

CLASSIFICATION OF INDEFINITE HYPER-KÄHLER SYMMETRIC SPACES*

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Abstract. We classify indefinite simply connected hyper-Kähler symmetric spaces. Any such space without flat factor has commutative holonomy group and signature $(4m, 4m)$. We establish a natural 1-1 correspondence between simply connected hyper-Kähler symmetric spaces of dimension $8m$ and orbits of the group $GL(m, \mathbb{H})$ on the space $(S^4 \mathbb{C}^n)^\tau$ of homogeneous quartic polynomials S in $n = 2m$ complex variables satisfying the reality condition $S = \tau S$, where τ is the real structure induced by the quaternionic structure of $\mathbb{C}^{2m} = \mathbb{H}^m$. We define and classify also complex hyper-Kähler symmetric spaces. Such spaces without flat factor exist in any (complex) dimension divisible by 4.

1. Introduction. We recall that a pseudo-Riemannian manifold (M, g) is called a symmetric space if any point $x \in M$ is an isolated fixed point of an involutive isometry s_x (called central symmetry with centre x). Since the product of two central symmetries s_x and s_y with sufficiently close centres is a shift along the geodesic (xy) , the group generated by central symmetries acts transitively on M and one can identify M with the quotient $M = G/K$, where G is the connected component of the isometry group $\text{Isom}(M, g)$ and K is the stabilizer of a point $o \in M$.

A symmetric space $(M = G/K, g)$ is called Kähler (respectively, hyper-Kähler) if its holonomy group $\text{Hol}(M, g)$ is a subgroup of the pseudo-unitary group $U(p, q)$ (respectively, of the pseudo-symplectic group $\text{Sp}(p, q) \subset \text{SU}(2p, 2q)$). Any hyper-Kähler symmetric space is in particular a homogeneous hypercomplex manifold. Homogeneous hypercomplex manifolds of compact Lie groups were constructed by Ph. Spindel, A. Sevrin, W. Troost, A. Van Proeyen [SSTVP] and by D. Joyce [J] and homogeneous hypercomplex structures on solvable Lie groups by M.L. Barberis and I. Dotti-Miatello [BD].

The classification of simply connected symmetric spaces reduces to the classification of involutive automorphisms σ of a Lie algebra \mathfrak{g} , such that the adjoint representation $\text{ad}_{\mathfrak{g}}|_{\mathfrak{m}}$ preserves a pseudo-Euclidean scalar product g , where

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{m}, \quad \sigma|_{\mathfrak{k}} = 1, \quad \sigma|_{\mathfrak{m}} = -1,$$

is the eigenspace decomposition of the involution σ . Note that the eigenspace decomposition of an involutive automorphism is characterized by the conditions

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}.$$

Such a decomposition is called a symmetric decomposition.

In fact, for any pseudo-Riemannian symmetric space $M = G/K$ the conjugation with respect to the central symmetry s_o with centre $o = eK$ is an involutive automorphism of the Lie group G , which induces an involutive automorphism σ of its Lie algebra \mathfrak{g} . The pseudo-Riemannian metric of M induces a \mathfrak{k} -invariant scalar product on $\mathfrak{m} \cong T_o M$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ is the symmetric decomposition defined by

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σ . Conversely, a symmetric decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ together with a \mathfrak{k} -invariant scalar product on \mathfrak{m} determines a pseudo-Riemannian symmetric space $M = G/K$, where G is the simply connected Lie group with the Lie algebra \mathfrak{g} , K is the connected (and closed) subgroup of G generated by \mathfrak{k} , the pseudo-Riemannian metric on M is defined by g and the central symmetry is defined by the involutive automorphism σ associated to the symmetric decomposition.

Naturally identifying the space \mathfrak{m} with the tangent space T_oM , the isotropy group is identified with $\text{Ad}_K|_{\mathfrak{m}}$ and the holonomy algebra is identified with $\text{ad}_{\mathfrak{h}}$, where $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}]$. If one assumes that the holonomy algebra is irreducible then one can prove that the Lie algebra \mathfrak{g} is semisimple. Hence the classification of pseudo-Riemannian symmetric spaces with irreducible holonomy reduces to the classification of involutive automorphisms of semisimple Lie algebras. Such a classification was obtained by M. Berger [B1, B2] and A. Fedenko [F]. It includes the classification of Riemannian symmetric spaces (obtained earlier by E. Cartan), since according to de Rham’s theorem any simply connected complete Riemannian manifold is a direct product of Riemannian manifolds with irreducible holonomy algebra and a Euclidean space.

A classification of pseudo-Riemannian symmetric spaces with non completely reducible holonomy is known only for signature $(1, n)$ (Cahen-Wallach [CW]) and for signature $(2, n)$ under the assumption that the holonomy group is solvable (Cahen-Parker [CP]). The classification problem for arbitrary signature looks very complicated and includes, for example, the classification of Lie algebras which admit a nondegenerate ad-invariant symmetric bilinear form. An inductive construction of solvable Lie algebras with such a form was given by V. Kac [K], see also [MR1], [Bo] and [MR2].

In this paper we give a classification of pseudo-Riemannian hyper-Kähler symmetric spaces. In particular, we prove that any simply connected hyper-Kähler symmetric space M has signature $(4m, 4m)$ and its holonomy group is commutative. The main result is the following, see Theorem 6.

Let (E, ω, j) be a complex symplectic vector space of dimension $4m$ with a quaternionic structure j such that $\omega(jx, jy) = \overline{\omega(x, y)}$ for all $x, y \in E$ and $E = E_+ \oplus E_-$ a j -invariant Lagrangian decomposition. Such a decomposition exists if and only if the Hermitian form $\gamma = \omega(\cdot, j\cdot)$ has real signature $(4m, 4m)$. We denote by τ the real structure in $S^{2r}E$ defined by $\tau(e_1 e_2 \dots e_{2r}) := j(e_1)j(e_2) \dots j(e_{2r})$, $e_i \in E$. Then any element $S \in (S^4 E_+)^{\tau}$ defines a hyper-Kähler symmetric space M_S which is associated with the symmetric decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} ,$$

where $\mathfrak{m} = (\mathbb{C}^2 \otimes E)^{\rho}$ is the fixed point set of the real structure ρ on $\mathbb{C}^2 \otimes E$ given by $\rho(h \otimes e) = j_H h \otimes j e$, where j_H is the standard quaternionic structure on $\mathbb{C}^2 = \mathbb{H}$, $\mathfrak{h} = \text{span}\{S_{e, e'} | e, e' \in E\}^{\tau} \subset \text{sp}(E)^{\tau} \cong \text{sp}(m, m)$ with the natural action on $\mathfrak{m} \subset \mathbb{C}^2 \otimes E$ and the Lie bracket $\mathfrak{m} \wedge \mathfrak{m} \rightarrow \mathfrak{h}$ is given by

$$[h \otimes e, h' \otimes e'] = \omega_H(h, h')S_{e, e'} ,$$

where ω_H is the standard complex symplectic form on \mathbb{C}^2 .

Moreover, we establish a natural 1-1 correspondence between simply connected hyper-Kähler symmetric spaces (up to isomorphism) and orbits of the group $GL(m, \mathbb{H})$ in $(S^4 E_+)^{\tau}$.

We define also the notion of complex hyper-Kähler symmetric space as a complex manifold (M, g) of complex dimension $4n$ with holomorphic metric g such that for any

point $x \in M$ there is a holomorphic central symmetry s_x with centre x and which has holonomy group $\text{Hol}(M, g) \subset \text{Sp}(n, \mathbb{C})$ ($\text{Sp}(n, \mathbb{C}) \hookrightarrow \text{Sp}(n, \mathbb{C}) \times \text{Sp}(n, \mathbb{C}) \subset \text{O}(4n, \mathbb{C})$ is diagonally embedded) and give a classification of such spaces. We establish a natural 1-1 correspondence between simply connected complex hyper-Kähler symmetric spaces and homogeneous polynomials of degree 4 in the vector space \mathbb{C}^n considered up to linear transformations from $GL(n, \mathbb{C})$.

2. Symmetric Spaces.

2.1. Basic facts about pseudo-Riemannian symmetric spaces. A **pseudo-Riemannian symmetric space** is a pseudo-Riemannian manifold (M, g) such that any point is an isolated fixed point of an isometric involution. Such a pseudo-Riemannian manifold admits a transitive Lie group of isometries L and can be identified with L/L_o , where L_o is the stabilizer of a point o . More precisely, any simply connected pseudo-Riemannian symmetric space $M = G/K$ is associated with a symmetric decomposition

$$(2.1) \quad \mathfrak{g} = \mathfrak{k} + \mathfrak{m}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$$

of the Lie algebra $\mathfrak{g} = \text{Lie } G$ together with an Ad_K -invariant pseudo-Euclidean scalar product on \mathfrak{m} . We will assume that G acts almost effectively on M , i.e. \mathfrak{k} does not contain any nontrivial ideal of \mathfrak{g} , that M and G are simply connected and that K is connected. Then, under the natural identification of the tangent space T_oM at the canonical base point $o = eK$ with \mathfrak{m} , the holonomy group $\text{Hol} \subset \text{Ad}_K|_{\mathfrak{m}}$. We will denote by \mathfrak{h} the holonomy Lie algebra. Since the isotropy representation is faithful it is identified with the subalgebra $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}] := \text{span}\{[x, y] \mid x, y \in \mathfrak{m}\} \subset \mathfrak{k}$. Recall that the curvature tensor R of a symmetric space M at o is \mathfrak{h} -invariant and determines the Lie bracket in the ideal $\mathfrak{h} + \mathfrak{m} \subset \mathfrak{g}$ as follows:

$$\mathfrak{h} = R(\mathfrak{m}, \mathfrak{m}) := \text{span}\{R(x, y) \mid x, y \in \mathfrak{m}\} \quad \text{and} \quad [x, y] = -R(x, y), \quad x, y \in \mathfrak{m}.$$

The following result is well known:

PROPOSITION 1. *The full Lie algebra of Killing fields of a symmetric space has the form*

$$\text{isom}(M) = \tilde{\mathfrak{h}} + \mathfrak{m},$$

where the full isotropy subalgebra is given by

$$(2.2) \quad \tilde{\mathfrak{h}} = \text{aut}(R) = \{A \in \text{so}(\mathfrak{m}) \mid A \cdot R = [A, R(\cdot, \cdot)] - R(A \cdot, \cdot) - R(\cdot, A \cdot) = 0\}.$$

2.2. Symmetric spaces of semisimple Lie groups. We will prove that in the case when $(M = G/K, g)$ is a pseudo-Riemannian symmetric space of a (connected) semisimple Lie group G then G is the maximal connected Lie group of isometries of M .

PROPOSITION 2. *Let $(M = G/K, g)$ be a pseudo-Riemannian symmetric space associated with a symmetric decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$. If G is semisimple and almost effective then*

- (i) *the restriction of the Cartan-Killing form B of \mathfrak{g} to \mathfrak{k} is nondegenerate and hence \mathfrak{k} is a reductive subalgebra of \mathfrak{g} and $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ is a B -orthogonal decomposition,*

(ii) $\mathfrak{k} = [\mathfrak{m}, \mathfrak{m}]$ and

(iii) $\mathfrak{g} = \text{isom}(M, g)$ is the Lie algebra of the full isometry group of M .

Proof: For (i) see [O-V] Ch. 3 Proposition 3.6.

