

WILLMORE HYPERSURFACES IN A SPHERE*

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Abstract. Let $x : M \rightarrow S^{n+1}$ be an n -dimensional hypersurface in S^{n+1} , $x : M \rightarrow S^{n+1}$ is called a Willmore hypersurface if it is an extremal hypersurface to the following Willmore functional:

$$\int_M (S - nH^2)^{\frac{n}{2}} dv,$$

where $S = \sum_{i,j} (h_{ij})^2$ is the square of the length of the second fundamental form, H is the mean curvature of M . In this paper, through study of the Euler-Lagrange equation of the Willmore functional, we obtain an integral inequality of Simons' type for Willmore hypersurfaces in S^{n+1} and give a characterization of *Willmore tori* by use of our integral formula. We also classify all isoparametric Willmore hypersurfaces in S^{n+1} .

1. Introduction. Let M be an n -dimensional compact hypersurface of the $(n + 1)$ -dimensional unit sphere S^{n+1} . If h_{ij} denotes the components of the second fundamental form of M , S denotes the square of the length of the second fundamental form and H denotes the mean curvature, then we have

$$S = \sum_{i,j} (h_{ij})^2, \quad \mathbf{H} = \frac{1}{n} \sum_k h_{kk} e_{n+1}, \quad H = |\mathbf{H}|,$$

where e_{n+1} is a unit normal vector field of M in S^{n+1} .

We define the following non-negative function on M

$$(1.1) \quad \rho^2 = S - nH^2,$$

which vanishes exactly at the umbilic points of M .

Willmore functional is the following non-negative functional (see [4], [20] or [22])

$$\int_M \rho^n dv = \int_M (S - nH^2)^{\frac{n}{2}} dv.$$

It was shown in [4] and [20] that this functional is an invariant under Moebius (or conformal) transformations of S^{n+1} . We use the term Willmore hypersurfaces to call its critical points, because when $n = 2$, the functional essentially coincides with the well-known Willmore functional and its critical points are the Willmore surfaces.

In this paper, we first prove the following theorem

THEOREM 1. *Let M be an n -dimensional hypersurface in an $(n + 1)$ -dimensional unit sphere S^{n+1} . Then M is a Willmore hypersurface if and only if*

$$(1.2) \quad \begin{aligned} & -\rho^{n-2} (2HS - nH^3 - \sum_{i,j,k} h_{ij} h_{jk} h_{ki}) \\ & + (n-1) \Delta(\rho^{n-2} H) - \sum_{i,j} (\rho^{n-2})_{,ij} (nH \delta_{ij} - h_{ij}) = 0, \end{aligned}$$

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where Δ is the Laplacian, $(\cdot)_{,ij}$ is the covariant derivative relative to the induced metric.

Remark 1.1. When $n = 2$, Theorem 1 was proved by R. Bryant [2] and J. Weiner [21], (1.2) reduces to the following well-known equation of Willmore surfaces (see [2,21])

$$(1.3) \quad \Delta H + H(S - 2H^2) = 0.$$

We note that Peter Li and S.-T. Yau [11] introduced a concept of conformal volume and obtained a partial solution of Willmore conjecture through estimating the eigenvalues of the Laplacian. We also note that Pinkall [14] constructed some compact non-minimal Willmore surfaces in S^3 .

Remark 1.2. We should note that for $n \geq 2$, C. P. Wang [20] got the Euler-Lagrange equation of Willmore functional for any n -dimensional submanifold in an $(n + p)$ -dimensional unit sphere S^{n+p} in terms of Moebius geometry.

In order to state our Theorem 3, we first give the following important example EXAMPLE (C.F.[6]). The tori

$$(1.4) \quad W_{m,n-m} = S^m \left(\sqrt{\frac{n-m}{n}} \right) \times S^{n-m} \left(\sqrt{\frac{m}{n}} \right), \quad 1 \leq m \leq n-1$$

are Willmore hypersurfaces. We call $W_{m,n-m}$, $1 \leq m \leq n-1$, *Willmore tori*. In fact, the principal curvatures k_1, \dots, k_n of $W_{m,n-m}$ are

$$(1.5) \quad k_1 = \dots = k_m = \sqrt{\frac{m}{n-m}}, \quad k_{m+1} = \dots = k_n = -\sqrt{\frac{n-m}{m}}.$$

We have from (1.5)

$$H = \frac{1}{n} \left(m \sqrt{\frac{m}{n-m}} - (n-m) \sqrt{\frac{n-m}{m}} \right), \quad S = \frac{m^2}{n-m} + \frac{(n-m)^2}{m},$$

$$\sum_{i,j,k} h_{ij} h_{jk} h_{ki} = \sum_i k_i^3 = m \left(\frac{m}{n-m} \right)^{\frac{3}{2}} - (n-m) \left(\frac{n-m}{m} \right)^{\frac{3}{2}},$$

thus we easily check that (1.2) holds, i.e., $W_{m,n-m}$ are Willmore hypersurfaces. In particular, We note that ρ^2 of $W_{m,n-m}$ for all $1 \leq m \leq n-1$ satisfy

$$(1.6) \quad \rho^2 = n.$$

We recall that well-known Clifford minimal tori are

$$(1.7) \quad C_{m,n-m} = S^m \left(\sqrt{\frac{m}{n}} \right) \times S^{n-m} \left(\sqrt{\frac{n-m}{n}} \right), \quad 1 \leq m \leq n-1.$$

It is remarkable that Willmore tori coincide with Clifford minimal tori if and only if $n = 2m$ for some m .

