pp. 269–291.

[134] B. Lian, K. Liu, and S.-T. Yau, *Mirror principle, I*, Asian J. Math., 1:4 (1997), pp. 729–763.

[135] B. H. Lian and S.-T. Yau, *Mirror symmetry, rational curves on algebraic manifolds and hypergeometric series*, in ICMP Proceedings (Paris 1994), IP, 1995, pp. 163–184.

[136] B. H. Lian and S.-T. Yau, *Mirror maps, modular relations and hypergeometric series, I.*.

[137] B. H. Lian and S.-T. Yau, *Mirror maps, modular relations and hypergeometric series, II.*, S-duality and mirror symmetry (Trieste, 1995), Nucl. Phys. B, Proc. Suppl., 46 (1996), pp. 248–262.

[138] B. H. Lian and S.-T. Yau, *Integrality of certain exponential series*, in Algebra and Geometry, Proceedings of the International Conference of Algebra and Geometry, IP, 1998, pp. 215–227.

[139] A. Lichnerowicz, *Spineurs harmoniques*, C. R. A. S. Série 1, 257 (1963), pp. 7–9.

[140] C.-S. Lin, *The local isometric embedding in* $\mathbb{R}^3$ *of* 2*-dimensional Riemannian manifolds with nonnegative curvature*, J. Diff. Geom., 21 (1985), pp. 213–230.

[141] C.-S. Lin, *The local isometric embedding in* $\mathbb{R}^3$ *of two-dimensional Riemannian manifolds with Gaussian curvature changing sign cleanly*, Comm. Pure Appl. Math., 39 (1986), pp. 867–887.

[142] A.-K. Liu, *Family blow up formula, admissible graphs and the counting of nodal curves*, Preprint.

[143] M. Lübke, *Stability of Einstein-Hermitian vector bundles*, Manuscripta Math., 42 (1983), pp. 245–257.

[144] Yu. I. Manin, *Generating functions in algebraic geometry and sums over trees*, in The Moduli Space of Curves, R. Dijkgraaf, C. Faber and G. van der Geer, ed., Progress in Math. 129, Birkhäuser, 1995, pp. 401–418.

[145] M. Maruyama, *Moduli of stable sheaves, I, II*, J. Math. Kyoto Univ., 17 (1977), pp. 91–126.

[146] Y. Matsushima, *Sur la structure du groupe d'homeomorphismes analytiques d'une certaine variété Kaehlerienne*, Nagoya Math. J., 11 (1957), pp. 145–150.

[147] R. C. McLean, *Deformation of calibrated submanifolds*, Comm. Anal. Geom., 6:4 (1998), pp. 705–747.

[148] W. H. Meeks III, J. Perez, and A. Ros, *Uniqueness of the Riemann minimal examples*, Invent. Math., 133:1 (1998), pp. 107–132.

[149] W. H. Meeks III and S.-T. Yau, *The equivariant loop theorem for three-dimensional manifolds and a review of the existence theorems for minimal surfaces*, in The Smith conjecture, Academic Press, 1984, pp. 153–163.

[150] M. Micallef and J. D. Moore, *Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes*, Ann. Math., 127 (1988), pp. 199–227.

[151] J. H. Michael and L. M. Simon, *Sobolev and mean-value inequalities on generalized submanifolds of* $\mathbb{R}^n$, Comm. Pure Appl. Math., 26 (1973), pp. 361–379.