(ii) It is clear that $\tilde{\mathfrak{g}} = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$ is an ideal of \mathfrak{g} . The B -orthogonal complement $\mathfrak{a} := \tilde{\mathfrak{g}}^\perp \subset \mathfrak{k}$ is a complementary ideal of \mathfrak{g} . Since $[\mathfrak{a}, \mathfrak{m}] = 0$ the Lie algebra \mathfrak{a} acts trivially on M . From the effectivity of \mathfrak{g} we conclude that $\mathfrak{a} = 0$.

(iii) By Proposition 1, $\tilde{\mathfrak{g}} = \text{isom}(M, g) = \tilde{\mathfrak{h}} + \mathfrak{m}$, where $\tilde{\mathfrak{h}} = \text{aut}(R) = \{A \in \text{so}(\mathfrak{m}) \mid A \cdot R = 0\}$. Now $\tilde{\mathfrak{h}}$ preserves \mathfrak{m} and by the identity $A \cdot R = [A, R(\cdot, \cdot)] - R(A \cdot, \cdot) - R(\cdot, A \cdot)$ it also normalizes \mathfrak{k} . This shows that $\tilde{\mathfrak{h}}$ normalizes \mathfrak{g} and hence $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ is an ideal. Since \mathfrak{g} is semisimple there exists a \mathfrak{g} -invariant complement \mathfrak{b} in $\tilde{\mathfrak{g}}$. Note that $[\mathfrak{g}, \mathfrak{b}] \subset \mathfrak{g} \cap \mathfrak{b} = 0$. We can decompose any $X \in \mathfrak{b}$ as $X = Y + Z$, where $Y \in \tilde{\mathfrak{h}}$ and $Z \in \mathfrak{m}$. From $[\mathfrak{g}, \mathfrak{b}] = 0$ it follows that $[\mathfrak{g}, Y] = [\mathfrak{g}, Z] = 0$ and in particular $[\mathfrak{m}, Y] = 0$. This implies that $Y = 0$ and $X = Z \in \mathfrak{b} \cap \mathfrak{m} = 0$. This shows that $\mathfrak{b} = 0$ proving (iii). \square

We recall that a pseudo-Riemannian Hermitian symmetric space is pseudo-Riemannian symmetric space $(M = G/K, g)$ together with an invariant (and hence parallel) g -orthogonal complex structure J .

PROPOSITION 3. *Let $(M = G/K, g, J)$ be a pseudo-Riemannian Hermitian symmetric space of a semisimple and almost effective Lie group G . Then the Ricci curvature of M is not zero.*

Proof: From Proposition 2 it follows that $\mathfrak{g} = \text{isom}(M, g) = \tilde{\mathfrak{h}} + \mathfrak{m}$, where $\tilde{\mathfrak{h}} = \mathfrak{k} = [\mathfrak{m}, \mathfrak{m}]$. It is well known that the curvature tensor R of any pseudo-Kähler manifold (and in particular of any pseudo-Riemannian Hermitian symmetric space) is invariant under the operator J . This shows that $J \in \tilde{\mathfrak{h}} = \text{aut}(R) = [\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$ (holonomy Lie algebra), which implies that the holonomy Lie algebra is not a subalgebra of $\text{su}(\mathfrak{m}) \cong \text{su}(p, q)$. Hence M is not Ricci-flat. In fact, we can write $J = \sum \text{ad}[X_i, Y_i]$, for $X_i, Y_i \in \mathfrak{m}$. Then using the formulas $-2\text{Ric}(X, JY) = \text{tr}JR(X, Y)$ for the Ricci curvature of a pseudo-Kähler manifold and $R(X, Y) = -\text{ad}_{[X, Y]}\mathfrak{m}$ for the curvature of a symmetric space we calculate:

$$-2 \sum \text{Ric}(X_i, JY_i) = \sum \text{tr}JR(X_i, Y_i) = - \sum \text{tr}J\text{ad}[X_i, Y_i] = -\text{tr}J^2 \neq 0.$$

\square

3. Structure of Hyper-Kähler Symmetric Spaces.

3.1. Definitions. A (possibly indefinite) **hyper-Kähler manifold** is a pseudo-Riemannian manifold (M^{4n}, g) of signature $(4k, 4l)$ together with a **compatible hypercomplex structure**, i.e. three g -orthogonal parallel complex structures $(J_1, J_2, J_3 = J_1 J_2)$. This means that the holonomy group $\text{Hol} \subset \text{Sp}(k, l)$. Two hyper-Kähler manifolds (M, g, J_α) ($\alpha = 1, 2, 3$) and (M', g', J'_α) are called **isomorphic** if there exists a triholomorphic isometry $\varphi : M \rightarrow M'$, i.e. $\varphi^* J'_\alpha = J_\alpha$ and $\varphi^* g' = g$.

A **hyper-Kähler symmetric space** is a pseudo-Riemannian symmetric space $(M = G/K, g)$ together with an invariant compatible hypercomplex structure. Consider now a simply connected hyper-Kähler symmetric space $(M = G/K, g, J_\alpha)$. Without restriction of generality we will assume that G acts almost effectively. M being hyper-Kähler is equivalent to $\text{Ad}_K \mathfrak{m} \subset \text{Sp}(k, l)$, or, since K is connected, to $\text{ad}_{\mathfrak{k}} \mathfrak{m} \subset \text{sp}(k, l)$. This condition means that \mathfrak{k} commutes with the Lie algebra $Q = \text{sp}(1) \subset \text{so}(\mathfrak{m}) = \text{so}(4k, 4l)$ spanned by three anticommuting complex structures J_1, J_2, J_3 .

3.2. Existence of a transitive solvable group of isometries and solvability of the holonomy. In this subsection we prove that any simply connected hyper-Kähler symmetric space (M, g, J_α) admits a transitive solvable Lie group $G \subset \text{Aut}(g, J_\alpha)$ of automorphisms and has solvable holonomy group.

PROPOSITION 4. *Let $(M = G/K, g, J_\alpha)$ be a simply connected hyper-Kähler symmetric space and $A = \text{Aut}_0(g, Q) \supset \text{Aut}_0(g, J_\alpha) \supset G$ the connected group of isometries which preserve the quaternionic structure $Q = \text{span}\{J_\alpha\}$. Then*

- (i) *the stabilizer A_o of a point $o \in M$ contains a maximal semisimple subgroup of A ,*
- (ii) *the radical R of A acts transitively and triholomorphically on M and*
- (iii) *the holonomy group of M is solvable.*

Proof: We consider the quaternionic Kähler symmetric space $(M = A/A_o, g, Q)$. The Lie algebra \mathfrak{a}_o of the stabilizer is given by

$$\mathfrak{a}_o = \text{aut}(R, Q) = \{A \in \text{so}(T_oM) \mid A \cdot R = 0, [A, Q] \subset Q\}.$$

Since the curvature tensor of a quaternionic Kähler manifold is invariant under the quaternionic structure Q we conclude that $Q \subset \mathfrak{a}_o$ and $\mathfrak{a}_o = Q \oplus Z_{\mathfrak{a}}(Q)$, where $Z_{\mathfrak{a}}(Q)$ denotes the centralizer of Q in \mathfrak{a} . Since $Q \cong \text{sp}(1)$ is simple, we may choose a Levi-Malcev decomposition $\mathfrak{a} = \mathfrak{s} + \mathfrak{r}$ such that the Levi subalgebra $\mathfrak{s} \supset Q$. We put $\mathfrak{m}_r := [Q, \mathfrak{r}]$ and denote by \mathfrak{m}_s a $Q \oplus Z_{\mathfrak{s}}(Q)$ -invariant complement of Q in $[Q, \mathfrak{s}]$. The stabilizer has the decomposition $\mathfrak{a}_o = Q \oplus (Z_{\mathfrak{s}}(Q) + Z_{\mathfrak{r}}(Q))$.

LEMMA 1. *The complement $\mathfrak{m} = \mathfrak{m}_s + \mathfrak{m}_r$ to \mathfrak{a}_o in \mathfrak{a} is \mathfrak{a}_o -invariant and the decomposition*

$$\mathfrak{a} = \mathfrak{a}_o + \mathfrak{m}$$

is a symmetric decomposition.

Proof: It is clear that \mathfrak{m}_r is \mathfrak{a}_o -invariant and \mathfrak{m}_s is invariant under $Q \oplus Z_{\mathfrak{s}}(Q)$ by construction. It remains to check that $[Z_{\mathfrak{r}}(Q), \mathfrak{m}_s] \subset \mathfrak{m}$. Since $\mathfrak{m}_s = [Q, \mathfrak{m}_s]$, we have

$$[Z_{\mathfrak{r}}(Q), \mathfrak{m}_s] = [Z_{\mathfrak{r}}(Q), [Q, \mathfrak{m}_s]] = [Q, [Z_{\mathfrak{r}}(Q), \mathfrak{m}_s]] \subset [Q, \mathfrak{r}] = \mathfrak{m}_r \subset \mathfrak{m}.$$

This shows that $\mathfrak{a} = \mathfrak{a}_o + \mathfrak{m}$ is an \mathfrak{a}_o -invariant decomposition. We denote by $\mathfrak{a} = \mathfrak{a}_o + \mathfrak{p}$ a symmetric decomposition. Any other \mathfrak{a}_o -invariant decomposition is of the form $\mathfrak{a} = \mathfrak{a}_o + \mathfrak{p}_\varphi$, where $\varphi : \mathfrak{p} \rightarrow \mathfrak{a}_o$ is an \mathfrak{a}_o -equivariant map and $\mathfrak{p}_\varphi = \{X + \varphi(X) \mid X \in \mathfrak{p}\}$. If such non-zero equivariant map φ exists then \mathfrak{p} and \mathfrak{a}_o contain non-trivial isomorphic Q -submodules. Since \mathfrak{p} is a sum of 4-dimensional irreducible Q -modules and \mathfrak{a}_o is the sum of the 3-dimensional irreducible Q -module Q and the trivial complementary Q -module $Z_{\mathfrak{a}_o}(Q)$, we infer that there exists a unique \mathfrak{a}_o -invariant decomposition, which coincides with the symmetric decomposition $\mathfrak{a} = \mathfrak{a}_o + \mathfrak{p}$. \square

To prove (i) we have to check that $\mathfrak{m}_s = 0$. We note that by the previous lemma $\mathfrak{s} = (Q \oplus Z_{\mathfrak{s}}(Q)) + \mathfrak{m}_s$ is a symmetric decomposition of the semisimple Lie algebra \mathfrak{s} . Since $[\mathfrak{m}_s, \mathfrak{m}_s] \subset Z_{\mathfrak{s}}(Q)$ it defines a hyper-Kähler symmetric space $N = L/L_o$, where L is the simply connected semisimple Lie group with Lie algebra $\mathfrak{l} = Z_{\mathfrak{s}}(Q) + \mathfrak{m}_s$ and L_o is the Lie subgroup generated by the subalgebra $Z_{\mathfrak{s}}(Q) \subset \mathfrak{l}$. Since N is in particular a Ricci-flat pseudo-Riemannian Hermitian symmetric space, from Proposition 3 we obtain that N is reduced to point. Therefore $\mathfrak{m}_s = 0$. This proves (i) and (ii). Finally, since the holonomy Lie algebra \mathfrak{h} is identified with $\mathfrak{h} = [\mathfrak{m}, \mathfrak{m}] = [\mathfrak{m}_r, \mathfrak{m}_r] \subset \mathfrak{r}$ it is solvable as subalgebra of the solvable Lie algebra \mathfrak{r} . \square

