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RIGIDITY OF A CLASS OF SPECIAL LAGRANGIAN FIBRATIONS SINGULARITY *

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1. Introduction. In [8], Strominger, Yau and Zaslow conjectured that the mirror pairs in mirror conjecture are pairs of dual special Lagrangian fibrations. Since then there has been a lot of research on special Lagrangian submanifolds or special Lagrangian fibrations. In this note, we will look at the possible singularities that can arise from a special Lagrangian fibrations. We will show that a fiber of a special Lagrangian fibration that has only isolated singularity of homogeneous type are essentially of the type given by Harvey-Lawson.

We let $\mathbb{C}^3$ be the complex 3-space endowed with the standard Kaehler metric with the associated Kaehler form ω_0 and the $(3,0)$ form $\Omega_0 = dz_1 \wedge dz_2 \wedge dz_3$. A submanifold $L \subset \mathbb{C}^3$ is called a special Lagrangian submanifold (in short SL-submanifold) if $\omega_0|_L = 0$ and $\mathrm{Im}(\Omega_0)|_L = 0$.

We let $S^5 \subset \mathbb{C}^3$ be the unit sphere. For $p = (z_1, z_2, z_3) \in \mathbb{C}^3$ and $t \in \mathbb{R}$ we use tp to denote the point $(tz_1, tz_2, tz_3) \in \mathbb{C}^3$. For any subset $\Sigma \subset S^5$ we define the cone supposed on Σ to be

$$C(\Sigma) = \{tp \mid t \in \mathbb{R}^+, p \in \Sigma\}.$$

We say $C(\Sigma)$ is an SL-cone if the smooth locus of $C(\Sigma)$ is dense in $C(\Sigma)$ and is an SL-submanifold of $\mathbb{C}^3$.

Now we introduce the notion of homogeneous SL-fibration of $\mathbb{C}^3$.

Definition 1. *Let $F : \mathbb{C}^3 \to \mathbb{R}^3$ be a smooth surjective map. We say F is an SL-fibration if the components f_1, f_2 and f_3 of F are real valued functions in $x_1, x_2, x_3, y_1, y_2, y_3$, where $z_k = x_k + iy_k$, so that all Poisson brackets $\{f_i, f_j\} = 0$ and the real part $Re\{\det_{\mathbb{C}}((\partial f_i/\partial \bar{z}_j))\} = 0$. We say the fibration is homogeneous if all f_i are homogeneous polynomials and we say the fiber $L_0 = F^{-1}(0)$ is a regular cone if L_0 has only isolated singularity 0 and all f_i are irreducible.*

Recall that the smooth locus of any fibers of F as in the Definition are automatically SL-submanifolds [3]. For convenience we denote the punctured cone $L_0 - \{0\}$ by L_0^*. We first observe that in case L_0 is a regular cone, then $\deg f_k \geq 2$ for all k. Indeed, let T_0 be the linear combination of all tangent spaces of points in L_0^*, after translating to the origin 0. We now show that $\dim T_0 = 6$. First, in case $\dim T_0 = 4$, then there are two points $p, q \in L_0^*$ so that $\dim(T_p L_0^* \cap T_q L_0^*) = 2$. Because $T_p L_0^*$ and $T_q L_0^*$ are special Lagrangian subspaces in $\mathbb{C}^3$, we must have $T_p L_0^* = T_q L_0^*$, a contradiction. Now assume $\dim T_0 = 5$. Then there is an unit vector $Jv \in T_0$. This implies that Jv is normal to L_0^* everywhere and hence v is a vector field of L_0^*. So we can write $L_0 = tv \times \Gamma$, where Γ is a cone of dimension 2 with singularity 0. Thus L_0 has at least singularity $\mathbb{R}$, a contradition. Now from $\dim T_0 = 6$, we can easily obtain $\deg f_k \geq 2$ for all k. Moreover we can obtain that $\Sigma = L_0 \cap S^5$ is full in $\mathbb{C}^3$.

The prototype of SL-fibration in $\mathbb{C}^3$ with homogeneous isolated singularity is the example of Harvey and Lawson [3].

