ASIAN J. MATH.
Vol. 4, No. 4, pp. 831–838, December 2000

007

A SIEVE APPLICATION*

H. HALBERSTAM†

The aim of this note is to prove the following

THEOREM. *Suppose $g \geq 3$ is fixed. There exist positive absolute constants A and B, and a g-term arithmetic progression in the interval $(1, 1+g^{Ag})$, so that the prime factors of all the terms of this progression are at least as large as g^B.*

Several years ago Professor W. M. Schmidt asked me for a result of this kind (he found later that he could prove what he needed in another way), and at the end of his plenary lecture at the 1997 Schinzel Conference in Zakopane, Poland, Professor H.-P. Schlickewei raised a similar question. I thought it might be of some interest to show how the question could be formulated as a sieve problem, and answered in explicit form by means of a simple sieve method.

We formulate the stated result as follows: Let

$$F(n) := (1+n)(1+2n)\dots(1+gn)$$

and

$$\mathcal{A} := \{F(n) : n \in \mathbb{N}, 1 \leq n \leq x\}.$$

Next, let $\mathcal{P}$ be the set of all primes p less than a number $z > 2$, and introduce the counting number

$$S(\mathcal{A}, \mathcal{P}) = |\{a \in \mathcal{A} : (a, \prod_{p<z} p) = 1\}|.$$

If there exist x and z such that

$$S(\mathcal{A}, \mathcal{P}) > 0, \tag{1}$$

this means that there does exist a g-term arithmetic progression

$$1 + rn \ (r = 1, 2, \dots, g)$$

with common difference n satisfying $1 \leq n \leq x$, that lies in the interval

$$(1, 1+gx]$$

and has all its g terms made up of primes at least as large as z. To prove the theorem it is enough to show that there exist constants A and B such that (1) holds with

$$x = g^{Ag-1}, \quad z = g^B. \tag{2}$$

We set about proving (1), with x and z as given in (2), by means of the Brun-Hooley sieve (see [HO]) as presented in [FH]; this method is especially easy to apply, but it should be said that any version of a fundamental lemma (see, for example,

*Received January 6, 2000; accepted for publication March 29, 2000.
†Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, Illinois 61801, USA (heini@math.uiuc.edu).

Theorem 2.5 of [HR] or Lemma 2.1 of [E]) would lead to a result of the same kind. We shall make free use of several numerically explicit prime number estimates from the invaluable [RS]. In this way we shall show that, for example,

$$A = 438 \quad \text{and} \quad B = 14$$

are admissible values.

It is necessary to introduce some notation. Let $\omega(p)$ denote the number of $\mod p$ incongruent solutions of $F(n) \equiv 0 \mod p$, where here, and throughout, p is a prime. It is well known that

$$\omega(p) \leq \min(p-1, g).$$

Let

$$2 = z_{r+1} < z_r < \cdots < z_1 = z$$

be a partition of the interval $[2, z)$, with

$$z_r = \frac{1}{126} \log x =: \xi,$$

say, and

$$z_j = z^{K^{1-j}} (j = 1, 2, \ldots r-1),$$

where $K > 1$ is a constant still to be chosen. Then

$$z^{K^{1-r}} \leq \xi < z^{K^{2-r}},$$

so that

$$\frac{1}{\log \xi} \leq \frac{K^{r-1}}{\log z} < \frac{K}{\log \xi};$$

note that

$$\xi \geq g+1 \quad \text{provided} \quad x \geq e^{126(g+1)}.$$

(By the definition of x in (2) and bearing in mind that $g \geq 3$, this requirement is satisfied if $A \geq 154$). Introduce the product

$$W(y_1, y_2) = \prod_{y_1 \leq p < y_2} (1 - \frac{\omega(p)}{p}), \quad 2 \leq y_1 < y_2,$$

and write

$$W = W(2, z), \quad W_j = W(z_{j+1}, z_j), \quad L_j = \log\ W_j^{-1}.$$

Then we quote from [FH] (see the Corollary to the Theorem in this article) the inequality

(3) $$S(\mathcal{A}, \mathcal{P}) \geq xW\{1 - E - \eta(1 + Ez)\},$$

where

$$E=\sum_{j=1}^{r-1}\frac{L_j^{b+2(j-1)+1}}{(b+2(j-1)+1)!}e^{L_j},$$

with b a (large)even integer still to be chosen, and

$$\eta=\Big(\prod_{j=1}^{r-1}z_j^{b+2(j-1)}\Big)\xi^{\pi(\xi)}x^{-1}W^{-2}.$$

For (3) to imply (1) we shall have to choose the various parameters in play so that

$$E<1\quad\text{and}\quad\eta(1+Ez)<1-E.$$

Before we can make these choices we need upper bounds for W^{-1} and $W_j^{-1}(j=1,\dots,r-1)$. Observe that

$$\begin{aligned}W^{-1}=W(2,z)^{-1}&\le\prod_{p\le g}(1-\frac{p-1}{p})^{-1}\prod_{g+1\le p<z}(1-\frac{g}{p})^{-1}\\&=e^{\theta(g)}\prod_{g+1\le p<z}(1-\frac{g}{p})^{-1}(1-\frac{1}{p})^{g}\prod_{g+1\le p<z}(1-\frac{1}{p})^{-g}\qquad(4)\\&\le e^{\theta(g)}G\prod_{g+1\le p<z}(1-\frac{1}{p})^{-g},\end{aligned}$$

