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HARMONIC MAPS OF DEGREE 1 INTO THE UNIT 2-SPHERE*

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1. Introduction. Let S^2 be the unit 2-sphere in $\mathbb{R}^3$ and M a closed Riemann surface of genus greater than 1. Let $H^1(M, S^2)$ be the set of all $u \in H^1(M, \mathbb{R}^3)$ with $u(x) \in S^2$ for a.e. $x \in M$. The Dirichlet energy functional E on $H^1(M, S^2)$ is given by

$$E(u) = \frac{1}{2}\int_M |\nabla v|^2 dVol_M, \qquad \forall u \in H^1(M, S^2)$$

The critical points of the energy functional are harmonic maps, which satisfy the following Euler-Lagrange equation

$$\triangle u + |\nabla u|^2 u = 0, \tag{1.1}$$

where $\triangle$ is the Laplace-Beltrami operator. Notice that the energy functional does not depend on the choice of the metrics on M, provided that they are compatible with the complex structure on M. So the weak harmonic maps from M to S^2 depend only on the complex structure on M. In this paper, we will consider the following problem: is there a harmonic map of degree 1 from M to S^2? There is definitely no E-minimum, since it would be holomorphic or anti-holomorphic [14]. And it is well known that there is no harmonic map of degree 1 from a torus to a 2-sphere (see [8]). In 1978, using a minimization procedure in a suitable space, L. Lemaire [14] obtained that there exists a harmonic map of degree one from M to S^2 provided that M has three planes of symmetry. Using the result due to J. Jost [13], G.F. Wang [21] improved this result under the condition that M has one plane of symmetry. Here, we will follow a strategy proposed by J.M. Coron [5]; using the Minimax Principle of J. Jost; to prove the existence of such a harmonic map in a different class of Riemann surfaces defined as below. In particular, we do not need any symmetry assumption and our existence result proves the existence of degree 1 harmonic map on all Riemann surfaces described by an open subset of the moduli space of Riemann surfaces. Let R_1 and R_2 be two closed Riemann surfaces with positive genus g_1 and g_2 respectively. For each j, fix a point $p_j \in R_j$, and a coordinate neighborhood (V_j, z_j) around p_j such that $z_j(p_j) = 0$ and $z_j(V_j) = B = \{z_j \in \mathbb{C} : |z_j| < 2\}$. For every complex ϵ_j, with $0 < |\epsilon_j| < 1$, we set

$$V_{j,\epsilon_j} = z_j^{-1}(\{z_j \in \mathbb{C} : |\epsilon_j| \leq |z_j| \leq 1\}), \qquad j = 1, 2.$$

Now, fix two points q_1 and $q_2 \in S^2$, and two coordinate neighborhoods (U'_j, z'_j) around q_j such that $z'_j(q_j) = 0$ and $z'_j(U'_j) = B(0,1) = \{z'_j \in \mathbb{C} : |z'_j| < 1\}$. We define

$$U'_{j,\epsilon_j} = U'_j - (z'_j)^{-1}(\{z'_j \in \mathbb{C} : |z'_j| \leq |\epsilon_j|\}), \qquad j = 1, 2.$$

Then, identifying U'_{j,ϵ_j} and V_{j,ϵ_j} for $j = 1, 2$ by the mappings

$$z'_j \cdot z_j = \epsilon_j,$$

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we obtain a Riemann surface $R_{\epsilon_1,\epsilon_2}$ of genus $g = g_1 + g_2$ with $g \geq 2$. In fact, these surfaces $R_{\epsilon_1,\epsilon_2}$ with specific conformal structure are close to the infinity in T_g, the Teichmüller space of genus $g \geq 2$. Therefore, we obtain the following theorem

THEOREM 1. *There exists $\alpha > 0$ such that if $|\epsilon_j| < \alpha$ for $j = 1, 2$, then we can find a harmonic map of degree 1 from $R_{\epsilon_1,\epsilon_2}$ to S^2.*

More generally, with the help of Schiffer's interior variation [12] (Appendix A), we establish our main result

THEOREM 1'. *There exists an open subset H in T_g such that for each Riemann surface $R \in H$, we can find a harmonic map of degree 1 from R to S^2.*

Here, our method follows those in [4] and in [9]. For this type of problems, the minimizing method fails. But the energy level sets near the minimum have a nontrivial topology, which allows us to look for a critical point with the help of a topological method. Our approach is the following. In Section 2, we will study minimizing sequences. We observe that concentration phenomena occur. In some way, our analysis is similar to P.L. Lions' concentration compactness results [15] on the best constant of Sobolev embedding for the limiting case, and to results of Brezis-Coron [2] and Struwe [19] on H-surfaces. More generally, C.Y. Wang [20] (see also [16]) showed that every Palais-Smale sequence may concentrate its energy at finitely many points where the sequence may generate a nontrivial harmonic mapping from S^2 to S^2 (called a bubble). This is the reason why Sacks-Uhlenbeck [17] and K. Uhlenbeck [23] studied a family of perturbed functionals and established a perturbed Morse theory for harmonic maps. M. Struwe [18] developped a similar theory using the heat flow for harmonic maps (see also K.C. Chang [3]) and J. Jost [13] carried out the bubbling process of a mini-max value for the Dirichlet energy of a mapping from a surface to a closed Riemannian manifold, a crucial result in our proof.