Remark 1.3. When $n = 2$, we can see from (1.3) that all minimal surfaces are Willmore surfaces. When $n \geq 3$, minimal hypersurfaces are not Willmore hypersurfaces in general, for example, Clifford minimal tori $C_{m,n-m} = S^m \left(\sqrt{\frac{m}{n}} \right) \times S^{n-m} \left(\sqrt{\frac{n-m}{n}} \right)$ are not Willmore hypersurfaces when $n \neq 2m$.

In the theory of minimal hypersurfaces, the following Simons' integral inequality is well-known

THEOREM 2. (Simons [17], Lawson [8], Chern-Do Carmo-Kobayashi [5]) Let M be an n -dimensional ($n \geq 2$) compact minimal hypersurface in $(n + 1)$ -dimensional unit sphere S^{n+1} . Then we have

$$(1.8) \quad \int_M S(n - S)dv \leq 0.$$

In particular, if

$$(1.9) \quad 0 \leq S \leq n,$$

then either $S = 0$ and M is totally geodesic, or $S = n$ and M is one of the Clifford tori $C_{m,n-m}$, which are defined by (1.7).

In this paper we prove the following integral inequality of Simons' type for Willmore hypersurfaces.

THEOREM 3. Let M be an n -dimensional ($n \geq 2$) compact Willmore hypersurface in $(n + 1)$ -dimensional unit sphere S^{n+1} . Then we have

$$(1.10) \quad \int_M \rho^n(n - \rho^2)dv \leq 0.$$

In particular, if

$$(1.11) \quad 0 \leq \rho^2 \leq n,$$

then either $\rho^2 = 0$ and M is totally umbilic, or $\rho^2 = n$ and M is one of the Willmore tori $W_{m,n-m}$, which are defined by (1.4).

2. Preliminaries. Let $x : M \rightarrow S^{n+1}$ be an n -dimensional hypersurface in an $(n + 1)$ -dimensional unit sphere S^{n+1} . Let $\{e_1, \dots, e_n\}$ be a local orthonormal basis of M with respect to the induced metric, $\{\theta_1, \dots, \theta_n\}$ their dual form. Let e_{n+1} be the local unit normal vector field. In this paper we make the following convention on the range of indices:

$$1 \leq i, j, k \leq n.$$

Then we have the structure equations

$$(2.1) \quad dx = \sum_i \theta_i e_i,$$

$$(2.2) \quad de_i = \sum_j \theta_{ij} e_j + \sum_j h_{ij} \theta_j e_{n+1} - \theta_i x,$$

$$(2.3) \quad de_{n+1} = - \sum_{i,j} h_{ij} \theta_j e_i.$$

The Gauss equations are

$$(2.4) \quad R_{ijkl} = (\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + (h_{ik} h_{jl} - h_{il} h_{jk}),$$

$$(2.5) \quad R_{ik} = (n - 1) \delta_{ik} + nHh_{ik} - \sum_j h_{ij} h_{jk},$$

$$(2.6) \quad n(n - 1)R = n(n - 1) + n^2 H^2 - S,$$

where R is the normalized scalar curvature of M and $S = \sum_{i,j} h_{ij}^2$ is the norm square of the second fundamental form and H is the mean curvature.

The Codazzi equations are

$$(2.7) \quad h_{ijk} = h_{ikj},$$

where the covariant derivative of h_{ij} is defined by

$$(2.8) \quad \sum_k h_{ijk} \theta_k = dh_{ij} + \sum_k h_{kj} \theta_{ki} + \sum_k h_{ik} \theta_{kj}.$$

The second covariant derivative of h_{ij} is defined by

$$(2.9) \quad \sum_l h_{ijkl} \theta_l = dh_{ijk} + \sum_l h_{ljk} \theta_{li} + \sum_l h_{ilk} \theta_{lj} + \sum_l h_{ijl} \theta_{lk}.$$

By exterior differentiation of (2.8), we have the following Ricci identities

$$(2.10) \quad h_{ijkl} - h_{ijlk} = \sum_m h_{mj} R_{mikl} + \sum_m h_{im} R_{mjkl}.$$

We define the following non-negative function on M

$$(2.11) \quad \rho^2 = S - nH^2,$$

which vanishes exactly at the umbilical points of M .