3.3. Hyper-Kähler symmetric spaces and second prolongation of symplectic Lie algebras. Let $(M = G/K, g, J_\alpha)$ be a simply connected hyper-Kähler symmetric space associated with a symmetric decomposition (2.1). Without restriction of generality we will assume that G acts almost effectively and that $\mathfrak{k} = [\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$ (holonomy Lie algebra). The complexification $\mathfrak{m}^\mathbb{C}$ as $\mathfrak{h}^\mathbb{C}$ -module can be written as $\mathfrak{m}^\mathbb{C} = H \otimes E$, such that $\mathfrak{h}^\mathbb{C} \subset \text{Id} \otimes \text{sp}(E) \cong \text{sp}(E)$, where $H = \mathbb{C}^2$ and $E = \mathbb{C}^{2n}$ are complex symplectic vector spaces with symplectic form ω_H and ω_E , respectively, such that $g^\mathbb{C} = \omega_H \otimes \omega_E$ is the complex bilinear metric on $\mathfrak{m}^\mathbb{C}$ induced by g . Note that the symplectic forms are unique up to the transformation $\omega_H \mapsto \lambda \omega_H$, $\omega_E \mapsto \lambda^{-1} \omega_E$, $\lambda \in \mathbb{C}^*$. We have also quaternionic structures j_H and j_E on H and E , such that $\omega_H(j_H x, j_H y) = \overline{\omega_H(x, y)}$ for all $x, y \in H$ and $\omega_E(j_E x, j_E y) = \overline{\omega_E(x, y)}$ for all $x, y \in E$, where the bar denotes complex conjugation. This implies that $\gamma_H := \omega_H(\cdot, j_H \cdot)$ and $\gamma_E := \omega_E(\cdot, j_E \cdot)$ are Hermitian forms on H and E . For fixed ω_H and ω_E the quaternionic structures j_H and j_E are uniquely determined if we require that γ_H is positive definite and that $\rho = j_H \otimes j_E$ is the real structure on $\mathfrak{m}^\mathbb{C}$, i.e. the complex conjugation with respect to \mathfrak{m} . The metric $g^\mathbb{C}$ and the Hermitian form $g^\mathbb{C}(\cdot, \rho) = \gamma_H \otimes \gamma_E$ restrict to a real valued scalar product g of some signature $(4k, 4l)$ on $\mathfrak{m} = (H \otimes E)^\rho$, where $(2k, 2l)$ is the (real) signature of the Hermitian form $\gamma_E = \omega_E(\cdot, j_E \cdot)$. Note that for the holonomy algebra we have the inclusion

$$\begin{aligned} \mathfrak{h} &= \text{Id} \otimes (\mathfrak{h}^\mathbb{C})^{j_E} \hookrightarrow \text{sp}(E)^{j_E} = \{A \in \text{sp}(E) \mid [A, j_E] = 0\} \\ &= \text{aut}(E, \omega_E, j_E) \cong \text{aut}(\mathfrak{m}, g, J_\alpha) \cong \text{sp}(k, l). \end{aligned}$$

Using the symplectic forms we identify $H = H^*$ and $E = E^*$. Then the symplectic Lie algebras are identified with symmetric tensors as follows:

$$\text{sp}(H) = S^2 H, \quad \text{sp}(E) = S^2 E.$$

Since the curvature tensor R of any hyper-Kähler manifold M^{4n} at a point $p \in M$ can be identified with an element $R \in S^2 \text{sp}(k, l)$ it is invariant under the Lie algebra $\text{sp}(1) = \text{span}\{J_1, J_2, J_3\}$. Let $M = G/K$ be a hyper-Kähler symmetric space as above. By Proposition 1 we can extend the Lie algebra $\mathfrak{g} = \mathfrak{k} + \mathfrak{m} = \mathfrak{h} + \mathfrak{m}$ to a Lie algebra

$$\tilde{\mathfrak{g}} = \text{sp}(1) + \mathfrak{h} + \mathfrak{m}$$

of Killing vector fields such that $[\text{sp}(1), \mathfrak{h}] = 0$. In the $H \otimes E$ -formalism the Lie algebra $\text{sp}(1)$ is identified with $\text{sp}(H)^{j_H} \otimes \text{Id} \subset \text{so}(\mathfrak{m})$.

LEMMA 2. Denote by $\tilde{\mathfrak{g}}^\mathbb{C} = \text{sp}(1, \mathbb{C}) + \mathfrak{h}^\mathbb{C} + \mathfrak{m}^\mathbb{C}$ the complexification of the Lie algebra $\tilde{\mathfrak{g}}$. Then the Lie bracket $[\cdot, \cdot] : \wedge^2 \mathfrak{m}^\mathbb{C} \rightarrow \mathfrak{h}^\mathbb{C}$ can be written as

$$(3.1) \quad [h \otimes e, h' \otimes e'] = \omega_H(h, h') S_{e, e'},$$

where $S \in (\mathfrak{h}^\mathbb{C})^{(2)} := \mathfrak{h}^\mathbb{C} \otimes S^2 E^* \cap E \otimes S^3 E^* = \mathfrak{h}^\mathbb{C} \otimes \mathfrak{h}^\mathbb{C} \cap S^4 E$. Moreover S is $\text{sp}(1, \mathbb{C}) \oplus \mathfrak{h}^\mathbb{C}$ -invariant and satisfies the following reality condition: $[S_{j_E e, e'} - S_{e, j_E e'}, j_E] = 0$.

Proof: The Lie bracket $[\cdot, \cdot] : \wedge^2 \mathfrak{m}^\mathbb{C} \rightarrow \mathfrak{h}^\mathbb{C}$ is an $\text{sp}(1, \mathbb{C}) \oplus \mathfrak{h}^\mathbb{C}$ -equivariant map, due to the Jacobi identity. We decompose the $\text{sp}(H) \oplus \text{sp}(E)$ -module $\wedge^2 \mathfrak{m}^\mathbb{C}$:

$$\wedge^2 \mathfrak{m}^\mathbb{C} = \wedge^2 (H \otimes E) = \wedge^2 H \otimes S^2 E \oplus S^2 H \otimes \wedge^2 E = \omega_H \otimes S^2 E \oplus S^2 H \otimes \wedge^2 E.$$

Since $\mathfrak{h}^{\mathbb{C}} \subset S^2 E$ the Lie bracket defines an $\mathfrak{sp}(1, \mathbb{C}) \oplus \mathfrak{h}^{\mathbb{C}}$ -invariant element of the space $\omega_H \otimes S^2 E \otimes S^2 E \oplus S^2 H \otimes \wedge^2 E \otimes S^2 E$. The second summand has no nontrivial $\mathfrak{sp}(1, \mathbb{C})$ -invariant elements. Hence the bracket is of the form (3.1), where $S \in S^2 E^* \otimes \mathfrak{h}^{\mathbb{C}} \subset S^2 E \otimes S^2 E$. The Jacobi identity reads:

$$\begin{aligned} 0 &= [h \otimes e, [h' \otimes e', h'' \otimes e'']] - [[h \otimes e, h' \otimes e'], h'' \otimes e''] - [h' \otimes e', [h \otimes e, h'' \otimes e'']] \\ &= -\omega_H(h', h'')h \otimes S_{e', e''}e - \omega_H(h, h')h'' \otimes S_{e, e'}e'' + \omega_H(h, h'')h' \otimes S_{e, e''}e'. \end{aligned}$$

Since $\dim H = 2$ we may assume that $h, h' = h''$ is a symplectic basis, i.e. $\omega_H(h, h') = 1$, and the equation implies: $S_{e, e''}e' = S_{e, e'}e''$, i.e. $S \in (\mathfrak{h}^{\mathbb{C}})^{(2)}$. The Lie bracket of two real elements $h \otimes e + j_H h \otimes j_E e$ and $h \otimes e' + j_H h \otimes j_E e' \in \mathfrak{m} \subset \mathfrak{m}^{\mathbb{C}}$ is an element of \mathfrak{h} . This gives:

$$[h \otimes e + j_H h \otimes j_E e, h \otimes e' + j_H h \otimes j_E e'] = \omega_H(h, j_H h)(S_{e, j_E e'} - S_{j_E e, e'}) \in \mathfrak{h}.$$

From the fact that the Hermitian form $\gamma_H = \omega_H(\cdot, j_H \cdot)$ is positive definite it follows that $\omega_H(h, j_H h) \neq 0$. This establishes the reality condition since $\mathfrak{h} = \{A \in \mathfrak{h}^{\mathbb{C}} \mid [A, j_E] = 0\}$. \square

In fact any tensor $S \in S^4 E$ satisfying the conditions of the above lemma can be used to define a hyper-Kähler symmetric space as the following theorem shows. We can identify $S^4 E$ with the space $\mathbb{C}[E]^{(4)}$ of homogeneous quartic polynomials on $E \cong E^*$.

THEOREM 1. *Let $S \in S^4 E$, $E = \mathbb{C}^{2n}$, be a quartic polynomial invariant under all endomorphisms $S_{e, e'} \in S^2 E = \mathfrak{sp}(E)$ and satisfying the reality condition*

$$(3.2) \quad [S_{j_E e, e'} - S_{e, j_E e'}, j_E] = 0.$$

Then it defines a hyper-Kähler symmetric space, which is associated with the following complex symmetric decomposition

$$(3.3) \quad \mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} + H \otimes E, \quad \mathfrak{h}^{\mathbb{C}} = \text{span}\{S_{e, e'} \mid e, e' \in E\} \subset \mathfrak{sp}(E).$$

The bracket $\wedge^2(H \otimes E) \rightarrow \mathfrak{h}^{\mathbb{C}}$ is given by (3.1). The real symmetric decomposition is defined as ρ -real form $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ of (3.3), where

$$\mathfrak{k} = \mathfrak{h} = \{A \in \mathfrak{h}^{\mathbb{C}} \mid [A, j_E] = 0\} = \text{span}\{S_{j_E e, e'} - S_{e, j_E e'} \mid e, e' \in E\}, \quad \mathfrak{m} = (H \otimes E)^\rho.$$

The hyper-Kähler symmetric space M associated to this symmetric decomposition is the quotient $M = M_S = G/K$, where G is the simply connected Lie group with Lie algebra \mathfrak{g} and $K \subset G$ is the connected (and closed) subgroup with Lie algebra $\mathfrak{k} = \mathfrak{h}$.

Moreover any simply connected hyper-Kähler symmetric space can be obtained by this construction. Two hyper-Kähler symmetric spaces M_S and $M_{S'}$ defined by quartics S and S' are isomorphic if and only if S and S' are in the same orbit of the group $\text{Aut}(E, \omega_E, j_E) = \{A \in \mathfrak{Sp}(E) \mid [A, j_E] = 0\} \cong \text{Sp}(k, l)$.

