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MAPPING OF NILPOTENT ORBITS UNDER EMBEDDINGS OF REAL FORMS OF EXCEPTIONAL COMPLEX LIE ALGEBRAS *

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Abstract. We consider Lie algebra monomorphisms $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ between various noncompact real forms $\mathfrak{g}_1$, $\mathfrak{g}_2$ of complex simple Lie algebras $\mathfrak{g}_1^c$, $\mathfrak{g}_2^c$. In all cases that we consider, $\mathfrak{g}_1^c$ or $\mathfrak{g}_2^c$ is of exceptional type, with one exception. For each adjoint nilpotent orbit $\mathcal{O}$ of $\mathfrak{g}_1$ we determine the adjoint nilpotent orbit of $\mathfrak{g}_2$ which contains the image $\varphi(\mathcal{O})$. The adjoint nilpotent orbits of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are themselves parametrized by using the Kostant–Sekiguchi correspondence.

1. Preliminaries. Let $\mathfrak{g}$ be a semisimple real Lie algebra and $\mathfrak{g}^c$ its complexification. Let θ be a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k}+\mathfrak{p}$ the corresponding $\mathbf{Z}_2$-gradation (a Cartan decomposition). By complexifying, we obtain the $\mathbf{Z}_2$-gradation $\mathfrak{g}^c = \mathfrak{k}^c + \mathfrak{p}^c$ and we extend θ to a complex linear automorphism θ^c of $\mathfrak{g}^c$. Denote by G (resp. G^c) the adjoint group of $\mathfrak{g}$ (resp. $\mathfrak{g}^c$). Thus G is the connected Lie subgroup of G^c with Lie algebra $\mathfrak{g}$. The group G (resp. G^c) acts on $\mathfrak{g}$ (resp. $\mathfrak{g}^c$) via the adjoint action: $(a, x) \to a \cdot x = \mathrm{Ad}(a)(x)$ where $a \in G^c$ (resp. G) and $x \in \mathfrak{g}^c$ (resp. $\mathfrak{g}$). A G^c (resp. G)-orbit is *nilpotent* if it consists of nilpotent elements of $\mathfrak{g}^c$ (resp $\mathfrak{g}$). There are only finitely many nilpotent G^c (resp. G)-orbits in $\mathfrak{g}^c$ (resp. $\mathfrak{g}$).

Let K^c (resp. K) be the connected Lie subgroup of G^c (resp. G) whose Lie algebra is $\mathfrak{k}^c$ (resp. $\mathfrak{k}$). By restricting the adjoint action of G^c, we obtain an action of K^c on $\mathfrak{p}^c$. The number of nilpotent K^c-orbits in $\mathfrak{p}^c$ is also finite [4].

Let $\mathcal{O}$ be a nilpotent G^c-orbit in $\mathfrak{g}^c$. The intersection $\mathcal{O} \cap \mathfrak{g}$ consists of finitely many connected components $\mathcal{A}_i$, $i = 1, \ldots, k$. Moreover, each of these connected components is a single nilpotent G-orbit, and $\dim_{\mathbf{R}}(\mathcal{A}_i) = \dim_{\mathbf{C}}(\mathcal{O})$ for each i. The intersection $\mathcal{O} \cap \mathfrak{p}^c$ also consists of k connected components, say $\mathcal{B}_i$, $i = 1, \ldots, k$, each of them is a single nilpotent K^c-orbit and $\dim_{\mathbf{C}}(\mathcal{B}_i) = \frac{1}{2}\dim_{\mathbf{C}}(\mathcal{O})$ for each i. The Kostant–Sekiguchi correspondence (see [10, 4]) establishes a bijection from $\{\mathcal{A}_i\}$ to $\{\mathcal{B}_i\}$.

If $E, H, F \in \mathfrak{g}^c$ are nonzero elements satisfying $[H, E] = 2E$, $[H, F] = -2F$, and $[F, E] = H$, then we say that (E, H, F) is a *standard triple*. If moreover $E, F \in \mathfrak{p}^c$ and $H \in \mathfrak{k}^c$, then we say that (E, H, F) is a *normal triple*. Let us fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$ and $\tilde{\mathfrak{h}}$ of $\mathfrak{g}$ such that $\tilde{\mathfrak{h}} \supseteq \mathfrak{h}$. Let $\mathfrak{h}^c$ and $\tilde{\mathfrak{h}}^c$ be their respective complexifications. Let R be the root system of $(\mathfrak{k}^c, \mathfrak{h}^c)$, and $\tilde{R}$ that of $(\mathfrak{g}^c, \tilde{\mathfrak{h}}^c)$. Finally, let W be the Weyl group of R, $\Pi = \{\beta_1, \beta_2, \ldots\}$ a base of R (a system of fundamental roots), and define $\tilde{W}$, $\tilde{R}$ and $\tilde{\Pi} = \{\alpha_1, \alpha_2, \ldots\}$ similarly. If $\tilde{R}$ (resp. R) is irreducible, we denote its highest root by $\tilde{\alpha}$ (resp. $\tilde{\beta}$).

We say that the $\mathbf{Z}_2$-graded Lie algebra $\mathfrak{g}^c$ is of *inner type* if $\mathfrak{k}^c$ and $\mathfrak{g}^c$ have the same rank, i.e., $\mathfrak{h}^c = \tilde{\mathfrak{h}}^c$, and otherwise of *outer type*. In the former case we may view R as a root subsystem of $\tilde{R}$, and we say that the roots in R are *compact* and the other roots in $\tilde{R}$ are *noncompact*. In the root diagrams, we shall represent compact (resp. noncompact) roots by black (resp. white) nodes.

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If $\mathfrak{g}^c$ is simple and of inner type, then one can choose a base $\tilde{\Pi}$ of $\tilde{R}$ such that there exists a unique base Π of R contained in $\tilde{\Pi} \cup \{-\tilde{\alpha}\}$. We assume that $\tilde{\Pi}$ and Π are chosen in this fashion (see Tables 4 and 8).

Let $\mathcal{A}$ be a nonzero nilpotent K^c-orbit in $\mathfrak{p}^c$. We can choose a normal triple (E, H, F) such that $E \in \mathcal{A}$, $H \in \mathfrak{h}^c$, and the numbers $\beta(H)$ are nonnegative integers for each $\beta \in \Pi$. The orbit $\mathcal{A}$ uniquely determines the element H and vice versa. We shall refer to H as the *characteristic* of the orbit $\mathcal{A}$. Among the algebras that we consider, there are two cases with $\mathfrak{k}^c$ non-semisimple. In these two cases $\tilde{\mathfrak{h}}^c = \mathfrak{h}^c$, $\tilde{\Pi} = \Pi \cup \{\beta'\}$, and we identify H by means of the labels $\beta_j(H)$, $\beta_j \in \Pi$, and the additional label $\beta'(H)$.

The containment relation between the closures of nilpotent G-orbits in $\mathfrak{g}$ defines a partial order on the set of these orbits. One obtains similarly a partial order on the set of nilpotent G^c-orbits in $\mathfrak{g}^c$ and the set of nilpotent K^c-orbits in $\mathfrak{p}^c$. We refer to these partial orders as the *closure orderings*.

It was shown by Barbasch and Sepanski [1] that the Kostant–Sekiguchi correspondence preserves the closure ordering of the two sets of orbits. Let $\mathcal{N}(\mathfrak{g}^c)$ denote the nilpotent variety of $\mathfrak{g}^c$ (an irreducible affine variety). We set $\mathcal{N}(\mathfrak{g}) = \mathfrak{g} \cap \mathcal{N}(\mathfrak{g}^c)$ and $\mathcal{N}(\mathfrak{p}^c) = \mathfrak{p}^c \cap \mathcal{N}(\mathfrak{g}^c)$. If we equip the quotients $\mathcal{N}(\mathfrak{g})/G$ and $\mathcal{N}(\mathfrak{p}^c)/K^c$ with their respective quotient topologies, then the result of Barbasch and Sepanski can be expressed by saying that the Kostant–Sekiguchi correspondence is a homeomorphism between these two finite topological spaces.