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Example 2. *Let* $F = (f_1, f_2, f_3)$ *be defined by*

$$f_1 = |z_1|^2 - |z_2|^2,\ f_2 = |z_1|^2 - |z_3|^2 \quad and \quad f_3 = \mathrm{Im}(z_1 z_2 z_3).$$

Then F *is a homogeneous SL-fibration of* $\mathbb{C}^3$ *and* $L_0 = F^{-1}(0)$ *is a regular SL-cone.*

So far this is the only known example of homogeneous SL-fibration of $\mathbb{C}^3$ whose central fiber is a regular cone. The fibration given by $(f_1, f_2, f_3){:}\mathbb{C}^3 \to \mathbb{R}^3$ with

$$f_1 = x_1 y_2 - x_2 y_1, \quad f_2 = x_1 y_1 + x_2 y_2 \quad \text{and} \quad f_3 = y_3.$$

is a homogeneous SL-fibration but its central fiber is not regular.

In this note, we will prove the following uniqueness result on homogeneous SL-fibrations with singular central fibers.

Theorem 3. *Let* $F = (f_1, f_2, f_3) : \mathbb{C}^3 \to \mathbb{R}^3$ *be a homogeneous SL-fibration so that its central fiber* L_0 *is a regular cone. We let* $n_i = \deg f_i$*, so arranged that* $n_1 \le n_2 \le n_3$*. Then we must have* $(n_1, n_2, n_3) = (2, 2, 3)$*. Furthermore, there is a unitary matrix* S *so that if we make the Darboux coordinates change*

$$(p_1, p_2, p_3, q_1, q_2, q_3)^T = S^{-1}(x_1, x_2, x_3, y_1, y_2, y_3)^T$$

and let $w_k = p_k + iq_k$*. Then* (f_1, f_2, f_3) *is linearly equivalent to*

$$\widetilde{f_1} = |w_1|^2 - |w_2|^2, \quad \widetilde{f_2} = |w_1|^2 - |w_3|^2 \quad and \quad \widetilde{f_3} = \mathrm{Im}(w_1 w_2 w_3).$$

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2. Harmonic 1-forms on SL cone. In this section, we collect a few facts concerning harmonic 1-forms on special Lagrangian cones in $\mathbb{C}^3$.

Let L_0 be an SL cone of $\mathbb{C}^3$ with isolated singularity 0. The question we will address in section is whether there is a family of smooth proper SL-submanifolds L_s of $\mathbb{C}^3$ such that $L_s \to L_0$ as $s \to 0$. In case such families exist, then on L_0^* we have the associated normal vector field $W(x)$ and the associated 1-form $\theta(x) = W(x)\rfloor\omega_0$. In the following, we will call such 1-form the deformation 1-form associated to the family L_s. By a result of McLean, θ is a harmonic 1-form on L_0^*. Further since L_s are smooth θ is singular at 0.

Lemma 4. *Let* $L_0 = C(\Sigma)$ *be an SL-cone with isolated singularity* 0 *and let* t *be the distance function* $t(x) = dist(x, 0)$ *on* $\mathbb{C}^3$*. Then the space of harmonic 1-forms on* L_0^* *is spanned by*

$$t^{-1}\eta; \quad d(t^{-1}); \quad d(t^{-\mu_i}\phi_i) \quad and \quad d(t^{\mu_i'}\phi_i)$$

where η *are harmonic 1-forms on* Σ*,* ϕ_i *are eigenfunctions on* Σ *with eigenvalues* $\lambda_i > 0$ *with* $\mu_i = (1 + \sqrt{1 + 4\lambda_i})/2$ *and* $\mu_i' = (-1 + \sqrt{1 + 4\lambda_i})/2$.