Willmore functional is the following functional (see [4] or [20])

$$(2.12) \quad \int_M \rho^n dv = \int_M (S - nH^2)^{\frac{n}{2}} dv.$$

By Gauss equation (2.6), this is equivalent to

$$(2.12)' \quad [n(n-1)]^{\frac{n}{2}} \int_M (H^2 - R + 1)^{\frac{n}{2}} dv.$$

It was shown in [20] and [22] that this functional is an invariant under Moebius (or conformal) transformations of S^{n+1} . We use the term *Willmore hypersurfaces* to call its critical points. When $n = 2$, the functional essentially coincides with the well-known Willmore functional and its critical points are *Willmore surfaces*.

In order to prove Theorem 1, we need the following Reilly's result

LEMMA 2.1 (Theorem A of Reilly [15]). *Suppose that f is any smooth function of n variables. $x : M \rightarrow S^{n+1}$ a compact hypersurface. Consider a one-parameter family $x_t : M \rightarrow S^{n+1}$ with $x_0 = x$, $t \in (-\epsilon, \epsilon)$. Let $\xi = \frac{\partial x_t}{\partial t} |_{t=0}$, $\lambda = \langle \xi, e_{n+1} \rangle$. Then*

$$(2.13) \quad \begin{aligned} & \frac{d}{dt} \Big|_{t=0} \int_M f(S_1, \dots, S_n) dv \\ &= \int_M \lambda \{ -S_1 f(S_1, \dots, S_n) + \sum_{r=1}^n (S_r S_1 - (r+1) S_{r+1}) D_r f(S_1, \dots, S_n) \\ & \quad + \sum_{i,j,r=1}^n (D_r f(S_1, \dots, S_n))_{,ij} T_{r-1}^{ij} + \sum_{r=1}^n D_r f(S_1, \dots, S_n) (n-r+1) S_{r-1} \} dv, \end{aligned}$$

where $D_r f(S_1, \dots, S_n)$ denotes the partial derivative of f with respect to variable S_r , $(\cdot)_{,ij}$ denotes the covariant derivative relative to the metric induced by x . $h = (h_{ij}) = (k_i \delta_{ij})$ and S_r the r -th elementary symmetric function of the the eigenvalues k_1, \dots, k_n of h , i.e.,

$$(2.14) \quad S_0 = 1, \quad S_1 = k_1 + \dots + k_n, \quad \dots, \quad S_n = k_1 \dots k_n.$$

The Newton transformation T_r is defined inductively by

$$(2.15) \quad T_0^{ij} = \delta_{ij}, \quad T_{r+1}^{ij} = S_{r+1} \delta_{ij} - \sum_k T_r^{ik} h_{kj}, \quad r = 0, \dots, n-1.$$

LEMMA 2.2 (see [16], c.f. [9,10]). *Let M be an n -dimensional ($n \geq 2$) hypersurface in S^{n+1} . Then we have*

$$(2.16) \quad \begin{aligned} \frac{1}{2} \Delta \rho^2 = & |\nabla h|^2 - n^2 |\nabla H|^2 + \sum_{i,j,k} (h_{ij} h_{kki})_j \\ & + nS - S^2 - n^2 H^2 + nH \sum_{i,j,k} h_{ij} h_{jk} h_{ki} - \frac{1}{2} \Delta(nH^2), \end{aligned}$$

where $|\nabla h|^2 = \sum_{i,j,k} h_{ijk}^2$ and $|\nabla H|^2 = \sum_i H_i^2$.

Proof. By the definition of Δ and ρ^2 , we have by use of (2.7) and (2.10)

$$(2.17) \quad \begin{aligned} & \frac{1}{2} \Delta \rho^2 \\ = & \frac{1}{2} \Delta \left(\sum_{i,j} h_{ij}^2 \right) - \frac{1}{2} \Delta(nH^2) \\ = & \sum_{i,j,k} h_{ijk}^2 + \sum_{i,j,k} h_{ij} h_{kij} - \frac{1}{2} \Delta(nH^2) \\ = & |\nabla h|^2 - n^2 |\nabla H|^2 + \sum_{i,j,k} (h_{ij} h_{kki})_j + \sum h_{ij} (h_{lk} R_{lij} + h_{il} R_{lj}) - \frac{1}{2} \Delta(nH^2). \end{aligned}$$

We can easily obtain (2.16) by putting (2.4) and (2.5) into (2.17)

3. Proof of Theorem 1. Choosing in Lemma 2.1

$$(3.1) \quad f(S_1, \dots, S_n) = Q^n := [(n-1)S_1^2 - 2nS_2]^{\frac{n}{2}} = n^{\frac{n}{2}} (S - nH^2)^{\frac{n}{2}} = n^{\frac{n}{2}} \rho^n,$$

and noting that

$$(3.2) \quad D_1 f = n(n-1)Q^{n-2}S_1, \quad D_2 f = -n^2 Q^{n-2}, \quad D_r f = 0, \quad r \geq 3.$$