We shall use the Cartan notation for the isomorphism types of noncompact real forms of the exceptional complex Lie algebras. Alternatively, these real forms may be distinguished by their Cartan indices $i = \dim(\mathfrak{p}) - \dim(\mathfrak{k})$ which are usually written in parentheses.

$$
\begin{aligned}
E_6 &: \mathrm{E\,I} = E_{6(6)},\ \mathrm{E\,II} = E_{6(2)},\ \mathrm{E\,III} = E_{6(-14)},\ \mathrm{E\,IV} = E_{6(-26)} \\
E_7 &: \mathrm{E\,V} = E_{7(7)},\ \mathrm{E\,VI} = E_{7(-5)},\ \mathrm{E\,VII} = E_{7(-25)} \\
E_8 &: \mathrm{E\,VIII} = E_{8(8)},\ \mathrm{E\,IX} = E_{8(-24)} \\
F_4 &: \mathrm{F\,I} = F_{4(4)},\ \mathrm{F\,II} = F_{4(-20)} \\
G_2 &: \mathrm{G\,I} = G_{2(2)}
\end{aligned}
$$

Let us now consider two semisimple real Lie algebras, say, $\mathfrak{g}_1$ and $\mathfrak{g}_2$. The notations θ, θ^c, G, G^c, W, Π, etc. will be used also for these algebras and the associated groups by adding subscripts 1 or 2, as appropriate. In particular, θ_1 and θ_2 are the Cartan involutions of $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively. We say that a Lie algebra monomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a $\mathbf{Z}_2$-*embedding* of $\mathfrak{g}_1$ in $\mathfrak{g}_2$ if $\varphi \circ \theta_1 = \theta_2 \circ \varphi$. From now on we assume that φ is such an embedding. Then $\varphi(\mathfrak{k}_1) \subseteq \mathfrak{k}_2$ and $\varphi(\mathfrak{p}_1) \subseteq \mathfrak{p}_2$. The complexification $\varphi^c : \mathfrak{g}_1^c \to \mathfrak{g}_2^c$ of φ will also be called a $\mathbf{Z}_2$-*embedding*. Then we have the commutative diagram:

$$
\begin{array}{ccc}
\mathfrak{g}_1 & \longrightarrow & \mathfrak{g}_1^c \\
\varphi \downarrow & & \downarrow \varphi^c \\
\mathfrak{g}_2 & \longrightarrow & \mathfrak{g}_2^c
\end{array}
\tag{1.1}
$$

where the horizontal arrows are the inclusion maps. From this diagram we obtain the following commutative diagram for orbit spaces:

$$
\begin{array}{ccc}
\mathcal{N}(\mathfrak{g}_1)/G_1 & \longrightarrow & \mathcal{N}(\mathfrak{p}_1^c)/K_1^c \\
\mu \downarrow & & \downarrow \nu \\
\mathcal{N}(\mathfrak{g}_2)/G_2 & \longrightarrow & \mathcal{N}(\mathfrak{p}_2^c)/K_2^c
\end{array}
\tag{1.2}
$$

where the horizontal arrows are homeomorphisms given by the Kostant–Sekiguchi correspondence and the vertical arrows μ and ν are the continuous maps induced by the $\mathbf{Z}_2$-embeddings φ and φ^c, respectively.

Our main objective is to give explicit description of the maps μ and ν for some interesting $\mathbf{Z}_2$-embeddings of real forms of complex simple Lie algebras. These embeddings are taken from an extensive list compiled by Berger [2]. In view of the commutativity of the above diagram, it suffices to determine the map ν. The main results are given in the tables of Sections 2 and 3.

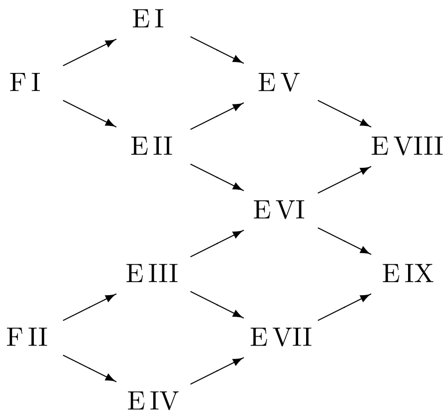

FIGURE 1. *Embeddings of real forms*

First of all we shall consider the $\mathbf{Z}_2$-embeddings shown schematically on Figure 1. We shall describe them in the next section. By analyzing the $\mathbf{Z}_2$-embedding E VI $\to$ E IX and by inspecting the closure diagrams for E VI [7, Figure 2] and E IX [8, Figure 3], we detected an error in the former diagram: The line joining the nodes 22 and 33 should be erased.

In addition to the $\mathbf{Z}_2$-embeddings shown on Figure 1, we shall also consider the following chain of $\mathbf{Z}_2$-embeddings of the split real forms:

$$\mathfrak{sl}(3,\mathbf{R}) \longrightarrow \text{G I} \longrightarrow \mathfrak{so}(4,3) \longrightarrow \mathfrak{so}(4,4) \longrightarrow \mathfrak{so}(5,4) \longrightarrow \text{F I} \tag{1.3}$$

For each of the arrows in Figure 1 and the diagram (1.3) we describe explicitly the map ν in tabular form. For the arrows in Figure 1 see Table 2 in the next section, and for those in the diagram (1.3) see the tables in Section 3. In order to make these tables user-friendly, we have included the necessary details about the enumeration of orbits. For the exceptional cases, these details are given in the Appendix.

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2. Embeddings from Figure 1. It is more convenient to work with complex Lie algebras than with the real ones. Hence in order to construct a commutative diagram (1.1) we shall start with the complex $\mathbf{Z}_2$-embedding $\varphi^c : \mathfrak{g}_1^c \to \mathfrak{g}_2^c$ and then construct the $\mathbf{Z}_2$-embedding of real froms $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ to obtain the diagram (1.1). This is indeed possible by the following result.

PROPOSITION 2.1. *Let $\mathfrak{g}_2^c$ be a semisimple complex Lie algebra and $\mathfrak{g}_1^c$ a semisimple subalgebra of $\mathfrak{g}_2^c$. Let θ_2^c be an involutorial automorphism of $\mathfrak{g}_2^c$, such that $\mathfrak{g}_1^c$ is*

θ_2^c-stable, and denote by θ_1^c the restriction of θ_2^c to $\mathfrak{g}_1^c$. Let $\mathfrak{g}_i^c = \mathfrak{k}_i^c \oplus \mathfrak{p}_i^c$ be the $\mathbf{Z}_2$-gradations induced by θ_i^c $(i = 1, 2)$. Then there exist real forms $\mathfrak{g}_i$ of $\mathfrak{g}_i^c$ such that $\mathfrak{g}_1 \subseteq \mathfrak{g}_2$ and $\mathfrak{g}_i$ is stable under θ_i^c. Moreover, the restriction $\theta_i = \theta_i^c|_{\mathfrak{g}_i}$ is a Cartan involution of $\mathfrak{g}_i$. Thus if $\mathfrak{k}_i = \mathfrak{g}_i \cap \mathfrak{k}_i^c$ and $\mathfrak{p}_i = \mathfrak{g}_i \cap \mathfrak{p}_i^c$, then $\mathfrak{g}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$ is a Cartan decomposition of $\mathfrak{g}_i$.

Proof. Let U_1 be a maximal compact subgroup of $\mathrm{Aut}(\mathfrak{g}_1^c)$ containing θ_1^c. The Lie algebra $\mathfrak{u}_1$ of U_1 is a compact real form of $\mathfrak{g}_1^c$ which is invariant under θ_1^c. Clearly, it is also invariant under θ_2^c. It follows that θ_2^c normalizes the connected compact Lie subgroup $\widehat{U}_1$ of $\mathrm{Aut}(\mathfrak{g}_2^c)$ having $\mathfrak{u}_1$ as its Lie algebra. Consequently, there exists a maximal compact subgroup U_2 of $\mathrm{Aut}(\mathfrak{g}_2^c)$ containing both $\widehat{U}_1$ and θ_2^c. Its Lie algebra, $\mathfrak{u}_2$, is a compact real form of $\mathfrak{g}_2^c$ invariant under θ_2^c and such that $\mathfrak{u}_2 \cap \mathfrak{g}_1^c = \mathfrak{u}_1$. We can now take

$$\mathfrak{g}_1 = (\mathfrak{u}_1 \cap \mathfrak{k}_1^c) \oplus i(\mathfrak{u}_1 \cap \mathfrak{p}_1^c), \quad \mathfrak{g}_2 = (\mathfrak{u}_2 \cap \mathfrak{k}_2^c) \oplus i(\mathfrak{u}_2 \cap \mathfrak{p}_2^c).$$

□

We may (and we do) assume that our $\mathbf{Z}_2$-embeddings φ^c are such that $\varphi^c(\mathfrak{h}_1^c) \subseteq \mathfrak{h}_2^c$ and $\varphi^c(\tilde{\mathfrak{h}}_1^c) \subseteq \tilde{\mathfrak{h}}_2^c$.

Assume that $\mathfrak{g}_1^c$ and $\mathfrak{g}_2^c$ are of inner type. Then we say that an embedding of root systems $\tilde{R}_1 \to \tilde{R}_2$ is a $\mathbf{Z}_2$*-embedding* if the compact roots are mapped to compact and noncompact to noncompact. Such embedding is uniquely determined by its restriction to a base of $\tilde{R}_1$. It is easy to see that every $\mathbf{Z}_2$-embedding $\tilde{R}_1 \to \tilde{R}_2$ can be lifted to a $\mathbf{Z}_2$-embedding $\mathfrak{g}_1^c \to \mathfrak{g}_2^c$.

