ASIAN J. MATH.
Vol. 5, No. 1, pp. 19–24, March 2001

002

GENERIC MODULES FOR EXTENSION ALGEBRAS*

XIANNENG DU†

Abstract. Let A be a tame hereditary algebra (finite-dimensional over an algebraically closed field), R_A^m ($m \geq 1$) the extension algebra of A. A generic R-module M over an arbitrary ring R is by definition an indecomposable R-module of infinite length, such that M considered as an End(M)-module, is of finite length (its endolength). In this paper we investigate the generic modules of $\widehat{A}$ (the repetitive algebra of A) and R_A^m. It is proved that R_A^m has at least $2m$ generic modules.

Introduction. The notion of generic module was introduced in [1] by Crawley-Boevey. The concept seems to be quite natural and important. The generic modules even have a dominating position in the category of modules. In [2], it was shown that whether a finite-dimensional algebra over an algebraically closed field is tame or wild is determined completely by the behaviour of the generic modules for that algebra.

In [3], Aronszajn and Fixman gave the concept of a divisible module for the Kronecker algebra and showed that for the Kronecker algebra there exists a unique indecomposable torsion-free divisile module. In [4], Ringel generalized the work of Aronszajn and Fixman and proved the same result for a tame hereditary algebra. Ringel's work, in fact, showed that for a tame hereditary algebra, there exists a unique generic module. In [6], we solved the existence and uniqueness of generic module for the tilted algebra determined by a tame hereditary algebra.

Following [1], A generic R-module M over an arbitrary ring R is by definition an indecomposable R-module of infinite length, such that M considered as an End(M)-module, is of finite length (its endolength). Of course, the generic modules with endomorphism ring a division ring just, form the vertices of the (Cohn) spectrum of R. By [1], the endomorphism ring of a generic module always is a local ring.

Our purpose here is to investigate the generic module of the extension algebra R_A^m (defined below) for a tame hereditary algebra A. In section 1, we investigate the ν-orbits of generic modules for a repetitive algebra. we shall prove that Mod $\widehat{A}$ has at least two ν-orbits of generic $\widehat{A}$-modules (Theorem 1.2). In section 2, we shall prove our main result on generic modules of R_A^m: R_A^m has at least $2m$ generic modules (Theorem 2.4 and Corollary 2.5).

Throughout this paper, we denote by k an algebraically closed field. An algebra means basic, connected and finite-dimensional k-algebra. For an algebra A we denote by Mod A the category of all right A-modules, by mod A the full subcategory of Mod A consisting of all finitely generated right A-modules and by $\underline{\text{mod}}\, A$ the corresponding stable category. We shall use freely properties of the Auslander-Reiten sequences, irreducible maps, Auslander-Reiten translation $\tau = D\text{Tr}$ and $\tau^{-1} = \text{Tr}D$, and the Auslander-Reiten quiver Γ_A of an algebra B, for which we refer to [5].

1. ν-orbits of Generic Modules of Repetitive Algebras. Let $A = k\vec{\Delta}$ be a tame hereditary algebra over an k. We denote by $D^b(A)$ the derived category $D^b(\text{mod}\, A)$ of bounded complexes over mod A. For the definition of derived category

* Received January 21, 1999; accepted for publication May 23, 1999.
† Department of Mathematics, Anhui University, Hefei, 230039, P. R. China (xndu@mars.ahu.edu.cn). Reserch was supported by the National Science Foundation of China.

we refer to [7]. By DA we denote the minimal injective cogenerator of A, where $D = \mathrm{Hom}_k(-, k)$ is the usual dual functor. Consider the repetitive algebra [7]:

$$\widehat{A} = \begin{bmatrix} \ddots & \ddots & & & \\ & A_{i-1} & DA_{i-1} & & \\ & & A_i & DA_i & \\ & & & A_{i+1} & \ddots \\ & & & & \ddots \end{bmatrix}$$

with $A_i = A$ on the main diagonal, $DA_i = DA$, and zeros elsewhere. The elements are all matrices with only a finite number of nonzero entries, and multiplication is given by the canonical bimodule structure of DA and the zero map $DA \otimes_A DA \to 0$. It is a Frobenius algebra and always infinite-dimendional. We know that the identity maps $A_i \to A_{i+1}$ and $DA_i \to DA_{i+1}$ induce an automorphism ν (Nakayama automorphism) of $\widehat{A}$, and also an automorphism of $\mathrm{mod}\, A$. Since A is of finite global dimension, we have $\underline{\mathrm{mod}}\, \widehat{A} \cong D^b(A)$ [7]. We may identity $\underline{\mathrm{mod}}\, \widehat{A}$ with $D^b(A)$. By [7], $\Gamma_{D^b(A)}$ is the union :

$$\cdots \vee \mathcal{T}[i] \vee \mathcal{Q}[i] \vee \mathcal{P}[i] \vee \mathcal{T}[i+1] \vee \mathcal{Q}[i+1] \vee \mathcal{P}[i+1] \vee \mathcal{T}[i+2] \vee \dots$$