In Section 3, we will analyze nonconstant harmonic maps from Riemann surfaces to the unit 2-sphere S^2, based on Schiffer's interior variation in Teichmüller space. Notice that E and harmonic maps depend only on the conformal structure of surfaces; we can consider the above problem on the set of all biholomorphic equivalence classes of closed Riemann surfaces of genus g, denoted by $\mathcal{M}_g$ (the Riemann's moduli space of genus g). Actually, we construct a family F of Riemann surfaces of genus > 1 containing the subfamily $A = \{R_{\epsilon_1,\epsilon_2} : |\epsilon_i| < 1, i = 1, 2\}$ and we show that the energy of nonconstant harmonic maps from each Riemann surface to S^2 is uniformly bounded from below.

In the last section, we construct a non-trivial loop in an energy level set near the minimum, which is contractible in higher level sets. Then, thanks to a result of J. Jost [13], based on ideas of Sacks-Uhlenbeck [17], we can apply a topological critical point theory in a level set of the energy functional where no blow-up can occur. In particular the result of Section 3 shows that no bubbling effect occurs, and this establishes our result.

2. Study of a minimizing sequence. First, we can label homotopy classes as the following connected components

$$
\begin{aligned}
H^1(M, S^2) \cap C^0(M, S^2) &= \bigcup_{-\infty}^{+\infty} \varepsilon_k(M) \\
&= \bigcup_{-\infty}^{+\infty} \left\{ u \in H^1(M, S^2) \cap C^0(M, S^2), \frac{1}{4\pi} \int_M u \cdot (u_x \times u_y) dVol_M = k \right\},
\end{aligned}
$$

where $Q(u) = \dfrac{1}{4\pi}\displaystyle\int_M u\cdot(u_x \times u_y)dVol_M$ is the degree and subscripts denote partial differentiation with respect to coordinates. We can also consider S^2 as $\mathbb{C}\cup\{\infty\}$ with the conformal metric $\frac{4}{(1+|z|^2)^2}dzd\bar{z}$, $u_z = \partial u$, $\partial = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$, and $u_{\bar{z}} = \bar{\partial}u$, $\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$. Then the energy functional is

$$E(u) = \int_M \frac{4}{(1+|u|^2)^2}(|u_z|^2 + |u_{\bar{z}}|^2)dzd\bar{z} \tag{2.1}$$

and

$$Q(u) = \frac{1}{\pi}\int_M \frac{1}{(1+|u|^2)^2}(|u_z|^2 - |u_{\bar{z}}|^2)dzd\bar{z}. \tag{2.2}$$

In the following, we will consider the energy functional on $\varepsilon_1(M)$. Our main result in this section is

THEOREM 2. *We have*

$$\phi_1(M) = \inf_{v\in\varepsilon_1(M)} E(v) = 4\pi,$$

and $\phi_1(M)$ is attained if and only if M is simply connected. Moreover, if M is multiply connected and if $\{u_n\}_{n\in\mathbb{N}}$ is a minimizing sequence of E, then there exists $z_0 \in M$ such that, modulo a subsequence,

$$\frac{1}{2}|\nabla u_n|^2 dVol_M \longrightarrow 4\pi\delta_{z_0}, \qquad in\ \mathcal{R}(M),$$

where δ_{z_0} is the Dirac mass concentrated at $z_0 \in M$ and $\mathcal{R}(M)$ is the space of Radon measures on M with finite masses.

In order to prove this result, we need the following lemmas.

LEMMA 1. (see [1] and [9]) *Assume that φ_n is a bounded sequence in $H^1(M,\mathbb{R})\cap L^\infty(M,\mathbb{R})$. Let $a_n \longrightarrow 0$ weakly in $H^1(M,\mathbb{R})$ and strongly in $L^2(M,\mathbb{R})$. Then for every $b \in H^1(M,\mathbb{R})$, we have*

$$\lim_{n\to\infty}\int_M \varphi_n((a_n)_x b_y - (a_n)_y b_x)dVol_M = 0.$$

LEMMA 2. (Isoperimetric inequality see [22] and [9]) *Assume that $\Psi = (\psi_1,\psi_2,\psi_3) \in C^0(M,\mathbb{R}^3)\cap H^1(M,\mathbb{R}^3)$. Then,*

$$\left|\int_M \Psi\cdot(\Psi_x\times\Psi_y)dVol_M\right|^2 \le \frac{1}{4\pi}\left(\int_M |\Psi_x\times\Psi_y|dVol_M\right)^3.$$

Proof of Theorem 2. Obviously,

$$E(u) \ge 4\pi Q(u) = 4\pi, \qquad \forall u \in \varepsilon_1(M). \tag{2.3}$$

On the other hand, let (U, z) be a coordinate neighborhood containing $p \in M$ such that $z(U) = B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$, we define for any $\epsilon > 0$

$$u_{p,\epsilon}(q) = \begin{cases} \frac{\epsilon}{z}, & \text{if } |z(q)| \leq \frac{1}{2}; \\ \frac{\epsilon}{z}(3 - 4|z|), & \text{if } \frac{1}{2} \leq |z(q)| \leq \frac{3}{4}; \\ 0, & \text{if } q \in M \setminus z^{-1}(B(0, \frac{3}{4})). \end{cases} \tag{2.4}$$

It is clear that

$$\lim_{\epsilon \to 0} E(u_{p,\epsilon}) = 4\pi.$$

Therefore, we prove the first part of the Theorem. Using the Uniformization Theorem (see [12]), we may prove that $\phi_1(M)$ is achieved provided that M is simply connected. Conversely, assume $u \in \varepsilon_1(M)$ such that $E(u) = 4\pi$. Then, u satisfies equation (1.1) and is regular (see [10]). From (2.1) to (2.3), we deduce that $u_{\bar{z}} = 0$; that is, u is holomorphic. But, the degree of u is one, and this is possible only if M is simply connected. In particular we deduce that if M is multiply connected, then Inequality (2.3) is strict.