We describe our embeddings in Table 1. In order to be able to distinguish the roots in R_1 or $\tilde{R}_1$ from those in R_2 or $\tilde{R}_2$, we shall use the superscript $^{(1)}$ for the former and $^{(2)}$ for the latter. We write $\beta_i^{(1)} \to \beta_j^{(2)}$ if φ^c maps the $\beta_i^{(1)}$-root space of $\mathfrak{g}_1^c$ into the $\beta_j^{(2)}$-root space of $\mathfrak{g}_2^c$. Similarly, we write $\beta_i^{(1)} \to \{\beta_j^{(2)}, \beta_k^{(2)}\}$ if φ^c embeds the $\beta_i^{(1)}$-root space of $\mathfrak{g}_1^c$ diagonally into the sum of the root spaces of $\mathfrak{g}_2^c$ corresponding to the roots $\beta_j^{(2)}$ and $\beta_k^{(2)}$. In some cases we give only the restriction of φ^c which embeds $\mathfrak{k}_1^c$ into $\mathfrak{k}_2^c$. For our choice of the simple roots of $\tilde{R}$ and R see Table 8 in the Appendix.

TABLE 1. *Description of the* $\mathbf{Z}_2$*-embeddings from Figure 1*

F I → E I	$\beta_k^{(1)} \to \beta_{k+1}^{(2)},\ k=1,2,3;\ \beta_4^{(1)} \to -\tilde{\beta}^{(2)}$
F I → E II	$\beta_1^{(1)} \to \{\beta_1^{(2)}, \beta_5^{(2)}\},\ \beta_2^{(1)} \to \{\beta_2^{(2)}, \beta_4^{(2)}\},$ $\beta_3^{(1)} \to \beta_3^{(2)},\ \beta_4^{(1)} \to \beta_6^{(2)}$
F II → E III	$\beta_k^{(1)} \to \beta_k^{(2)},\ k=1,2,3;\ \beta_4^{(1)} \to \{\beta_4^{(2)}, \beta_5^{(2)}\}$
F II → E IV	$\beta_1^{(1)} \to -\tilde{\beta}^{(2)};\ \beta_k^{(1)} \to \beta_{k-1}^{(2)},\ k=2,3,4$
E I → E V	$\beta_k^{(1)} \to \{\beta_k^{(2)}, \beta_{8-k}^{(2)}\},\ k=1,2,3;\ \beta_4^{(1)} \to \beta_4^{(2)}$
E II → E V	$\beta_k^{(1)} \to \beta_k^{(2)},\ 1 \le k \le 5;\ \beta_6^{(1)} \to \beta_7^{(2)},$ $\alpha_2^{(1)} \to \alpha_2^{(2)} + \alpha_4^{(2)} + \alpha_5^{(2)} + \alpha_6^{(2)}$
E II → E VI	$\beta_k^{(1)} \to \beta_k^{(2)},\ 1 \le k \le 5;\ \beta_6^{(1)} \to \beta_7^{(2)},$ $\alpha_2^{(1)} \to \alpha_2^{(2)} + \alpha_4^{(2)} + \alpha_5^{(2)} + \alpha_6^{(2)}$
E III → E VI	$\beta_k^{(1)} \to \beta_{k+1}^{(2)},\ 1 \le k \le 5;\ \beta_6^{(1)} \to \alpha_6^{(2)}$
E III → E VII	$\beta_1^{(1)} \to \beta_1^{(2)}, \beta_k^{(1)} \to \beta_{k+1}^{(2)},\ k=2,3,4;$ $\beta_5^{(1)} \to \beta_2^{(2)},\ \beta_6^{(1)} \to \alpha_6^{(2)} + \alpha_7^{(2)}$
E IV → E VII	$\beta_1^{(1)} \to \beta_2^{(2)},\ \beta_2^{(1)} \to \beta_4^{(2)},\ \beta_3^{(1)} \to \{\beta_3^{(2)}, \beta_5^{(2)}\},$ $\beta_4^{(1)} \to \{\beta_1^{(2)}, \beta_6^{(2)}\}$
E V → E VIII	$\beta_k^{(1)} \to \beta_k^{(2)},\ 1 \le k \le 7;$ $\alpha_2^{(1)} \to \alpha_1^{(2)} + \alpha_2^{(2)} + 2\alpha_3^{(2)} + 2\alpha_4^{(2)} + \alpha_5^{(2)}$
E VI → E VIII	$\beta_k^{(1)} \to \beta_k^{(2)},\ 1 \le k \le 4;\ \beta_5^{(1)} \to \beta_8^{(2)},$ $\beta_6^{(1)} \to \beta_7^{(2)},\ \beta_7^{(1)} \to \beta_1^{(2)},$ $\alpha_6^{(1)} \to \alpha_1^{(2)} + \alpha_3^{(2)} + \alpha_4^{(2)} + \alpha_5^{(2)} + \alpha_6^{(2)} + \alpha_7^{(2)} + \alpha_8^{(2)}$
E VI → E IX	$\beta_k^{(1)} \to \beta_{8-k}^{(2)},\ 1 \le k \le 4;\ \beta_5^{(1)} \to \beta_2^{(2)},$ $\beta_6^{(1)} \to \beta_3^{(2)},\ \beta_7^{(1)} \to \beta_8^{(2)},$ $\alpha_6^{(1)} \to \alpha_1^{(2)} + \alpha_3^{(2)} + \alpha_4^{(2)} + \alpha_5^{(2)} + \alpha_6^{(2)} + \alpha_7^{(2)} + \alpha_8^{(2)}$
E VII → E IX	$\beta_k^{(1)} \to \beta_k^{(2)},\ 1 \le k \le 6;\ \beta_7^{(1)} \to \alpha_7^{(2)} + \alpha_8^{(2)}$

Let $\mathcal{O}_1^i \subseteq \mathfrak{p}_1^c$ be the i-th nonzero nilpotent K_1^c-orbit and let $H_1^i \in \mathfrak{h}_1^c$ be its characteristic. Define similarly $\mathcal{O}_2^j \subseteq \mathfrak{p}_2^c$ and $H_2^j \in \mathfrak{h}_2^c$. We shall write $i \to j$ if $\varphi^c(\mathcal{O}_1^i) \subseteq \mathcal{O}_2^j$.

THEOREM 2.2. *Consider the* $\mathbf{Z}_2$*-embeddings* $\varphi^c : \mathfrak{g}_1^c \to \mathfrak{g}_2^c$ *from Figure 1 and described by Table 1. Then the nonzero nonempty fibers of the map* ν *(see the diagram (1.2)) are as given in Table 2.*

In Table 2, for each value of $j \neq 0$, we have recorded the superscripts i (if any) such that $i \to j$. The i's are listed first (on the left hand side) and then the j (on the right hand side). For instance, we have $11, 12 \to 7$ under the embedding F I $\to$ E I. The arrows are omitted. If $i \to j$ then, in general, $\varphi^c(H_1^i)$ is not equal to H_2^j, but they belong to the same orbit of the Weyl group W_2.

The above theorem is a simple consequence of the following proposition. Indeed the labels of $\varphi^c(H_1^i)$ can be computed by using the transformation rules given in Table 3, and then, by using the action of W_2, one can determine the superscript j such that $i \to j$.

PROPOSITION 2.3. *Consider the* $\mathbf{Z}_2$*-embeddings* $\varphi^c : \mathfrak{g}_1^c \to \mathfrak{g}_2^c$ *from Figure 1 and described by Table 1. Let* $\mathcal{O}_1^i \subseteq \mathfrak{p}_1^c$ *and* $\mathcal{O}_2^j \subseteq \mathfrak{p}_2^c$ *be nonzero nilpotent orbits such that* $\varphi^c(\mathcal{O}_1^i) \subseteq \mathcal{O}_2^j$*. If* $H_1^i \in \mathfrak{h}_1^c$ *is the characteristic of the orbit* $\mathcal{O}_1^i$*, then the labels* $\beta_k^{(2)}(\varphi^c H_1^i)$ *of the element* $\varphi^c(H_1^i) \in \mathfrak{h}_2^c$ *can be computed from the labels* $\beta_k^{(1)}(H_1^i)$ *by using the transformation rules given in Table 3.*

Proof. The proofs are different in each case but they are of similar nature. We shall give the details for four cases only.

We derive first the transformation rule for the $\mathbf{Z}_2$-embedding F I $\to$ E I. Thus $\mathfrak{g}_1$ is of type F I. The labels of H_1^i are given in column 2 of Table 3 as "$abc\ d$". Using the first row of Table 1, this means that

$$\beta_2^{(2)}(\varphi^c H_1^i) = \beta_1^{(1)}(H_1^i) = a, \quad \beta_3^{(2)}(\varphi^c H_1^i) = \beta_2^{(1)}(H_1^i) = b,$$
$$\beta_4^{(2)}(\varphi^c H_1^i) = \beta_3^{(1)}(H_1^i) = c, \quad -\tilde{\beta}^{(2)}(\varphi^c H_1^i) = \beta_4^{(1)}(H_1^i) = d.$$

Since $\tilde{\beta}^{(2)} = 2\beta_1^{(2)} + 2\beta_2^{(2)} + 2\beta_3^{(2)} + \beta_4^{(2)}$, we have $-d = 2x + 2a + 2b + c$ where $x = \beta_1^{(2)}(\varphi^c H_1^i)$. This gives the required formula for the label x.

