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CANONICAL KÄHLER CLASSES*

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Abstract. On a compact complex manifold (M, J) of Kähler type, consider the functional defined by the square of the L^2-norm of the scalar curvature with domain the space of Kähler metrics of fixed total volume. The infimum of this functional over metrics that represent a given class defines the energy of the class. We find the Euler-Lagrange equation for critical classes of this energy, and compute its Hessian at a critical class. We use these results to conclude that there exist a unique critical class for any del Pezzo surface as well as for any manifold with negative first Chern class. It follows that if the first Chern class is positive and not a critical class of the energy, it cannot be represented by an Einstein metric. We also draw some conclusions on the generic case of a manifold without holomorphic vector fields.

1. Introduction. Let (M, J) be a compact complex manifold of Kähler type. We denote by $\mathfrak{M}$ the set of Kähler forms on (M, J), an open cone in the infinite-dimensional affine space of closed $(1,1)$-forms. We will identify the Kähler form ω in $\mathfrak{M}$ with the corresponding Kähler metric g. Then, if the complex dimension of (M, J) is n and $\omega \in \mathfrak{M}$, the associated volume form is $d\mu_\omega := \omega^n/n!$.

It this context, it is quite natural to analyze the functional given by the square of the L^2-norm of the scalar curvature of a metric,

$$\omega \overset{\Phi}{\to} \int_M s_\omega^2 d\mu_\omega \,,$$

taking $\mathfrak{M}$ as its domain. However, this functional is homogeneous of degree $n-2$ and, in general, will not admit critical points without the specification of a suitable normalization for its domain. Calabi [2, 3] solved this problem by fixing a class $\Omega \in H^{1,1}(M, \mathbb{C}) \subset H^2(M, \mathbb{R})$ in the Kähler cone, and searching for critical metrics of the restriction of Φ to $\mathfrak{M}_\Omega = \{\omega \in \mathfrak{M} : \ [\omega] = \Omega\}$:

$$\begin{array}{ccc} \mathfrak{M}_\Omega & \overset{\Phi_\Omega}{\to} & \mathbb{R} \\ \omega & \mapsto & \displaystyle\int_M s_\omega^2 d\mu_\omega \,. \end{array} \tag{1.1}$$

All metrics in $\mathfrak{M}_\Omega$ have fixed volume $\Omega^n/n!$. Those which are critical points of Φ_Ω are called *extremal*, and their mere existence is far from being a settled issue to this date.

On the other hand, we may also consider the set $\mathfrak{M}_v$ of Kähler metrics of fixed total volume v, and define the functional

$$\begin{array}{ccc} \mathfrak{M}_v & \overset{\Phi}{\to} & \mathbb{R} \\ \omega & \mapsto & \displaystyle\int_M s_\omega^2 d\mu_\omega \,. \end{array} \tag{1.2}$$

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The value of the constant v is inessential to our considerations, and can be taken to be 1 if so desired. Critical points of (1.2) are a particular type of extremal metrics, that we termed *strongly extremal* in [12].

The energy $\Phi_\Omega(\omega)$ of an extremal Kähler metric ω that represents Ω is a continuous function of Ω, and may be calculated from *a priori* data. Its value, which we shall call the *potential energy* $E(\Omega)$ of Ω, is an optimal lower bound for the functional Φ among metrics in $\mathfrak{M}_\Omega$. In other words, we have $\Phi_\Omega(\omega) \geq E(\Omega)$ for any Kähler metric ω, and the equality is achieved only for those metrics in $\mathfrak{M}_\Omega$ that are extremal. As a function of the Kähler class, the potential energy involves only finitely many parameters, and is also homogeneous of degree $n-2$. Interestingly enough, both Φ_Ω and E are scale-invariant on complex surfaces.

In this paper we study the cohomology classes that are critical points of the potential energy E. The connection between these classes and the problem of finding extremal metrics is rather clear: the critical classes for E are the only ones that may be represented by strongly extremal metrics, critical points of the functional (1.2).

The analysis of these classes is a delicate issue, as we cannot assume that the Kähler classes are extremal. In fact, we cannot even assume that any Kähler class is extremal, and that imposes some difficulties on the problem. We may easily uncover a relationship between $E(\Omega)$ and a suitable function in the space of holomorphy potentials, the space of functions that give rise to holomorphic vector fields when raising the indices of their $\bar{\partial}$-derivatives. This suitable function is the L^2-projection onto the space of holomorphy potentials of the scalar curvature of any Kähler metric that represents g. Therefore, the analysis of the Euler-Lagrange equations for E must be done through a careful study of the relationship between this projection and the Ricci curvature, and how these two quantities vary when deforming the cohomology class while infinitesimally preserving the volume.

The classes represented by Kähler-Einstein metrics are all critical. Hence, for those manifolds where one could find a Kähler Einstein metric, the functional E has a critical class that is, up to a multiple, the first Chern class. But interestingly enough, there are cases where E does have critical classes that are not equal to the canonical one. In particular, by deriving the Hessian of E at a critical class, we prove that for any del Pezzo surface there exists a unique such class. This singles out cohomology classes in the manifolds $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$, respectively, which are not equal to the first Chern class and that minimize the functional E. The class in the first of these manifolds was proven to exist and to be unique in [6], using a direct approach based upon our knowledge that in this case every Kähler class is extremal. On the other hand, for the latter manifold the existence of that class was shown in [8], and our work gives the first proof of its uniqueness. Strong evidence of this uniqueness was given earlier in [9, 12].

Our Euler-Lagrange equation shows that if the first Chern class is not a critical class of E, it cannot be represented by an Einstein metric. Thus, for a Kähler manifold to carry Einstein metrics, its first Chern class must be a critical point of E, and so this condition represents an obstruction to the existence of Einstein metrics. We do not know how this obstruction relates to others. But it becomes rather clear that critical classes of E play a significant rôle in those cases where the manifold (M,J) has a positive first Chern class and carries no Kähler Einstein metrics.