Now let $\{u_n\}_{n \in \mathbb{N}}$ be a minimizing sequence of E in $\varepsilon_1(M)$ for a Riemann surface M of genus greater than 1. Since H^1 is reflexive, we may assume that, modulo a subsequence, $u_n \longrightarrow u$ weakly in H^1 for some $u \in H^1$. But $H^1(M, S^2)$ is weakly closed, therefore $u \in H^1(M, S^2)$ and by weak lower semi-continuity, we have

$$E(u) \leq \liminf_{n \to \infty} E(u_n) = 4\pi.$$

Hence $u \in \varepsilon_0(M)$ because of (2.3). Set $\alpha_n = u_n - u$ so that (as in [9]) by lemma 1

$$\begin{aligned} 1 = Q(u_n) &= \frac{1}{4\pi} \int_M u_n \cdot ((u_n)_x \times (u_n)_y) dVol_M \\ &= \frac{1}{4\pi} \int_M u_n \cdot ((\alpha_n)_x \times (\alpha_n)_y) dVol_M + \frac{1}{4\pi} \int_M u \cdot (u_x \times u_y) dVol_M + o(1) \\ &= \frac{1}{4\pi} \int_M u_n \cdot ((\alpha_n)_x \times (\alpha_n)_y) dVol_M + o(1). \end{aligned}$$

Therefore,

$$E(\alpha_n) = \frac{1}{2} \int_M |\nabla \alpha_n|^2 dVol_M \geq \left| \int_M u_n \cdot ((\alpha_n)_x \times (\alpha_n)_y) dVol_M \right| \geq 4\pi + o(1).$$

On the other hand, we have

$$4\pi + o(1) = E(u_n) = E(u) + E(\alpha_n) + o(1),$$

which implies

$$E(u) = 0 \qquad \text{and} \qquad E(\alpha_n) = 4\pi + o(1).$$

Hence, u is constant. Set $\mu_n = \frac{1}{2}|\nabla \alpha_n|^2 dVol_M$ and $\nu_n = \alpha_n \cdot ((\alpha_n)_x \times (\alpha_n)_y) dVol_M$. Clearly, $\mu_n(M) = 4\pi + o(1)$, $\nu_n(M) = 4\pi$ and $|\nu_n| \ll \mu_n$. Therefore $\{\mu_n\}$ and $\{\nu_n\}$

are bounded in $\mathcal{R}(M)$. Modulo a subsequence, we may assume that $\mu_n \rightharpoonup \mu$, $\nu_n \rightharpoonup \nu$ weakly in the sense of measures where μ and ν are bounded measures on M. Moreover, $\mu(M) = 4\pi$ and $\nu(M) = 4\pi$ since M is compact. Choose $\xi \in C^\infty(M, \mathbb{R})$. Applying lemma 2, we have

$$\left|\int_M \xi\alpha_n \cdot ((\xi\alpha_n)_x \times (\xi\alpha_n)_y) dVol_M\right|^2 \leq \frac{1}{32\pi}\left(\int_M |\nabla(\xi\alpha_n)|^2 dVol_M\right)^3.$$

It is clear that

$$\begin{aligned}\int_M \xi\alpha_n \cdot ((\xi\alpha_n)_x \times (\xi\alpha_n)_y) dVol_M &= \int_M \xi^3\alpha_n \cdot ((\alpha_n)_x \times (\alpha_n)_y) dVol_M + o(1)\\ \int_M |\nabla(\xi\alpha_n)|^2 dVol_M &= \int_M \xi^2 |\nabla\alpha_n|^2 dVol_M + o(1)\end{aligned}$$

since $\alpha_n \longrightarrow 0$ strongly in L^2 and $\|\alpha_n\|_{L^\infty} \leq 2$. Consequently, we get

$$\left|\int_M \xi^3\alpha_n \cdot ((\alpha_n)_x \times (\alpha_n)_y) dVol_M\right|^2 \leq \frac{1}{32\pi}\left(\int_M \xi^2|\nabla\alpha_n|^2 dVol_M + o(1)\right)^3.$$

Passing to the limit as $n \to \infty$, there holds

$$\left|\int_M \xi^3 d\nu\right|^2 \leq \frac{1}{4\pi}\left(\int_M \xi^2 d\mu\right)^3, \qquad \forall \xi \in C^\infty(M, \mathbb{R}). \tag{2.5}$$