Proof: First of all we note that $\mathfrak{h}^{\mathbb{C}} = S_{E, E} := \text{span}\{S_{e, e'} \mid e, e' \in E\}$ is a subalgebra of $\mathfrak{sp}(E)$ because

$$[S_{e, e'}, S_{f, f'}] = (S_{e, e'} \cdot S)_{f, f'} - S_{S_{e, e'} f, f'} - S_{f, S_{e, e'} f'} = -S_{S_{e, e'} f, f'} - S_{f, S_{e, e'} f'} \in \mathfrak{h}^{\mathbb{C}}.$$

Since S is $\mathfrak{h}^{\mathbb{C}}$ -invariant and completely symmetric we can check, as in Lemma 2, that the Jacobi identity is satisfied and that (3.3) defines a complex symmetric decomposition. We prove that $\mathfrak{h} := \text{span}\{S_{j_E e, e'} - S_{e, j_E e'} | e, e' \in E\} \subset \{A \in \mathfrak{h}^{\mathbb{C}} | [A, j_E] = 0\}$ defines a real form of $\mathfrak{h}^{\mathbb{C}}$. Indeed for $e, e' \in E$ we have

$$\begin{aligned} S_{e, e'} &= \frac{1}{2}(S_{e, e'} + S_{j_E e, j_E e'}) - \frac{\sqrt{-1}}{2}(\sqrt{-1}S_{e, e'} - \sqrt{-1}S_{j_E e, j_E e'}) \\ &= \frac{1}{2}(S_{j_E e'', e'} - S_{e'', j_E e'}) - \frac{\sqrt{-1}}{2}(S_{j_E e'', \sqrt{-1}e'} - S_{e'', j_E \sqrt{-1}e'}), \end{aligned}$$

where $e'' = -j_E e$. Due to the reality condition the restriction of the Lie bracket $[\cdot, \cdot] : \wedge^2 \mathfrak{m}^{\mathbb{C}} \rightarrow \mathfrak{h}^{\mathbb{C}}$ to $\wedge^2 \mathfrak{m}$ has values in \mathfrak{h} and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ is a symmetric decomposition with $[\mathfrak{m}, \mathfrak{m}] = \mathfrak{h}$. The metric $g^{\mathbb{C}} = \omega_H \otimes \omega_E$ defines a real valued scalar product g of some signature (p, q) on $\mathfrak{m} = (H \otimes E)^{\rho}$, which is invariant under the Lie algebra \mathfrak{h} . Since $[\mathfrak{h}, j_E] = 0$ the holonomy algebra $\mathfrak{h} \subset \text{sp}(k, l)$, $p = 4k$, $q = 4l$. Hence this symmetric decomposition defines a hyper-Kähler symmetric space.

By Lemma 2 any hyper-Kähler symmetric space can be obtained by this construction. It is well known that a simply connected symmetric space M of signature (p, q) is determined by its abstract curvature tensor $R \in S^2(\wedge^2 V)$, $V = \mathbb{R}^{p, q}$, and two tensors R and R' define isometric symmetric spaces if and only if they belong to the same $O(V)$ orbit. Similarly a simply connected hyper-Kähler symmetric space is determined up to isometry by its abstract curvature tensor $R \in S^2(\wedge^2 V)$, where $V = \mathbb{R}^{4k, 4l}$ is the pseudo-Euclidean vector space with fixed hypercomplex structure $J_{\alpha} \in O(V)$. For a hyper-Kähler symmetric space the complexified curvature tensor has the form

$$R(h \otimes e, h' \otimes e') = -\omega_H(h, h')S_{e, e'},$$

where $S \in S^4 E$ is the quartic form of Lemma 2. Two such curvature tensors define isomorphic hyper-Kähler symmetric spaces if and only if they belong to the same orbit of $\text{Aut}(\mathbb{R}^{4k, 4l}, J_{\alpha}) = \text{Sp}(k, l)$. The group $\text{Sp}(k, l)$ acts on $V^{\mathbb{C}} = H \otimes E$ as $\text{Id} \otimes \text{Sp}(E)^{j_E} = \text{Id} \otimes \text{Aut}(E, \omega_E, j_E)$. Hence two curvature tensors $R = -\omega_H \otimes S$ and $R' = -\omega_H \otimes S'$ are in the same $\text{Sp}(k, l)$ -orbit if and only if S and S' are in the same $\text{Sp}(k, l)$ -orbit on $S^4 E$. \square

4. Complex Hyper-Kähler Symmetric Spaces.

4.1. Complex hyper-Kähler manifolds. A **complex Riemannian manifold** is a complex manifold M equipped with a complex metric g , i.e. a holomorphic section $g \in \Gamma(S^2 T^* M)$ which defines a nondegenerate complex quadratic form. As in the real case any such manifold has a unique holomorphic torsionfree and metric connection (Levi-Civita connection). A **complex hyper-Kähler manifold** is a complex Riemannian manifold (M^{4n}, g) of complex dimension $4n$ together with a compatible hypercomplex structure, i.e. three g -orthogonal parallel complex linear endomorphisms $(J_1, J_2, J_3 = J_1 J_2)$ with $J_{\alpha}^2 = -1$. This means that the holonomy group $\text{Hol} \subset \text{Sp}(n, \mathbb{C}) = Z_{O(4n, \mathbb{C})}(\text{Sp}(1, \mathbb{C}))$. The linear group $\text{Sp}(n, \mathbb{C})$ is diagonally embedded into $\text{Sp}(n, \mathbb{C}) \times \text{Sp}(n, \mathbb{C}) \subset \text{GL}(4n, \mathbb{C})$. Two complex hyper-Kähler manifolds (M, g, J_{α}) ($\alpha = 1, 2, 3$) and (M', g', J'_{α}) are called **isomorphic** if there exists a holomorphic isometry $\varphi : M \rightarrow M'$ such that $\varphi^* J'_{\alpha} = J_{\alpha}$ and $\varphi^* g' = g$.

We will show that the complex hyper-Kähler structure can be described as a half-flat Grassmann structure of a certain type. A **Grassmann structure** on a complex Riemannian manifold (M, g) is a decomposition of the (holomorphic) tangent bundle $TM \cong H \otimes E$ into the tensor product of two holomorphic vector bundles H and E of rank $2m$ and $2n$ with holomorphic nondegenerate 2-forms ω_H and ω_E such that $g = \omega_H \otimes \omega_E$. The Grassmann structure will be called **parallel** if the Levi-Civita connection $\nabla = \nabla^{TM}$ can be decomposed as:

$$\nabla = \nabla^H \otimes \text{Id} + \text{Id} \otimes \nabla^E,$$

where ∇^H and ∇^E are (uniquely defined) symplectic connections in the bundles H and E . A parallel Grassmann structure will be called **half-flat** if ∇^H is flat. Note that a parallel Grassmann structure on a simply connected manifold is half-flat if and only if the holonomy group of the Levi-Civita connection is contained in $\text{Id} \otimes \text{Sp}(n, \mathbb{C}) \subset \text{Sp}(m, \mathbb{C}) \otimes \text{Sp}(n, \mathbb{C}) \subset \text{O}(\mathbb{C}^{2m} \otimes \mathbb{C}^{2n})$.

PROPOSITION 5. *A complex hyper-Kähler structure (g, J_α) on a simply connected complex manifold M is equivalent to the following geometric data:*

- (i) *a half-flat Grassmann structure $(TM, g, \nabla) \cong (H, \omega_H, \nabla^H) \otimes (E, \omega_E, \nabla^E)$ and*
- (ii) *an isomorphism of flat symplectic vector bundles $H \cong M \times \mathbb{C}^2$. Under this isomorphism $\omega_H = h_1^* \wedge h_2^*$, where (h_1, h_2) is the standard basis of \mathbb{C}^2 considered as parallel frame of the trivial bundle $H = M \times \mathbb{C}^2$.*

More precisely,

$$(4.1) \quad J_1 = R_i \otimes \text{Id}, \quad J_2 = R_j \otimes \text{Id}, \quad \text{and} \quad J_3 = R_k \otimes \text{Id},$$

where we have identified $\mathbb{C}^2 = \mathbb{C}h_1 \oplus \mathbb{C}h_2$ with $\mathbb{H} = \text{span}_{\mathbb{R}}\{1, i, j, k\} = \text{span}_{\mathbb{C}}\{1, j\} = \mathbb{C}1 \oplus \mathbb{C}j$ with the complex structure defined by left-multiplication by i and R_x denotes the right-multiplication by the quaternion $x \in \mathbb{H}$.

Proof: It is easy to check that the geometric data (i) and (ii) define a complex hyper-Kähler structure on M . Conversely let (g, J_α) be a complex hyper-Kähler structure on M . The endomorphism J_1 has eigenvalues $\pm i$ and the tangent space can be decomposed into a sum of eigenspaces

$$TM = E_+ \oplus E_-.$$

From the J_1 -invariance of the metric g it follows that $g(E_\pm, E_\pm) = 0$ and we can identify $E_- = E^*$ with the dual space of $E = E_+$. Since J_2 anticommutes with J_1 it interchanges E and E^* and hence defines an isomorphism $E \xrightarrow{\sim} E^*$. Now $g(\cdot, J_2 \cdot)$ defines a symplectic form ω_E on E . Let $H = M \times \mathbb{C}^2 = M \times (\mathbb{C}h_1 \oplus \mathbb{C}h_2)$ be the trivial bundle with 2-form $\omega_H = h_1^* \wedge h_2^*$. Then we can identify

$$TM = E \oplus E^* = E \oplus E = h_1 \otimes E \oplus h_2 \otimes E = H \otimes E.$$

We check that under this identification we have $g = \omega_H \otimes \omega_E$. Note that both sides vanish on $h_1 \otimes E$ and $h_2 \otimes E$ and $\omega_H(h_1, h_2) = 1$. We calculate for $e, e' \in E = E_+ = h_1 \otimes E$:

$$g(e, J_2 e') = \omega_E(e, e') = \omega_H(h_1, h_2)\omega_E(e, e') = (\omega_H \otimes \omega_E)(h_1 \otimes e, h_2 \otimes e').$$

Hence we have a Grassmann structure. The eigenspaces E_\pm of the parallel endomorphism J_1 are invariant under parallel transport. Therefore the Levi-Civita connection

∇ induces a connection ∇^E in the bundle E . Since $\nabla g = 0$ and $\nabla J_2 = 0$ we have $\nabla^E \omega_E = 0$. We define a flat connection ∇^H on the trivial bundle $H = M \times \mathbb{C}^2$ by the condition $\nabla^H h_1 = \nabla^H h_2 = 0$. Then $\nabla = \nabla^H \otimes \text{Id} + \text{Id} \otimes \nabla^E$. So the Grassmann structure is half-flat.

Finally, using the standard identification $\mathbb{C}^2 = \mathbb{H}$, one can easily check that the J_α are given by (4.1). \square

4.2. Complexification of real hyper-Kähler manifolds. Let (M, g, J_α) be a (real) hyper-Kähler manifold. We will assume that it is real analytic. This is automatically true if the metric g is positive definite since it is Ricci-flat and a fortiori Einstein. Using analytic continuation we can extend (M, g, J_α) to a complex hyper-Kähler manifold $(M^{\mathbb{C}}, g^{\mathbb{C}}, J_\alpha^{\mathbb{C}})$ equipped with an antiholomorphic involution T . In complex local coordinates $z^j = x^j + iy^j$ which are extension of real analytic coordinates x^j, y^j the involution is given by the complex conjugation $z^j \rightarrow \bar{z}^j = x^j - iy^j$. We can reconstruct the (real) hyper-Kähler manifold as the fixed point set of T . We will call (M, g, J_α) a **real form** of $(M^{\mathbb{C}}, g^{\mathbb{C}}, J_\alpha^{\mathbb{C}})$ and $(M^{\mathbb{C}}, g^{\mathbb{C}}, J_\alpha^{\mathbb{C}})$ the complexification of (M, g, J_α) .