2. Extremal Metrics. The Kähler condition makes a metric quite special. Its scalar curvature may be calculated by the formula

$$s\,\omega^{\wedge n} = 2n\,\rho\wedge\omega^{\wedge(n-1)}\,, \tag{2.1}$$

where ρ is the Ricci form of ω, and since $\rho/2\pi$ represents the *First Chern class* c_1, assuming that M is compact, we get the relation

$$\int_M s\,d\mu_g = \frac{4\pi}{(n-1)!}c_1\cup[\omega]^{\cup(n-1)}\,.$$

Therefore, the average scalar curvature

$$s_0 = \frac{\int_M s\,d\mu_g}{\frac{1}{n!}[\omega]^{\cup n}} = 4\pi n\frac{c_1\cup[\omega]^{\cup(n-1)}}{[\omega]^{\cup n}} \tag{2.2}$$

is a topological invariant.

Acting on functions, the Laplace-Beltrami operator of g will be

$$\Delta = d^*d = 2\overline{\partial}^*\overline{\partial} = -2g^{j\overline{k}}\frac{\partial^2}{\partial z^j\partial\overline{z}^k}\,, \tag{2.3}$$

so we also have

$$s = -2g^{j\overline{k}}\frac{\partial^2}{\partial z^j\partial\overline{z}^k}\log\det(g_{p\overline{q}})\,. \tag{2.4}$$

Given a complex valued function f, we define the vector field $\partial_g^{\#}f$ by the identity $g(\partial_g^{\#}f,\,\cdot) = \overline{\partial}f$. This is a vector field of type $(1,0)$, and as such, it is rarely *holomorphic*. For that, we need to require that f be in the kernel of the operator

$$(\overline{\partial}\partial^{\#})^*\overline{\partial}\partial^{\#}f = \frac{1}{4}\Delta^2 f + \frac{1}{2}r^{\mu\nu}\nabla_\mu\nabla_\nu f + \frac{1}{2}(\nabla^{\overline{\ell}}s)\nabla_{\overline{\ell}}f\,, \tag{2.5}$$

where the adjoint, Ricci tensor r, scalar curvature s and other relevant quantities are those of the metric g. Every complex valued function f in the kernel of this operator is thus associated with a holomorphic vector field $\Xi = \partial^{\#}f$, and since the operator is elliptic, the space of such functions is finite dimensional. The function f, determined up to a constant, is called the *holomorphy potential* of Ξ. A holomorphic vector field Ξ is defined by a holomorphy potential f iff Ξ vanishes at some point.

The Ricci form of g may be written as

$$\rho = \rho_H + i\partial\overline{\partial}\psi_\omega\,, \tag{2.6}$$

where ρ_H is harmonic and ψ_ω is L^2-perpendicular to the constants. The function ψ_ω so obtained is called the *Ricci potential* of the metric, and in terms of the Green's operator $\mathbb{G}$ and the projection s_0 of s onto the constants, it can be written as $\psi_\omega = -2\mathbb{G}(s-s_0) = -2\mathbb{G}s$.

Let $\mathfrak{h}(M)$ be the complex Lie algebra of holomorphic vector fields of the complex manifold (M,J); by compactness of M, this is precisely the Lie algebra of the group of biholomorphism of (M,J). The *Futaki character* is defined to be the map

$$\begin{aligned}\mathfrak{F}:\mathfrak{h}(M)\times H^{1,1}(M,\mathbb{R})^+ &\longrightarrow \mathbb{C}\\ \mathfrak{F}(\Xi,[\omega]) = \int_M \Xi(\psi_\omega)d\mu &= -2\int_M \Xi(\mathbb{G}s)d\mu\,.\end{aligned} \tag{2.7}$$

It is independent of the particular metric in the class $[\omega]$ chosen to calculate it [5], and it is such that when applied to a holomorphic vector field of the form $\Xi = \partial^{\#} f$, it produces

$$\mathfrak{F}(\Xi, [\omega]) = -\int_M f(s - s_0)\, d\mu\,. \tag{2.8}$$

We now summarize some known facts about extremal metrics. A Kähler metric g is a critical point of Φ_Ω if, and only if, the gradient of its scalar curvature is a real-holomorphic vector field. If M supports no non-trivial holomorphic vector fields, every extremal Kähler metric must have constant scalar curvature; in fact, in the presence of holomorphic vector fields, the Futaki character $\mathfrak{F}$ measures the extent to which any such metric fails to be of constant scalar curvature. In that case, by (2.8) we have that $\mathfrak{F}(\partial^{\#} s, [\omega]) = -\|s - s_0\|^2$. After Calabi's initial examples [2], many compact Kähler manifolds are now known which carry extremal metrics of non-constant scalar curvature [10, 11].

Given a Kähler class Ω, there exists a holomorphic vector field X_Ω [4] obtained as the vector field defined by the holomorphy potential $\pi_g s$, the L^2-projection of the scalar curvature s onto the space of holomorphy potentials. This vector field would coincide with $\partial^{\#}_g s_g$ if there were an extremal metric g representing Ω. It is always independent of the metric $g \in \mathfrak{M}_\Omega$ used to calculate it.

THEOREM 1. *Let (M, J, Ω) be a polarized Kähler manifold. A Kähler metric $g \in \mathfrak{M}_\Omega$ is extremal if, and only if, the gradient of its scalar curvature s is a holomorphic vector field*:

$$Ls = (\overline{\partial}\partial^{\#})^* \overline{\partial}\partial^{\#} s = \frac{1}{4}\Delta^2 s + \frac{1}{2} r^{\mu\nu} \nabla_\mu \nabla_\nu s + \frac{1}{2}(\nabla^{\overline{\ell}} s)\nabla_{\overline{\ell}} s = 0\,.$$

Furthermore, the Calabi energy has a lower bound –the potential energy of the class– which is a continuous function of Ω,

$$\Phi_\Omega(\omega) \geq E(\Omega) := s_0^2 \frac{\Omega^n}{n!} - \mathfrak{F}(X_\Omega, \Omega)\,, \tag{2.9}$$

and this lower bound is achieved if, and only if, the metric is extremal.