where

$$\theta(g)=\sum_{p\le g}\log p,$$

and

$$G:=\prod_{p\ge g+1}(1-\frac{g}{p})^{-1}(1-\frac{1}{p})^{g}.$$

Similarly, since $z_{j+1}\ge z_r=\xi\ge g+1(j=1,\dots,r-1)$,

$$W_j^{-1}\le G\prod_{z_{j+1}\le p<z_j}(1-\frac{1}{p})^{-g}.\qquad(5)$$

LEMMA 1. *We have*

$$G<\frac{2}{3}ge^{1.01624g}.$$

Proof. The product G is equal to

$$\exp\{\sum_{m=2}^{\infty}\frac{g^m-g}{m}\sum_{p\ge g+1}\frac{1}{p^m}\}<\exp\{g\sum_{m=2}^{\infty}\frac{g^{m-1}}{m}\sum_{p\ge g+1}\frac{1}{p^m}\}.$$

By Theorem 9, Lemma 9 and the calculation on p. 87 of [RS],

$$\sum_{p>y} p^{-m} < 1.01624 \frac{m}{m-1} y^{1-m} (\log y)^{-1}, \quad m \geq 2,\ y > 2,$$

whence

$$\begin{aligned} G &< \exp\{g \sum_{m=2}^{\infty} \frac{g^{m-1}}{m} (\frac{1}{(g+1)^m} + 1.01624 \frac{m}{m-1} \frac{1}{(g+1)^{m-1}} \frac{1}{\log(g+1)})\} \\ &= \exp\{\log(g+1) - \frac{g}{g+1} + 1.01624 g\} < \frac{g+1}{2} e^{1.01624g} \leq \frac{2}{3} g e^{1.01624g}. \end{aligned}$$

□

Lemma 2. *We have*

$$\theta(g) < g(1 + \frac{1}{2 \log g}) < 1.45512g, \quad g \geq 3,$$

and

$$\prod_{y_1 \leq p < y_2} (1 - \frac{1}{p})^{-1} < (1 + \frac{1}{g}) \exp(\frac{1.5}{\log^2(g+1)}) \cdot \frac{\log y_2}{\log y_1}, \quad y_1 \geq g+1.$$

Proof. The first inequality is (3.15) of [RS].
Next, by (3.26) and (3.30) of [RS], if $g + 1 \leq y_1 < y_2$,

$$\begin{aligned} \prod_{y_1 \leq p < y_2} (1 - \frac{1}{p})^{-1} &= \prod_{p<y_1} (1 - \frac{1}{p}) \cdot \prod_{p<y_2} (1 - \frac{1}{p})^{-1} \\ &< \frac{y_1}{y_1 - 1} \frac{e^{-\gamma}}{\log y_1} (1 + \frac{1}{2 \log^2 y_1}) \cdot e^{\gamma} \log y_2 (1 + \frac{1}{\log^2 y_2}) \\ &\leq \frac{g+1}{g} \exp(\frac{1.5}{\log^2 y_1}) \cdot \frac{\log y_2}{\log y_1}. \end{aligned}$$

(Here γ is Euler's constant.) □

Corollary. *We have*

$$W^{-1} < 2g \exp\{(3.252 + \log B) g\}$$

and

$$W_j^{-1} < 2g \exp\{(1.8 + \log K) g\}, \quad 1 \leq j \leq r - 1.$$

Proof. By (4) and Lemmas 1 and 2, since $g \geq 3$,

$$\begin{aligned} W^{-1} &< \frac{2}{3} g e^{2.47136g} (1 + \frac{1}{g})^g \exp(\frac{1.5g}{\log^2(g+1)}) (\frac{\log z}{\log(g+1)})^g \\ &< \frac{2}{3} e g \exp(2.47136g + \frac{1.5g}{\log^2 4} + g \log B) \\ &< 2g \exp\{(3.25188 + \log B) g\}. \end{aligned}$$

Similarly, by (5) and Lemmas 1 and 2, for $j \leq r-1$

$$\begin{aligned} W_j^{-1} &< \frac{2}{3} g e^{1.01624g} (1+\frac{1}{g})^g \exp(\frac{1.5g}{\log^2 4})(\frac{\log z_j}{\log z_{j+1}})^g \\ &< \frac{2e}{3} g e^{1.79676g} K^g < 2g \exp\{(1.79676 + \log K)g\}. \end{aligned}$$