Proof. Let $\tau = \frac{\partial}{\partial t}$ be the unit tangent vector field on L_0^*. Let $m_0 \in \Sigma$ be any point. We pick an orthonormal vector fields (e_1, e_2) on Σ near m_0 that is *covariantly* constant at m_0 with respect to the connection on Σ. We then extend the vector fields e_k, considered as vector fields of $\Sigma \subset S^5 \subset \mathbb{C}^3$, to the cone L_0^* by parallel translation along rays in the cone. Combined with τ, we obtain an orthonormal frame of L_0^* near $\mathbb{R}m_0$. Now let $l : \mathbb{R}^+ \times \Sigma \to C(\Sigma) \subset \mathbb{C}^3$ be defined via $l(t,m) = tm$. We give $\mathbb{R}^+ \times \Sigma$ the metric $\widetilde{g} = dt^2 + t^2 d_\Sigma^2$, where d_Σ^2 is the metric on Σ. We let $\tau : \mathbb{R}^+ \times \Sigma \to \Sigma$ be the project. Then $(\tau, E_1(r,m), E_2(t,m))$ forms an orthonormal frame on $\mathbb{R}^+ \times \Sigma$ with $\tau = \frac{\partial}{\partial t}$, $E_1(t,m) = \frac{1}{t}\tau^* e_1(m)$ and $E_2(t,m) = \frac{1}{t}\tau^* e_2(m)$. Its dual frame is given by dt, $\omega_1(t,m) = t\tau^*\omega_1(m)$ and $\omega_2(t,m) = t\tau^*\omega_2(m)$. Clearly, we have

$$l_* E_i(t,m) = e_i(tm) = e_i(m) \quad \text{and} \quad l^*\omega_i(tm) = l^*\omega_i(m) = \omega_i(t,m)$$

On L_0^* we have the structure equation

$$d\omega_i(tm) = -\omega_{ij}(tm) \wedge \omega_j(tm) - t^{-1}\omega_i(tm) \wedge dt$$

Our convention is that we use (t,m) to denote the point in $\mathbb{R}^+ \times \Sigma$ while we use tm to denote the corresponding point in $C(\Sigma)$. Note $l(t,m) = tm$. Over $\mathbb{R}^+ \times \Sigma$ we have the structure equation

$$d\omega_i(t,m) = d(l^*\omega_i(tm)) = -l^*(\omega_{ij}(tm)) \wedge \omega_j(t,m) - t^{-1}\omega_i(t,m) \wedge dt$$

Let harmonic 1-form

$$\theta = f(tm)dt + \omega(tm) = f(tm)dt + \sum \alpha_i(tm)\omega_i(tm)$$

then

$$l^*\theta = f(t,m)dt + \sum \alpha_i(t,m)\omega_i(t,m).$$

From $dl^*\theta = 0$, we obtain

$$t \bigtriangledown_{E_i(t,m)} f(t,m) - \frac{\partial}{\partial t}(t\alpha_i(t,m)) = 0, \text{ for } i = 1,2$$

Using the above equation, a straight forward computation shows that

$$d\Big(\int_1^t f(r,m)dr\Big) = l^*\theta - \tau^*\eta(m),$$

where $\eta(m) = \sum_{i=1}^2 \alpha_i(1,m)\omega_i(m)$, a $1-$form on Σ. Because $l^*\theta$ is harmonic on $\mathbb{R}^+ \times \Sigma$, $\tau^*\eta$ is closed on $\mathbb{R}^+ \times \Sigma$ and hence η must be closed on Σ. Now let η_h be the harmonic part of η, namely, $\eta = \eta_h + dk$. Then

$$l^*\theta = d\Big(\int_1^t f(r,\cdot)dr + \tau^*k\Big) + \tau^*\eta_h = dF + \tau^*\eta_h,$$

where $F = \int_1^t f(r,\cdot)dr + \tau^*k$. Clearly, $\tau^*\eta_h$ is harmonic on $\mathbb{R}^* \times \Sigma$. Since $l^*\theta$ is harmonic, dF must be harmonic. Hence by [7, page 98]

$$\Delta_\Sigma(F(t,m)) + 2t\frac{\partial}{\partial t}F(t,m) + t^2\frac{\partial^2}{\partial t^2}F(t,m) = 0.$$

Now let ϕ_i be the eigenfunctions on Σ with eigenvalues λ_i. Then

$$F(t,m) = \sum_{i=0}^{\infty} f_i(t)\phi_i(m)$$

for some functions $f_i(t)$. From this we obtain

$$\sum_{i=0}^{\infty}(-\lambda_i f_i(t) + 2tf_i^{'}(t) + t^2 f_i^{''}(t))\phi_i(m) = 0.$$

Therefore f_i satisfies the equation

$$-\lambda_i f_i(t) + 2tf_i^{'}(t) + t^2 f_i^{''}(t) = 0,$$

whose general solutions are $f_i = C_{i1}t^{\mu_i'} + C_{i2}t^{-\mu_i}$ with $\mu_i' = (-1 + \sqrt{1+4\lambda_i})/2$ and $\mu_i = (1 + \sqrt{1+4\lambda_i})/2$.