3. Critical Classes. If a class Ω is represented by an extremal metric ω, then $\Psi_\Omega(\omega) = E(\Omega)$. It is quite natural to attempt to identify the cohomology classes for which this occurs, but we are going to content ourselves with the study of those classes which are critical points of E. These are defined to be critical points of the restriction of E to the base of the Kähler cone.

The potential energy function is given by

$$E(\Omega) = s_0^2 \frac{\Omega^n}{n!} - \mathfrak{F}(X_\Omega, \Omega)\,,$$

and can be computed for any class in the entire Kähler cone. In order to ensure that E has critical points, we *normalized* this domain and restric our attention to the classes that represent Kähler metrics of fixed volume.

Under the assumption that a critical class is represented by an extremal metric, one can easily write down the equation that the class must satisfy [12]. However, though the set of extremal classes is open [11] in the Kähler cone, little else is known about its complement, and we cannot at this point in time expect to say very much about critical classes presuming that they can be represented by extremal metrics.

3.1. Holomorphic fields with holomorphy potentials. The isometry group of any extremal metric is a maximal compact subgroup of the identity component $\mathcal{A}$ of the biholomorphism group [3]. Any two such groups are conjugate [7]. Thus, up to biholomorphism, the search for extremal metrics can be carried out among those metrics which are invariant under the action of a maximal compact subgroup G of $\mathcal{A}$. We may also search for critical classes of E among those that are represented by G-invariant Kähler metrics.

Let (M, J) be any compact complex manifold of Kähler type. We assume it has complex dimension n. Let G be a maximal compact subgroup of the biholomorphism group of (M, J), and g be a Kähler metric on M, representing some Kähler class Ω. Without loss of generality, we assume that g is G-invariant and consider the Hilbert space space $L^2_{k,G}$ of G-invariant real-valued functions of class L^2_k. The space $\mathfrak{H}^{1,1}(M)$ of g-harmonic (1,1)-forms is G-invariant, and if the form $\alpha \in \mathfrak{H}^{1,1}(M)$ is sufficiently small, the Kähler form

$$\tilde{\omega} = \omega + \alpha$$

is G-invariant, and so will be its scalar curvature $\tilde{s}$. Let $\mathfrak{z} \subset \mathfrak{g}$ denote the center of $\mathfrak{g}$, the Lie algebra of G, and let $\mathfrak{z}_0 = \mathfrak{z} \cap \mathfrak{g}_0$, where $\mathfrak{g}_0 \subset \mathfrak{g}$ is the ideal of Killing fields which have zeroes. For any G-invariant Kähler metric $\tilde{g}$ on (M, J), each element of $\mathfrak{z}_0$ is of the form $J\nabla_{\tilde{g}} f$ for a real-valued solution of $(\overline{\partial}\partial^{\#}_{\tilde{g}})^*\overline{\partial}\partial_{\tilde{g}} f = 0$, and $\mathfrak{z}_0$ precisely corresponds to the set of real solutions f which are *invariant under* G.

The restriction of $\ker(\overline{\partial}\partial^{\#}_{\tilde{g}})^*\overline{\partial}\partial_{\tilde{g}}$ to $L^2_{k+4,G}$ depends smoothly on the G-invariant metric $\tilde{g}$. Indeed, choose a basis $\{X_1, \ldots, X_m\}$ for $\mathfrak{z}_0$, and, for each $(1,1)$-form χ on (M, J), consider the set of functions

$$\begin{aligned} p_0(\chi) &= 1 \\ p_j(\chi) &= 2i\mathbb{G}_g\overline{\partial}^*_g((JX_j + iX_j)\lrcorner\chi)\,, \quad j = 1, \ldots, m \end{aligned}$$

where $\mathbb{G}_g$ is the Green's operator of the metric g.

If $\tilde{\omega}$ is the Kähler form of a G-invariant metric $\tilde{g}$, then the $p_j(\tilde{\omega})$'s are real-valued and constitute a basis of $\ker(\overline{\partial}\partial^{\#}_{\tilde{g}})^*\overline{\partial}\partial_{\tilde{g}}$. The map $\alpha \mapsto p_j(\omega + \alpha)$ is, for each j, a bounded linear map $\mathfrak{H}^{1,1}(M) \to L^2_{k+3,G}$. With respect to the background L^2 inner product, let

$$\{f^0_{\tilde{\omega}}, \ldots, f^m_{\tilde{\omega}}\} \tag{3.1}$$

be the orthonormal set extracted from $\{p_j(\tilde{\omega})\}$ by the Gram-Schmidt procedure. The set $\{f^j_{\tilde{\omega}}\}_{j=0}^m$ is a basis of the vector space of real holomorphy potentials. We then let

$$\begin{aligned} \pi_{\tilde{\omega}} : L^2_{k,G} &\to L^2_{k,G} \\ u &\mapsto \sum_{j=0}^{m} \langle f^j_{\tilde{\omega}}, u\rangle_{L^2} f^j_{\tilde{\omega}} \end{aligned} \tag{3.2}$$

denote the associated projector. By the regularity of the functions $\{p_1, \ldots, p_m\}$, this projection can be defined on $L^2_{k+j,G}$ for $j = 0, 1, 2, 3$, and the map $\alpha \mapsto \pi_{\tilde{\omega}} \in \mathrm{End}(L^2_{k+j,G}) \cong \bigotimes^2 L^2_{k+j,G}$ is smooth on a suitable neighborhood of the origin in $\mathfrak{H}^{1,1}(M)$.