□

It follows from the second of the results in the corollary that

$$\begin{aligned} L_j = \log W_j^{-1} &< (1.8 + \log K)g + \log(2g) < (2.4 + \log K)g \\ &< 3.5g =: L, \end{aligned}$$

say, on choosing

$$K = 3.$$

We are now in a position to estimate E. We have

$$E < e^L L^{b+1} \sum_{i=0}^{\infty} \frac{L^{2i}}{(2i)!} \frac{(2i)!(b+1)!}{(b+1+2i)!} \frac{1}{(b+1)!} \leq e^{2L} \frac{L^{b+1}}{(b+1)!} < (\frac{eL}{b+1})^{b+1} e^{2L}$$

and if we choose

$$b = 2([eL] + 1),$$

then $b > 2eL$ and

$$\begin{aligned} E &< 2^{-2eL} e^{2L} = \exp\{-(e \log 2 - 1)2L\} = \exp\{-7(e \log 2 - 1)g\} \\ &< e^{-6.189g} \leq e^{-18.567} < 10^{-8}, \ g \geq 3. \end{aligned}$$

It follows that

$$Ez = Eg^B < \exp(-6.189g + B \log g);$$

since $-6.189g + B \log g$ is decreasing in g if $B < 6.189g$, and $g \geq 3$, we see that

$$Ez < 1$$

for all $g \geq 3$ provided $B \leq 16.9$. Altogether we place on record that, in those circumstances,

$$1 - E > 1 - 10^{-8} \quad \text{and} \quad 1 + Ez < 2,$$

so that, from (3),

$$S(\mathcal{A}, \mathcal{P}) \geq xW\{1 - 10^{-8} - 2\eta\}$$

for any $g \geq 3$ provided only that $B \leq 16.9$. It remains to bound η. We have

$$\prod_{j=1}^{r-1} z_j^{b+2(j-1)} = z^{\Gamma} = g^{B\Gamma}$$

where

$$\Gamma = \sum_{j=1}^{r-1} \frac{b+2(j-1)}{K^{j-1}} < \frac{K}{K-1}b + \frac{2K}{(K-1)^2} = \frac{3}{2}(b+1)$$

at $K = 3$. Next, by relation (3.6) of [RS],

$$\xi^{\pi(\xi)} = \exp(\pi(\xi)\log\xi) < \exp(1.25506\xi) < x^{\frac{1}{100}}$$

since $\xi = \frac{\log x}{126}$. Hence, from the definition of η and by Lemma 2, Corollary,

$$\begin{aligned}\eta &< g^{1.5(b+1)B}x^{-0.99}\cdot 4g^2\exp\{(6.504+2\log B)g\}\\ &= 4\exp\{-0.99(Ag-1)\log g + 1.5(b+1)B\log g + (6.504+2\log B)g + 2\log g\}.\end{aligned}$$

Having chosen $L = 3.5g$ and $b = 2([eL]+1)$, we have

$$b < 7eg + 2,$$

and hence that

$$\eta < 4\exp\{-(0.99A - 10.5eB)g\log g + (6.504+2\log B)g + (4.5B+2.99)\log g\}.$$

The exponent on the right is of the form

$$-\alpha g\log g + \beta g + \gamma\log g$$

where, provided that we insist that $0.99A > 10.5eB$, α, β and γ are positive constants. This function of g is strictly decreasing (in g) so long as

$$\alpha > \frac{\beta+\gamma/g}{1+\log g},$$

and for $g \geq 3$ the expression on the right is largest at $g = 3$. Hence we require that

$$\alpha > \frac{\beta+\gamma/3}{1+\log 3}.$$

With α satisfying this inequality, we have

$$-\alpha g\log g + \beta g + \gamma\log g \leq -3\alpha\log 3 + 3\beta + \gamma\log 3 < 0$$

if

$$\alpha > \frac{\beta}{\log 3} + \frac{\gamma}{3}.$$

Suppose even that

$$\alpha > \beta + \gamma/3.$$

Then

$$-\alpha g\log g + \beta g + \gamma\log g < -3(\log 3 - 1)\beta < -0.2958\beta,$$

and we conclude that

$$\eta < 4\exp\{-0.2958(6.504 + 2\log B)\}$$

provided that

$$0.99A - 10.5eB > 6.504 + 2\log B + 1.5B + 1$$

i.e.

$$0.99A > (10.5e + 1.5)B + 2\log B + 7.504.$$

When we take $B = 14$, $A = 438$ satisfies this inequality (as does any $A > 438$). We conclude that, on making the choice $A = 438$ and $B = 14$,

$$\eta < 4\exp(-3.485) < 0.123,$$

and therefore

$$S(\mathcal{A}, \mathcal{P}) > xW(0.754 - 10^{-8}) > \frac{3}{4}xW.$$

This completes the proof of our result.

It will be clear from these calculations that there are other admissible choices of the parameters A and B. For example, taking $B = 5$ one finds that any $A \geq 163$ is in order.

I am grateful to the referee for a very careful reading of this note, with the result that some clarification was added and a mistake corrected.

REFERENCES

[E] P. D. T. A. Elliott, *Probabilistic Number Theory*, Vol. I, Springer 1980.
[HO] C. Hooley, *On an almost-pure sieve*, Acta Arith., LXVI (1994), pp. 359–368.
[HR] H. Halberstam and H. E. Richert, *Sieve Methods*, Academic Press, 1974.
[FH] K. Ford and H. Halberstam, *The Brun-Hooley sieve*, J. of Number Theory, 81(2000), pp. 335–350.
[RS] J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math 6 (1962), pp. 64–94.