When $i = 0$ then $\mu_0 = 1$ and ϕ_0 is a constant. In this case $d(t^{-1}\phi_0)$ reduces to the 1-form $d(\frac{1}{t})$. □

We now compute the deformation 1-forms of two examples of smoothing of SL-cones. We begin with the deformation 1-forms of Harvey-Lawson's example. Let θ_k be the deformation 1-form on associated to the family $L_s^{[k]}$ defined by $f_k = s$ and $f_j = 0$ for $j \neq k$. Here (f_1, f_2, f_3) is the defining equation in Example 2. Then by a direct computation we have $\theta_1 = \frac{3}{2}(d\alpha_1 - 2d\alpha_2 + d\alpha_3)$, $\theta_2 = \frac{3}{2}(d\alpha_1 + d\alpha_2 - 2d\alpha_3)$ and $\theta_3 = \frac{1}{3}d(\frac{1}{t})$. Here α_k is the function on Σ defined by $x_k = r\cos\alpha_k$, $y_k = r\sin\alpha_k$. Note that $\theta_1|_\Sigma$ and $\theta_2|_\Sigma$ are harmonic 1−forms on the 2-torus Σ.

Now we consider the case of a homogeneous SL-fibration $F = (f_1, f_2, f_3) : \mathbb{C}^3 \to \mathbb{R}^3$ whose central fiber is a regular cone, as defined in Definition 1. We let $L_s^{[i]}$ be the family $\{f_i = s, f_j = 0 \text{ for } j \neq i\}$ and let W_i be the deformation vector field associated to the family $L_s^{[i]}$.

Lemma 5. *Let $W_1 = \sum_i c_{1i}\partial_{x_i} + c_{1i+3}\partial_{y_i}$ be the normal deformation vector field of the family $L_s^{[1]}$ at L_0. Then c_{1i} can be written as $c_{1i} = h_i/g_i$, where h_i and g_i are homogeneous polynomials with $\deg h_i - \deg g_i = 1 - n_1$ for $1 = 1, \cdots, 6$.*

Proof. Normal deformation vector field W_1 satisfies the following equation:

$$A \cdot (c_{11}, c_{12}, \cdots, c_{16})^T = (1, 0, \cdots, 0)^T$$

where

$$A = \begin{bmatrix} \frac{\partial f_i}{\partial x_j} & \frac{\partial f_i}{\partial y_j} \\ -\frac{\partial f_i}{\partial y_j} & \frac{\partial f_i}{\partial x_j} \end{bmatrix}_{1\le i,j\le 3}$$

So,

$$(c_{11}, c_{12}, \cdots, c_{16})^T = A^{-1}(1, 0, \cdots, 0)^T = (\det A)^{-1}(A_{11}, A_{12}, \cdots, A_{16})^T$$

Because $\deg(\det A) = 2(n_1+n_2+n_3) - 6$ and $\deg(A_{1i}) = 2(n_2+n_3) - 5$, we can write $c_{1i} = h_i/g_i$ with h_i and g_i are homogeneous and $\deg h_i - \deg g_i = 1 - n_1$. □

In this case we say the deformation 1-form $\theta_1 = W_1 \rfloor \omega_0$ has order $n_1 - 1$.

The previous examples show that deformation 1-forms associated to the families L_s are spanned by 1-forms $t^{-1}\eta$ or $d(t^{-1})$. In following, we will study a class of SL-submanifolds that has similar property.

Definition 6. *Let L be a smooth SL-submanifold in $\mathbb{C}^3$ and let L_0 be an SL-cone in $\mathbb{C}^3$ with isolated singularity 0. We say L is asymptotically conical (in short AC) to*

L_0 and call L_0 the asymptotic cone of L if L is asymptotic to L_0 to order $O(t^{-1})$ as $t \to \infty$.

Clearly, when L is AC to L_0 than sL is also AC to L_0 for $s \in \mathbb{R}^+$ and $sL \to L_0$ as $s \to 0$. To obtain a deformation 1-form on L_0^* we need certain convergence condition on sL.