Next, we consider the embedding F I $\to$ E II. Let σ be the diagram automorphism of $\mathfrak{g}_2^c$ of order two which fixes the root spaces for $\alpha_2^{(2)}$ and $\alpha_4^{(2)}$ and interchanges those for $\alpha_1^{(2)}$ and $\alpha_6^{(2)}$ as well as those for $\alpha_3^{(2)}$ and $\alpha_5^{(2)}$ (see e.g. [3, Chapter 8, §5, Exercise 13]). The fixed point subalgebra of σ is a simple Lie algebra of type F_4 which we can identify with our $\mathfrak{g}_1^c$, and so we take φ_1^c to be the inclusion map. The automorphism θ_2^c of $\mathfrak{g}_2^c$ defined by the $\mathbf{Z}_2$-gradation of $\mathfrak{g}_2^c$ exhibited in Table 8 (with $\mathfrak{k}_2^c$ of type $A_5 + A_1$) leaves $\mathfrak{g}_1^c$ invariant. We denote by θ_1^c the restriction of θ_2^c to $\mathfrak{g}_1^c$. Then $\mathfrak{k}_1^c$ is of type $C_3 + A_1$. By Proposition 2.1 this gives a $\mathbf{Z}_2$-embedding F I $\to$ E II. The Cartan subalgebra $\mathfrak{h}_1^c$ is the subspace of $\mathfrak{h}_2^c$ defined by the equations $\alpha_1^{(2)}(H) = \alpha_6^{(2)}(H)$ and $\alpha_3^{(2)}(H) = \alpha_5^{(2)}(H)$. If $i \to j$ and the labels of H_1^i are "$abc\,d$", i.e.,

$$\beta_1^{(1)}(H_1^i) = a, \quad \beta_2^{(1)}(H_1^i) = b, \quad \beta_3^{(1)}(H_1^i) = c, \quad \beta_4^{(1)}(H_1^i) = d,$$

then the labels of $H_2^j = H_1^i$ are given by "$abcba\,d$". This follows from the fact that

$$\beta_1^{(2)}|_{\mathfrak{h}_1^c} = \beta_5^{(2)}|_{\mathfrak{h}_1^c} = \beta_1^{(1)}, \quad \beta_3^{(2)}|_{\mathfrak{h}_1^c} = \beta_3^{(1)},$$
$$\beta_2^{(2)}|_{\mathfrak{h}_1^c} = \beta_4^{(2)}|_{\mathfrak{h}_1^c} = \beta_2^{(1)}, \quad \beta_6^{(2)}|_{\mathfrak{h}_1^c} = \beta_4^{(1)}.$$

We shall now derive the transformation rules for the $\mathbf{Z}_2$-embeddings E III $\to$ E VI and E III $\to$ E VII. Thus $\mathfrak{g}_1$ is of type E III. The labels of H_1^i are given in the second

TABLE 2. *Mapping of nilpotent orbits*

F I → E I									
1	1	2,3	2	4,5	3	8	4	6,7	5
9	6	11,12	7	10	10	13	11	18	12
19,20	13	21	14	14,15	15	23	19	26	20
24,25	21	22	22	16,17	23				
F I → E II									
1	1	2	2	3	3	4	4	5	5
6	6	7	7	8	8	9	11	10	14
11	15	12	16	13	17	14	18	15	19
16	20	17	21	18	22	19	23	20	24
21	31	23	32	22	33	24	34	25	35
26	37								
F II → E III									
1	5	2	9						
F II → E IV									
1	1	2	2						
E I → E V									
1	1	2	2	3	5	5	6	4	7
8	12	10	15	7	20	6	21	11	24
15	25	12	26	23	27	13	30	9	43
16	50	17	53	14	59	19	62	22	63
21	66	18	81	20	84				
E II → E V									
1	1	2,3	2	4,5	5	6,7	6	8	7
9	10	10	11	12	13	13	14	14	15
15,16	20	11	21	17	24	18,19	25	22	26
20,21	27	23,24	30	25,26	38	28	48	27	49
29	51	30	52	31	59	32	62	33	63
34,35	66	36	80	37	84				
E II → E VI									
1	1	2	2	3	3	4	4	5	5
6	6	8	7	7	8	9,10	9	12,13	10
14	11	15	12	16	13	11	15	17	16
18	17	19	18	20	19	21	20	22	21
23	22	24	23	25	25	26	26	27,28	27
29,30	28	31	30	32	31	33	32	34	33
35	34	36	36	37	37				
E III → E VI									
1,2	1	3,4	2	5	3	6	8	7,8	9
10,11	13	9	14	12	26				
E III → E VII									
1	1	2	2	3	3	4	4	5	5
6	10	7	11	8	12	10	13	11	14
9	15	12	20						

Table 2. (continued)

E IV → E VII									
1	5	2	15						
E V → E VIII									
1	1	2	2	3,4,5	3	6	4	7	5
8,9	6	10,11	7	12	8	13,14	9	15	10
20	11	16,17	12	18,19	13	21	16	24	17
22,23,25	18	26	19	27	20	30	21	28,29	23
31,32	24	33,34	25	35,36	27	37	28	38	29
43	30	39,40	31	41,42	32	46,47	33	48,49	37
50	38	51,52	40	53	41	54	46	44,45,59	47
55,56	48	57,58	49	62	54	63	55	66	56
60,61	58	64,65	61	69,70	64	67,68	65	74,75	66
71	71	72,73	73	76,77	75	78,79	76	80	77
81	78	84	82	82,83	83	85,86	89	87,88	90
89,90	97	91,92	103	93,94	108				
E VI → E VIII									
1	1	2,3	2	4,5	3	6,8	4	7	5
9	7	10	9	11	10	12,13	11	14	14
15	16	16	17	17,18	18	21	19	19,20	20
22,23	21	24	26	25,26	29	27	37	28	40
29	45	30	47	31	54	32	55	33,34	56
35	70	36	77	37	102				
E VI → E IX									
1	1	2	2	3	3	4	4	5	5
6	6	7	7	8	8	9	9	11	10
10	11	12	12	13	13	14,15	14	16	15
18	16	17	17	20	18	19	19	21	20
22	21	23	22	24	23	25	24	26	25
27	26	28	27	29	28	30	29	31	30
32	31	33	32	34	33	35	34	36	35
37	36								
E VII → E IX									
1,2	1	3,4	2	5	3	6,7	4	8,9	5
10	8	11,12	9	13,14	13	15	14	16,19	16
17,18	17	20	25	21,22	29				

column of Table 3 as "*abcdef*". This means that

$$\beta_1^{(1)}(H_1^i) = a, \quad \beta_2^{(1)}(H_1^i) = b, \quad \beta_3^{(1)}(H_1^i) = c,$$
$$\beta_4^{(1)}(H_1^i) = d, \quad \beta_5^{(1)}(H_1^i) = e, \quad \beta_6^{(1)}(H_1^i) = f$$

(see the diagram for $\mathfrak{g} = \mathrm{E\,III}$ in Table 8 for the definition of the β_j's). The fundamental weights of $(\mathfrak{g}_2^c, \mathfrak{h}_2^c)$ will be denoted by $\omega_k^{(2)}$, $1 \le k \le 7$.

TABLE 3. *Transformation rules for $\mathbf{Z}_2$-embeddings in Figure 1*

$\mathfrak{g}_1 \to \mathfrak{g}_2$	H_1^i	$\varphi^c(H_1^i)$	x, y, z
F I → E I F I → E II	$abc\ d$	$xabc$ $abcba\ d$	$x = -(a+b) - \frac{1}{2}(c+d)$
F II → E III F II → E IV	$abcd$	$abcdd\ x$ $bcdx$	$x = -(b+2d) - \frac{1}{2}(a+3c)$
E I → E V	$abcd$	$abcdcba$	
E II → E V E II → E VI	$abcde\ f$	$abcdexf$ $abcdey\ f$	$x = -\frac{1}{6}(a+2b+3c+4d+5e+3f)$ $y = e+f+2x$
E III → E VI E III → E VII	$abcde\ f$	$yabcde\ x$ $aebcdz\ -x$	$x = -2c - e - \frac{1}{3}(2a+4b+5d+4f)$ $y = -2c - e - \frac{1}{3}(4a+5b+4d+2f)$ $z = f + x$
E IV → E VII	$abcd$	$dacbcd\ x$	$x = -(a+2b+3c+2d)$
E V → E VIII	$abcdefg$	$abcdefxg$	$x = -d$ $-\frac{1}{4}(a+2b+3c+5e+6f+3g)$
E VI → E VIII E VI → E IX	$abcdef\ g$	$gxabcdfe$ $yefdcba\ g$	$x = -(a+b+c+d) - \frac{1}{2}(e+f+g)$ $y = -(b+2d+e) - \frac{1}{2}(a+3c+3f)$
E VII → E IX	$abcdef\ g$	$abcdefx\ z$	$x = -(a+2c+3d+2f)$ $-\frac{1}{2}(3b+5e+g)$ $z = x - g$