Putting (3.1) and (3.2) into (2.13), we obtain by use of (2.14)

$$(3.3) \quad \begin{aligned} & n^{\frac{n}{2}} \frac{d}{dt} \Big|_{t=0} \int_M (S - nH^2)^{\frac{n}{2}} dv = \frac{d}{dt} \Big|_{t=0} \int_M ((n-1)S_1^2 - 2nS_2)^{\frac{n}{2}} dv \\ = & \int_M \lambda [-S_1 Q^n + (S_1^2 - 2S_2)n(n-1)Q^{n-2}S_1 - n^2 Q^{n-2}(S_2 S_1 - 3S_3) \\ & + n(n-1)\Delta(Q^{n-2}S_1) - n^2 \sum_{i,j} (Q^{n-2})_{,ij} (S_1 \delta_{ij} - h_{ij})] dv. \end{aligned}$$

Thus the Euler-Lagrange equation of Willmore functional (2.12) is

$$(3.4) \quad \begin{aligned} & -S_1Q^n + (S_1^2 - 2S_2)n(n-1)Q^{n-2}S_1 - n^2Q^{n-2}(S_2S_1 - 3S_3) \\ & + n(n-1)\Delta(Q^{n-2}S_1) - n^2\sum_{i,j}(Q^{n-2})_{,ij}(S_1\delta_{ij} - h_{ij}) = 0. \end{aligned}$$

From $Q = n^{\frac{1}{2}}\rho$, we know that (3.4) is equivalent to

$$(3.4)' \quad \begin{aligned} & -nS_1\rho^n + (S_1^2 - 2S_2)n(n-1)\rho^{n-2}S_1 - n^2\rho^{n-2}(S_2S_1 - 3S_3) \\ & + n(n-1)\Delta(\rho^{n-2}S_1) - n^2\sum_{i,j}(\rho^{n-2})_{,ij}(S_1\delta_{ij} - h_{ij}) = 0. \end{aligned}$$

Noting that

$$S_1 = nH, \quad S_2 = \frac{1}{2}(S_1^2 - S), \quad S_3 = \frac{1}{3}\left(\sum_i k_i^3 - S_1S + S_2S_1\right),$$

we know that (3.4)' is equivalent to (1.2). This completes the proof of Theorem 1.

4. Lemmas and Proof of Theorem 3. We first prove the following lemma (c.f. [7])

LEMMA 4.1. *Let M be an n -dimensional ($n \geq 2$) hypersurface in S^{n+1} , then we have*

$$(4.1) \quad |\nabla h|^2 \geq \frac{3n^2}{n+2}|\nabla H|^2,$$

where $|\nabla h|^2 = \sum_{i,j,k} h_{ijk}^2$, $|\nabla H|^2 = \sum_i H_i^2$, $H_i = \nabla_i H$.

Proof. We decompose the tensor ∇h :

$$(4.2) \quad h_{ijk} = E_{ijk} + F_{ijk},$$

where

$$E_{ijk} = \frac{n}{n+2}(H_i\delta_{jk} + H_j\delta_{ik} + H_k\delta_{ij}).$$

Then we can easily compute that

$$|E|^2 = \sum_{i,j,k} E_{ijk}^2 = \frac{3n^2}{n+2}|\nabla H|^2, \quad \langle E_{ijk}, F_{ijk} \rangle = \langle E_{ijk}, h_{ijk} - E_{ijk} \rangle = 0,$$

i.e., E and F are orthogonal components of ∇h . Then

$$|\nabla h|^2 \geq |E|^2 = \frac{3n^2}{n+2}|\nabla H|^2,$$

which proves the Lemma 4.1.

Define trace-free tensor

$$(4.3) \quad \tilde{h}_{ij} = h_{ij} - H\delta_{ij}.$$

We have by a direct calculation

$$(4.4) \quad \sum_k \tilde{h}_{kk} = 0, \quad \sum_{i,j} \tilde{h}_{ij}^2 = \rho^2 = S - nH^2,$$

$$(4.5) \quad \sum_{i,j,k} h_{ij}h_{jk}h_{ki} = \sum_{i,j,k} \tilde{h}_{ij}\tilde{h}_{jk}\tilde{h}_{ki} + 3H\rho^2 + nH^3.$$

From (4.4), (4.5) and Theorem 1, we have

LEMMA 4.2. *Let M be an n -dimensional hypersurface in the $(n + 1)$ -dimensional unit sphere S^{n+1} . Then M is a Willmore hypersurface if and only if*

$$(4.6) \quad (n - 1)\Delta(\rho^{n-2}H) + \rho^{n-2}(H\rho^2 + \sum_{i,j,k} \tilde{h}_{ij}\tilde{h}_{jk}\tilde{h}_{ki}) - \sum_{i,j} (\rho^{n-2})_{,ij}(nH\delta_{ij} - h_{ij}) = 0,$$

where Δ is the Laplacian, $(\cdot)_{,ij}$ is the covariant derivative relative to the induced metric.