By approximation, therefore,

$$|\nu(E)|^{\frac{2}{3}} \leq \left(\frac{1}{4\pi}\right)^{\frac{1}{3}} \mu(E) \qquad (E \subset M, E \text{ borel}), \tag{2.6}$$

which implies

$$|\nu(A)|^{\frac{2}{3}} \leq \left(\frac{1}{4\pi}\right)^{\frac{1}{3}} \mu(A) \text{ and } |\nu(M \setminus A)|^{\frac{2}{3}} \leq \left(\frac{1}{4\pi}\right)^{\frac{1}{3}} \mu(M\setminus A) \text{ for any Borel set } A.$$

However, $\mu(M) = \nu(M) = 4\pi$. Thus, we deduce that

$$\nu(A) = \mu(A) = 0 \qquad \text{or} \qquad \nu(M \setminus A) = \mu(M \setminus A) = 0.$$

Since it is true for all A, that implies the assertion of Theorem 2. □

Now let $\tilde{g}$ be a metric compatible with the complex structure on M. By Nash-Moser's Theorem, $(M, \tilde{g})$ can be isometrically embedded in some Euclidean space $\mathbb{R}^k$ and we denote this embedding by I. We analyze the topology of the energy level sets $E_M^{4\pi+\gamma} = \{u \in \varepsilon_1(M) : E(u) \leq \gamma + 4\pi\}$ for small $\gamma > 0$. For this purpose, we introduce a map C from $\varepsilon_1(M)$ into $\mathbb{R}^k$

$$\begin{aligned} C: \quad \varepsilon_1(M) &\longrightarrow \mathbb{R}^k\\ u &\longmapsto \frac{1}{8\pi}\int_M I(q)|\nabla u|^2(q) dVol_M.\end{aligned}$$

There is a neighborhood $U_\delta = \{x \in \mathbb{R}^k : dist(x, M) \leq \delta\}$ of M in $\mathbb{R}^k$ on which the nearest projection P from U_δ onto M is continuous. According to Theorem 2, we can construct a map $\bar{\pi}$ for all small $\gamma > 0$

$$\begin{aligned} \bar{\pi}: \quad E_M^{4\pi+\gamma} &\longrightarrow M\\ u &\longmapsto \bar{\pi}(u) = P(C(u)).\end{aligned}$$

Note that, the construction given in (2.4) shows that $\bar{\pi}$ is surjective. For some γ fixed, we choose a suitable ϵ_0 such that for any $p \in M$, we have $u_{p,\epsilon_0} \in E_M^{4\pi+\gamma}$. So we define another continuous map τ from M to $E_M^{4\pi+\gamma}$

$$\tau(p) = u_{p,\epsilon_0}.$$

Using Theorem 2, we deduce that $\tau \circ \bar{\pi}$ and $Id_{E_M^{4\pi+\gamma}}$ are homotopic and that $\bar{\pi} \circ \tau$ and Id_M are so if γ is sufficiently small; i.e., $E_M^{4\pi+\gamma}$ and M are of the same homotopy type.

3. Study of nonconstant harmonic maps. In this section, we will analyze the energy of nonconstant harmonic maps from closed Riemann surface of genus g to S^2. For this purpose, we consider Schiffer's interior variation in T_g. Assume distinct points $r_1, \ldots, r_{3g-3}$, are given on R_1. We take coordinate neighborhoods (U_j, z_j^*) of every r_i so that

$$\begin{aligned}
&z_j^*(r_j) = 0;\\
&z_j^*(U_j) = \{z \in \mathbb{C} : |z| < 2\}, \qquad j = 1, \ldots, 3g-3;\\
&\overline{U_j} \cap \overline{U_k} = \emptyset, \qquad j \neq k;\\
&\overline{U_j} \cap \overline{V_1} = \emptyset,
\end{aligned}$$

where V_1 is defined in §1. Set $D_j = (z_j^*)^{-1}(\{z \in \mathbb{C} : |z| < 1\})$. For any complex number ϵ'_j with $|\epsilon'_j| < \frac{1}{2}$, consider the mapping

$$z^*_{j,\epsilon'_j}(p) = z_j^*(p) + \frac{\epsilon'_j}{z_j^*(p)}, \qquad \forall p \in D_j.$$

When $|\epsilon'_j|$ is sufficiently small, $z^*_{j,\epsilon'_j}(\partial D_j)$ is a simple closed curve in the z^*_{j,ϵ'_j}-plane, which is denoted by C_{j,ϵ'_j}, and z^*_{j,ϵ'_j} gives a conformal mapping of a suitable neighborhood A_{j,ϵ'_j} of ∂D_j. Now exclude D_j from R_1 and paste the domain D_{j,ϵ'_j} in the z^*_{j,ϵ'_j}-plane surrounded by C_{j,ϵ'_j}. Thus, for ϵ_1 and ϵ_2 fixed, we can construct a family $\{R_{\epsilon_1,\epsilon_2,\epsilon'}\}$ of Riemann surfaces depending on complex parameters $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_{3g-3})$ (see [12]).