Assume first that $\mathfrak{g}_2$ is of type E VI. Then the transformed labels, i.e., the labels of $\varphi^c(H_1^i)$ are given in Table 3 as "$yabcde\,x$". This means that

$$\beta_1^{(2)}(\varphi^c H_1^i) = y, \quad \beta_2^{(2)}(\varphi^c H_1^i) = a, \quad \beta_3^{(2)}(\varphi^c H_1^i) = b, \quad \beta_4^{(2)}(\varphi^c H_1^i) = c,$$
$$\beta_5^{(2)}(\varphi^c H_1^i) = d, \quad \beta_6^{(2)}(\varphi^c H_1^i) = e, \quad \beta_7^{(2)}(\varphi^c H_1^i) = x.$$

This is illustrated on Figure 2. These data are in agreement with Table 1 which says that φ^c maps the root spaces of $\mathfrak{g}_1^c$ corresponding to the roots $\beta_k^{(1)}$ to those of $\mathfrak{g}_2^c$ corresponding to the roots $\beta_{k+1}^{(2)}$ for $1 \le k \le 5$, and the root space of $\beta_6^{(1)} = \alpha_6^{(1)}$ to that of $\alpha_6^{(2)}$. We still need to compute the labels $\beta_1^{(2)}(\varphi^c H_1^i) = y$ and $\beta_7^{(2)}(\varphi^c H_1^i) = x$. Observe that $\varphi^c(\mathfrak{h}_1^c)$ is precisely the kernel of the fundamental weight $\omega_7^{(2)}$. As

$$2\omega_7^{(2)} = 2\alpha_1^{(2)} + 3\alpha_2^{(2)} + 4\alpha_3^{(2)} + 6\alpha_4^{(2)} + 5\alpha_5^{(2)} + 4\alpha_6^{(2)} + 3\alpha_7^{(2)},$$

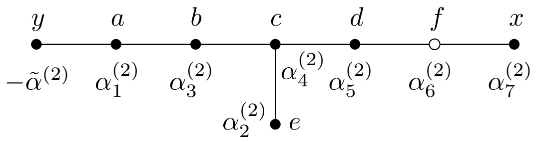
we obtain the equation

$$2a + 3e + 4b + 6c + 5d + 4f + 3x = 0.$$

Since $\tilde{\alpha}^{(2)}$ is the highest root of $\tilde{R}_2$, we also have

$$x + y + 2(a+e+f) + 3(b+d) + 4c = 0.$$

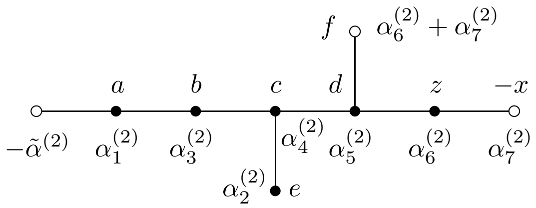
From these two equations we obtain the formulae for x and y given in Table 3.



E III → E VI

$\alpha_k^{(1)} \to \alpha_k^{(2)} \quad (1 \le k \le 6)$

abcde	*f*
yabcde	*x*



E III → E VII

$\alpha_k^{(1)} \to \alpha_k^{(2)} \quad (1 \le k \le 5)$

$\alpha_6^{(1)} \to \alpha_6^{(2)} + \alpha_7^{(2)}$

abcde	*f*
aebcdz	$-x$

FIGURE 2. *Two* $\mathbf{Z}_2$*-embeddings of inner type*

Next assume that $\mathfrak{g}_2$ is of type E VII. Then the labels of $\varphi^c(H_1^i)$ are given in Table 3 as "*aebcdz* $-x$". This means that

$$\beta_1^{(2)}(\varphi^c H_1^i) = a, \quad \beta_2^{(2)}(\varphi^c H_1^i) = e, \quad \beta_3^{(2)}(\varphi^c H_1^i) = b, \quad \beta_4^{(2)}(\varphi^c H_1^i) = c,$$
$$\beta_5^{(2)}(\varphi^c H_1^i) = d, \quad \beta_6^{(2)}(\varphi^c H_1^i) = z, \quad \beta_7^{(2)}(\varphi^c H_1^i) = -x$$

and that the root spaces of $\mathfrak{g}_1^c$ corresponding to $\beta_1^{(2)}, \beta_2^{(2)}, \beta_3^{(2)}, \beta_4^{(2)}, \beta_5^{(2)}$ are mapped to those of $\mathfrak{g}_2^c$ corresponding to $\alpha_1^{(2)}, \alpha_3^{(2)}, \alpha_4^{(2)}, \alpha_5^{(2)}, \alpha_2^{(2)}$, respectively. As indicated in Table 1 (see also Figure 2), the root space of $\beta_6^{(1)}$ is mapped to that of $\alpha_6^{(2)} + \alpha_7^{(2)}$. Consequently, $f = z - x$.

In this case, $\varphi^c(\mathfrak{h}_1^c)$ is the kernel of $\omega_6^{(2)} - \omega_7^{(2)}$. As

$$2\left(\omega_6^{(2)} - \omega_7^{(2)}\right) = 2\alpha_1^{(2)} + 3\alpha_2^{(2)} + 4\alpha_3^{(2)} + 6\alpha_4^{(2)} + 5\alpha_5^{(2)} + 4\alpha_6^{(2)} + \alpha_7^{(2)},$$

we obtain the equation

$$2a + 3e + 4b + 6c + 5d + 4z - x = 0.$$

As $z = f + x$, we obtain the same formula for x as in the previous case. □

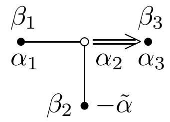
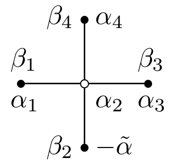
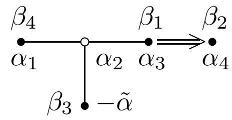
REMARK 2.4. *It follows from the transformation rules given in Table 3 that the labels given in column 2 of this table satisfy the following arithmetic conditions:*

(i) $c \equiv d$ *(mod 2)* *if* $\mathfrak{g}$ *is of type F I,*
(ii) $a \equiv c$ *(mod 2)* *if* $\mathfrak{g}$ *is of type F II,*

(iii) $a + 2b + 3c \equiv e + 2d + 3f$ *(mod 6)* *if* $\mathfrak{g}$ *is of type E II,*
(iv) $a + d \equiv b + f$ *(mod 3)* *if* $\mathfrak{g}$ *is of type E III,*
(v) $a + 2b + e \equiv c + 2f + g$ *(mod 4)* *if* $\mathfrak{g}$ *is of type E V, and*
(vi) $e + f \equiv g$ *(mod 2)* *if* $\mathfrak{g}$ *is of type E VI.*
Of course they can be easily verified by inspecting the tables given in the Appendix.

3. Embeddings from the diagram (1.3). We consider first the $\mathbf{Z}_2$-embedding $\mathfrak{g}_1 \to \mathfrak{g}_2$ where $\mathfrak{g}_1$ is of type $\mathfrak{so}(5,4)$ and $\mathfrak{g}_2$ of type F I. The nilpotent G_1^c-orbits in $\mathfrak{g}_1^c$ are parametrized by the partitions of 9 in which the even parts occur in pairs. They are listed in the first column of Table 5. The second and third columns give the ab-diagram and the right superscript (when needed) which parametrize the K_1^c-orbits in $\mathfrak{p}_1^c$. For more details about this notation we refer the reader to [9]. We warn the reader that the group K in that paper is disconnected, but the orbits of its identity component are the same as the orbits of the group K_1^c of this paper.

TABLE 4. *The simple roots of $\tilde{R}$ and R*

$\mathfrak{g}$	$\mathfrak{k}^c$	$\tilde{R}$ and R
$\mathfrak{so}(4,3)$	$2A_1 + \tilde{A}_1$	
$\mathfrak{so}(4,4)$	$4A_1$	
$\mathfrak{so}(5,4)$	$B_2 + 2A_1$	

In the fourth column of Table 5 we assign a number i to each of the nonzero nilpotent K_1^c-orbits in $\mathfrak{p}_1^c$, and in the fifth column we list the labels of the characteristics H^i of these orbits. These labels are written as "$ab\ c\ d$" where $a = \beta_1(H^i)$, $b = \beta_2(H^i)$, $c = \beta_3(H^i)$, $d = \beta_4(H^i)$. See the last diagram in Table 4 for the definition of these β_j's.

The number in the last column indicates the nilpotent K_2^c-orbit (see Table 10 for the enumeration of these orbits) that contains the image of the given K_1^c-orbit. These numbers are computed by using the same technique as in the previous section. The transformation rule in this case is "$ab\,c\,d$" $\to$ "$xba\,d$" where $x = -b + \frac{1}{2}(c - a)$.