[152] H. Minkowski, *Volumen und Oberfläche*, Math. Ann., 57 (1903), pp. 447–495.
[153] N. Mok, *Compactification of complete Kähler surfaces of finite volume satisfying certain curvature conditions*, Ann. of Math., 129:2 (1989), pp. 383–425.
[154] N. Mok and J. Q. Zhong, *Compactifying complete Kähler-Einstein manifolds of finite topological type and bounded curvature*, Ann. of Math., 129:3 (1989), pp. 427–470.
[155] G. Moore, *Attractors and arithmetic*, hep-th/9807056; *Arithmetic and attractors*, hep-th/9807087.
[156] G. Moore and E. Witten, *Integration over the u-plane in Donaldson theory*, Adv. Theor. Math. Phys., 1:2 (1997), pp. 298–387.
[157] D. Morrison and R. Plesser, *Summing the instantons: quantum cohomology and mirror symmetry in toric varieties*, alg-geom/9412236.
[158] S. Mukai, *Symplectic structure of the moduli space of sheaves on an abelian or $K3$ surface*, Invent. Math., 77:1 (1984), pp. 101–116.
[159] D. Mumford, *Geometric Invariant Theory*, Academic Press, New York, 1965.
[160] N. Nadirashvili, *Hadamard's and Calabi-Yau's conjectures on negatively curved and minimal surfaces*, Invent. Math., 126 (1996), pp. 457–465.
[161] J. Nash, *The imbedding problem for Riemannian manifolds*, Ann. Math., 63 (1956), pp. 20–63.
[162] L. Nirenberg, *The Weyl and Minkowski problems in differential geometry in the large*, Comm. Pure Appl. Math., 6 (1953), pp. 337–394.
[163] L. Nirenberg, *Rigidity of a class of closed surfaces*, in Nonlinear Problems, R. E. Langer, ed., Univ. of Wisconsin Press, Madison, 1963, pp. 177–193.
[164] R. Pandharipande, *Rational curves on hypersurfaces (after A. Givental)*, math. AG/9806133.
[165] T. Parker and J. Wolfson, *A compactness theorem for Gromov's moduli space*, J. Geom. Anal., 3 (1993), pp. 63–98.
[166] U. Persson and H. Pinkham, *Degeneration of surfaces with trivial canonical bundle*, Ann. of Math., 113:1 (1981), pp. 45–66.
[167] I. I. Piatetski-Shapiro and I. R. Shafarevich, *Torelli's theorem for algebraic surfaces of type* K3, (Russian), Izv. Akad. Nauk SSSR Ser. Mat., 35 (1971), pp. 530–572.
[168] A. V. Pogorelov, *The rigidity of general convex surfaces*, Doklady, Akademia naul, SSSR, (N.S.), 79 (1951), pp. 739–742.
[169] A. V. Pogorelov, *Extrinsic Geometry of Convex Surfaces*, "Nauka", Moscow, 1969; English trans., Math. Monographs 35, Amer. Math. Soc., Providence, RI, 1973.
[170] H. E. Rauch, *A contribution to differential geometry in the large*, Ann. of Math., 54 (1951), pp. 38–55.
[171] D. B. Ray and I. M. Singer, *Analytic torsion for complex manifolds*, Ann. of Math., 98 (1973), pp. 154–177.
[172] M. Reid, *The moduli space of 3-folds with $K = 0$ may nevertheless be irreducible*, Math. Ann., 278:1-4 (1987), pp. 329–334.
[173] M. Roček, *Modified Calabi-Yau manifolds with torsion*, in Essays on mirror manifolds, Internat. Press, Hong Kong, 1992, pp. 480–488.
[174] Y. Ruan, *Topological sigma model and Donaldson-type invariants in Gromov theory*, Duke Math. J., 83:2 (1996), pp. 461–500.
[175] Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology*, J. Diff. Geom., 42:2 (1995), pp. 259–367.
[176] Y. Ruan and G. Tian, *Higher genus symplectic invariants and sigma madels coupled with gravity*, Invent. Math., 130:3 (1997), pp. 455–516.
[177] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. Math., 113 (1981), pp. 1–24.
[178] T. Schick, *A counterexample to the (unstable) Gromov-Lawson-Rosenberg conjecture*, Topology, 37:6 (1998), pp. 1165–1168.
[179] R. Schoen, *A remark on minimal hypercones*, Proc. Natl. Acad. Sci. USA, 79 (1982), pp. 4523–4524.
[180] R. Schoen, *Conformal deformation of a Riemannian metric to constant scalar curvature*, J. Diff. Geom., 20:2 (1984), pp. 479–495.
[181] R. Schoen and J. Wolfson, *Minimizing volume among Lagrangian submanifolds,* in Differential equations: La Pietra 1996 (Florence), Proc. Symp. Pure Math. 65, A.M.S., Providence, RI, 1999, pp. 181–199.
[182] R. Schoen and S.-T. Yau, *Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with nonnegative scalar curvature*, Ann. Math., 110 (1979), pp. 127–142.