This projection can be defined at the level of $(1,1)$-forms. Indeed, let us denote by $\wedge^{1,1}_{k,G}$ the space of real forms of type $(1,1)$, invariant under G and of class L^2_k. Then, given any G-invariant metric $\tilde{g}$, there exists a unique continuous linear map

$$\Pi_{\tilde{\omega}} : \wedge^{1,1}_{k+2,G} \mapsto \wedge^{1,1}_{k+2,G} \, , \tag{3.3}$$

which intertwines the trace and the projection map $\pi_{\tilde{\omega}}$ in (3.2), and such that $\eta - \Pi_{\tilde{\omega}}\eta$ is cohomologous to zero for all $\eta \in \wedge^{1,1}_{k+2,G}$. Once again, where defined, the map $\alpha \mapsto \Pi_{\omega+\alpha}$ from $L^2_{k+4,G}$ to $\mathrm{End}(\wedge^{1,1}_{k+2,G})$ is smooth.

The operators π and Π can also be defined when varying the metric ω while preserving its Kähler class, that is to say, when considering deformations of the form $\tilde{\omega} = \omega + i\partial\overline{\partial}\varphi$, $\varphi \in L^2_{k+4}$, $k > n$. In this case, a metric $\tilde{\omega}$ is extremal if and only if $\tilde{s} = \pi_{\tilde{\omega}}\tilde{s}$, or equivalently, if and only if $\tilde{\rho} = \Pi_{\tilde{\omega}}\tilde{\rho}$. For arbitrary G-invariant metrics one only has

$$\rho = \Pi\rho + i\partial\overline{\partial}\psi \, ,$$

where

$$\psi = -\mathbb{G}_g(s_g - \pi_g s_g) \, ,$$

$\mathbb{G}_g$ the Green's operator of g. The extremality condition implies that ψ is a constant.

Notice that we have the identity

$$\pi s \, \omega^n = 2n \, \Pi\rho \wedge \omega^{n-1} \, , \tag{3.4}$$

and since $\Pi\rho$ represents the same cohomology class as that represented by ρ, we obtain that

$$\int \pi s \, d\mu = \frac{4\pi}{(n-1)!} c_1 \cdot [\omega]^{n-1} \, . \tag{3.5}$$

3.2. The equation for critical classes. Given any Kähler metric ω representing the class Ω, we have that $X_\Omega = \partial^{\#}_g \pi_g s$, where π_g is the L^2-inner product projection onto the space of holomorphy potentials. The dimension of this space is independent of g, and the constants are always part of it. By (2.9), it follows that

$$E(\Omega) = \int (\pi_g s_g)^2 d\mu_g \, .$$

We proceed to compute the variation of this functional along a curve Ω_t represented by a one-parameter family of Kähler metrics ω_t whose tangent vector α_t at ω_t is a ω_t-harmonic (1,1)-form. This curve is assumed to start at ω when $t = 0$. Using the sub-index t to denote the corresponding geometric quantity associated to the metric ω_t, we obtain that

$$\frac{d}{dt} E([\omega_t]) = 2 \int (\pi_t s_t)(\frac{d}{dt}(\pi_t s_t)) d\mu_t + \int (\pi_t s_t)^2 (\omega_t, \alpha_t)_t d\mu_t \, .$$

Let us assume first that $\pi_t s_t$ is a constant (that may depend upon t). Since

$$d\mu_t = (1 + t(\omega, \alpha) + O(t^2)) d\mu = (1 + O(t^2)) d\mu \, ,$$

by (3.5) we would then conclude that

$$\pi_t s_t \mu_\omega(M) = \frac{4\pi}{(n-1)!} c_1 \cdot [\omega_t]^{n-1} + O(t^2)\,,$$

where $\mu_\omega(M)$ is the ω-volume of M. Taking the derivative and setting $t = 0$, we obtain that

$$\int (\pi_t s_t)(\frac{d}{dt}(\pi_t s_t)) d\mu_t \mid_{t=0} = -2\pi s \int (\Pi\rho, \alpha) d\mu = -2 \int \pi s (\Pi\rho, \alpha) d\mu\,,$$

and so

$$\frac{d}{dt} E([\omega_t]) \mid_{t=0} = -4 \int \pi s\, (\Pi\rho, \alpha)\, d\mu + \int (\pi s)^2 (\omega, \alpha)\, d\mu\,, \tag{3.6}$$

where the geometric quantities in the right side are those of the metric $\omega = \omega_0$.

We would like to prove a similar result even when $\pi_t s_t$ is not constant. Observe that

$$\begin{aligned} \frac{d}{dt} E([\omega_t]) &= 2 \int (\pi_t s_t)(\frac{d}{dt}(\pi_t s_t)) d\mu_t + \int (\pi_t s_t)^2 (\omega_t, \alpha_t)_t d\mu_t \\ &= 2 \int (\pi_t s_t)(\pi_t \dot{s}_t + \dot{\pi}_t s_t) d\mu_t + \int (\pi_t s_t)^2 (\omega_t, \alpha_t)_t d\mu \\ &= 2 \int (\pi_t s_t)(\Delta_t (\omega_t, \alpha_t)_t - 2(\rho_t, \alpha_t)_t) d\mu_t + 2 \int (\pi_t s_t)(\dot{\pi}_t s_t) d\mu_t \\ &\qquad + \int (\pi_t s_t)^2 (\omega_t, \alpha_t)_t d\mu_t\,. \end{aligned}$$

Here we have made use of the fact that π is a projection operator, and of a fairly well known formula for the variation of s_t.