The following lemma is a key step of the proof of Theorem 3

LEMMA 4.3. *Let M be the n -dimensional hypersurface in the $(n + 1)$ -dimensional unit sphere S^{n+1} , then we have*

$$(4.7) \quad \begin{aligned} & \frac{1}{2}\Delta(\rho^n) \\ &= \frac{1}{2}n(n - 2)\rho^{n-2}|\nabla\rho|^2 + \frac{n}{2}\rho^{n-2}\{(|\nabla h|^2 - \frac{3n^2}{n+2}|\nabla H|^2) + (\frac{3n^2}{n+2} - n)|\nabla H|^2 \\ & \quad - n(n - 1)|\nabla H|^2 + \sum_{i,j,k} (h_{ij}h_{kki})_j + \rho^2(n + nH^2 - \rho^2) \\ & \quad + nH \sum_{i,j,k} \tilde{h}_{ij}\tilde{h}_{jk}\tilde{h}_{ki} - \frac{1}{2}\Delta(nH^2)\} \\ & \geq \frac{n}{2}\rho^{n-2}\{-n(n - 1)|\nabla H|^2 + \sum_{i,j,k} (h_{ij}h_{kki})_j + \rho^2(n + nH^2 - \rho^2) \\ & \quad + nH \sum_{i,j,k} \tilde{h}_{ij}\tilde{h}_{jk}\tilde{h}_{ki} - \frac{1}{2}\Delta(nH^2)\} \end{aligned}$$

Proof. First it is easy to check the following identity

$$(4.8) \quad \frac{1}{2}\Delta(\rho^n) = \frac{1}{2}n(n - 2)\rho^{n-2}|\nabla\rho|^2 + \frac{n}{4}\rho^{n-2}\Delta(\rho^2).$$

By use of (4.4) and (4.5), (2.16) can be written as

$$(4.9) \quad \begin{aligned} \frac{1}{2}\Delta\rho^2 &= |\nabla h|^2 - n^2|\nabla H|^2 + \sum_{i,j,k} (h_{ij}h_{kki})_j \\ & \quad + \rho^2(n + nH^2 - \rho^2) + nH \sum_{i,j,k} \tilde{h}_{ij}\tilde{h}_{jk}\tilde{h}_{ki} - \frac{1}{2}\Delta(nH^2). \end{aligned}$$

Putting (4.9) into (4.8) and noting $\frac{3n^2}{n+2} - n = \frac{2n(n-1)}{n+2} > 0$, we obtain (4.7) by use of Lemma 4.1.

LEMMA 4.4. *Let M be an n -dimensional compact Willmore hypersurface in the*

$(n + 1)$ -dimensional unit sphere S^{n+1} , then we have
(4.10)

$$\begin{aligned} & -n(n-1) \int_M \rho^{n-2} |\nabla H|^2 \\ = & -n(n-1) \int_M \rho^{n-2} (HH_i)_i + n \int_M H(\rho^{n-2})_{,ij} (nH\delta_{ij} - h_{ij}) - n(n-1) \int_M H^2 \Delta(\rho^{n-2}) \\ & - 2n(n-1) \int_M H \nabla(\rho^{n-2}) \cdot \nabla H - n \int_M H \rho^{n-2} (H\rho^2 + \sum_{i,j,k} \tilde{h}_{ij} \tilde{h}_{jk} \tilde{h}_{ki}). \end{aligned}$$

Proof. We first note the following identity

$$\begin{aligned} & -n(n-1)\rho^{n-2} |\nabla H|^2 \\ (4.11) \quad = & -n(n-1)(\rho^{n-2}(HH_i)_i - \rho^{n-2}H\Delta H) \\ = & -n(n-1)[\rho^{n-2}(HH_i)_i - H\Delta(\rho^{n-2}H) + H^2\Delta(\rho^{n-2}) + 2H\nabla(\rho^{n-2}) \cdot \nabla H]. \end{aligned}$$

Integrating (4.11) over M , we have (4.10) by use of (4.6).

LEMMA 4.5. *Let M be an n -dimensional compact hypersurface in the $(n + 1)$ -dimensional unit sphere S^{n+1} , then we have*

$$(4.12) \quad \int_M \rho^{n-2} \sum_{i,j,k} (h_{ij}h_{kki})_j = n \int_M H \sum_{i,j} h_{ij}(\rho^{n-2})_{,ij} + n^2 \int_M H \nabla(\rho^{n-2}) \cdot \nabla H.$$