Set $\mathcal{C}_i = \mathcal{Q}[i] \vee \mathcal{P}[i+1], i \in \mathbb{Z}$. Let Δ_0 and Δ_1 be the complete sections of $\mathcal{C}_0$ and $\mathcal{C}_1$ respectively. Let $\mathcal{S}_i = \mathrm{add}_{\widehat{A}}\Delta_i$, that is, $\mathcal{S}_i$ are the module classes in $\mathrm{mod}\, \widehat{A}$ generated by $\Delta_i (i = 0, 1)$. $\mathcal{S}_{j+2m} = \nu^m \mathcal{S}_j, m \in \mathbb{Z}, j = 0, 1$. The support of $\mathcal{S}_i$, denote by $\mathrm{Supp}\mathcal{S}_i$ is the set:

$$\{P(x) |\, P(x) \text{is indecomposable projective}\, \widehat{A} - \text{module with } \mathrm{Hom}(P(x), \mathcal{S}_i) \neq 0\}.$$

Let, for each $i \in \mathbb{Z}, A_i$ be the support algebra of $\mathcal{S}_i$, i.e.,

$$A_i = \mathrm{End}(\underset{P(x)\in \mathrm{Supp}\mathcal{S}_i}{\oplus} P(x)).$$

Then

$$A_i = \widehat{A} / \langle P(x) \notin \mathrm{Supp}\, \mathcal{S}_i \rangle$$

and A_i are tilted algebras of Euclidean type. Since the main purpose of the paper is to investigate the generic modules of R_A^m, we may assume that each A_i is representation-infinite. By [6], for each $i \in \mathbb{Z}$, there exists a unique generic A_i-module M_i. Of course, all M_i are also the generic $\widehat{A}$-modules.

Lemma 1.1. *For each $i \in \mathbb{Z}$ we have*

$$Supp(\nu\mathcal{S}_i) = \nu Supp(\mathcal{S}_i).$$

Proof. Let $P(x) \in \mathrm{Supp}(\nu\mathcal{S}_i)$. Then there exists an $\widehat{A}$-module $S \in \mathcal{S}_i$ such that $\mathrm{Hom}_{\widehat{A}}(P(x), \nu S) \neq 0$. Hence we get that $\mathrm{Hom}_{\widehat{A}}(\nu^- P(x), S) \neq 0$. This means $\nu^- P(x) \in \mathrm{Supp}(\nu\mathcal{S}_i)$ and hence $P(x) \in \nu\mathrm{Supp}(\mathcal{S}_i)$. So, $\mathrm{Supp}(\nu\mathcal{S}_i) \sqsubseteq \nu\mathrm{Supp}(\mathcal{S}_i)$. Similarly, we have $\nu\mathrm{Supp}(\mathcal{S}_i) \sqsubseteq \mathrm{Supp}(\nu\mathcal{S}_i)$.

The main result of this section is the follow

THEOREM 1.2. *Let A be a tame hereditary algebra, then Mod$\widehat{A}$ has at least two ν-orbits of generic $\widehat{A}$-modules.*

Proof. Suppose A_i and M_i $(i \in \mathbb{Z})$ are as before. Then for each $i \in \mathbb{Z}$, M_i is also a generic $\widehat{A}$-module.

For each $P(x) \notin \mathrm{Supp}(\mathcal{S}_{i+2})$, we show $\mathrm{Hom}_{\widehat{A}}(P(x), \nu M_i) = 0$. If not, we have $\mathrm{Hom}_{\widehat{A}}(\nu^- P(x), M_i) \neq 0$. This gives $\nu^- P(x) \in \mathrm{Supp}(M_i) \sqsubseteq \mathrm{Supp}(\mathcal{S}_i)$. By Lemma 1.1, $P(x) \in \nu\mathrm{Supp}(\mathcal{S}_i) = \mathrm{Supp}(\nu\mathcal{S}_i) = \mathrm{Supp}(\mathcal{S}_{i+2})$. This is a contradiction. Thus νM_i is a A_{i+2}-module. Since νM_i is a generic A_{i+2}-module and A_{i+2} has a unique generic module, we get $\nu M_i = M_{i+2}$. In general, we have $\nu^m M_i = M_{i+2m}$, $i \in \mathbb{Z}$, $m \in \mathbb{Z}$. By the structure of Mod $\widehat{A}$ we know that $M_i \neq M_j$ $(i \neq j)$ as $\widehat{A}$-modules. We get two distinct ν-orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ of generic $\widehat{A}$-modues:

$$\mathcal{O}_0 = \{\nu^m M_0 |\, m \in \mathbb{Z}\},\ \mathcal{O}_1 = \{\nu^n M_1 |\, n \in \mathbb{Z}\}.$$

2. Generic Modules for Extension Algebra $R_A{}^m$. Let $A = k\vec{\Delta}$ be a tame hereditary algebra over k. $\widehat{A}$ the repetitive algebra of A. We consider, for each $m \geq 1$, the algebra R_A^m:

$$R_A^m = \left\{ \left(\begin{matrix} \lambda_1 & x_1 & & & \\ & \lambda_2 & x_2 & & \\ & & \ddots & \ddots & \\ & & & \lambda_m & x_m \\ & & & & \lambda_1 \end{matrix} \right) \middle| \, \lambda_i \in A,\ x_i \in DA \right\}.$$

As above, the multiplication is given by the bimodule structure of DA and zero map $DA \otimes_A DA \to 0$. In particular, R_A^1 is the trivial extension $A \ltimes DA$. The category R_A^m is just the quotient category $\widehat{A}/(\nu^m)$.

For a fixed $m \geq 1$, we consider the canonical Galois covering functor $F^m : \widehat{A} \to \widehat{A}/(\nu^m) = R_A^m$, and the associated pushdown functor F_λ^m : Mod $\widehat{A} \to$ Mod R_A^m and the pull-up functor $F_{\cdot}^m$: Mod $R_A^m \to$ Mod $\widehat{A}$ [8].

From now on we fix some m. In this section we show that R_A^m has at least $2m$ generic modules.

We first prove some lemmas

LEMMA 2.1 [8]. *For each $N \in$ Mod$\widehat{A}$ and each $r \in \mathbb{Z}$, we have*

$$F_\lambda^m((\nu^m)^r N) \cong F_\lambda^m(N).$$

LEMMA 2.2. *Let M be a generic $\widehat{A}$-module, N an indecomposable $\widehat{A}$-module. If $F_\lambda^m N \cong F_\lambda^m M$, then $N \cong \nu^{mr} M$ for some $r \in \mathbb{Z}$.*

Proof. Assume that $F_\lambda^m N \cong F_\lambda^m M$. Then by [8], we have

$$\bigoplus_{r\in\mathbb{Z}} \nu^{mr} N \cong F_{\cdot}^m F_\lambda^m N \cong F_{\cdot}^m F_\lambda^m M \cong \bigoplus_{l\in\mathbb{Z}} \nu^{ml} M.$$

Since M is a generic $\widehat{A}$-module, $\nu^{ms} M$ $(s \in \mathbb{Z})$ are also generic $\widehat{A}$-modules, it follows from [1] that every ring $\mathrm{End}(\nu^{mt} M)$ is local, we infer that $N = \nu^{mr} M$ for some $r \in \mathbb{Z}$.

Lemma 2.3. *Suppose that M is a generic $\widehat{A}$-modole. Then $F_\lambda^m M$, as a left $End_{\widehat{A}}(M)$-module, is of finite length.*

Proof. Since we have an imbeding map $\mathrm{End}_{\widehat{A}}(M) \to \mathrm{End}_{R_A^m}(F_\lambda^m M)$, we infer that $F_\lambda^m M$ is also a left $\mathrm{End}_{\widehat{A}}(M)$-module. Suppose that

$$(*) \qquad 0 = N_0 \le N_1 \le N_2 \le \cdots \le N_i \le \dots$$

be a composition series for left $\mathrm{End}_{\widehat{A}}(M)$-module $F_\lambda^m M$. Since $F_\lambda^m M = M$ as a k vector space, every N_i in (*) is a subspace of M. For each $f \in \mathrm{End}_{\widehat{A}}(M)$, we have $fN_i \sqsubseteq N_i$ and hence each N_i is a left $\mathrm{End}_{\widehat{A}}(M)$-submodule of M. Hence we may regard (*) as a composition series for left $\mathrm{End}_{\widehat{A}}(M)$-module M. Since M is a generic $\widehat{A}$-module, it follows that (*) has only finite terms. Therefore $F_\lambda^m M$, as a left $\mathrm{End}_{\widehat{A}}(M)$-module, is of finite length.