In general a complex hyper-Kähler manifold has no real form. A necessary condition is that the holonomy group of ∇^E is contained in $\text{Sp}(k, l)$, $n = k + l$, and hence preserves a quaternionic structure. Then we can define a parallel antilinear endomorphism field $j_E : E \rightarrow E$ such that $j_E^2 = -1$ and $\omega_E(j_E x, j_E y) = \overline{\omega_E(x, y)}$ for all $x, y \in E$, where the bar denotes complex conjugation. We define a parallel antilinear endomorphism field $j_H : H \rightarrow H$ as the left-multiplication by the quaternion j on $H = M \times \mathbb{H}$. Then $\rho = j_H \otimes j_E$ defines a field of real structures in $TM = H \otimes E$. We denote by $\mathcal{D} \subset TM$ the real eigenspace distribution of ρ with eigenvalue 1. Here TM is considered as real tangent bundle of the real manifold M . If $M^\rho \subset M$ is a leaf of \mathcal{D} of real dimension $4n$ then the data (g, J_α) induce on M^ρ a (real) hyper-Kähler structure.

4.3. Complex hyper-Kähler symmetric spaces. A complex Riemannian symmetric space is a complex Riemannian manifold (M, g) such that any point is an isolated fixed point of an isometric holomorphic involution. Like in the real case one can prove that it admits a transitive complex Lie group of holomorphic isometries and that any simply connected complex Riemannian symmetric M is associated to a complex symmetric decomposition

$$(4.2) \quad \mathfrak{g} = \mathfrak{k} + \mathfrak{m}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] = \mathfrak{k}$$

of a complex Lie algebra \mathfrak{g} together with an $\text{ad}_{\mathfrak{k}}$ -invariant complex scalar product on \mathfrak{m} . More precisely $M = G/K$, where G is the simply connected complex Lie group with the Lie algebra \mathfrak{g} and K is the (closed) connected subgroup associated with \mathfrak{k} . The holonomy group of such manifold is $H = \text{Ad}_K | \mathfrak{m}$. Any pseudo-Riemannian symmetric space $M = G/K$ associated with a symmetric decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ has a canonical complexification $M^{\mathbb{C}} = G^{\mathbb{C}}/K^{\mathbb{C}}$ defined by the complexification $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{m}^{\mathbb{C}}$ of the symmetric decomposition. Proposition 1 remains true for complex Riemannian symmetric spaces. Ignoring the reality condition we obtain the following complex version of Theorem 1.

THEOREM 2. *Let $S \in S^4 E$, $E = \mathbb{C}^{2n}$, be a quartic polynomial invariant under all endomorphisms $S_{e, e'} \in S^2 E = \text{sp}(E)$. Then it defines a complex hyper-Kähler symmetric space, which is associated with the following complex symmetric decomposition*

$$(4.3) \quad \mathfrak{g} = \mathfrak{h} + H \otimes E, \quad \mathfrak{h} = S_{E, E} = \text{span}\{S_{e, e'} | e, e' \in E\} \subset \text{sp}(E).$$

The bracket $\wedge^2(H \otimes E) \rightarrow \mathfrak{h}$ is given by (3.1). The complex hyper-Kähler symmetric space M associated to this symmetric decomposition is the quotient $M = M_S = G/K$, where G is the (complex) simply connected Lie group with Lie algebra \mathfrak{g} and $K \subset G$ is the connected (and closed) subgroup with Lie algebra $\mathfrak{k} = \mathfrak{h}$.

Moreover any simply connected complex hyper-Kähler symmetric space can be obtained by this construction. Two complex hyper-Kähler symmetric spaces M_S and $M_{S'}$ defined by quartics S and S' are isomorphic if and only if S and S' are in the same orbit of $\text{Aut}(E, \omega_E) = \text{Sp}(E) \cong \text{Sp}(n, \mathbb{C})$.

COROLLARY 1. *There is a natural bijection between simply connected complex hyper-Kähler symmetric spaces of dimension $4n$ up to isomorphism and $\text{Sp}(n, \mathbb{C})$ -orbits on the space of quartic polynomials $S \in S^4E$ in the symplectic vector space $E = \mathbb{C}^{2n}$ such that*

$$(4.4) \quad S_{e,e'} \cdot S = 0 \quad \text{for all } e, e' \in E.$$

4.4. Classification of complex hyper-Kähler symmetric spaces. The following complex version of Proposition 4 (with similar proof) will be a crucial step in the classification of complex hyper-Kähler symmetric spaces.

PROPOSITION 6. *Let $(M = G/K, g, J_\alpha)$ be a simply connected complex hyper-Kähler symmetric space. Then the holonomy group of M is solvable and M admits a transitive solvable Lie group of automorphisms.*

Due to Corollary 1 the classification of simply connected complex hyper-Kähler symmetric spaces reduces to the determination of quartic polynomials S satisfying (4.4). Below we will determine all such polynomials. We will prove that the following example gives all such polynomials.

EXAMPLE 1: Let $E = E_+ \oplus E_-$ be a Lagrangian decomposition, i.e. $\omega(E_\pm, E_\pm) = 0$, of the symplectic vector space $E = \mathbb{C}^{2n}$. Then any polynomial $S \in S^4E_+ \subset S^4E$ satisfies the condition (4.4) and defines a simply connected complex hyper-Kähler symmetric space M_S with Abelian holonomy algebra $\mathfrak{h} = S_{E_+,E_+} \subset S^2E_+ \subset S^2E = \mathfrak{sp}(E)$.

In fact, since E_+ is Lagrangian the endomorphisms from S^2E_+ form an Abelian subalgebra of $\mathfrak{sp}(E)$, which acts trivially on E_+ and hence on S^4E_+ .

THEOREM 3. *Let $S \in S^4E$ be a quartic polynomial satisfying (4.4). Then there exists a Lagrangian decomposition $E = E_+ \oplus E_-$ such that $S \in S^4E_+$.*

Proof: According to Theorem 2 the quartic S defines a hyper-Kähler symmetric space with holonomy Lie algebra $\mathfrak{h} = S_{E,E}$. Since, by Proposition 6, \mathfrak{h} is solvable, Lie’s theorem implies the existence of a one-dimensional \mathfrak{h} -invariant subspace $P = \mathbb{C}p \subset E$. There exists an ω -nondegenerate subspace $W \subset E$ such that the ω -orthogonal complement of P is $P^\perp = P \oplus W$. We choose a vector $q \in E$ such that $\omega(p, q) = 1$ and $\omega(W, q) = 0$ and put $Q := \mathbb{C}q$. Then we have

$$E = P \oplus W \oplus Q.$$

Since \mathfrak{h} preserves P we have the following inclusion

$$\mathfrak{h} \subset PE + W^2 = P^2 + PW + PQ + W^2,$$

where we use the notation $XY = X \vee Y$ for the symmetric product of subspaces $X, Y \subset E$. Then the second prolongation $\mathfrak{h}^{(2)} = \{T \in S^4E \mid T_{e,e'} \in \mathfrak{h} \text{ for all } e, e' \in E\}$

has the following inclusion

$$(4.5) \quad \begin{aligned} \mathfrak{h}^{(2)} &\subset P^3E + P^2W^2 + PW^3 + W^4 \\ &= P^4 + P^3Q + P^3W + P^2W^2 + PW^3 + W^4. \end{aligned}$$

Indeed $\mathfrak{h}^{(2)} \subset \mathfrak{h}^2 = P^4 + P^3Q + P^3W + P^2Q^2 + P^2WQ + P^2W^2 + PQW^2 + PW^3 + W^4$. The projection $\mathfrak{h}^{(2)} \rightarrow P^2Q^2 + P^2WQ + PQW^2$ is zero because otherwise $S_{q,q} \in \mathfrak{h} \subset PE + W^2$ would have a nonzero projection to $Q^2 + WQ$ or $S_{w,q} \in \mathfrak{h}$ would have a nonzero projection to QW for appropriate choice of $w \in W$. By (4.5) we can write the quartic S as

$$S = p^3(\lambda p + \mu q + w_0) + p^2B + pC + D,$$

where $\lambda, \mu \in \mathbb{C}$, $w_0 \in W$, $B \in S^2W$, $C \in S^3W$ and $D \in S^4W$. From now on we will identify S^dE with the space $\mathbb{C}[E^*]^{(d)}$ of homogeneous polynomials on E^* of degree d . Then the ω -contraction $T_x = \iota_{\omega x}T = T(\omega x, \dots)$ of a tensor $T \in S^dE$ with a vector $x \in E$ is identified with the following homogeneous polynomial of degree $d - 1$:

$$T_x = \frac{1}{d}\partial_{\omega x}T,$$

where $\partial_{\omega x}T$ is the derivative of the polynomial $T \in \mathbb{C}[E^*]^{(d)}$ in the direction of $\omega x = \omega(x, \cdot) \in E^*$. For example $p_q = \langle p, \omega q \rangle = \omega(q, p) = \partial_{\omega q}p = -\partial_{p^*}p = -1 = -q_p$.

From $S_{p,q} = -\frac{1}{4}\mu p^2$ and the condition $S_{p,q} \cdot S = 0$ we obtain $\mu = 0$, since $p^2 \cdot S = \mu p^4$. This implies $S_{p,\cdot} = 0$. Next we compute:

$$\begin{aligned} S_{q,q} &= \frac{1}{6}(6\lambda p^2 + 3pw_0 + B) \\ S_{q,w} &= -\frac{1}{12}(-3p^2\omega(w_0, w) + 2p\partial_{\omega w}B + \partial_{\omega w}C) \\ &= -\frac{1}{12}(-3p^2\omega(w_0, w) + 4pB_w + 3C_w) \\ S_{w,w'} &= \frac{1}{6}(p^2B_{w,w'} + 3pC_{w,w'} + 6D_{w,w'}) \end{aligned}$$

for any $w, w' \in W$.

Now the condition (4.4) can be written as follows:

$$\begin{aligned} 0 &= 6S_{q,q} \cdot S = (3pw_0 + B) \cdot S = \left(\frac{3}{2}(p \otimes w_0 + w_0 \otimes p) + B\right) \cdot S \\ &= \frac{3}{2}(2p^3B_{w_0} + 3p^2C_{w_0} + 4pD_{w_0}) + p^3Bw_0 + p^2B \cdot B + pB \cdot C + B \cdot D \\ &= -2p^3Bw_0 + \frac{9}{2}p^2C_{w_0} + p(6D_{w_0} + B \cdot C) + B \cdot D. \end{aligned}$$

Note that $Bw_0 = -B_{w_0}$ and $B \cdot B = [B, B] = 0$.

$$\begin{aligned} 0 &= -12S_{q,w} \cdot S = (4pB_w + 3C_w) \cdot S \\ &= 2(p^4\omega(B_w, w_0) + 2p^3B^2w - 3p^2C_{Bw} - 4pD_{Bw}) \\ &\quad + 3(p^3C_w w_0 + p^2C_w \cdot B + pC_w \cdot C + C_w \cdot D) \\ &= 2p^4\omega(B_w, w_0) + p^3(4B^2w + 3C_w w_0) + p^2(-6C_{Bw} + 3C_w \cdot B) \end{aligned}$$

$$\begin{aligned}
 & +p(-8D_{Bw} + 3C_w \cdot C) + 3C_w \cdot D \\
 0 = & 2S_{w,w'} \cdot S = \frac{1}{2}(p \otimes C_{w,w'} + C_{w,w'} \otimes p) \cdot S + 2D_{w,w'} \cdot S \\
 = & \frac{1}{2}(p^4 \omega(C_{w,w'}, w_0) - 2p^3 BC_{w,w'} + 3p^2 C_{C_{w,w'}} + 4pD_{C_{w,w'}}) + \\
 & 2(p^3 D_{w,w'} w_0 + p^2 D_{w,w'} \cdot B + pD_{w,w'} \cdot C + D_{w,w'} \cdot D) \\
 = & \frac{1}{2}p^4 \omega(C_{w,w'}, w_0) + p^3(-BC_{w,w'} + 2D_{w,w'} w_0) + \\
 & p^2(\frac{3}{2}C_{C_{w,w'}} + 2D_{w,w'} \cdot B) + p(2D_{C_{w,w'}} + 2D_{w,w'} \cdot C) + 2D_{w,w'} \cdot D.
 \end{aligned}$$