Since the form α_t is ω_t harmonic, the inner-product $(\omega_t, \alpha_t)_t$ is a constant (which may depend upon t) and its Laplacian is therefore zero. Thus,

$$\frac{d}{dt} E([\omega_t]) = -4 \int (\pi_t s_t)(\rho_t, \alpha_t)_t d\mu_t + 2 \int (\pi_t s_t)(\dot{\pi}_t s_t) d\mu_t + \int (\pi_t s_t)^2 (\omega_t, \alpha_t)_t d\mu_t\,,$$

and so, when $t = 0$, we obtain

$$\frac{d}{dt} E([\omega_t]) \mid_{t=0} = -4 \int (\pi s)(\rho, \alpha) d\mu + 2 \int (\pi s)(\dot{\pi} s) d\mu + \int (\pi s)^2 (\omega, \alpha) d\mu\,. \tag{3.7}$$

We proceed to compute the term involving $\dot{\pi}$ in further detail. And for this, we make a rather judicious choice of basis for the space of holomorphy potentials. Something similar had been done in [11].

Recall that $\pi_t s_t$ is assumed not to be a constant. Let us take a basis $\{X_1, \ldots, X_m\}$ for $\mathfrak{z}_0$ such that $JX_1 + iX_1 = \partial^{\#}\pi s$, where π and s are the projection operator and scalar curvature of ω. Applying the procedure of §3.1 to the metric $\omega_t = \omega + t\alpha$, we obtain a family of t-dependent potentials $p_j(\omega_t) = 2i\mathbb{G}_g \overline{\partial}_g^*((JX_j + iX_j) \lrcorner \omega_t)$, from which we obtain an ω-orthonormal set of functions $\{f^0_{\omega_t}, \ldots, f^m_{\omega_t}\}$. With this choice, $f^0_{\omega_t} = (\mu_\omega(M))^{-1/2}$, while

$$f^1_{\omega_t} = \frac{p_1(\omega_t)}{\|p_1(\omega_t)\|}\,,$$

where $p_1(\omega_t) = 2i\mathbb{G}_g\overline{\partial}_g^*((JX_1+iX_1)\lrcorner\omega_t)$. Notice that $p_1(\omega_0) = \pi s - s_0$, where s_0 is the ω-average of s, and that $\frac{d}{dt}p_1(\tilde{\omega}_t)\mid_{t=0} = 2i\mathbb{G}_g\overline{\partial}_g^*(\partial^{\#}\pi s\lrcorner\alpha)$. Since $f^j_{\omega_t}$ belongs to the range of π_t and is perpendicular to s for $j>1$, we see that

$$\pi_t s = s_0 + \langle f^1_{\omega_t}, s\rangle f^1_{\omega_t}\,,$$

and, therefore,

$$\frac{d\pi_t}{dt}\mid_{t=0} s = \langle \dot{f}^1_{\omega_t}\mid_{t=0}, s\rangle f^1_{\omega_0} + \langle f^1_{\omega_0}, s\rangle \dot{f}^1_{\omega_t}\,, \tag{3.8}$$

so, by (3.2), we obtain

$$\int (\pi s)\dot{\pi} s\, d\mu = \langle \dot{f}^1_{\tilde{\omega}_t}\mid_{t=0}, s\rangle\langle f^1_{\tilde{\omega}_0}, \pi s\rangle + \langle f^1_{\tilde{\omega}_0}, s\rangle\langle \dot{f}^1_{\tilde{\omega}_t}\mid_{t=0}, \pi s\rangle\,.$$

An elementary calculation shows that this is equal to $\langle \dot{p}_1, s-\pi s+2s_0\rangle$. We then conclude that

$$\int (\pi s)\dot{\pi} s\, d\mu = \int (s-\pi s)2i\mathbb{G}_g\overline{\partial}_g^*(\partial^{\#}\pi s\lrcorner\alpha)d\mu\,. \tag{3.9}$$

If ω were an extremal metric, $s=\pi s$ and the integrand of this expression will be zero identically. As there are Kähler manifolds with classes that cannot be represented by extremal metrics, this is too stringent a condition to assume. Plugging (3.9) into (3.7), we obtain

$$\begin{aligned}\frac{d}{dt}E([\omega+t\alpha])\mid_{t=0} \;&=\; -4\int(\pi s)(\rho,\alpha)d\mu + 4i\int(s-\pi s)\mathbb{G}_g\overline{\partial}^*(\partial^{\#}\pi s\lrcorner\alpha)d\mu\\ &\quad + \int(\pi s)^2(\omega,\alpha)d\mu\,,\end{aligned}$$

for any g harmonic $(1,1)$-form α. Using this property of α, integration by parts, the self-adjointness of $\mathbb{G}_g$, and dualizing $\overline{\partial}^*$, we rewrite the second integral to obtain

$$\begin{aligned}\frac{d}{dt}E([\omega+t\alpha])\mid_{t=0} \;&=\; -4\int(\pi s)(\rho,\alpha)d\mu - 4\int \pi s(i\partial\overline{\partial}\mathbb{G}_g(s-\pi s),\alpha)d\mu\\ &\quad + \int(\pi s)^2(\omega,\alpha)d\mu\,.\end{aligned}$$

But

$$\rho + i\partial\overline{\partial}\mathbb{G}_g(s-\pi s) = \Pi_g\rho\,. \tag{3.10}$$

Therefore,

$$\frac{d}{dt}E([\omega+t\alpha])\mid_{t=0} = -4\int(\pi s)(\Pi\rho,\alpha)d\mu + \int(\pi s)^2(\omega,\alpha)d\mu\,. \tag{3.11}$$

that coincides with the earlier expression (3.6), obtained under the assumption that πs was a constant.