Definition 7. *Let L be a smooth SL-submanifold in $\mathbb{C}^3$ that is AC to L_0. We assume $L_0 = C(\Sigma)$ for a smooth $\Sigma \subset S^5$. We say L is strongly AC to L_0 if there is a constant $q > 0$ and a smooth maps $\Phi(\cdot, s): \Sigma \times [0,1] \to S^5$ such that*
(1) $\Phi(\Sigma, s) = (s^{1/1+q}L) \cap S^5$ and $\Phi(\cdot, s)$ is a diffeomorphism from Σ to its image;
(2) The family of maps $H_s(p,t) = t\Phi(p, s/t^{1+q})$ from $\Sigma \times [1,\infty) \to \mathbb{C}^3$ C^1-converges to the standard map $\Sigma \times [1,\infty) \to L_0 - B_1$ when $s \to 0$, where B_1 is the unit ball in $\mathbb{C}^3$;
(3) The vector field $\mathbf{v}(p) = \frac{d}{ds}\mid_{s=0} \Phi(p,s)$ is a non-trivial vector field on each connected component of $\Sigma \subset S^5$.

The notion of SL-submanifolds AC to a cone was introduced in [5].

Example. [3] Let

$$L_s = \{(x,y) \in \mathbb{C}^3 \mid |x|y = |y|x \text{ and } \operatorname{Im}(|x| + i|y|)^3 = s\}$$

Then L_s is a SL-submanifold strongly AC to a cone L_0 that is the union of two linear subspaces singular at 0. The associated q is 2 in this case.

Example. Let $F = (f_1, f_2, f_3) : \mathbb{C}^3 \to \mathbb{R}^3$ be a homogeneous SL-fibration whose central fiber is a regular cone. We let $n_i = \deg f_i$ with f_i so arranged that $n_1 \leq n_2 \leq n_3$. By Sard theorem, there is a point $\xi \neq 0 \in \mathbb{R}^3$ such that $L = F^{-1}(\xi)$ is a smooth complete SL-submanifold. Then L is strongly AC to the cone $L_0 = F^{-1}(0)$. The associated q is $1/n_i$ where i is the smallest index so that $\xi_i \neq 0$.

Let L be an SL-submanifold which is strongly AC to an SL-cone L_0 with the associated constant q. It follows from the definition that the deformation 1-form on L_0^* associated to the family $L_s = s^{1/1+q}L$ is a smooth non-trivial 1-form on L_0^*.

Lemma 8. *Let L, L_0 and q be as before and let θ be the deformation 1-form on L_0^* associated to the family L_s. Then q can only take values 1, 2 or $\mu_i + 1$, where $\mu_i = (1 + \sqrt{1 + 4\lambda_i})/2$. Further, when $q = 1$ (resp. $q = 2$; resp. $q = \mu_i + 1$) the form $\theta = t^{-1}\eta$ (resp. $\theta = dt^{-1}$; resp. $\theta = d(t^{-\mu_i}\phi_i)$), where η and ϕ_i are as in Lemma 4.*

Proof. Let $\mathbf{v}(p)$ be the vector field on L_0^* that is the limit of $\frac{d}{ds}L_s$ when $s \to 0$. Then by definition, $\mathbf{v}(p)$ is non-trivial on each component of L_0^*. By the definition of H_s, it is direct to check that $\mathbf{v}(tp) = t^{-q}\mathbf{v}(p)$. Hence the deformation 1-form θ on L_0 is also homogeneous, which must be of the forms $\frac{1}{t}\eta$, $d(\frac{1}{t})$, or $d(t^{-\mu_i}\phi_i)$. □

3. Harmonic 1-forms on SL-submanifolds. Let L be an SL-submanifold that is strongly AC to an SL-cone L_0. We first give L a new metric that is quasi-isometric to its induced metric g.