Proof. We have the following calculation

$$\begin{aligned} \int_M \rho^{n-2} \sum_{i,j,k} (h_{ij}h_{kki})_j &= \int_M \sum_{i,j,k} (\rho^{n-2}h_{ij}h_{kki})_j - \int_M \sum_{i,j,k} (\rho^{n-2})_{,j} h_{ij}h_{kki} \\ &= - \int_M \sum_{i,j,k} (\rho^{n-2})_{,j} h_{ij}h_{kki} \\ &= - \int_M \sum_{i,j,k} ((\rho^{n-2})_{,j} h_{ij}h_{kk})_i + \int_M \sum_{i,j,k} (\rho^{n-2})_{,ji} h_{ij}h_{kk} \\ &\quad + n^2 \int_M \sum_j H(\rho^{n-2})_{,j} H_j \\ &= n \int_M H \sum_{i,j} h_{ij}(\rho^{n-2})_{,ij} + n^2 \int_M H \nabla(\rho^{n-2}) \cdot \nabla H. \end{aligned}$$

Proof of Theorem 3. Integrating (4.7) over M , we have

$$\begin{aligned} (4.13) \quad 0 \geq & \frac{n}{2} \{-n(n-1) \int_M \rho^{n-2} |\nabla H|^2 + \int_M \rho^{n-2} \sum_{i,j,k} (h_{ij}h_{kki})_j + \int_M \rho^n (n+nH^2 - \rho^2) \\ & + n \int_M H \rho^{n-2} \sum_{i,j,k} \tilde{h}_{ij} \tilde{h}_{jk} \tilde{h}_{ki} - \frac{1}{2} \int_M \rho^{n-2} \Delta(nH^2)\}. \end{aligned}$$

Putting (4.10) and (4.12) into (4.13), we get

$$\begin{aligned}
 (4.14) \quad & 0 \geq \frac{n}{2} \left\{ [-n(n-1) \int_M \rho^{n-2} (HH_i)_i + n \int_M H \sum_{i,j} (\rho^{n-2})_{,ij} (nH\delta_{ij} - h_{ij}) \right. \\
 & - n(n-1) \int_M H^2 \Delta(\rho^{n-2}) - 2n(n-1) \int_M H \nabla(\rho^{n-2}) \cdot \nabla H - n \int_M H \rho^{n-2} (H\rho^2 \\
 & + \sum_{i,j,k} \tilde{h}_{ij} \tilde{h}_{jk} \tilde{h}_{ki})] + [n \int_M H \sum_{i,j} h_{ij} (\rho^{n-2})_{,ij} + n^2 \int_M H \nabla(\rho^{n-2}) \cdot \nabla H] \\
 & + \int_M \rho^n (n + nH^2 - \rho^2) + n \int_M H \rho^{n-2} \sum_{i,j,k} \tilde{h}_{ij} \tilde{h}_{jk} \tilde{h}_{ki} - \frac{1}{2} \int_M \rho^{n-2} \Delta(nH^2) \left. \right\} \\
 & = \frac{n}{2} \left\{ -n(n-1) \int_M \rho^{n-2} (HH_i)_i + n^2 \int_M H^2 \Delta(\rho^{n-2}) - n(n-1) \int_M H^2 \Delta(\rho^{n-2}) \right. \\
 & - 2n(n-1) \int_M H \nabla(\rho^{n-2}) \cdot \nabla H + n^2 \int_M H \nabla(\rho^{n-2}) \cdot \nabla H + \int_M \rho^n (n - \rho^2) \\
 & - \left. \frac{1}{2} \int_M \rho^{n-2} \Delta(nH^2) \right\} \\
 & = \frac{n}{2} \int_M \rho^n (n - \rho^2).
 \end{aligned}$$

Thus we reach the following integral inequality of Simons' type

$$(4.15) \quad \int_M \rho^n (n - \rho^2) \leq 0.$$

Therefore we have proved the integral inequality (1.10) in Theorem 3.

If (1.11) holds, then we conclude from (4.15) that either $\rho^2 \equiv 0$, or $\rho^2 \equiv n$. In the first case, we know that $S \equiv nH^2$, i.e. M is totally umbilic; in the latter case, i.e., $\rho^2 \equiv n$, we have from (4.7)

$$\int_M \rho^{n-2} \left(\frac{3n^2}{n+2} - n \right) |\nabla H|^2 = 0,$$

we have $H = constant$, thus we have again from (4.7)

$$\nabla h = 0.$$

It easily follows that M is an isoparametric hypersurface with two distinct constant principal curvatures, M is one of the Willmore tori (see Theorem 5.1), that is, $M = W_{m,n-m}$ for some m with $1 \leq m \leq n - 1$. We complete the proof of Theorem 3.

5. Isoparametric Willmore hypersurfaces. In this section, we give the classification of isoparametric hypersurfaces in S^{n+1} . We need the following result

LEMMA 5.1 (see [1,3,12,18,19]). *Let M be an n -dimensional compact isoparametric hypersurface (i.e. hypersurface with constant principal curvatures) in S^{n+1} . Let $k_1 > k_2 > \dots > k_g$ be the distinct principal curvatures with multiplicities m_1, \dots, m_g (so that $n = m_1 + m_2 + \dots + m_g$). Then*

- (a) g is either 1, 2, 3, 4, or 6.
- (b) If $g = 1$, M is totally umbilic.
- (c) If $g = 2$, $M = S^m(r_1) \times S^{n-m}(r_2)$, $r_1^2 + r_2^2 = 1$.