TABLE 5. *Nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$ for $\mathfrak{g} = \mathfrak{so}(5,4)$*

Partition	ab-diagram	r.s.	No.	Labels	F I
$1^5 2^2$	ab, ba, a^3, b^2		1	10 1 1	1
$1^6 3$	bab, a^4, b^2		2	00 2 2	2
	aba, a^3, b^3		3	20 0 0	3
$1 \cdot 2^4$	$(ab, ba)^2, a$	I	4	01 0 2	2
		II	5	01 2 0	3
$1^2 2^2 3$	bab, ab, ba, a^2	I	6	10 1 3	4
		II	7	10 3 1	5
	aba, ab, ba, a, b		8	11 1 1	5
$1^3 3^2$	$(bab)^2, a^3$	I	9	00 0 4	6
		II	10	00 4 0	7
	aba, bab, a^2, b		11	20 2 2	8
	$(aba)^2, a, b^2$		12	02 0 0	7
3^3	$(aba)^2, bab$		13	02 2 2	10
$1^4 5$	$(ba)^2 b, a^3, b$		14	20 4 4	11
	$(ab)^2 a, a^2, b^2$		15	40 2 2	12
$1 \cdot 4^2$	$(ab)^2, (ba)^2, a$	I	16	21 2 4	11
		II	17	21 4 2	12
$2^2 5$	$(ab)^2 a, ab, ba$	I	18	31 1 3	15
		II	19	31 3 1	14
$1 \cdot 3 \cdot 5$	$(ba)^2 b, aba, a$		20	02 4 4	17
	$(ab)^2 a, bab, a$	I	21	40 0 4	17
		II	22	40 4 0	16
	$(ab)^2 a, aba, b$		23	22 2 2	18
$1^2 7$	$(ba)^3 b, a^2$	I	24	40 4 8	19
		II	25	40 8 4	20
	$(ab)^3 a, a, b$		26	42 4 4	20
9	$(ab)^4 a$	I	27	44 4 8	25
		II	28	44 8 4	24

Next we consider the $\mathbf{Z}_2$-embedding $\mathfrak{g}_1 \to \mathfrak{g}_2$ where $\mathfrak{g}_1$ is of type $\mathfrak{so}(4,4)$ and $\mathfrak{g}_2$ of type $\mathfrak{so}(5,4)$. The nilpotent G_1^c-orbits in $\mathfrak{g}_1^c$ are parametrized by the partitions of 8 in which the even parts occur in pairs, except that to each of the two very even partitions (2^4 and 4^2) there correspond two orbits. These partitions are listed in the first column of Table 6. The next three columns give the ab-diagrams and the left and/or right superscripts (when needed) which parametrize the K_1^c-orbits in $\mathfrak{p}_1^c$ (see [9] for details).

In the fifth column of Table 6 we assign a number i to each of the nonzero nilpotent K_1^c-orbits in $\mathfrak{p}_1^c$, and in the sixth column we list the labels $\beta_j(H^i)$, $1 \le j \le 4$, of the characteristics H^i (see Table 4 for the definition of the β_j's).

TABLE 6. *Nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$ for $\mathfrak{g} = \mathfrak{so}(4,4)$*

Partition	l.s.	*ab*-diagram	r.s.	No.	Labels	$\mathfrak{so}(5,4)$
$1^4 2^2$		ab, ba, a^2, b^2		1	1 1 1 1	1
$1^5 3$		aba, a^2, b^3		2	2 2 0 0	3
		bab, a^3, b^2		3	0 0 2 2	2
2^4	I	$(ab, ba)^2$	I	4	0 2 0 2	4
	I		II	5	0 2 2 0	5
	II		I	6	2 0 0 2	4
	II		II	7	2 0 2 0	5
$1 \cdot 2^2 3$	I	aba, ab, ba, b		8	1 3 1 1	8
	II			9	3 1 1 1	8
		bab, ab, ba, a	I	10	1 1 1 3	6
			II	11	1 1 3 1	7
$1^2 3^2$	I	$(aba)^2, b^2$		12	0 4 0 0	12
	II			13	4 0 0 0	12
		aba, bab, a, b		14	2 2 2 2	11
		$(bab)^2, a^2$	I	15	0 0 0 4	9
			II	16	0 0 4 0	10
$1^3 5$		$(ab)^2 a, a, b^2$		17	4 4 2 2	15
		$(ba)^2 b, a^2, b$		18	2 2 4 4	14
4^2	I	$(ab)^2, (ba)^2$	I	19	2 4 2 4	16
	I		II	20	2 4 4 2	17
	II		I	21	4 2 2 4	16
	II		II	22	4 2 4 2	17
$3 \cdot 5$		$(ab)^2 a, bab$	I	23	4 4 0 4	21
			II	24	4 4 4 0	22
	I	$(ba)^2 b, aba$		25	0 4 4 4	20
	II			26	4 0 4 4	20
$1 \cdot 7$	I	$(ab)^3 a, b$		27	4 8 4 4	26
	II			28	8 4 4 4	26
		$(ba)^3 b, a$	I	29	4 4 4 8	24
			II	30	4 4 8 4	25

The last column gives the number (from Table 5) of the nilpotent K_2^c-orbit that contains the given K_1^c-orbit. They were computed by using the transformation rule "$a\,b\,c\,d$" $\to$ "$ax\,c\,d$" where $x = \frac{1}{2}(b-a)$.

Next we consider $\mathfrak{g}_1 \to \mathfrak{g}_2$ where $\mathfrak{g}_1$ is of type $\mathfrak{so}(4,3)$ and $\mathfrak{g}_2$ of type $\mathfrak{so}(4,4)$. The nilpotent G_1^c-orbits in $\mathfrak{g}_1^c$ are parametrized by the partitions of 7 in which the even parts occur in pairs. These partitions are listed in the first column of Table 7. The next two columns give the *ab*-diagrams and the left superscripts (when needed) which parametrize the K_1^c-orbits in $\mathfrak{p}_1^c$.

In the fourth column we assign a number i to each of the nonzero nilpotent K_1^c-orbits in $\mathfrak{p}_1^c$, and in the fifth column we list the labels $\beta_j(H^i)$, $1 \le j \le 3$, of the characteristics H^i (see Table 4).

TABLE 7. *Nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$ for $\mathfrak{g} = \mathfrak{so}(4,3)$*

Partition	l.s.	ab-diagram	No.	Labels	$\mathfrak{so}(4,4)$
$1^3 2^2$		ab, ba, a^2, b	1	1 1 1	1
$1^4 3$		aba, a^2, b^2	2	2 2 0	2
		bab, a^3, b	3	0 0 2	3
$2^2 3$	I	aba, ab, ba	4	1 3 1	8
	II		5	3 1 1	9
$1 \cdot 3^2$	I	$(aba)^2, b$	6	0 4 0	12
	II		7	4 0 0	13
		aba, bab, a	8	2 2 2	14
$1^2 5$		$(ab)^2 a, a, b$	9	4 4 2	17
		$(ba)^2 b, a^2$	10	2 2 4	18
7	I	$(ab)^3 a$	11	4 8 4	27
	II		12	8 4 4	28

The last column gives the number (from Table 6) of the nilpotent K_2^c-orbit that contains the given K_1^c-orbit. The transformation rule in this case is "$a\,b\,c$" $\rightarrow$ "$a\,b\,c\,c$".

There is a $\mathbf{Z}_2$-embedding $\mathfrak{g}_1 \rightarrow \mathfrak{g}_2$ with $\mathfrak{g}_1$ of type G I and $\mathfrak{g}_2 = \mathfrak{so}(4,3)$, with the transformation rule "$a\,b$" $\rightarrow$ "$a\,b\,a$". Under this embedding the nonzero nilpotent K_1^c-orbits in $\mathfrak{p}_1^c$ are mapped to those of K_2^c in $\mathfrak{p}_2^c$ as follows:

$$1 \rightarrow 1, \quad 2 \rightarrow 4, \quad 3 \rightarrow 8, \quad 4 \rightarrow 6, \quad 5 \rightarrow 11.$$

Finally for the $\mathbf{Z}_2$-embedding of $\mathfrak{sl}(3, \mathbf{R})$ into the algebra $\mathfrak{g}_2$ of type G I, the minimal nonzero nilpotent K_1^c-orbit in $\mathfrak{p}_1^c$ is mapped into the orbit 1 and the principal orbit is mapped into the orbit 3.

4. Appendix: Enumeration of the K^c-orbits in $\mathfrak{p}^c$. For the reader's convenience, we give here the parametrization of the nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$ for $\mathfrak{g}$ of exceptional type, which is taken from [5, 6] but is presented here in a different form (using Bala–Carter symbols). For the sake of consistency, we use the same numbering of orbits as in these two papers. They are also listed in [4].

For each of the real forms $\mathfrak{g}$ we give in Table 8 the Dynkin diagram of the root system R of $(\mathfrak{k}^c, \mathfrak{h}^c)$. The nodes of this Dynkin diagram are the black nodes. They are labeled by the simple roots $\beta_1, \beta_2, \ldots$.

If $\mathfrak{g}$ is of inner type, then this diagram is embedded in the extended Dynkin diagram of the root system $\tilde{R}$ of $(\mathfrak{g}^c, \mathfrak{h}^c)$. The simple roots of $\tilde{R}$ are denoted by $\alpha_1, \alpha_2, \ldots$, and its highest root by $\tilde{\alpha}$. If to the simple roots of $\tilde{R}$ we assign the weights: 0 for black nodes and 1 for the single white node, then we obtain a $\mathbf{Z}$-gradation of $\mathfrak{g}^c$ whose associated $\mathbf{Z}_2$-gradation is $\mathfrak{g}^c = \mathfrak{k}^c \oplus \mathfrak{p}^c$.