THEOREM 2. *Let Ω be a cohomology class that is represented by a Kähler metric g, assumed to be invariant under the maximal compact subgroup G of the biholomorphism group of (M,J). Then Ω is critical class if and only if*

$$\int_M (\pi_g s_g)(\Pi_g\rho,\alpha)d\mu_g = 0$$

for any trace-free harmonic $(1,1)$*-form* α*. In this expression,* ρ *is the Ricci form of the metric* g*,* π *is the* L^2 *projection* (3.2) *onto the space of holomorphy potentials, and* Π *is its version* (3.3) *at the level of* (1,1)*-forms.*

Proof. By (3.11), if we further assume that α is a trace-free form, we have that

$$\frac{d}{dt}E([\omega + t\alpha]) \mid_{t=0} = -4\int (\pi s)(\Pi\rho,\alpha)d\mu\,,$$

proving the desired critical class equation. □

The following result is now obvious. Its significance makes it worthwhile to state it separately.

Corollary 3. *For the first Chern class to be represented by an Einstein metric, it must be a critical class of* E*.*

Before continuing any further, let us compute the Hodge star of the $(1,1)$-form $\Pi\rho$. We know that

$$*\rho = \frac{s}{2}\frac{\omega^{n-1}}{(n-1)!} - \rho\wedge\frac{\omega^{n-2}}{(n-2)!}\,.$$

On the other hand, since for any primitive $(1,1)$-form γ we have that [14]

$$*\gamma = -\gamma\wedge\frac{\omega^{n-2}}{(n-2)!}\,,$$

then we find that

$$*i\partial\overline{\partial}f = -\frac{\Delta f}{2}\frac{\omega^{n-1}}{(n-1)!} - i\partial\overline{\partial}f\wedge\frac{\omega^{n-2}}{(n-2)!}\,.$$

Therefore, by (3.10), we obtain that

$$*\Pi\rho = *\rho + *i\partial\overline{\partial}G_g(s-\pi s) = \frac{\pi s}{2}\frac{\omega^{n-1}}{(n-1)!} - \Pi\rho\wedge\frac{\omega^{n-2}}{(n-2)!}\,.$$

The expression above may be used to re-write the first variational formula (3.11) in a more convenient manner. We get

$$\frac{d}{dt}E([\omega + t\alpha]) \mid_{t=0} = -\int (\pi s)^2\frac{\omega^{n-1}}{(n-1)!}\wedge\alpha + 4\int(\pi s)\Pi\rho\wedge\frac{\omega^{n-2}}{(n-2)!}\wedge\alpha\,.$$

We now use this result to calculate the Hessian of the potential energy E at a critical class Ω. We do so by representing the class by a Kähler metric ω, and considering deformations of the class of the form $[\omega_t] = [\omega + t\beta]$ where β is a trace-free ω-harmonic (1,1)-form. We insert this curve into the first variational formula, and compute the t derivative of the result at $t=0$. Since $(\omega,\alpha)=0$, we obtain:

$$\begin{aligned}
&D^2E_{[\omega]}([\alpha],[\beta])\\
&= \frac{d}{dt}\left(4\int(\pi_t s_t)\Pi_t\rho_t\wedge\frac{\omega_t^{n-2}}{(n-2)!}\wedge\alpha - \int(\pi_t s_t)^2\frac{\omega_t^{n-1}}{(n-1)!}\wedge\alpha\right)_{t=0}\\
&= -4\int\frac{d(\pi_t s_t)}{dt}(\Pi\rho,\alpha)d\mu + 4\int(\pi s)\frac{d(\Pi_t\rho_t)}{dt}\wedge\frac{\omega^{n-2}}{(n-2)!}\wedge\alpha\\
&\qquad +4\int(\pi s)\,\Pi\rho\wedge\frac{\omega^{n-3}}{(n-3)!}\wedge\beta\wedge\alpha - \int(\pi s)^2\wedge\frac{\omega^{n-2}}{(n-2)!}\wedge\beta\wedge\alpha\,.
\end{aligned}$$

By (3.10) we may obtain the Taylor series expansion of $\Pi_t \rho_t$. Indeed, we know that $\rho_t = \rho + O(t^2)$ and, since $(1-\pi)^2 = 1-\pi$, we have that

$$\Pi_t \rho_t = \Pi\rho + ti\partial\overline{\partial}\left(\dot{\mathbb{G}}(1-\pi)s - 2\mathbb{G}(1-\pi)\dot{\pi}s + \mathbb{G}(1-\pi)\dot{s}\right) + O(t^2)\,.$$

By (3.8) we have

$$\dot{\pi}s = \langle \dot{f}^1, s\rangle f^1 + \langle f^1, s\rangle \dot{f}^1\,,$$

where

$$\frac{df^1}{dt} = \frac{\dot{p}_1}{\|p_1\|} - \frac{p_1}{\|p_1\|^3}\int p_1\dot{p}_1\,.$$

This implies that

$$\dot{\pi}s = \dot{p}_1 + \frac{p_1}{\|p_1\|^2}\int \dot{p}_1(s - 2\pi s + 2s_0)d\mu\,,$$

and, therefore, $(1-\pi)\dot{\pi}s = 0$ because both p_1 and $\dot{p}_1$ are holomorphy potentials. Since β has zero trace, we also have that $\dot{\mathbb{G}} = -\mathbb{G}\dot{\Delta}\mathbb{G}$. Therefore, after a small simplification, we obtain that

$$\Pi_t\rho_t = \Pi\rho + ti\partial\overline{\partial}\mathbb{G}\left(-2(\Pi\rho,\beta) + 2\pi(\rho,\beta)\right) + O(t^2)\,. \tag{3.12}$$

By (3.4), we see that

$$\frac{d(\pi_t s_t)}{dt} = 2\left(\frac{d(\Pi_t\rho_t)}{dt}, \omega\right) - 2(\Pi\rho,\beta)\,.$$

Hence, using (3.12), we obtain

$$\frac{d(\pi_t s_t)}{dt} = -2\pi(\rho,\beta)\,,$$

and that allows us to work out the first integral in the right side of the expression for $D^2E_{[\omega]}([\alpha],[\beta])$. Indeed, by computing the inner product with holomorphy potentials, we can easily show that $\pi(\rho,\beta) = \pi(\Pi\rho,\beta)$, and, therefore, the integral in question is nothing but $8\langle \pi(\Pi\rho,\beta), \pi(\Pi\rho,\alpha)\rangle$.