This gives the following system of equations:

- (1) $Bw_0 = 0$
- (2) $C_{w_0} = 0$
- (3) $6D_{w_0} + B \cdot C = 0$
- (4) $B \cdot D = 0$
- (5) $\omega(B_w, w_0) = 0$
- (6) $4B^2 w + 3C_w w_0 = 0$
- (7) $-2C_{Bw} + C_w \cdot B = 0$
- (8) $-8D_{Bw} + 3C_w \cdot C = 0$
- (9) $C_w \cdot D = 0$
- (10) $\omega(C_w w', w_0) = 0$
- (11) $-BC_w w' + 2D_{w,w'} w_0 = 0$
- (12) $\frac{3}{2}C_{C_w w'} + 2D_{w,w'} \cdot B = 0$
- (13) $D_{C_w w'} + D_{w,w'} \cdot C = 0$
- (14) $D_{w,w'} \cdot D = 0.$

Note that (5) and (10) follow from (1) and (2) and that using (2) equation (6) says that the endomorphism B has zero square:

$$(6') \quad B^2 = 0.$$

Eliminating D_{w_0} in equations (3) and (11) we obtain:

$$\begin{aligned}
 (15) \quad 0 = & (B \cdot C)_{w'} w' + 3BC_w w' = BC_w w' - C_{Bw} w' - C_w Bw' + 3BC_w w' \\
 & = 4BC_w w' - C_{Bw} w' - C_w Bw'.
 \end{aligned}$$

We can rewrite (7) as:

$$(7') \quad -2C_{Bw} w' + C_w Bw' - BC_w w' = 0.$$

Eliminating $C_{Bw} w'$ in (7') and (15) we obtain:

$$(16) \quad -3BC_w w' + C_w Bw' = 0.$$

Since the first summand is symmetric in w and w' we get

$$(17) \quad C_w Bw' = C_w Bw = C_{Bw} w'.$$

Now using (17) we can rewrite (15) as:

$$(15') \quad 2BC_w w' - C_w Bw' = 0.$$

The equations (15') and (16) show that $BC_w w' = C_w B w' = C_{B w} w' = 0$ and hence also $B \cdot C = 0$. This implies $D_{w_0} = 0$, by (3). Now we can rewrite (1-14) as:

$$(4.6) \quad B w_0 = C_{w_0} = D_{w_0} = 0$$

$$(4.7) \quad B \cdot C = B \cdot D = 0$$

$$(4.8) \quad B^2 = 0$$

$$(4.9) \quad C_{B w} = C_w B = B C_w = 0$$

$$(4.10) \quad -8D_{B w} + 3C_w \cdot C = 0$$

$$(4.11) \quad C_w \cdot D = 0$$

$$(4.12) \quad \frac{3}{2}C_{C_w w'} + 2D_{w, w'} \cdot B = 0$$

$$(4.13) \quad D_{C_w w'} + D_{w, w'} \cdot C = 0$$

$$(4.14) \quad D_{w, w'} \cdot D = 0.$$

Now to proceed further we decompose $K := \ker B = W_0 \oplus W'$, where $W_0 = \ker \omega|_K$ and W' is a (nondegenerate) complement. Let us denote by W_1 a complement to K in W such that $\omega(W', W_1) = 0$. Then $W_0 + W_1$ is the ω -orthogonal complement to the B -invariant nondegenerate subspace W' . This shows that $BW_1 \subset (W_0 + W_1) \cap K = W_0$. Moreover since $W_1 \cap K = 0$ the map $B : W_1 \rightarrow W_0$ is injective and hence $\dim W_1 \leq \dim W_0$. On the other hand $\dim W_1 \geq \dim W_0$, since W_0 is an isotropic subspace of the symplectic vector space $W_0 + W_1$. This shows that $B : W_1 \rightarrow W_0$ is an isomorphism.

LEMMA 3. $C \in S^3 K$ and $D \in S^4 K$.

Proof: Since $W_0 = BW$ the equation (4.9) shows that $C_{W_0} = 0$, which proves the first statement. From (4.10) and the identity

$$(4.15) \quad (C_x \cdot C)_y = [C_x, C_y] - C_{C_x y}$$

we obtain

$$(4.16) \quad D_{Bx, y} + D_{By, x} = \frac{3}{8}((C_x \cdot C)_y + (C_y \cdot C)_x) = -\frac{3}{4}C_{C_x y}.$$

The equation $B \cdot D = 0$ (4.7) reads:

$$0 = (B \cdot D)_{x, y} = [B, D_{x, y}] - D_{Bx, y} - D_{x, By}.$$

Using this (4.12) yields:

$$(4.17) \quad D_{Bx, y} + D_{By, x} = [B, D_{x, y}] = -D_{x, y} \cdot B = \frac{3}{4}C_{C_x y}.$$

Now from (4.16) and (4.17) we obtain that

$$0 = C_{C_x y} z = C_z C_x y$$

for all $x, y, z \in W$. This implies $[C_x, C_y] = 0$ for all $x, y \in W$ and hence

$$(4.18) \quad C_x \cdot C = 0$$

for all $x \in W$, by (4.15). Finally this shows that $D_{W_0} = 0$ by (4.10). This proves the second statement. \square

LEMMA 4. $D_{x,y} C_z = C_z D_{x,y} = 0$ for all $x, y, z \in W$.

Proof: Using (4.13) we compute:

$$(4.19) \quad D_{x,y} C_z w = D_{C_z w, x} y = -(D_{z,w} \cdot C)_{x} y = -([D_{z,w}, C_x] y - C_{D_{z,w} x} y).$$

From (4.11) we get:

$$\begin{aligned} 0 &= (C_x \cdot D)_{z,w} y = [C_x, D_{z,w}] y - D_{C_x z, w} y - D_{z, C_x w} y \\ &= C_x D_{z,w} y - D_{z,w} C_x y - D_{y,w} C_x z - D_{z,y} C_x w, \end{aligned}$$

and hence:

$$[D_{z,w}, C_x] y = -D_{y,w} C_x z - D_{z,y} C_x w,$$

and

$$C_{D_{z,w} x} y = C_y D_{z,w} x = D_{z,w} C_x y + D_{x,w} C_y z + D_{z,x} C_y w.$$

Now we eliminate the CD -terms from (4.19) arriving at:

$$(4.20) \quad D_{x,y} C_z w = (D_{y,w} C_x z + D_{z,y} C_x w + D_{z,w} C_x y + D_{x,w} C_y z + D_{z,x} C_y w).$$

Considering all the permutations of (x, y, z, w) we get 6 homogeneous linear equations for the 6 terms of equation (4.20) with the matrix:

$$\begin{pmatrix} -1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 & -1 \end{pmatrix}$$

This is the matrix of the endomorphism $-2 \text{Id} + e \otimes e$ in the arithmetic space \mathbb{R}^6 , where $e = e_1 + \dots + e_6$; (e_i) the standard basis. It has eigenvalues $(4, -2, -2, -2, -2, -2)$. This shows that the matrix is nondegenerate and proves the lemma. \square

For a symmetric tensor $T \in S^d W$ we denote by

$$\Sigma_T := \text{span}\{T_{x_1, x_2, \dots, x_{d-2}, x_{d-1}} | x_1, x_2, \dots, x_{d-1} \in W\} \subset W$$

the **support** of T .

LEMMA 5. *The supports of the tensors $B \in S^2W$, $C \in S^3W$ and $D \in S^4W$ admit the following inclusions*

$$\Sigma_B + \Sigma_C \subset \ker B \cap \ker C \cap \ker D, \quad \Sigma_D \subset \ker B \cap \ker C.$$

Moreover $\Sigma_B + \Sigma_C$ is isotropic and $\omega(\Sigma_D, \Sigma_B + \Sigma_C) = 0$.

Proof: The first statement follows from $B^2 = BC_x = BD_{x,y} = C_xB = C_xC_y = C_xD_{y,z} = D_{x,y}B = D_{x,y}C_z = 0$ for all $x, y, z \in W$. The second statement follows from the first and the definition of support, e.g. if $z = C_xy \in \Sigma_C$ and $w \in \Sigma_B + \Sigma_C + \Sigma_D \subset \ker C$ we compute:

$$\omega(z, w) = \omega(C_xy, w) = -\omega(y, C_xw) = 0.$$

□

LEMMA 6. *The Lie algebra $D_{W,W} \subset S^2W \cong \mathfrak{sp}(W)$ is solvable.*

Proof: This follows from Proposition 6, since $D \in S^4W$ satisfies (4.4) and hence defines a complex hyper-Kähler symmetric space with holonomy Lie algebra $D_{W,W}$. It also follows from the solvability of $S_{E,E}$ as we show now. In terms of the decomposition $E = P + W + Q$ an endomorphism

$$S_{x,y} = (\lambda p^4 + p^3 w_0 + p^2 B + pC + D)_{x,y} = (p^2 B + pC + D)_{x,y} = B(x, y)p^2 - pC_xy + D_{x,y},$$

where $x, y \in W$, is represented by

$$\begin{pmatrix} 0 & -(C_xy)^t & B(x, y) \\ 0 & D_{x,y} & -C_xy \\ 0 & 0 & 0 \end{pmatrix}.$$

Since the Lie algebra $S_{E,E}$ is solvable this implies that the Lie algebra $D_{W,W}$, which corresponds to the induced representation of $S_{E,E}$ on $P^\perp/P \cong W$, is also solvable. □

LEMMA 7.

$$\omega(w_0, \Sigma_B + \Sigma_C + \Sigma_D) = 0.$$

Proof: Note that $w_0 \in \ker B \cap \ker C \cap \ker D$, due to equations (1-3) and (4.7). This implies the lemma. In fact, if e.g. $y = Bx \in \Sigma_B$ then

$$\omega(w_0, y) = \omega(w_0, Bx) = -\omega(Bw_0, x) = 0,$$

which shows that $\omega(w_0, \Sigma_B) = 0$ □

Now to finish the proof of Theorem 3 we will use induction on the dimension $\dim E = 2n$. If $n = 1$ the (solvable) holonomy algebra \mathfrak{h} is a proper subalgebra of $S^2E \cong \mathfrak{sl}(2, \mathbb{C})$. Without loss of generality we may assume that either

- a) $\mathfrak{h} = \mathbb{C}p^2$ or
- b) $\mathfrak{h} = \mathbb{C}pq$ or
- c) $\mathfrak{h} = \mathbb{C}p^2 + \mathbb{C}pq$,

where (p, q) is a symplectic basis of E . In the all three cases the Lie algebra $S_{E,E} = \mathfrak{h} \subset \mathbb{C}p^2 + \mathbb{C}pq$ and hence

$$S = \lambda p^4 + \mu p^3 q.$$

In the cases b) and c) we have that $pq \in \mathfrak{h}$ and since

$$pq \cdot S = \frac{1}{2}(p \otimes q + q \otimes p) \cdot S = -2\lambda p^4 - \mu p^3 q = 0$$

it follows that $S = 0$. In the case a) from $S_{E,E} = \mathfrak{h} = \mathbb{C}p^2$ we have that $S = \lambda p^4$. This tensor is invariant under $\mathfrak{h} = \mathbb{C}p^2$ and belongs to the fourth symmetric power of the Lagrangian subspace $\mathbb{C}p \subset E$. This establishes the first step of the induction. Now by induction using equation (4.14) and Lemma 6 we may assume that Σ_D is isotropic. Now Lemma 5 and Lemma 7 show that $\mathbb{C}w_0 + \Sigma_B + \Sigma_C + \Sigma_D$ is isotropic and hence is contained in some Lagrangian subspace $E_+ \subset E$. This implies that $S \in S^4 E_+$. \square

Now we give a necessary and sufficient condition for a symmetric manifold $M = M_S, S \in S^4 E_+$, to have no flat de Rham factor.