[183] R. Schoen and S.-T. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. in Math. Phys., 65:1 (1979), pp. 45–76.

[184] R. Schoen and S.-T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math., 28 (1979), pp. 159–183.

[185] R. Schoen and S.-T. Yau, *Complete three-dimensional manifolds with positive Ricci curvature and scalar curvature*, in Seminar on Differential Geometry, Princeton University Press, 1982, pp. 209–228.

[186] R. Schoen and S.-T. Yau, *The existence of a black hole due to condensation of matter*, Comm. Math. Phys., 90 (1983), pp. 575–579.

[187] R. Schoen and S.-T. Yau, *Lectures on Differential Geometry*, Conference Proceedings and Lecture Notes in Geometry and Topology, I, International Press, Cambridge, MA, 1994.

[188] B. Siebert, *Algebraic and symplectic Gromov-Witten invariants coincide*, math. AG/9804108.

[189] L. Simon, *Existence of surfaces minimizing the Willmore functional*, Comm. Anal. Geom., 1:2 (1993), pp. 281–326.

[190] Y. T. Siu, *Every* K3 *surface is Kähler*, Invent. Math., 73:1 (1983), pp. 139–150.

[191] Y. T. Siu and S.-T. Yau, *Compactification of negatively curved complete Kähler manifolds of finite volume*, in Seminar on Differential Geometry, Ann. of Math. Stud. 102, Princeton Univ. Press, Princeton, NJ, 1982, pp. 363–380.

[192] S. Stolz, *Simply connected manifolds of positive scalar curvature*, Annals of Math., 136:3 (1992), pp. 511–540.

[193] S. Stolz, *A conjecture concerning positive Ricci curvature and the Witten genus*, Math. Ann., 304:4 (1996), pp. 785–800.

[194] A. Strominger, *Special geometry*, Comm. Math. Phys., 133:1 (1990), pp. 163–180.

[195] A. Strominger, S.-T. Yau, and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B, 479:1-2 (1996), pp. 243–259.

[196] D. Sullivan, *The Dirichlet problem at infinity for a negatively curved manifold*, J. Diff. Geo., 18 (1983), pp. 723–732.

[197] C. H. Taubes, *Self-dual Yang-Mills connections on non-self-dual* 4*-manifolds*, J. Diff. Geom., 17 (1982), pp. 139–170.

[198] C. H. Taubes, *The existence of anti-self-dual conformal structures*, J. Diff. Geom., 36:1 (1992), pp. 163–253.

[199] C. H. Taubes, *$SW \Rightarrow Gr$: from the Seiberg-Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc., 9:3 (1996), pp. 845–918.

[200] G. Tian, *Smoothness of the universal deformation space of compact Calabi-Yau manifolds and its Petersson-Weil metric*, in Mathematical aspects of string theory (San Diego, Calif., 1986), Adv. Ser. Math. Phys. 1, World Sci. Publishing, Singapore, 1987, pp. 629–646.

[201] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds*, J. Diff. Geom., 32:1 (1990), pp. 99–130.

[202] G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math., 101:1 (1990), pp. 101–172.

[203] G. Tian, *Smoothing* 3*-folds with trivial canonical bundle and ordinary double points*, in Essays on mirror manifolds, Internat. Press, Hong Kong, 1992, pp. 458–479.

[204] G. Tian, *Kähler-Einstein metrics with positive scalar curvature*, Invent. Math., 130:1 (1997), pp. 1–37.

[205] G. Tian, *Yang-Mills equations and calibrated submanifolds*, Preprint.

[206] G. Tian and S.-T. Yau, *Three-dimensional algebraic manifolds with $C_1 = 0$ and $\chi = -6$*, in Math. Aspects of String Theory (San Diego, Calif., 1986), Adv. Ser. Math. Phys. 1, World Sci. Publ., Singapore, 1986, pp. 543–559.

[207] G. Tian and S.-T. Yau, *Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry*, in Math. Aspects of String Theory (San Diego), Adv. Ser. Math. Phys. 1, World Sci. Publ., Singapore, 1986, pp. 574–628.

[208] G. Tian and S.-T. Yau, *Complete Kähler manifolds with zero Ricci curvature I, II*, J. A. M. S., 3 (1990), pp. 579–610; Invent. Math., 106 (1991), pp. 27–60.