We can now prove our main result

Theorem 2.4. *Let M be a generic $\widehat{A}$-module. Then $F_\lambda^m M$ is a generic R_A^m-module.*

Proof. Since F_λ^m is left adjoint to $F_\cdot^m$, it follows from [8] that

$$\begin{aligned}\mathrm{End}_{R_A^m}(F_\lambda^m M) &= \mathrm{Hom}_{R_A^m}(F_\lambda^m M,\, F_\lambda^m M)\\ &\cong \mathrm{Hom}_{\widehat{A}}(M,\, F_\cdot^m F_\lambda^m M)\\ &\cong \mathrm{Hom}_{\widehat{A}}(M,\, \bigoplus_{l\in\mathbb{Z}} \nu^{ml} M).\end{aligned}$$

If $m = 1$, then, for $s \neq 0, 1$, we have $\mathrm{Hom}_{\widehat{A}}(M,\, \nu^s M) = 0$, and hence

$$\mathrm{End}_{R_A^1}(F_\lambda^1 M) \cong \mathrm{Hom}_{\widehat{A}}(M,\, M \oplus \nu M) = \mathrm{End}_{\widehat{A}}(M) \oplus \mathrm{Hom}_{\widehat{A}}(M,\, \nu M).$$

Since $\mathrm{Hom}_{\widehat{A}}(M,\, \nu M)$ is an $\mathrm{End}_{\widehat{A}}(M)$-bimodule: $g \circ f = gf$ is the ordinary composition and $(f \circ g)(x) = \nu f(g(x))$ for $f \in \mathrm{End}_{\widehat{A}}(M), g \in \mathrm{Hom}_{\widehat{A}}(M,\, \nu M)$ and $x \in M$. It follows from [9] that we have the following ring isomorphism

$$\mathrm{End}_{R_A^1}(F_\lambda^1 M) \cong \mathrm{End}_{\widehat{A}}(M) \ltimes \mathrm{Hom}_{\widehat{A}}(M,\, \nu M).$$

From the definition of trivial extension of algebra we know

$$\mathrm{End}_{R_A^1}(F_\lambda^1 M)/\mathrm{rad}\,\mathrm{End}_{R_A^1}(F_\lambda^1 M) \cong \mathrm{End}_{\widehat{A}}(M)/\mathrm{rad}\,\mathrm{End}_{\widehat{A}}(M).$$

Suppose that $m \ge 2$. By the structure of $\mathrm{Mod}\,\widehat{A}$ we have that $\mathrm{Hom}_{\widehat{A}}(M,\, \nu^{ms} M) = 0$ for $s \neq 0$. Hence

$$\mathrm{End}_{R_A^m}(F_\lambda^m M)/\mathrm{rad}\,\mathrm{End}_{R_A^m}(F_\lambda^m M) \cong \mathrm{End}_{\widehat{A}}(M)/\mathrm{rad}\,\mathrm{End}_{\widehat{A}}(M).$$

Since M is a generic $\widehat{A}$-module, we infer that $\mathrm{End}_{\widehat{A}}(M)/\mathrm{rad}\,\mathrm{End}_{\widehat{A}}(M)$ is a division ring and hence $\mathrm{End}_{R_A^m}(F_\lambda^m M)$ is local for $m \ge 1$. Therefore, $F_\lambda^m M$ is an indecomposable R_A^m-module.

Write $C = \mathrm{End}_{\widehat{A}}(M)$, $\quad D = \mathrm{End}_{R_A^m}(F_\lambda^m M)$, $\quad \overline{C} = C/\mathrm{rad}C$, $\quad \overline{D} = D/\mathrm{rad}D$. Let $l_A(M)$ denote the lenght of A-module M. We have

$$\begin{aligned} l_D(F_\lambda^m M) &= l_D(F_\lambda^m M/(\mathrm{rad}D)\, F_\lambda^m M) + l_D((\mathrm{rad}\, D) F_\lambda^m M/(\mathrm{rad}^2\, D)\, F_\lambda^m M) + \dots\\ &\quad + l_D((\mathrm{rad}^i\, D)\, F_\lambda^m M/(\mathrm{rad}^{i+1}\, D)\, F_\lambda^m M) + \dots .\end{aligned}$$