We also need the Hodge star operator. For each $\alpha \in \wedge^p(M)$, we associate to α a $(2-p)$-form $*\alpha$, called the adjoint of α, defined as follows:

$$*1 = \eta, *dx = dy, *dy = -dx, *\eta = 1, \tag{3.1}$$

where η is the oriented volume element on M. And we define $\delta\alpha$ by

$$\delta\alpha = (-1)^p *^{-1} d * \alpha, \qquad \text{where } p = deg(\alpha). \tag{3.2}$$

Then, the Laplace operator Δ is defined by

$$\Delta = d\delta + \delta d. \tag{3.3}$$

THEOREM 3. *Under the above assumptions, there exists $\beta > 0$ such that for any Riemann surface $R_{\epsilon_1,\epsilon_2,\epsilon'}$ with $|\epsilon_i| < 1$ and $|\epsilon'_j| < \frac{1}{2}$ and a nonconstant harmonic map u from $R_{\epsilon_1,\epsilon_2,\epsilon'}$ to S^2, we have*

$$E(u) \geq \beta.$$

As a direct consequence of Schiffer's interior variation, we obtain the following corollary

Corollary. *The energy of nonconstant harmonic maps to S^2 is uniformly bounded from below on a open set of T_g.*

Remark. $\mathcal{M}_g$ is identified with the quotient space T_g/Mod_g, where Mod_g is the Teichmüller modular group. Hence, our result is also true on a open set of $\mathcal{M}_g$ equipped with the quotient topology.

In order to prove the theorem, we need a technical lemma.

Lemma 3. (see [10] and [11]) *Let $u : B(0,1) = \{z \in \mathbb{C} : |z| < 1\} \longrightarrow S^2$ be a harmonic map. Then there exists a positive constant C such that for any z_1, $z_2 \in B(0, \frac{1}{2})$,*

$$|u(z_1) - u(z_2)| \leq C \left(\int_B |\nabla u|^2 dz d\bar{z} \right)^{\frac{1}{2}} \left(\left(\int_B |\nabla u|^2 dz d\bar{z} \right)^{\frac{1}{2}} + 1 \right). \tag{3.4}$$

Proof. In view of (1.1), there exists G from B to $\mathbb{R}^3$ satisfying

$$\frac{\partial G}{\partial x} = u \times \frac{\partial u}{\partial y}, \qquad \frac{\partial G}{\partial y} = -u \times \frac{\partial u}{\partial x}.$$

By the Courant-Lebesgue Lemma, there exists some $r \in (\frac{1}{2}, 1)$ for which $u|_{\partial B(0,r)}$ is absolutely continuous and

$$|u(z_3) - u(z_4)| \leq \left(\frac{8\pi}{\log 2} \right)^{\frac{1}{2}} \left(\int_B |\nabla u|^2 dz d\bar{z} \right)^{\frac{1}{2}}, \tag{3.5}$$

for all z_3, $z_4 \in \partial B(0,r)$. In $B(0,r)$, we will decompose u into its harmonic (u_0) and non harmonic (u_1) components

$$u = u_0 + u_1,$$

where

$$\begin{cases} \triangle u_0 &= 0 \quad \text{in } B(0,r) \\ u_0 &= u \quad \text{on } \partial B(0,r), \end{cases} \tag{3.6}$$

$$\begin{cases} \triangle u_1 &= \triangle u = u_x \times G_y + G_x \times u_y \quad \text{in } B(0,r) \\ u_1 &= 0 \quad \text{on } \partial B(0,r). \end{cases} \tag{3.7}$$

Thanks to the Wente's inequality (see [1] and [22]), we obtain

$$\|u_1\|_{C^0} \leq C \|\nabla u\|_{L^2} \|\nabla G\|_{L^2}. \tag{3.8}$$

And by the Maximum Principle, we deduce that for all z_1, $z_2 \in B(0,r)$

$$|u_0(z_1) - u_0(z_2)| \leq \sup_{z_3, z_4 \in \partial B(0,r)} |u_0(z_3) - u_0(z_4)| \tag{3.9}$$

Combining (3.5)-(3.9), we establish (3.4). □

Proof of Theorem 3. Let us decompose $R_{\epsilon_1,\epsilon_2,\epsilon'} = F \cup (R_{\epsilon_1,\epsilon_2,\epsilon'} \setminus F)$, where $F = \{z \in R_{\epsilon_1,\epsilon_2,\epsilon'} : rank(\nabla u)(z) \leq 1\}$ is the set of degenerated points. Denote by $\mathcal{H}^2$ the 2-dimensional Hausdorff measure in $\mathbb{R}^3$. Applying Sard's Lemma,

$$\mathcal{H}^2(u(F)) = 0. \tag{3.10}$$

For each $z \in R_{\epsilon_1,\epsilon_2,\epsilon'} \setminus F$, there exists a closed neighborhood G_z of z such that $u|_{G_z} : G_z \longrightarrow \overline{B(u(z), r_z)}$ is a diffeomorphism. Notice that $u(R_{\epsilon_1,\epsilon_2,\epsilon'} \setminus F)$ is open. Then, using Vitali's covering theorem, we can choose a countable family of disjoint discs $\overline{B(u(z_i), r_{z_i})}$ satisfying

$$u(R_{\epsilon_1,\epsilon_2,\epsilon'} \setminus F) \subset \bigcup_{i=1}^{\infty} B(u(z_i), 5r_{z_i}). \tag{3.11}$$