Lemma 9. *Let the notation be as before. Then there is a metric $\bar{g}$ on L that is quasi-isomorphic to (L, g) so that $\bar{g}$ is isomorphic to a cone metric away from a compact subset of L.*

Proof. The proof is standard. It follows from the C^1 convergence of H_t in Definition 7. □

Let M be a complete non-compact Riemannian manifold. We recall several groups associated to M. Here we use $\|\theta\|_2$ to mean the L_2 norm $\int_M |\theta|^2$ and use L^2 to denote the space of L^2 finite functions or forms. We define

$$\mathcal{H}_D(M) = \{f \mid \Delta f = 0,\ df \in L^2\} \quad \text{and} \quad \mathcal{H}_D^\infty(M) = \{f \in \mathcal{H}_D(M) \mid f \in L^\infty\}$$

and

$$\mathcal{H}^1(M) = \{\theta \mid d\theta = \delta\theta = 0,\ \theta \in L^2\} \quad \text{and} \quad \mathcal{H}_0^1(M) = \{df \mid \Delta f = 0,\ df \in L^2\}.$$

Here f are functions and θ are one forms on M. We also define

$$H^1_{(2)}(M) = \{\theta \mid d\theta = 0,\ \theta \in L^2\}/\{df \mid f \in L^2,\ df \in L^2\}.$$

First, recall that we have the Hodge decomposition $\mathcal{H}^1(M) \cong H^1_{(2)}(M)$ [4]. In the following, we will apply a theorem P.Li and L-H.Tam [6, Thm 4.2] to prove the following fact.

Lemma 10. *Let L be an SL-submanifold strongly AC to an SL-cone L_0 endowed with the induced metric g and let $\bar{L}$ be L endowed with the metric $\bar{g}$ given in Lemma 9. Then we have*

$$\dim \mathcal{H}_D^\infty(\bar{L}) = \dim \mathcal{H}_D^\infty(L) = \#(\textit{ends of } L).$$

Proof. Let $K(R)$ be $B(R) \cap L$. By the construction, for large enough R the compliment $L - K(R)$ with metric $\bar{g}$ is a union of cones. Our strategy is to apply [6, Theorem 4.2] to $(\bar{L}, \bar{g})$.

We now check that this theorem can be applied in our situation. First, $\bar{L}$ is large because L is strongly AC to the cone L_0. Now let E be an end of $L - K(R)$. To proceed, we need to check that there is a constant C so that the Ricci curvature of $(E, \bar{g})$ satisfies

$$Ric(x) \geq -\frac{2C}{(1 + r(x))^2} \tag{3.1}$$

where $r(x) = \mathrm{dist}(p, x)$ for a p in E. Because $C(\Sigma)$ is minimal, we have

$$R_{ii} = \sum_{\alpha,j} (h^\alpha_{jj} h^\alpha_{ii} - (h^\alpha_{ij})^2) = -\sum_{\alpha,j} (h^\alpha_{ij})^2$$

where $h^\alpha_{ij} = h^\alpha_{ij}(t, m)$ is the second fundamental form of $C(\Sigma)$. Now let $h^\alpha_{ij}(m)$ be the second fundamental form of Σ in S^5. Because $h^\alpha_{ij}(t, m) = \frac{1}{t} h^\alpha_{ij}(m)$ (on $C(\Sigma)$), we have

$$R_{ii}(m, t) = -t^{-2} \sum_{\alpha,j} (h^\alpha_{ij}(m))^2 \geq t^{-2} C_1$$

for $C_1 = \sup_{i,m} \sum_{j,\alpha} (h^\alpha_{ij}(m))^2$. Because $t^2 \sim r^2(x)$, there is a constant C_2 so that $R_{ii}(x) \geq -\frac{C_2}{r(x)^2}$ on E. This shows that there is a constant C so that (3.1) holds.

Finally, we need to check that E satisfies the condition (VC) in [6, p.282]. Namely, there is a constant $\zeta > 0$ such that for all r and all $x \in \partial B_p(r) \cap E$, we have $V_{p,E}(r) < \zeta V_{x,E}(\frac{r}{2})$. First, it is clear that it suffices to check this condition for $r \geq R'$

for some constant R'. We choose an R' so that $\partial B_p(R') \subset E$ and that for any $x \in \partial B_p(R')$, $B_x(R'/2) \cap E \subset \bar{L} - K(R)$. Then when $r \geq R'$, we have

$$V_{x,E}(\frac{r}{2}) = V_{\frac{R'}{r}x,E}(R'/2) \cdot \frac{r^3}{R'^3}.$$

Let $\zeta_1 = \min_{x \in \partial B_p(R') \cap E} V_{x,E}(\frac{R'}{2})$. Then $V_{x,E}(r/2) > \zeta_1 R'^{-3} r^3$. Since $V_{p,E}(r) \approx Area(\Sigma)(r^3 - (2R)^3)$, therefore there is a constant ζ so that $V_{p,E}(r) < \zeta V_{x,E}(r/2)$.