PROPOSITION 7. *The complex hyper-Kähler symmetric space $M_S, S \in S^4 E_+$, has no flat de Rham factor if and only if the support $\Sigma_S = E_+$.*

Proof: If $M = M_S = G/K$ has a flat factor M_0 , such that $M = M_1 \times M_0$, then this induces a decomposition $E = E^1 \oplus E^0$ and $S \in S^4 E_1$; hence $\Sigma_S \subset E_1 \cap E_+ \neq E_+$. Conversely let $S \in S^4 E_+$, assume that $E_+^1 = \Sigma_S \subset E_+$ is a proper subspace and choose a complementary subspace E_+^0 . We denote by E_+^1 and E_+^0 the annihilator of ωE_+^0 and ωE_+^1 respectively. Let us denote $E^1 = E_+^1 \oplus E_-^1, E^0 = E_+^0 \oplus E_-^0, \mathfrak{m}^1 = H \otimes E^1$ and $\mathfrak{m}^0 = H \otimes E^0$. Then $E^0, E^1 \subset E$ are ω -nondegenerate complementary subspaces and $\mathfrak{m}^0, \mathfrak{m}^1 \subset \mathfrak{m} = T_o M$ are g -nondegenerate complementary subspaces. Since $S \in S^4 E_+^1$ the Lie algebra $\mathfrak{g} = \mathfrak{h} + \mathfrak{m} = (\mathfrak{h} + \mathfrak{m}^1) \oplus \mathfrak{m}^0$ has the Abelian direct summand \mathfrak{m}^0 , see (3.1), which gives rise to a flat factor $M^0 \subset M_S = M^1 \times M^0$.

THEOREM 4. *Any simply connected complex hyper-Kähler symmetric space without flat de Rham factor is isomorphic to a complex hyper-Kähler symmetric space of the form M_S , where $S \in S^4 E_+$ and $E_+ \subset E$ is a Lagrangian subspace of the complex symplectic vector space $E = \mathbb{C}^{2n}$. Moreover there is a natural 1-1 correspondence between simply connected complex hyper-Kähler symmetric spaces without flat factor up to isomorphism and orbits \mathcal{O} of the group $\text{Aut}(E, \omega, E_+) |_{E_+} = \{A \in \text{Sp}(E) | AE_+ = E_+\} |_{E_+} \cong \text{GL}(E_+) \cong \text{GL}(n, \mathbb{C})$ on the space $S^4 E_+$ such that $\Sigma_S = E_+$ for all $S \in \mathcal{O}$.*

Proof: This is a corollary of Theorem 2, Theorem 3 and Proposition 7. \square

Let $M = G/K$ be a simply connected complex hyper-Kähler symmetric space without flat factor. By Theorem 3 and Proposition 7 it is associated to quartic polynomial $S \in S^4 E_+$ with support $\Sigma_S = E_+$. Now we describe the Lie algebra $\text{aut}(M_S)$ of the full group of automorphisms, i.e. isometries which preserve the hypercomplex structure, of M_S .

THEOREM 5. *Let $M_S = G/K$ be as above. Then the full automorphism algebra is given by*

$$\text{aut}(M_S) = \text{aut}(S) + \mathfrak{g},$$

where $A \in \text{aut}(S) = \{B \in \mathfrak{gl}(E_+) | B \cdot S = 0\}$ acts on $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ as follows. It preserves the decomposition and acts on $\mathfrak{h} = S_{E,E}$ by

$$[A, S_{x,y}] = S_{Ax,y} + S_{x,Ay}$$

for all $x, y \in E$ and on $\mathfrak{m} = H \otimes E$ by

$$[A, h \otimes e] = h \otimes Ae,$$

where $\mathfrak{gl}(E_+)$ is canonically embedded into $\mathfrak{sp}(E)$.

Proof: By (the complex version of) Proposition 1 it is sufficient to determine the centralizer \mathfrak{c} of $\mathfrak{sp}(1, \mathbb{C})$ in the full isotropy algebra $\tilde{\mathfrak{h}} = \text{aut}(R) \supset \mathfrak{sp}(1, \mathbb{C}) \oplus \mathfrak{h}$. Equation (2.2) shows that

$$\mathfrak{c} = \{\text{Id} \otimes A \mid A \in \mathfrak{sp}(E), A \cdot S = [A, S(\cdot, \cdot)] - S(A \cdot, \cdot) - S(\cdot, A) = 0\}.$$

From $A \cdot S = 0$ we obtain that the commutator $[A, S_{x,y}] = S_{Ax,y} + S_{x,Ay}$ for all $x, y \in E$ and $A\Sigma_S = AE_+ \subset E_+$. This implies $\mathfrak{c} = \text{aut}(S)$. \square

5. Classification of Hyper-Kähler Symmetric Spaces. Using the description of complex hyper-Kähler symmetric spaces given in Theorem 4 we will now classify (real) hyper-Kähler symmetric spaces. Recall that a simply connected pseudo-Riemannian manifold is called **indecomposable** if it is not a Riemannian product of two pseudo-Riemannian manifolds. Any simply connected pseudo-Riemannian manifold can be decomposed into the Riemannian product of indecomposable pseudo-Riemannian manifolds. By Wu’s theorem [W] a simply connected pseudo-Riemannian manifold is indecomposable if and only if its holonomy group is **weakly irreducible**, i.e. has no invariant proper nondegenerate subspaces. Therefore it is sufficient to classify (real) hyper-Kähler symmetric spaces with *indecomposable* holonomy.

Let $(M = G/K, g, J_\alpha)$ be a hyper-Kähler symmetric space associated to a symmetric decomposition (2.1). The complexified tangent space of M is identified with $\mathfrak{m}^\mathbb{C} = H \otimes E$, the tensor product of complex symplectic vector spaces with quaternionic structure j_H and j_E such that $\rho = j_H \otimes j_E$ is the complex conjugation of $\mathfrak{m}^\mathbb{C}$ with respect to \mathfrak{m} . By Theorem 1 it is defined by a quartic polynomial $S \in S^4E$ satisfying the conditions of the theorem. Moreover the holonomy algebra \mathfrak{h} acts trivially on H and is identified with the real form of the complex Lie algebra $S_{E,E} \subset \mathfrak{sp}(E)$ given by $\mathfrak{h} = \text{span}\{S_{j_e, e'} - S_{e, j_{e'}} \mid e, e' \in E\} = \{A \in S_{E,E} \mid [A, j_E] = 0\} \subset \mathfrak{sp}(E)^{j_E}$.

The quartic polynomial S defines also a complex hyper-Kähler symmetric space $M^\mathbb{C} = G^\mathbb{C}/K^\mathbb{C}$, which is the complexification of $M = G/K$. By Theorem 3, $S \in S^4L$ for some Lagrangian subspace $L \subset E$. Recall that the symplectic form $\omega = \omega_E$ together with the quaternionic structure $j = j_E$ define a Hermitian metric $\gamma = \gamma_E = \omega_E(\cdot, j_E \cdot)$ of (real) signature $(4k, 4l)$, $n = k + l$, which coincides with the signature of the pseudo-Riemannian metric g (we normalize $\gamma_H = \omega_H(\cdot, j_H \cdot)$ to be positive definite). We may decompose γ -orthogonally $L = L^0 \oplus L^+ \oplus L^-$, such that γ vanishes on L^0 is positive definite on L^+ and negative definite on L^- .

LEMMA 8.

- (i) $jL^0 = L^0$ and
- (ii) $L^+ + L^- + jL^+ + jL^- \subset E$ is an ω -nondegenerate and γ -nondegenerate \mathfrak{h} -invariant subspace (with trivial action of \mathfrak{h}).

Proof: We show first that $L + jL^0$ is ω -isotropic and hence $L + jL^0 = L$ since L is Lagrangian. Indeed $L \supset L^0$ is ω -isotropic and also jL^0 because ω is j -invariant. So it suffices to remark that $\omega(L, jL^0) = 0$:

$$\omega(L, jL^0) = \gamma(L, L^0) = 0.$$

This implies that $jL^0 \subset L$. Since $\gamma(L, jL^0) = -\omega(L, L^0) = 0$, we conclude that $jL^0 \subset \ker \gamma|_L = L^0$. This proves (i).

To prove (ii) it is sufficient to check that the subspace $L^+ + L^- + jL^+ + jL^- \subset E$ is nondegenerate with respect to γ , since it is j -invariant. First we remark that γ is

positive definite on L^+ and jL^+ and negative definite on L^- and jL^- , due to the j -invariance of γ : $\gamma(jx, jx) = \gamma(x, x)$, $x \in E$. So to prove (ii) it is sufficient to check that $jL^+ \oplus jL^-$ is γ -orthogonal to the γ -nondegenerate vector space $L^+ + L^-$:

$$\gamma(L^+ + L^-, jL^+ + jL^-) = \omega(L^+ + L^-, L^+ + L^-) = 0.$$

□

By Theorem 1 the quartic polynomial S must satisfy the reality condition $[S_{j_e, e'} - S_{e, j_{e'}}, j] = 0$. Now we describe all such polynomials.

The quaternionic structure j on E is **compatible** with ω , i.e. $\omega(jx, jy) = \overline{\omega(x, y)}$ for all $x, y \in E$ and it induces a real structure (i.e. an antilinear involution) $\tau := j \otimes j \otimes \dots \otimes j$ on all even powers $S^{2r}E \subset E \otimes E \otimes \dots \otimes E$. For $S \in S^{2r}E$ and $x_1, \dots, x_{2r} \in E$ we have

$$(\tau S)(x_1, \dots, x_{2r}) = \overline{S(jx_1, \dots, jx_{2r})}.$$

Note that the fixed point set $\text{sp}(E)^\tau = \{A \in \text{sp}(E) \mid [A, j] = 0\} \cong \text{sp}(k, l)$.

PROPOSITION 8. *Let (E, ω, j) be a complex symplectic vector space with a quaternionic structure j such that $\omega(jx, jy) = \overline{\omega(x, y)}$ for all $x, y \in E$. Then a quartic polynomial $S \in S^4E$ satisfies the reality condition $[S_{j_e, e'} - S_{e, j_{e'}}, j] = 0$ if and only if $S \in (S^4E)^\tau = \text{span}\{T + \tau T \mid T \in S^4E\}$.*

Proof: The reality condition for $S \in S^4E$ can be written as

$$[S_{j_x, j_y} + S_{x, y}, j]z = 0$$

for all $x, y, z \in E$. Contracting this vector equation with $jw \in E$ by means of ω and using the compatibility between j and ω we obtain the equivalent condition

$$(5.1) \quad 0 = -\omega(jw, [S_{j_x, j_y} + S_{x, y}, j]z) \\ = S(jx, jy, jz, jw) - \overline{S(jx, jy, z, w)} + S(x, y, jz, jw) - \overline{S(x, y, z, w)}.$$

Now putting $x = y = z = w = u$ we obtain:

$$0 = S(ju, ju, ju, ju) - \overline{S(ju, ju, u, u)} + S(u, u, ju, ju) - \overline{S(u, u, u, u)}$$

and putting $x = iu$ and $y = z = w = u$ we obtain:

$$0 = -iS(ju, ju, ju, ju) - \overline{iS(ju, ju, u, u)} + iS(u, u, ju, ju) + \overline{iS(u, u, u, u)}.$$

Comparing these two equations we get $S(ju, ju, ju, ju) = \overline{S(u, u, u, u)}$, i.e. $S = \tau S$. This shows that the reality condition implies that $S \in (S^4E)^\tau$. Conversely the condition $S = \tau S$ can be written as

$$S(jx, jy, jz, jw) = \overline{S(x, y, z, w)} \quad \text{for all } x, y, z, w \in E.$$

Changing $z \rightarrow jz$ and $w \rightarrow jw$ in this equation we obtain

$$S(jx, jy, z, w) = \overline{S(x, y, jz, jw)} \quad \text{for all } x, y, z, w \in E.$$

These two equations imply (5.1) and hence the reality condition. □

Now we are ready to classify simply connected hyper-Kähler symmetric spaces. We will show that the following construction gives all such symmetric spaces.