Consequently, there exists a positive constant C such that

$$\begin{aligned} \mathcal{H}^2(u(R_{\epsilon_1,\epsilon_2,\epsilon'} \setminus F)) &\leq \mathcal{H}^2(\bigcup_{i=1}^{\infty} B(u(z_i), 5r_{z_i})) \\ &\leq \sum_{i=1}^{\infty} \mathcal{H}^2(B(u(z_i), 5r_{z_i})) \\ &\leq C \sum_{i=1}^{\infty} \mathcal{H}^2(B(u(z_i), r_{z_i})) \\ &\leq C \sum_{i=1}^{\infty} \int_{G_{z_i}} |u \cdot (u_x \times u_y)| dz d\bar{z} \\ &\leq CE(u). \end{aligned} \tag{3.12}$$

On the other hand, according to Lemma 3, for some $s_1, s_2 \in S^2$ and for any sufficiently small $E(u)$, we have

$$u\left(R_1 \setminus \left(V_1 \cup \left(\bigcup_{j=1}^{3g-3} U_j\right)\right)\right) \subset B(s_1, C\sqrt{E(u)}) \tag{3.13}$$

and

$$u(R_2 \setminus V_2) \subset B(s_2, C\sqrt{E(u)}) \tag{3.14}$$

Let Q be the antipodal mapping on S^2; that is, $Q(v) = -v$ for all $v \in S^2$. Combining equations (3.10)-(3.14), we have

$$\mathcal{H}^2(K \cup Q(K)) \leq CE(u), \qquad \text{for sufficiently small } E(u),$$

where $K = u(R_{\epsilon_1,\epsilon_2,\epsilon'}) \cup \overline{B(s_1, C\sqrt{E(u)})} \cup \overline{B(s_2, C\sqrt{E(u)})}$. However, $K \cup Q(K)$ is closed since $u(R_{\epsilon_1,\epsilon_2,\epsilon'})$ is compact. Thus, choosing $E(u)$ sufficiently small, we can find a pair of antipodal points v_1 and $-v_1$ on S^2 such that $v_1 \notin K \cup Q(K)$ and $-v_1 \notin K \cup Q(K)$. Without loss of generality, we suppose that $v_1 = (0, 0, 1)$ is the north pole. Denote $K_1 = S^2 \setminus \{v_1, -v_1\}$ and let

$$u_\sharp : \pi_1(R_{\epsilon_1,\epsilon_2,\epsilon'}) \longrightarrow \pi_1(K_1) = \mathbb{Z}$$

be the induced map on the fundamental groups. We claim that if $E(u) \le \beta$ (here β is a uniform constant), then $u_\sharp$ is trivial. Indeed, if $E(u)$ is sufficiently small, we can choose a simply connected subset K_2 of K_1 containing $\overline{B(s_i, C\sqrt{E(u)})}$, for $i = 1, 2$, and a deformation map of K_1 such that

$$\begin{aligned} & f : K_1 \times [0,1] \longrightarrow K_1, \\ & f(0,\cdot) = Id, \\ & f(K_2, t) \subset K_2, \qquad \text{for all } t \in [0,1], \\ & f(K_2, 1) = \{s_1\}. \end{aligned}$$

Denote $f_t = f(t,\cdot)$. Hence, $u_\sharp = (f_1 \circ u)_\sharp$. We can write $f_1 \circ u = \check{f} \circ P_1$, where

$$\begin{array}{llll} P_1 : & R_{\epsilon_1,\epsilon_2,\epsilon'} & \longrightarrow & K_3 = R_{\epsilon_1,\epsilon_2,\epsilon'} \Big/ \left(\left(R_1 \setminus \left(V_1 \cup \bigcup_{i=1}^{3g-3} U_i \right) \right) \cup (R_2 \setminus V_2) \right) \\ & z & \longrightarrow & [z] \end{array}$$

is a projection from $R_{\epsilon_1,\epsilon_2,\epsilon'}$ onto K_3 which is equipped with quotient topology, and $\check{f}$ is a continuous mapping from K_3 to K_1 since $f_1(K_2) = \{s_1\}$. Notice that K_3 is simply connected, the claim is proved.

Denote $u = (u^1, u^2, u^3)$. Set $u^1 + iu^2 = \sqrt{(u^1)^2 + (u^2)^2} e^{i\psi}$. In fact, ψ can be defined locally or on a simply connected subset of $R_{\epsilon_1,\epsilon_2,\epsilon'}$. But $u_\sharp(\pi_1(R_{\epsilon_1,\epsilon_2,\epsilon'}))$ is trivial in $\pi_1(K_1)$. So ψ is defined on the whole $R_{\epsilon_1,\epsilon_2,\epsilon'}$. It follows from (1.1) that

$$div(((u^1)^2 + (u^2)^2)\nabla\psi) = div(u^1 \nabla u^2 - u^2 \nabla u^1) = 0,$$

or equivalently, using the language of differential forms,

$$\delta(((u^1)^2 + (u^2)^2)d\psi) = 0.$$

Consequently,

$$0 = \int_{R_{\epsilon_1,\epsilon_2,\epsilon'}} \delta(((u^1)^2 + (u^2)^2)d\psi)\psi dz d\bar{z} = \int_{R_{\epsilon_1,\epsilon_2,\epsilon'}} ((u^1)^2 + (u^2)^2)|d\psi|^2 dz d\bar{z},$$

which implies that ψ is a constant mapping. That is, $u(R_{\epsilon_1,\epsilon_2,\epsilon'})$ is contained in a great circle. Replacing $\{v_1, -v_1\}$ by another pair of antipodal points $\{v_1', -v_1'\}$ and proceeding similarly, we deduce that u is a constant mapping. This completes the proof.