If $\mathfrak{g}$ is of type E III or E VII, then $\mathfrak{k}^c$ is not semisimple and we need another root from $\tilde{R}$ to specify the characteristics $H \in \mathfrak{h}^c$. For that purpose we use the root β_6 for E III and β_7 for E VII. Note that both of these nodes are white.

TABLE 8. *The simple roots of $\tilde{R}$ and R*

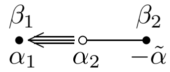
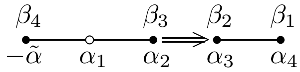
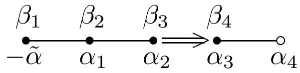
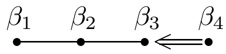
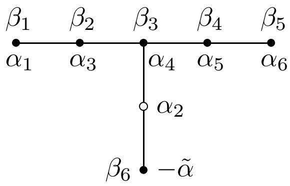
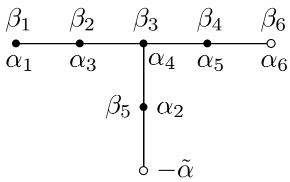
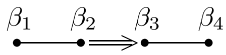
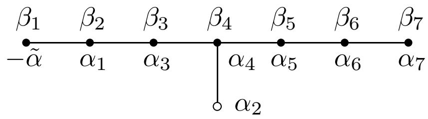
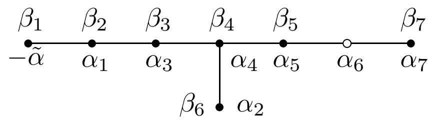
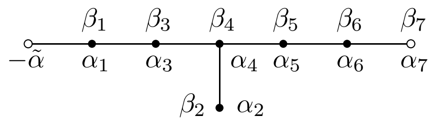
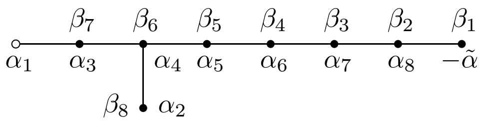
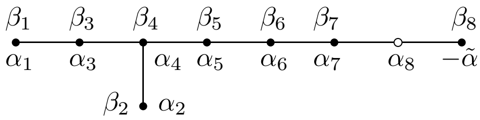
$\mathfrak{g}$	$\mathfrak{k}^c$	$\tilde{R}$ and R
G I	$\tilde{A}_1 + A_1$	
F I	$C_3 + A_1$	
F II	B_4	
E I	C_4	
E II	$A_5 + A_1$	
E III	$D_5 + T_1$	
E IV	F_4	
E V	A_7	

TABLE 8. (continued)

$\mathfrak{g}$	$\mathfrak{k}^c$	$\tilde{R}$ and R
E VI	$D_6 + A_1$	
E VII	$E_6 + T_1$	
E VIII	D_8	
E IX	$E_7 + A_1$	

If H^i is the characteristic of the i-th orbit, then the labels $\beta_j(H^i)$ determine H^i uniquely. All these labels are nonnegative integers except the one corresponding to the white β node which may be a negative integer.

There is only one misprint in the list of characteristics in [5, 6], namely the one for the orbit 31 of [5, Table XII]. These labels should be 020220 2 instead of 020220 0. This misprint was also copied into [4, p. 158]. We warn the reader that the Dynkin diagram of F_4 in [4, p. 152] should have its direction arrow reversed.

The B-C columns in Tables 9–20 of this appendix give the Bala–Carter symbols for nonzero nilpotent G^c-orbits $\mathcal{O}$ in $\mathfrak{g}^c$. As in the introduction, let $\mathcal{B}_1, \ldots, \mathcal{B}_k$ be the connected components of $\mathcal{O} \cap \mathfrak{p}^c$. Each of these components is given a number, say i, which is followed by the labels $\beta_j(H^i)$ of the characteristic H^i of that component.

For instance if $\mathfrak{g}$ is of type E II and $\mathcal{O}$ has the Bala–Carter symbol $E_6(a_3)$, then $k = 2$ and the two connected components are given the numbers 32 and 33 (see Table 13). Since $\mathfrak{k}^c$ is of type $A_5 + A_1$, we have separated the first five labels (corresponding to A_5) from the last one (corresponding to A_1) by a blank space. For the orbit 32, all labels $\beta_j(H^{32})$, $1 \le j \le 6$, are equal to 2. (See the diagram E II in Table 8 for the definition of the roots β_j.) For the orbit 33, the labels $\beta_j(H^{33})$ are 0 for j odd and 4 for j even.

TABLE 9. *Nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$ for $\mathfrak{g}$ of type GI*

B-C symbol	No.	Labels	B-C symbol	No.	Labels
A_1	1	1 1	$G_2(a_1)$	4	0 4
$\tilde{A}_1$	2	1 3	G_2	5	4 8
$G_2(a_1)$	3	2 2			

TABLE 10. *Nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$ for $\mathfrak{g}$ of type FI*

B-C symbol	No.	Labels	B-C symbol	No.	Labels
A_1	1	001 1	$C_3(a_1)$	14	103 1
$\tilde{A}_1$	2	100 2		15	111 3
	3	010 0	$F_4(a_3)$	16	004 0
$A_1+\tilde{A}_1$	4	001 3		17	020 4
	5	101 1		18	202 2
A_2	6	000 4	B_3	19	004 8
	7	200 0		20	204 4
	8	002 2	C_3	21	131 3
$\tilde{A}_2$	9	020 0	$F_4(a_2)$	22	040 4
$A_2+\tilde{A}_1$	10	110 2		23	222 2
B_2	11	102 4	$F_4(a_1)$	24	224 4
	12	012 2		25	404 8
$\tilde{A}_2+A_1$	13	111 1	F_4	26	444 8

TABLE 11. *Nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$ for $\mathfrak{g}$ of type FII*

B-C symbol	No.	Labels
$\tilde{A}_1$	1	0001
$\tilde{A}_2$	2	4000

TABLE 12. *Nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$ for $\mathfrak{g}$ of type EI*

B-C symbol	No.	Labels	B-C symbol	No.	Labels
A_1	1	0001	$D_4(a_1)$	23	0020
$2A_1$	2	0100	A_4	9	0202
$3A_1$	3	1001	D_4	13	2004
A_2	4	0002	A_4+A_1	16	1111
	5	2000	$D_5(a_1)$	17	1112
A_2+A_1	8	0101	A_5	14	1211
$2A_2$	6	0200	$E_6(a_3)$	19	2202
A_2+2A_1	10	1010		22	0220
A_3	7	0102	D_5	21	2204
$2A_2+A_1$	11	1101	$E_6(a_1)$	18	2222
A_3+A_1	15	1011	E_6	20	4224
$D_4(a_1)$	12	2002			

TABLE 13. *Nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$ for $\mathfrak{g}$ of type EII*

B-C symbol	No.	Labels	B-C symbol	No.	Labels
A_1	1	00100 1	$D_4(a_1)$	20	00400 0
$2A_1$	2	10001 2		21	02020 4
	3	01010 0		22	20202 2
$3A_1$	4	00100 3	A_4	25	40004 4
	5	10101 1		26	22022 0
A_2	6	00000 4	D_4	23	00400 8
	7	20002 0		24	20402 4
	8	00200 2	$A_4 + A_1$	27	12113 1
$A_2 + A_1$	9	21001 1		28	31121 1
	10	10012 1	$D_5(a_1)$	29	31310 4
$2A_2$	11	02020 0		30	01313 4
$A_2 + 2A_1$	12	30100 0	A_5	31	13131 3
	13	00103 0	$E_6(a_3)$	32	22222 2
	14	11011 2		33	04040 4
A_3	15	10201 4	D_5	34	22422 4
	16	01210 2		35	40404 8
$2A_2 + A_1$	17	11111 1	$E_6(a_1)$	36	44044 4
$A_3 + A_1$	18	10301 1	E_6	37	44444 8
	19	11111 3			

TABLE 14. *Nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$ for $\mathfrak{g}$ of type EIII*

B-C symbol	No.	Labels		B-C symbol	No.	Labels	
A_1	1	00001	0	$A_2 + A_1$	7	11010	−2
	2	00010	−2		8	11001	−3
$2A_1$	3	10000	1	$2A_2$	9	40000	−2
	4	10000	−2	A_3	10	00013	−2
	5	00011	−2		11	00031	−6
A_2	6	02000	−2	A_4	12	02022	−6

TABLE 15. *Nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$ for $\mathfrak{g}$ of type EIV*

B-C symbol	No.	Labels
A_1	1	0001
$2A_1$	2	0002

TABLE 16. *Nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$ for $\mathfrak{g}$ of type $E\,V$*