Let (E, ω, j) be a complex symplectic vector space of dimension $2n$ with a quaternionic structure j such that $\omega(jx, jy) = \omega(x, y)$ for all $x, y \in E$ and $E = E_+ \oplus E_-$ a j -invariant Lagrangian decomposition. Such a decomposition exists if and only if the Hermitian form $\gamma = \omega(\cdot, j\cdot)$ has real signature $(4m, 4m)$, where $\dim_{\mathbb{C}} E = 2n = 4m$. Then any polynomial $S \in (S^4 E_+)^{\tau} = S^4 E_+ \cap (S^4 E)^{\tau}$ satisfies the condition (4.4) and the reality condition, by Proposition 8. Hence by Theorem 1 it defines a (real) simply connected hyper-Kähler symmetric space M_S with Abelian holonomy algebra $\mathfrak{h} = (S_{E_+, E_+})^{\tau} = S_{E_+, E_+} \cap (S^2 E)^{\tau} = \text{span}\{S_{j e, e'} - S_{e, j e'} | e, e' \in E\} \subset \text{sp}(E)^{\tau} \cong \text{sp}(m, m)$.

THEOREM 6. *Any simply connected hyper-Kähler symmetric space without flat de Rham factor is isomorphic to a hyper-Kähler symmetric space of the form M_S , where $S = T + \tau T$, $T \in S^4 E_+$ and $E_+ \subset E$ is a j -invariant Lagrangian subspace of the complex symplectic vector space E with compatible quaternionic structure j . A hyper-Kähler symmetric space of the form M_S has no flat factor if and only if its complexification has no flat factor, which happens if and only if the support $\Sigma_S = E_+$. Moreover there is a natural 1-1 correspondence between simply connected hyper-Kähler symmetric spaces without flat factor up to isomorphism and orbits \mathcal{O} of the group $\text{Aut}(E, \omega, j, E_+) |_{E_+} = \{A \in \text{Sp}(E) | [A, j] = 0, A E_+ = E_+\} |_{E_+} \cong \text{GL}(m, \mathbb{H})$ on the space $(S^4 E_+)^{\tau}$ such that $\Sigma_S = E_+$ for all $S \in \mathcal{O}$.*

Proof: Let M be a simply connected hyper-Kähler symmetric space. We first assume that it is indecomposable. Then the holonomy algebra \mathfrak{h} is weakly irreducible. By Theorem 1, $M = M_S$ for some quartic polynomial $S \in S^4 E$ satisfying (4.4) and the reality condition (3.2). By Proposition 8 the reality condition means that $S \in (S^4 E)^{\tau}$. On the other hand, by Theorem 3 $S \in S^4 L$ for some Lagrangian subspace L of E . Now the weak irreducibility of \mathfrak{h} and Lemma 8 imply that $L = L^0$ is j -invariant. This proves that $S \in (S^4 E_+)^{\tau}$, where $E_+ = L = L^0$ is a j -invariant Lagrangian subspace of E . This shows that M is obtained from the above construction. Any simply connected hyper-Kähler symmetric space M without flat factor is the Riemannian product of indecomposable ones, say $M = M_1 \times M_2 \times \dots \times M_r$, and we may assume that $M_i = M_{S_i}$, $S_i \in S^4 E_i$. Therefore M is associated to the quartic polynomial $S = S_1 \oplus S_2 \oplus \dots \oplus S_r \in S^4 E$, $E = E_1 \oplus E_2 \oplus \dots \oplus E_r$. Moreover S satisfies (4.4) and (3.2) if the S_i satisfy (4.4) and (3.2). This shows that any simply connected hyper-Kähler symmetric space is obtained from the above construction.

It is clear that the complexification $M_S^{\mathbb{C}}$ has a flat factor if M_S has a flat factor. Conversely let us assume that $M_S^{\mathbb{C}}$ has a flat factor, hence $\Sigma_S \subset E_+$ is a proper subspace. Since $j \Sigma_S = j S_{E, E} E = S_{E, E} j E = S_{E, E} E = \Sigma_S$ there exists a j -invariant complementary subspace E'_+ in E_+ . Denote by E'_- the annihilator of Σ_S in E_- then $E' = E'_+ \oplus E'_-$ is an ω -nondegenerate and j -invariant subspace of E on which the holonomy $\mathfrak{h}^{\mathbb{C}} = S_{E, E} \subset S^2 \Sigma_S$ acts trivially. Then the corresponding real subspace $(H \otimes E')^{\rho} \subset \mathfrak{m} = (H \otimes E)^{\rho}$ is a g -nondegenerate subspace on which the holonomy \mathfrak{h} acts trivially. By Wu's theorem [W] it defines a flat de Rham factor.

Now the last statement follows from the corresponding statement in Theorem 1.

□

COROLLARY 2. *Any hyper-Kähler symmetric space without flat factor has signature $(4m, 4m)$. In particular its dimension is divisible by 8.*

COROLLARY 3. *Let $M = M_S$ be a complex hyper-Kähler symmetric space without flat factor associated with a quartic $S \in S^4 E_+$, where $E_+ \subset E$ is a Lagrangian subspace. It admits a real form if and only if there exists a quaternionic structure j*

on E compatible with ω preserving E_+ such that $\tau S = S$, where τ is the real structure on $S^4 E$ induced by j . In particular $\dim_{\mathbb{C}} M$ has to be divisible by 8.

Let $M = G/K$ be a simply connected hyper-Kähler symmetric space without flat factor. By Theorem 6 it is associated to a quartic polynomial $S \in (S^4 E_+)^{\tau}$ with support $\Sigma_S = E_+$. Now we describe the Lie algebra $\text{aut}(M_S)$ of the full group of automorphisms, i.e. isometries which preserve the hypercomplex structure of M_S .

THEOREM 7. *Let $M_S = G/K$ be as above. Then the full automorphism algebra is given by*

$$\text{aut}(M_S) = \text{aut}(S) + \mathfrak{g},$$

where $\text{aut}(S) = \{A \in \mathfrak{gl}(E_+) \mid [A, j] = 0, A \cdot S = 0\}$ acts on

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m}, \mathfrak{h} = \{A \in S_{E,E} \mid [A, j] = 0\} = \text{span}\{S_{jx,y} - S_{x,jy} \mid x, y \in E\}, \mathfrak{m} = (H \otimes E)^{\rho}$$

as in Theorem 5.

Proof: The proof is similar to that of Theorem 5. \square

6. Low Dimensional Hyper-Kähler Symmetric Spaces.

6.1 Complex hyper-Kähler symmetric spaces of dimension ≤ 8 .

Dimension 4

Assume that M is a simply connected complex hyper-Kähler symmetric space of dimension 4. Applying Theorem 4 we conclude that $M = M_S$ for some $S \in S^4 E_+$, where $E_+ \subset E$ is a one-dimensional subspace $E_+ = \mathbb{C}e$. This proves:

THEOREM 8. *There exists up to isomorphism only one non-flat simply connected complex hyper-Kähler symmetric space of dimension 4: $M = M_S$ associated with the quartic $S = e^4$.*

Dimension 8

Any eight-dimensional simply connected complex hyper-Kähler symmetric space is associated with a quartic $S \in S^4 E_+$, where $E_+ \subset E$ is a Lagrangian subspace of $E = \mathbb{C}^4$. We denote by (e, e') a basis of E_+ .

THEOREM 9. *Eight-dimensional simply connected complex hyper-Kähler symmetric space are in natural 1-1 correspondence with the orbits of the group $\text{CO}(3, \mathbb{C}) = \mathbb{C}^* \cdot \text{SO}(3, \mathbb{C})$ on the space $S_0^2 \mathbb{C}^3$ of traceless symmetric matrices. The complex hyper-Kähler symmetric space associated with a traceless symmetric matrix A is the manifold $M_{S(A)}$, where $S(A) \in S^4 \mathbb{C}^2$ is the quartic polynomial which corresponds to A under the $\text{SO}(3, \mathbb{C})$ -equivariant isomorphism $S_0^2 \mathbb{C}^3 \cong S_0^2 \wedge^2 \mathbb{C}^3 \cong S_0^2 S^2 \mathbb{C}^2 = S^4 \mathbb{C}^2$.*

The classification of $\text{SO}(3, \mathbb{C})$ -orbits on $S_0^2 \mathbb{C}^3$ was given by Petrov [P] in his classification of Weyl tensors of Lorentzian 4-manifolds.

Proof: By Theorem 4 the classification of eight-dimensional simply connected complex hyper-Kähler symmetric spaces reduces to the description of orbits of the group $\text{GL}(E_+) = \text{GL}(2, \mathbb{C})$ on $S^4 \mathbb{C}^2 \subset S^2 S^2 \mathbb{C}^2$. Fixing a volume form σ on \mathbb{C}^2 we can identify $S^2 \mathbb{C}^2$ with $\mathfrak{sp}(1, \mathbb{C}) \cong \mathfrak{so}(3, \mathbb{C})$. Then the Killing form B is an $\text{SL}(2, \mathbb{C})$ -invariant and we have the $\text{GL}(2, \mathbb{C})$ -invariant decomposition: $S^2 S^2 \mathbb{C}^2 = S_0^2 S^2 \mathbb{C}^2 \oplus \mathbb{C}B$. The action of $\text{SL}(2, \mathbb{C})$ on $S^2 \mathbb{C}^2$ is effectively equivalent to the adjoint action of $\text{SO}(3, \mathbb{C})$. The problem thus reduces essentially to the determination of the orbits of $\text{SO}(3, \mathbb{C})$ on $S_0^2 \mathbb{C}^3$. \square

5.1. Hyper-Kähler symmetric spaces of dimension ≤ 8 . By Corollary 3 the minimal dimension of non-flat hyper-Kähler symmetric spaces is 8.

THEOREM 10. *Eight-dimensional simply connected hyper-Kähler symmetric space are in natural 1-1 correspondence with the orbits of the group $\mathbb{R}^+ \cdot \text{SO}(3)$ on the space $S_0^2 \mathbb{R}^3$ of traceless symmetric matrices. The hyper-Kähler symmetric space associated with a traceless symmetric matrix A is the manifold $M_{S(A)}$, where $S(A) \in (S^4 \mathbb{C}^2)^\tau$ is the quartic polynomial which corresponds to A under the $\text{SO}(3)$ -equivariant isomorphism $S_0^2 \mathbb{R}^3 \cong (S^4 \mathbb{C}^2)^\tau$.*

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