4. Proof of Theorem 1. First, we recall some facts which can be found in [13] (§4.2), primarily due to Sack-Uhlenbeck [17].

LEMMA 4. *Let M be a compact Riemann surface without boundary, and N be a compact Riemannian manifold without boundary. Let A be a compact parameter space and let $h_0 : M \times A \longrightarrow N$ be continuous. Let H be the class of all maps homotopic to h_0 and put*

$$\kappa = \inf_{h \in H} \sup_{t \in A} E(h(\cdot, t)).$$

In case $\partial A \neq \emptyset$, $h|_{M \times \partial A}$ is fixed in such a way that the above supremum cannot be attained on ∂A. Then there exists a harmonic map

$$u_0 : M \longrightarrow N$$

and possibly also some nontrivial conformal harmonic maps

$$u_i : S^2 \longrightarrow N \qquad (i = 1, ..., m)$$

with

$$E(u_0) + \sum_{i=1}^{m} E(u_i) = \kappa.$$

Here $(u_0; u_1, ..., u_m)$ *represents a saddle point corresponding to* H *in the sense that there exist sequences* $\{h_n\} \subset H$, $\{t_n\} \subset A$ *and points* $x_1, .., x_k \in M$, $k \leq m$ *(if* $m \geq 1$*) with*

$$\begin{array}{ll} E(h_n(\cdot, t_n)) \longrightarrow \kappa & ; \\ h_n(\cdot, t_n) \longrightarrow u_0 & \textit{weakly in } H^1; \\ h_n(\cdot, t_n) \longrightarrow u_0 & \textit{uniformly on each compact subset of } M \setminus \{x_1, .., x_k\}. \end{array}$$

Furthermore, for each $i \in \{1, ..., m\}$*, there exists a sequence* $\{\lambda_n^i\}_{n\in\mathbb{N}} \subset \mathbb{R}^+$, $\lambda_n^i \to 0$ *as* $n \to \infty$ *with,*

$$h_n\left(\left(\frac{\rho}{\lambda_n^i}, \varphi\right), t_n\right) \longrightarrow u_i,$$

where (ρ, φ) *are polar coordinates centered at some* $x_{j,n}$ *with* $x_{j,n} \to x_j$ $(1 \leq j \leq k)$.

Lemma 5. *Let* u *be a harmonic map from* S^2 *to* S^2*. Then* u *is conformal and*

$$E(u) = 4\pi deg(u). \tag{4.1}$$

In view of the results in §2, we know that the energy level sets near the minimum have a nontrivial topology. Thus, we can construct a nontrivial loop in such level sets. However, it will be contractible in higher energy level sets. With the help of the above lemmas, we obtain a critical value of E using the Minimax Principle for the energy E based on this loop. In case no blow-up can occur, we prove the assertion.

For simplicity, we use conformal coordinates $z \in \mathbb{C} \cup \{\infty\}$ on S^2, such that $q_1 = 0$ and $q_2 = \infty$. Choose a function $\psi(z) \in C_0^\infty(\mathbb{R}^2, \mathbb{R})$ such that $supp(\psi) \subset B(0,1)$, $0 \leq \psi \leq 1$ and $\psi(z) = 1$ if $|z| \leq \frac{1}{2}$. We write $S^2 = S_+^2 \cup S_-^2$ where $S_+^2 = \{v = (v^1, v^2, v^3) \in S^2 : v^3 \geq 0\} \simeq \overline{B(0,1)}$ and $S_-^2 = \{v = (v^1, v^2, v^3) \in S^2 : v^3 \leq 0\} \simeq \mathbb{C} \cup \{\infty\} \setminus B(0,1)$. Similarly, we write $R_{\epsilon_1,\epsilon_2,\epsilon'} = R^+_{\epsilon_1,\epsilon_2,\epsilon'} \cup R^-_{\epsilon_1,\epsilon_2,\epsilon'}$ where $R^+_{\epsilon_1,\epsilon_2,\epsilon'}$ (resp. $R^-_{\epsilon_1,\epsilon_2,\epsilon'}$) is obtained by pasting R_1 (resp. R_2) to S_+^2 (resp. S_-^2). For $z^* \in S^1$ and $0 \leq t < 1$, we set $\sigma_{z^*,t}(z) = \dfrac{z + tz^*}{1 + t\overline{z^*}z}$. We construct a continuous map h from $B(0,1) \times R_{\epsilon_1,\epsilon_2,\epsilon'}$ to S^2 by

$$h(z^*, t, z) = \begin{cases} tz^*, & \text{if } z \in R^+_{\epsilon_1,\epsilon_2,\epsilon'} \setminus U'_{1,\epsilon_1} \\ (\sigma_{z^*,t}(z) - tz^*)\left(1 - \psi\left(\dfrac{z}{2|\epsilon_1|}\right)\right) + tz^*, & \text{if } |\epsilon_1| \leq |z| \leq 2|\epsilon_1| \\ \sigma_{z^*,t}(z), & \text{if } 2|\epsilon_1| \leq |z| \leq \dfrac{1}{2|\epsilon_2|} \\ \left(\sigma_{z^*,t}(z) - \dfrac{1}{t\overline{z^*}}\right)\left(1 - \psi\left(\dfrac{1}{2|\epsilon_2|z}\right)\right) + \dfrac{1}{t\overline{z^*}}, & \text{if } \dfrac{1}{2|\epsilon_2|} \leq |z| \leq \dfrac{1}{|\epsilon_2|} \\ \dfrac{1}{t\overline{z^*}}, & \text{if } z \in R^-_{\epsilon_1,\epsilon_2,\epsilon'} \setminus U'_{2,\epsilon_2}. \end{cases} \tag{4.2}$$