- (d) If $g = 3$, $m_1 = m_2 = m_3 = 2^k$, ($k = 0, 1, 2, 3$).
- (e) If $g = 4$, $m_1 = m_3$ and $m_2 = m_4$. Moreover, $(m_1, m_2) = (2, 2)$ or $(4, 5)$, or $m_1 + m_2 + 1$ is a multiple of $2^{\phi(m_1-1)}$. Here $\phi(l)$ is the number of integers s with $1 \leq s \leq l$ and $s \equiv 0, 1, 2, 4 \pmod 8$.
- (f) If $g = 6$, $m_1 = m_2 = \dots = m_6 = 1$ or 2 .
- (g) There exists an angle θ , $0 < \theta < \frac{\pi}{g}$, such that

$$(5.1) \quad k_\alpha = \cot(\theta + \frac{\alpha - 1}{g}\pi), \quad \alpha = 1, \dots, g.$$

In the isoparametric case, $\rho^2 = \text{constant}$, $H = \text{constant}$, we get from Theorem 1

LEMMA 5.2. *Let M be an n -dimensional isoparametric Willmore hypersurface in S^{n+1} . Then*

$$(5.2) \quad 2HS - nH^3 - \sum_i k_i^3 = 0,$$

where $h_{ij} = k_i \delta_{ij}$.

THEOREM 5.1. *Let M be an n -dimensional compact isoparametric Willmore hypersurface in S^{n+1} . Then*

- (1) If $g = 1$, M is a totally umbilic hypersphere, satisfying $\rho^2 = 0$.
- (2) If $g = 2$, M is one of the Willmore tori $W_{m,n-m}$, which are defined by (1.4), satisfying $\rho^2 = n$.
- (3) If $g = 3$, $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$. $n = 3, 6, 12$ or 24 . M are Cartan minimal hypersurfaces, satisfying $\rho^2 = 2n$.
- (4) If $g = 4$,

$$(5.3) \quad k_1 = \lambda, \quad k_2 = \frac{\lambda - 1}{\lambda + 1}, \quad k_3 = -\frac{1}{\lambda}, \quad k_4 = -\frac{\lambda + 1}{\lambda - 1},$$

where $A = (\lambda - \frac{1}{\lambda})^2$ is the positive solution of the following algebraic equation

$$(5.4) \quad m_1(m_1 + 2m_2)^2 x^2 + 4m_1 m_2 (m_2 - m_1)x - 16m_2(m_2 + 2m_1)^2 = 0,$$

and $(m_1, m_2) = (2, 2)$ or $(4, 5)$, or $m_1 + m_2 + 1$ is a multiple of $2^{\phi(m_1-1)}$. Here $\phi(l)$ is the number of integers s with $1 \leq s \leq l$ and $s \equiv 0, 1, 2, 4 \pmod 8$.

Among these isoparametric Willmore hypersurfaces, the only isoparametric minimal hypersurfaces are case with $m_1 = m_2 = m_3 = m_4 = 2$, principal curvatures are

$$k_1 = 1 + \sqrt{2}, \quad k_2 = \sqrt{2} - 1, \quad k_3 = 1 - \sqrt{2}, \quad k_4 = -(1 + \sqrt{2}).$$

(5) If $g = 6$, then

$$k_1 = 2 + \sqrt{3}, \quad k_2 = 1, \quad k_3 = 2 - \sqrt{3}, \quad k_4 = -(2 - \sqrt{3}), \quad k_5 = -1, \quad k_6 = -(2 + \sqrt{3}).$$

In this case, $n = 6$ or 12 . These Willmore hypersurfaces are minimal and satisfying $\rho^2 = 5n$.

Proof. (1) Case $g = 1$ is trivial.

(2) If $g = 2$, let distinct principal curvatures are k_1 (multiplicity m) and k_2 (multiplicity $n - m$). Then by (c) of Lemma 5.1 and (5.2), we have

$$(5.5) \quad 1 + k_1 k_2 = 0,$$

$$(5.6) \quad \frac{2(mk_1+(n-m)k_2)}{n}(mk_1^2+(n-m)k_2^2)-\frac{1}{n^2}(mk_1+(n-m)k_2)^3-(mk_1^3+(n-m)k_2^3) = 0.$$

Putting (5.5) into (5.6), we have

$$(5.7) \quad (n-m)k_1^6 + (2n-3m)k_1^4 + (n-3m)k_1^2 - m = 0,$$

that is,

$$(5.7)' \quad ((n-m)k_1^2 - m)(k_1^2 + 1)^2 = 0.$$

Thus

$$k_1^2 = \frac{m}{n-m}, \quad M = W_{m,n-m} = S^m \left(\sqrt{\frac{n-m}{n}} \right) \times S^{n-m} \left(\sqrt{\frac{m}{n}} \right), \quad 1 \leq m \leq n-1.$$

(3) If $g = 3$, by (d) of Lemma 5.1, $m_1 = m_2 = m_3 := m$, $n = 3m$.