Since each $(\mathrm{rad}^i D)\, F_\lambda^m M/(\mathrm{rad}^{i+1} D)\, F_\lambda^m M$ is a $\overline{D}$-module and

$$l_D((\mathrm{rad}^i D)\, F_\lambda^m M/(\mathrm{rad}^{i+1} D)\, F_\lambda^m M) = l_{\overline{D}}((\mathrm{rad}^i D)\, F_\lambda^m M/(\mathrm{rad}^{i+1} D)\, F_\lambda^m M).$$

We have

$$\begin{aligned} l_D(F_\lambda^m M) &= l_{\overline{D}}(F_\lambda^m M/(\mathrm{rad}\, D)\, F_\lambda^m M) + l_{\overline{D}}((\mathrm{rad}\, D) F_\lambda^m M/(\mathrm{rad}^2 D)\, F_\lambda^m M) \\ &+ \cdots + l_{\overline{D}}((\mathrm{rad}^i D)\, F_\lambda^m M/(\mathrm{rad}^{i+1} D)\, F_\lambda^m M) + \ldots \\ &\overset{\overline{C}\cong\overline{D}}{=} l_{\overline{C}}(F_\lambda^m M/(\mathrm{rad}\, D)\, F_\lambda^m M) + l_{\overline{C}}((\mathrm{rad}\, D)\, F_\lambda^m M/(\mathrm{rad}^2 D)\, F_\lambda^m M) \\ &+ \cdots + l_{\overline{C}}((\mathrm{rad}^i D)\, F_\lambda^m M/(\mathrm{rad}^{i+1} D)\, F_\lambda^m M) + \ldots \\ &= l_C(F_\lambda^m M/(\mathrm{rad}\, D)\, F_\lambda^m M) + l_C((\mathrm{rad}\, D)\, F_\lambda^m M/(\mathrm{rad}^2 D)\, F_\lambda^m M) \\ &+ \cdots + l_C((\mathrm{rad}^i D)\, F_\lambda^m M/(\mathrm{rad}^{i+1} D)\, F_\lambda^m M) + \ldots \\ &= l_C(F_\lambda^m M). \end{aligned}$$

By lemma 2.3, we have $l_C(F_\lambda^m M) \leq \infty$ and hence $l_D(F_\lambda^m M) \leq \infty$. $F_\lambda^m M$ is clearly of infinite -dimension since M is so.

Therefore $F_\lambda^m M$ is a generic R_A^m-module.

Corollary 2.5. *R_A^m has at least $2m$ generic modules.*

Proof. By Theorem 1.2, Mod $\widehat{A}$ has two ν-orbits $\mathcal{O}_0$ and $\mathcal{O}_1$ of generic $\widehat{A}$-modules.

$$\mathcal{O}_0 = \{\nu^m M_0 |\, m \in \mathbb{Z}\},\ \mathcal{O}_1 = \{\nu^n M_1 |\, n \in \mathbb{Z}\}.$$

From Lemma 2.1 and 2.2, it is easy to know that $F_\lambda^m(\nu^l N)$, $F_\lambda^m(\nu^t N)\, (l, t = 0, 1, 2, \ldots, m-1)$ are different generic R_A^m-modules.

REFERENCES

[1] W. W. Crawley-Boevey, *Tame algebras and generic modules*, Proc. London Math. Soc., 63 (1991), pp. 241–265.

[2] W. W. Crawley-Boevey, *Modules of finite length over their endomorphism rings*, in London Math. Soc. Lecture Notes Series 168, 1992, pp. 127-184.

[3] N. Aronszajn and U. Fixman, *Algebraic spectral problems*, Studia Math., 30 (1968), pp. 273–338.

[4] C. M. Ringel, *Infinite dimensional representations of finite dimensional hereditary algebras*, Symp. Math., 23 (1979), pp. 321–412.

[5] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, LNM 1099, Springer, 1984.

[6] Du Xianneng, *Generic modules for tilted algebras*, Chinese Science Bulletin, 42:3 (1997), pp. 177–180.

[7] D. Happel, *Triangulated Categories in the Representation Theory of Finite Dimensional Algebra*, LNS 119, Cambridge Univ. Press, 1988.

[8] P. Gabriel, *The universal cover of a representation-finite algebra*, in LNM 903, Springer, 1981, pp. 68–105.

[9] R. M. Fossum, P. A. Griffith, and I. Reiten, *Trivial Extensions of Abelian Categories*, LNM 456, Springer, 1975.