In order to study the second integral in the expression defining $D^2E_{[\omega]}([\alpha],[\beta])$, we use (3.12) to see that

$$\begin{aligned}\frac{d(\Pi_t\rho_t)}{dt} &= i\partial\overline{\partial}\mathbb{G}\left(-2(\Pi\rho,\beta) + 2\pi(\rho,\beta)\right)\\ &= i\partial\overline{\partial}\mathbb{G}\left(-2(1-\pi)(\rho,\beta) - 2(\beta, i\partial\overline{\partial}\mathbb{G}(1-\pi)s)\right)\,.\end{aligned}$$

Since $\mathbb{G}\bar{\partial}^*\partial^*(\pi s\alpha)$ is a holomorphy potential, we have that

$$4\int(\pi s)\frac{d(\Pi_t\rho_t)}{dt}\wedge\frac{\omega^{n-2}}{(n-2)!}\wedge\alpha = 8\int(\beta, i\partial\overline{\partial}\mathbb{G}(1-\pi)s)\mathbb{G}\bar{\partial}^*\partial^*(\pi s\alpha)d\mu\,,$$

and this is zero because $\mathbb{G}\bar{\partial}^*\partial^*(\beta\mathbb{G}\bar{\partial}^*\partial^*(\pi s\alpha))$ is also a holomorphy potential.

THEOREM 4. *Let Ω be a critical cohomology class of the energy functional E, represented by a Kähler metric g that is invariant under a maximal compact subgroup*

G of the biholomorphism group of (M,J). Then the Hessian of E at Ω in the direction of trace-free harmonic (1,1)-forms α and β, is given by

$$\begin{aligned} D^2E_{[\omega]}([\alpha],[\beta]) &= 8\int \pi(\Pi\rho,\beta)\pi(\Pi\rho,\alpha)d\mu + \frac{4-n}{n}\int (\pi s)^2(\alpha,\beta)d\mu \\ &\quad -4(n-2)(n-3)!\int (\pi s)\,((\Pi\rho)_0,\Lambda(\alpha\wedge\beta))d\mu\,. \end{aligned}$$

Here π is the L^2 projection (3.2) onto the space of holomorphy potentials, Π is its version (3.3) at the level of (1,1)-forms, ρ is the Ricci form of the metric, $(\Pi\rho)_0$ is the trace-free component of $\Pi\rho$, and $\Lambda(\alpha\wedge\beta)$ is the contraction of $\alpha\wedge\beta$ by the Kähler form.

Proof. We have seen that

$$\begin{aligned} D^2E_{[\omega]}([\alpha],[\beta]) &= 8\int \pi(\Pi\rho,\beta)\pi(\Pi\rho,\alpha)d\mu + 4\int (\pi s)\,\Pi\rho\wedge\frac{\omega^{n-3}}{(n-3)!}\wedge\beta\wedge\alpha \\ &\quad -\int (\pi s)^2\wedge\frac{\omega^{n-2}}{(n-2)!}\wedge\beta\wedge\alpha\,, \end{aligned}$$

and so the result follows if we show that

$$\begin{aligned} \int (\pi s)\,\Pi\rho\wedge\frac{\omega^{n-3}}{(n-3)!}\wedge\beta\wedge\alpha &= -\frac{n-2}{2n}\int (\pi s)^2(\alpha,\beta)d\mu \\ &\quad -(n-2)(n-3)!\int (\pi s)\,((\Pi\rho)_0,\Lambda(\alpha\wedge\beta))d\mu\,. \end{aligned}$$

We may easily show that

$$*\left(\alpha\wedge\beta\wedge\frac{\omega^{n-3}}{(n-3)!}\right) = -\frac{n-2}{n}(\alpha,\beta)\omega+\gamma\,,$$

where γ is a trace-free (1,1)-form. Therefore,

$$\alpha\wedge\beta\wedge\frac{\omega^{n-3}}{(n-3)!} = -\frac{n-2}{n}(\alpha,\beta)\frac{\omega^{n-1}}{(n-1)!}-\gamma\frac{\omega^{n-2}}{(n-2)!}\,.$$

Let L be the operator dual to Λ. We know that $[\Lambda,L^r]=\sum_q^n r(n-q-r+1)L^{r-1}P_q$, P_q the projection onto the q-component of a covector. Repeated application of this identity in the expression above leads to the conclusion that

$$\gamma = -(n-2)(n-3)!\Lambda(\alpha\wedge\beta)+c\omega\,,$$

for some scalar c. Consequently,

$$\begin{aligned} \int (\pi s)\,\Pi\rho\wedge\frac{\omega^{n-3}}{(n-3)!}\wedge\beta\wedge\alpha &= \int (\pi s)\left(\Pi\rho,*\left(\alpha\wedge\beta\wedge\frac{\omega^{n-3}}{(n-3)!}\right)\right)d\mu \\ &= -\frac{n-2}{2n}\int (\pi s)^2(\alpha,\beta)d\mu+\int (\pi s)((\Pi\rho)_0,\gamma)d\mu\,, \end{aligned}$$

and the desired result follows from the expression above for γ because its ω-component does not contribute to the last integral on the right. ∎