[209] G. Tian and X. H. Zhu, *Uniqueness of Kähler-Ricci solitons*, Preprint.

[210] A. N. Todorov, *Applications of the Kähler-Einstein-Calabi-Yau metric to moduli of* K3 *surfaces*, Invent. Math., 61:3 (1980), pp. 251–265.

[211] A. N. Todorov, *The Weil-Petersson geometry of the moduli space of* $\mathrm{SU}(n \geq 3)$ *(Calabi-Yau) manifolds, I*, Comm. Math. Phys., 126:2 (1989), pp. 325–346.

[212] F. Tomi, *On the finite solvability of Plateau's problem*, in Lect. Notes Math. 597, Springer, Berlin-Heidelberg-New York, 1977, pp. 679–695.

[213] V. A. Topogonov, *Riemannian spaces which contain straight lines*, AMS Translations, 37:2

(1964), pp. 287–290.

[214] N. S. Trudinger and X.-J. Wang, *Bernstein problem for affine maximal hypersurfaces*, Preprint.

[215] K. Uhlenbeck and S.-T. Yau, *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Comm. Pure Appl. Math., 39 (1986), pp. S257–S293; *A note on our previous paper: On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Comm. Pure Appl. Math., 42 (1989), pp. 703–707.

[216] C. Vafa, *Extending mirror conjecture to Calabi-Yau with bundles*, hep-th/9804131.

[217] C.-L. Wang, *On the incompleteness of the Weil-Petersson metric along degenerations of Calabi-Yau manifolds*, Math. Res. Lett., 4:1 (1997), pp. 157–171.

[218] C.-P. Wang, *Some examples of complete hyperbolic affine* 2*-spheres in* $\mathbb{R}^3$, in Global Differential Geometry and Global Analysis, (Proceedings, Berlin, 1990), Lecture Notes in Math. 1481, Springer, Berlin, 1991, pp. 272–280.

[219] M. Y. Wang and W. Ziller, *Einstein metrics on principal torus bundles*, J. Diff. Geom., 31:1 (1990), pp. 215–248.

[220] H. Wente, *Counterexample to a question of H. Hopf*, Pacific J. Math., 121 (1986), pp. 193–243.

[221] H. Weyl, *Über die Bestimmung einer geschlossenen konvexen Fläche durch ihr Linienelement*, Vierteljahrsschrift der naturforschenden Gesellschaft, Zürich, 61 (1916), pp. 40–72.

[222] T. J. Willmore, *Note on embedded surfaces*, Anal. Stüntifice ale Univ., Iasi Sect. I a Mat., 11 (1965), pp. 493–496.

[223] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surveys in Diff. Geom., 1 (1991), pp. 243–310.

[224] S.-T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. USA, 74:5 (1977), pp. 1798–1799.

[225] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the Monge-Ampere equation*, I, Comm. Pure Appl. Math., 31 (1978), pp. 339–411.

[226] S.-T. Yau, *Compact three dimensional Kähler manifolds with zero Ricci curvature*, in Proc. of the Symposium on Anomalies, Geometry and Topology, Argonne, World Scientific, 1985, pp. 395–406.

[227] S.-T. Yau, *Nonlinear analysis in geometry*, Enseign. Math., 33:1-2 (1987), pp. 108–159 (reprinted in [182]).

[228] S.-T. Yau, *A splitting theorem and an algebraic geometric characterization of locally Hermitian symmetric spaces*, Comm. Anal. Geom., 1:3-4 (1993), pp. 473–486.

[229] S.-T. Yau and E. Zaslow, *BPS States, String Duality, and Nodal Curves on* $K3$, Nuclear Physics B, 471 (1996), pp. 503–512; hep-th/9512121.

[230] S.-K. Yeung, *Compactification of Kähler manifolds with negative Ricci curvature*, Invent. Math., 106:1 (1991), pp. 13–25.

[231] S. Zelditch, *Szegö kernels and a theorem of Tian*, Internat. Math. Res. Notices, 6 (1998), pp. 317–331.

[232] S. Zhang, *Positive line bundles on arithmetic surfaces*, Ann. of Math., 136:3 (1992), pp. 569–587.