Notice that, for any z^*, t, the map $z \longmapsto h(z^*, t, z)$ is constant on the glued surfaces R_1 and R_2; if $t \longrightarrow 1$, it concentrates at some point on the equator of S^2, parametrized by z^*. A direct computation shows

$$\lim_{t\to 1} E(h(z^*, t, \cdot)) = 4\pi \qquad \text{uniformly on } S^1 \tag{4.3}$$

and

$$\lim_{\epsilon_1,\epsilon_2\to 0} \sup_{z^* t\in B(0,1)} E(h(z^*, t, \cdot)) = 4\pi. \tag{4.4}$$

Thus, there is $\alpha > 0$ such that

$$\sup_{B(0,1)} E(h(z^*, t, \cdot)) < 4\pi + \beta < 8\pi \qquad \text{for any } |\epsilon_i| < \alpha \text{ and } |\epsilon'_j| < \frac{1}{2}.$$

Now, fix ϵ_i and ϵ'. We choose some small μ_0 in such a way that $\bar{\pi}$ in §2 can be defined continuously from $E^{4\pi+\mu_0}_{R_{\epsilon_1,\epsilon_2,\epsilon'}}$ to $R_{\epsilon_1,\epsilon_2,\epsilon'}$. By (4.3) and Theorem 2, we can find $0 < t_0 < 1$ such that $E(h(z^*, t_0, \cdot)) < 4\pi + \mu_0$ for all $z^* \in S^1$ and $z^* \to \bar{\pi}(h(z^*, t_0, \cdot))$ is a nontrivial loop in $\pi_1(R_{\epsilon_1,\epsilon_2,\epsilon'})$ since $\lim_{t\to 1} \bar{\pi}(h(z^*, t, \cdot)) = z^* \in R_{\epsilon_1,\epsilon_2,\epsilon'}$. We consider h as a function defined on $B(0, t_0) \times R_{\epsilon_1,\epsilon_2,\epsilon'}$ and let H be the homotopy classes of h. Set

$$\kappa_{\epsilon_1,\epsilon_2,\epsilon'} = \inf_{\bar{f}\in H} \sup_{z^* t\in B(0,t_0)} E(\bar{f}(z^*, t, \cdot)).$$

LEMMA 6. *We have*

$$\kappa_{\epsilon_1,\epsilon_2,\epsilon'} \geq 4\pi + \mu_0.$$

Proof. We suppose that $\kappa_{\epsilon_1,\epsilon_2,\epsilon'} < 4\pi + \mu_0$. Then for some $\bar{f} \in H$, we have

$$E(\bar{f}(z^*, t, \cdot)) < 4\pi + \mu_0, \qquad \text{for all } t \in [0, t_0] \text{ and } z^* \in S^1.$$

Hence, we can construct a deformation map $\tilde{f}$ from $B(0, t_0)$ to $R_{\epsilon_1,\epsilon_2,\epsilon'}$ by

$$\begin{aligned} \tilde{f} :\ \ & B(0, t_0) && \longrightarrow R_{\epsilon_1,\epsilon_2,\epsilon'} \\ & (z^*, t) && \longmapsto \bar{\pi}(\bar{f}(z^*, t, \cdot)), \end{aligned}$$

that is, $\tilde{f}$ is a contraction of S^1 in $R_{\epsilon_1,\epsilon_2,\epsilon'}$. This contradiction completes our proof. □

Proof of Theorem 1'. Obviously, $\kappa_{\epsilon_1,\epsilon_2,\epsilon'} < 4\pi + \beta$, where β is chosen as in Theorem 3. Now, according to Lemma 4 and Lemma 5, there exists a harmonic map

$$u_0 : R_{\epsilon_1,\epsilon_2,\epsilon'} \longrightarrow S^2$$

and possibly also some non-trivial conformal harmonic maps

$$u_i : S^2 \longrightarrow S^2, \qquad (i = 1, ..., m)$$

with

$$\kappa_{\epsilon_1,\epsilon_2,\epsilon'} = E(u_0) + \sum_{i=1}^{m} E(u_i) = 4k\pi + E(u_0), \qquad \text{for some } k \in \mathbb{N}.$$

Finally, notice that $4\pi + \mu_0 \leq \kappa_{\epsilon_1,\epsilon_2,\epsilon'} < 4\pi + \beta < 8\pi$. So $k = 0$ or $k = 1$. But, in view of Theorem 3, the latter implies that u_0 is a constant map, which contradicts $\kappa_{\epsilon_1,\epsilon_2,\epsilon'} \geq 4\pi + \mu_0$. Thus, u_0 is harmonic map of degree 1 from $R_{\epsilon_1,\epsilon_2,\epsilon'}$ to S^2. □

REMARK. In general, if we attach several small handles to the unit sphere S^2 as in §1, we have the same result.

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