4. Existence and Uniqueness of Critical Classes. Let us observe that for any Kähler metric g, the holomorphic vector field $X = \partial_g^{\#}\pi_g s$ only depends upon the cohomology class of the metric and not upon its particular representative. Since the projection onto the constants of πs and s coincide, if $\tilde{g}$ is any other metric representing the same class as that represented by g, then $X = \partial_{\tilde{g}}^{\#}\pi_{\tilde{g}} s_{\tilde{g}}$ and, by the invariance of the Futaki character (2.7), we then have that

$$\begin{aligned}
\mathfrak{F}(X, [\tilde{\omega}]) = -\|\pi_{\tilde{g}} s_{\tilde{g}} - s_0\|^2 \quad &= \quad -\int_M \pi_{\tilde{g}} s_{\tilde{g}}(s_{\tilde{g}} - s_0)\, d\mu_{\tilde{g}} \\
&= \quad -\int_M \pi_g s_g (s_g - s_0)\, d\mu_g \\
&= \quad -\int_M \pi_g s_g (\pi_g s_g - s_0)\, d\mu_g \\
&= \quad -\|\pi_g s_g - s_0\|^2 = \mathfrak{F}(X, [\omega])\,.
\end{aligned}$$

Therefore, if for some Kähler metric g we have $\pi_g s_g$ a constant, the same will be true of any other metric $\tilde{g}$ representing the same cohomology class as that of g, and $\pi_{\tilde{g}} s_{\tilde{g}} = \pi_g s_g$. This observation leads to some special properties exhibited by critical classes of E.

PROPOSITION 5. *Let Ω be a critical class for E, and assume that g is a Kähler representative such that πs is a non-zero constant. Then, this also holds for any Kähler representative of Ω, and furthermore,*

$$\Pi\rho = \frac{\pi s}{2n}\omega$$

so ω is in the canonical class, which therefore must have a sign. If πs is the constant zero, the same result is true if we assume that the canonical class is zero.

Proof. By the critical condition of Theorem 2, we must have that $\langle \rho, \alpha \rangle = 0$ for any g-trace-free harmonic (1,1)-form α. The form $\Pi\rho$ is harmonic, and its Lefschetz decomposition will be

$$\Pi\rho = \frac{\pi s}{2n}\omega + \rho_0\,,$$

with ρ_0 a harmonic trace-free form. Hence, $\langle \Pi\rho, \rho_0 \rangle = \langle \rho_0, \rho_0 \rangle = 0$, proving that $\rho_0 = 0$. Consequently,

$$\Pi\rho = \frac{\pi s}{2n}\omega\,,$$

and since the cohomology class of $\Pi\rho$ is that of c_1, the results follows for g.

If $\tilde{g}$ is another metric representing the same cohomology class as that represented by g, the desired result for $\Pi_{\tilde{g}}\rho_{\tilde{g}}$ follows from the remark preceding the statement of the Proposition.

Finally, if $\pi_g s_g = 0$, since $\Pi\rho$ represents the first Chern class, the assumption implies that $\Pi\rho = 0$. This completes the proof. ☐

Generically, a complex manifold carries no holomorphic vector field other than the trivial one. Therefore, if such a manifold is of Kähler type, any representative g of a critical class of the energy functional E will necessarily have $\pi_g s_g$ constant. If this constant is not zero, the proposition above implies that the critical class is the first Chern class, which therefore must have a sign, either positive or negative depending upon the sign of $\pi_g s_g$. This imposes a topological condition on the complex manifold (M, J).

Corollary 6. *Let (M, J) be a complex manifold of Kähler type without non-trivial holomorphic vector fields. Assume that the first Chern class $c_1 = c_1(M, J)$ does not have a sign. Then, the only critical classes of the energy functional E, if any, are those represented by metrics whose Ricci tensors have harmonic components that are trace-free.*

Theorem 7. *Let (M, J) be a compact manifold of Kähler type. Suppose that $c_1 = c_1(M, J) < 0$. Then the energy functional E has only one critical class, and that class is the first Chern class c_1.*

Proof. When $c_1 < 0$, the manifold does not carry holomorphic vector fields other than the trivial one. Therefore, for any Kähler metric g, we must have πs a constant, and by (3.5), this constant is not zero. The cohomology class of an Einstein metric is a critical class of E, and by Aubin-Yau result [1, 15], the canonical class of (M, J) can be represented by a unique Einstein metric. So E has critical points. The uniqueness follows from Proposition 5. □

Theorem 8. *Let (M, J) be any del Pezzo surface. Then the energy functional E admits one an only one critical class, that coincides with the canonical class except for the cases $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$.*

Proof. By the Hessian formula for E, all critical classes are local minima. If there were more than one critical class, there would have to be a critical class that is either degenerate or a local maximum, contradicting this fact.

Now, by Tian's work [13], each del Pezzo surface of the form $\mathbb{CP}^2\#k\overline{\mathbb{CP}}^2$, $3 \leq k \leq 8$, admits an Einstein metric. Hence, in those cases E has the canonical class as its only critical point. On the other hand, both $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ are Einstein. So the statement is also true for these two del Pezzo surfaces as well.

On $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$ every Kähler class is extremal. Using this fact, we have verified earlier [6] by an explicit calculation that the energy functional E admits a unique critical class.

For $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$, LeBrun [8] has proven that E admits a critical class. □

This result provides the first proof of the uniqueness of the critical class on $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$ found by LeBrun [8]. Numerical evidence for that was given in [9], and the result proven in [12] under the strong assumption that all critical classes were extremal. The fact that this property holds remains an open problem, and we strongly believe that to be the case for this particular manifold.

Notice that the one point and two points blow up of $\mathbb{CP}^2$ are exceptional in that the critical class is not c_1. In the first case, we have proven [6] directly that the extremal metric representing the critical class is conformally equivalent to the Page metric. As for the LeBrun's class on the two points blow up, if it were represented by an extremal metric, such a metric will necessarily have positive scalar curvature. The main theorem in [12] —on complex surfaces, a critical point of (1.2) is, away from the locus of its scalar curvature, conformally equivalent to an Einstein metric— would then imply that it can be conformally deformed to an Einstein metric of constant positive curvature. We conjecture this to be the case, fact that will follow if we merely verify that the critical class of the functional E in $\mathbb{CP}^2\#2\overline{\mathbb{CP}}^2$ is extremal.

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