From (g) of Lemma 5.1, we have

$$(5.8) \quad k_1 = \cot\theta, \quad k_2 = \cot\left(\theta + \frac{\pi}{3}\right) = \frac{k_1 - \sqrt{3}}{1 + \sqrt{3}k_1}, \quad k_3 = \cot\left(\theta + \frac{2}{3}\pi\right) = \frac{k_1 + \sqrt{3}}{1 - \sqrt{3}k_1}.$$

Putting (5.8) into (5.2) and noting $n = 3m$, we obtain

$$(5.9) \quad k_1(k_1^2 - 3)(k_1^2 + 1)^3 = 0.$$

Thus we have $k_1 = \sqrt{3}, k_2 = 0, k_3 = -\sqrt{3}$.

(4) If $g = 4$, in this case

$$m_1 = m_3, \quad m_2 = m_4.$$

By (g) of Lemma 5.1,

$$(5.10) \quad \begin{aligned} k_1 = \cot\theta := \lambda, \quad k_2 = \cot\left(\theta + \frac{\pi}{4}\right) &= \frac{\lambda - 1}{\lambda + 1}, \\ k_3 = \cot\left(\theta + \frac{2\pi}{4}\right) &= -\frac{1}{\lambda}, \quad k_4 = \cot\left(\theta + \frac{3\pi}{4}\right) = -\frac{\lambda + 1}{\lambda - 1}. \end{aligned}$$

Write

$$(5.11) \quad A = \lambda - \frac{1}{\lambda}, \quad B = k_2 - \frac{1}{k_2}.$$

Noting $nH = m_1A + m_2B$ and

$$(5.12) \quad S = m_1A^2 + m_2B^2 + 2(m_1 + m_2), \quad \sum_i k_i^3 = m_1A^3 + m_2B^3 + 3(m_1A + m_2B),$$

we have by putting (5.12) into (5.2)

$$(5.13) \quad \begin{aligned} \frac{2}{n}(m_1A + m_2B)(m_1A^2 + m_2B^2 + 2m_1 + 2m_2) - \frac{1}{n^2}(m_1A + m_2B)^3 \\ - (m_1A^3 + m_2B^3 + 3m_1A + 3m_2B) = 0. \end{aligned}$$

Putting $n = 2(m_1 + m_2)$ into (5.13) and noting $AB = -4$, we get that

$$(5.14) \quad m_1(m_1 + 2m_2)^2 A^6 + 4m_1(m_1^2 + 3m_1m_2 + 5m_2^2)A^4 - 16m_2(m_2^2 + 3m_1m_2 + 5m_1^2) - 64m_2(m_2 + 2m_1)^2 = 0,$$

which can be written as

$$(5.15) \quad (A^2 + 4)[m_1(m_1 + 2m_2)^2 A^4 + 4m_1m_2(m_2 - m_1)A^2 - 16m_2(m_2 + 2m_1)^2] = 0,$$

that is equivalent to

$$(5.16) \quad m_1(m_1 + 2m_2)^2 A^4 + 4m_1m_2(m_2 - m_1)A^2 - 16m_2(m_2 + 2m_1)^2 = 0.$$

(5) If $g = 6$, in this case, by (f) of Lemma 5.1 we have

$$m_1 = m_2 = m_3 = m_4 = m_5 = m_6 = 1, \text{ or } 2.$$

By (g) of Lemma 5.1,

$$(5.17) \quad \begin{aligned} k_1 &= \cot\theta, & k_2 &= \frac{\sqrt{3}k_1 - 1}{k_1 + \sqrt{3}}, & k_3 &= \frac{k_1 - \sqrt{3}}{1 + \sqrt{3}k_1}, \\ k_4 &= -\frac{1}{k_1}, & k_5 &= \frac{k_1 + \sqrt{3}}{1 - \sqrt{3}k_1}, & k_6 &= \frac{1 + \sqrt{3}k_1}{\sqrt{3} - k_1}. \end{aligned}$$

Putting (5.17) into (5.2), we can get by a direct calculation

$$(5.18) \quad (k_1^2 - 1)(k_1^4 - 14k_1^2 + 1)[(k_1^2 - 1)^2(k_1^4 - 14k_1^2 + 1)^2 + 4k_1^2(3k_1^4 - 10k_1^2 + 3)^2] = 0.$$

Thus we obtain

$$k_1 = 2 + \sqrt{3}, \quad k_2 = 1, \quad k_3 = 2 - \sqrt{3}, \quad k_4 = -(2 - \sqrt{3}), \quad k_5 = -1, \quad k_6 = -(2 + \sqrt{3}).$$

We complete the proof of Theorem 5.1.

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