B-C symbol	No.	Labels	B-C symbol	No.	Labels
A_1	1	0001000	A_4+A_1	48	3101021
$2A_1$	2	0100010		49	1201013
$(3A_1)''$	3	0200000		50	1111111
	4	0000020	$D_5(a_1)$	51	3013010
$(3A_1)'$	5	1001001		52	0103103
A_2	6	2000002		53	1112111
	7	0002000	A_4+A_2	54	2020202
$4A_1$	8	1100100	$D_5(a_1)+A_1$	55	4004000
	9	0010011		56	0004004
A_2+A_1	10	2010001		57	2022020
	11	1000102		58	0202202
	12	0101010	$(A_5)'$	59	1211121
A_2+2A_1	13	3000100	A_5+A_1	60	1311111
	14	0010003		61	1111131
	15	1010101	$E_6(a_3)$	62	2202022
A_2+3A_1	16	4000000		63	0220220
	17	0000004	$D_6(a_2)$	64	1310301
	18	2000200		65	1030131
	19	0020002	D_5	66	2204022
A_3	20	0102010	$E_7(a_5)$	67	2220202
$2A_2$	21	0200020		68	2020222
$(A_3+A_1)''$	22	0202000		69	0400400
	23	0002020		70	0040040
$2A_2+A_1$	24	1101011	A_6	71	2220222
$(A_3+A_1)'$	25	1011101	$D_6(a_1)$	72	3013131
$D_4(a_1)$	26	2002002		73	1313103
	27	0020200	D_5+A_1	74	3113121
A_3+2A_1	28	1111010		75	1213113
	29	0101111	$E_7(a_4)$	76	2222202
D_4	30	2004002		77	2022222
$D_4(a_1)+A_1$	31	2101101		78	4004040
	32	1011012		79	0404004
	33	0120101	$E_6(a_1)$	80	4220224
	34	1010210		81	2222222
A_3+A_2	35	1030010	D_6	82	3413131
	36	0100301		83	1313143
	37	1110111	E_6	84	4224224
A_4	38	2200022	$E_7(a_3)$	85	2422222
	43	0202020		86	2222242
$A_3+A_2+A_1$	39	0040000		87	4404040
	40	0000400		88	0404044
	41	2020020	$E_7(a_2)$	89	4404404
	42	0200202		90	4044044
$(A_5)''$	44	0402020	$E_7(a_1)$	91	4444044
	45	0202040		92	4404444

TABLE 16. (continued)

B-C symbol	No.	Labels	B-C symbol	No.	Labels
$D_4 + A_1$	46	2103101	E_7	93	8444444
	47	1013012		94	4444448

TABLE 17. *Nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$ for $\mathfrak{g}$ of type E VI*

B-C symbol	No.	Labels	B-C symbol	No.	Labels
A_1	1	000010 1	$D_4(a_1)$	19	000040 0
$2A_1$	2	010000 2		20	000200 4
	3	000100 0		21	020020 2
$(3A_1)'$	4	000010 3	D_4	22	000040 8
	5	010010 1		23	020040 4
A_2	6	000000 4	$A_3 + A_2$	24	201011 2
	7	000020 2	A_4	25	040000 4
	8	020000 0		26	020200 0
$A_2 + A_1$	9	110001 1	$A_4 + A_1$	27	111110 1
$A_2 + 2A_1$	10	200100 0	$D_5(a_1)$	28	201031 4
	11	010100 2	$A_4 + A_2$	29	004000 0
A_3	12	010020 4	$(A_5)'$	30	010310 3
	13	000120 2	$E_6(a_3)$	31	020220 2
$2A_2$	14	400000 0		32	000400 4
	15	000200 0	D_5	33	020240 4
$2A_2 + A_1$	16	010110 1		34	040040 8
$(A_3 + A_1)'$	17	010030 1	A_6	35	400400 0
	18	010110 3	$E_6(a_1)$	36	040400 4
			E_6	37	040440 8

TABLE 18. *Nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$ for $\mathfrak{g}$ of type E VII*

B-C symbol	No.	Labels		B-C symbol	No.	Labels	
A_1	1	100000	0	$A_2 + A_1$	12	011000	−3
	2	000001	−2	A_3	13	300001	−2
$2A_1$	3	000001	0		14	100003	−6
	4	100000	−2	$2A_2$	15	200002	−4
	5	100001	−2	$(A_3 + A_1)''$	16	200002	−2
$(3A_1)''$	6	000000	2		17	400000	−2
	7	000000	−2		18	000004	−6
	8	000002	−2		19	200002	−6
	9	200000	−2	A_4	20	220002	−6
A_2	10	020000	−2	$(A_5)''$	21	400004	−6
$A_2 + A_1$	11	010010	−2		22	400004	−10

TABLE 19. *Nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$ for $\mathfrak{g}$ of type $E\,VIII$*

B-C symbol	No.	Labels	B-C symbol	No.	Labels
A_1	1	00000010	$D_5(a_1)+A_2$	59	11010111
$2A_1$	2	00010000	$D_6(a_2)$	60	21011011
$3A_1$	3	01000010		61	10102100
A_2	4	02000000	$E_6(a_3)+A_1$	62	11110110
	5	00000020		63	01011101
$4A_1$	6	10001000	$E_7(a_5)$	64	01003001
A_2+A_1	7	11000001		65	11101101
	8	00010010	D_5+A_1	66	11110130
A_2+2A_1	9	20010000	$E_8(a_7)$	67	20200200
	10	01000100		68	00004000
A_3	11	00010020		69	02002002
A_2+3A_1	12	30000001	A_6	70	40040000
	13	10010001		71	40040000
$2A_2$	14	40000000	$D_6(a_1)$	72	21011031
	15	20000002		73	01201031
	16	00020000	A_6+A_1	74	11111101
$2A_2+A_1$	17	01010010	$E_7(a_4)$	75	11101121
A_3+A_1	18	01000110		76	10300130
$D_4(a_1)$	19	02000020	$E_6(a_1)$	77	04020200
	20	00000200		78	02020220
D_4	21	02000040	D_5+A_2	79	02002022
$2A_2+2A_1$	22	10100100		80	00400040
A_3+2A_1	23	10010011		81	20200220
$D_4(a_1)+A_1$	24	11001010	E_6	82	04020240
	25	00100101	D_6	83	21031031
A_3+A_2	26	20100011	$D_7(a_2)$	84	31010211
	27	10001002		85	11111111
	28	01010100	A_7	86	12111111
A_4	29	02020000	$E_6(a_1)+A_1$	87	13111101
	30	00020020		88	11111121
$A_3+A_2+A_1$	31	00100003	$E_7(a_3)$	89	11121121
	32	10101001		90	30130130
D_4+A_1	33	11001030	$E_8(b_2)$	91	20202022
$D_4(a_1)+A_2$	34	00000004		92	04004000
	35	20002000	$D_7(a_1)$	93	02022022
	36	00200002		94	40040040
A_4+A_1	37	11110010		95	20220220
	38	01010110	E_6+A_1	96	13111141
$2A_3$	39	10110100	$E_7(a_2)$	97	13103041
$D_5(a_1)$	40	20100031	$E_8(a_6)$	98	00400400
	41	01010120		99	22202022
A_4+2A_1	42	21010100	D_7	100	31131211
	43	01200100	$E_8(b_5)$	101	22202042
	44	10101011		102	04004040

TABLE 19. (continued)

B-C symbol	No.	Labels	B-C symbol	No.	Labels
$A_4 + A_2$	45	00400000	$E_7(a_1)$	103	13131043
	46	02000200	$E_8(a_5)$	104	22222022
A_5	47	01020110		105	40040400
$D_5(a_1) + A_1$	48	30001030	$E_8(b_4)$	106	22222042
	49	10101021		107	04040044
$A_4 + A_2 + A_1$	50	11010101	E_7	108	34131341
$D_4 + A_2$	51	40000040	$E_8(a_4)$	109	22222222
	52	00200022		110	44040400
	53	20002020	$E_8(a_3)$	111	24222242
$E_6(a_3)$	54	02020020		112	44040440
	55	00020200	$E_8(a_2)$	113	44040440
D_5	56	02020040	$E_8(a_1)$	114	44440444
$A_4 + A_3$	57	11101011	E_8	115	84444444
$A_5 + A_1$	58	10111011			

TABLE 20. *Nonzero nilpotent K^c-orbits in $\mathfrak{p}^c$ for $\mathfrak{g}$ of type EIX*

B-C symbol	No.	Labels	B-C symbol	No.	Labels
A_1	1	0000001 0	$D_4(a_1)$	19	0000004 0
$2A_1$	2	1000000 2		20	2000002 2
	3	0000010 0	D_4	21	0000004 8
$3A_1$	4	0000001 3		22	2000004 4
	5	1000001 1	$A_3 + A_2$	23	0110001 2
A_2	6	0000000 4	A_4	24	4000000 4
	7	0000002 2		25	2000020 0
	8	2000000 2	$A_4 + A_1$	26	1010011 1
$A_2 + A_1$	9	1100000 1	$D_5(a_1)$	27	0110003 4
$A_2 + 2A_1$	10	1000010 2	$A_4 + A_2$	28	0002000 0
	11	0001000 0	A_5	29	1000031 3
A_3	12	1000002 4	$E_6(a_3)$	30	2000022 2
	13	0000012 2		31	0000040 4
$2A_2$	14	0000020 0	D_5	32	2000024 4
$2A_2 + A_1$	15	1000011 1		33	4000004 8
$A_3 + A_1$	16	1000011 3	A_6	34	0002020 0
	17	1000003 1	$E_6(a_1)$	35	4000040 4
$D_4(a_1)$	18	0000020 4	E_6	36	4000044 8

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