This shows that we can apply [6, Theorem 4.2] to $\bar{L}$ to conclude that the dimension of $\mathcal{H}_D^\infty(\bar{L})$ is equal to the number of ends of $\bar{L}$. Since L is quasi-isometric to$\bar{L}$, $\dim \mathcal{H}_D^\infty(L) = \dim \mathcal{H}_D^\infty(\bar{L})$ [2]. Therefore $\dim \mathcal{H}_D^\infty(L)$ is equal to the number of ends on L. □

LEMMA 11. *Let L be a SL submanifold which is strongly AC to SL cone L_0, then* $\dim\, \mathcal{H}_0^1(\bar{L}) = \#\{$*ends of* $L\} - 1$.

Proof. From Lemma 4, we obtain $\mathcal{H}_D(\bar{L}) = \mathcal{H}_D^\infty(\bar{L})$. Then the Lemma follows from $\dim \mathcal{H}_0^1(\bar{L}) = \dim \mathcal{H}_D(\bar{L}) - 1$ and Lemma 10. □

Let

$$H_c^1(L) = \{\theta \mid d\theta = 0, \theta \text{ has compact support}\}/\{df \mid f \text{ has compact support}\}$$

and let $H^1(L)$ be the first de Rham cohomology group. Consider the natural map $i_1^* : H_c^1(L) \to H^1(L)$. Then

$$\mathrm{Ker}(i_1^*) = \{dg \mid dg \text{ has compact support}\}/\{df \mid f \text{ has compact support}\}$$

We continue to assume that L is an SL-submanifold strongly AC to an SL-cone L_0.

LEMMA 12. *$Ker(i_1^*) = \#($ends of $L) - 1$.*

Proof. Let $[dg] \neq 0 \in \mathrm{Ker}(i_1^*)$. Because L is strongly AC to L_0, without loss of generality, we can assume that the compact subset $K \subset L$ of dg is so large that $L - K$ is diffeomorphic to the disjoint union of $\Sigma_i \times (0, \infty)$. Hence when restricted to the ends $L - K$ g is locally constant but not constant. Now we see that $\mathrm{Ker}(i_1^*) = \#(\text{ends of } L) - 1$. □

Now we consider $\mathrm{Im}(i_1^*)$. From [4, Page 9], any compactly supported cohomology class on a complete Riemannian manifold that defines a non-trivial de Rham cohomology class is automatically represented by an L^2-harmonic form. This defines a natural Hodge projective

$$\pi : \mathrm{Im}(i_1^*) \longrightarrow \mathcal{H}^1(\bar{L}).$$

Define

$$[\pi] : \mathrm{Im}(i_1^*) \longrightarrow \mathcal{H}^1(\bar{L})/\mathcal{H}_0^1(\bar{L}).$$

LEMMA 13. *Let L be an SL-submanifold strongly AC to an SL-cone L_0, then*

$$\mathrm{Im}(i_1^*) \cong \mathcal{H}^1(\bar{L})/\mathcal{H}_0^1(\bar{L})$$

and the isomorphism is induced by above Hodge projection.

Proof. Clearly, $[\pi]$ is injective. And we must prove $[\pi]$ is surjective.

Let $[\omega] \in \mathcal{H}^1(\bar{L})/\mathcal{H}_0^1(\bar{L})$. Because L is strongly AC to L_0, we can pick a sufficiently large compact subset $K(2r_0) \subset L$ so that $L - K(2r_0)$ is diffeomorphic to the disjoint union of $\Sigma_i \times (0, \infty)$. We let Λ_i be the ith component of $\bar{L} - K(2r_0)$. Then from lemma 4, we can write

$$\omega \mid_{\Lambda_i} = C_i \cdot \frac{1}{t}\eta_i + df_i,$$

where C_i is some constant and η_i is some harmonic 1-form on Σ_i, f_i is a harmonic function on Λ_i. If $C_i \neq 0$ for some i, then

$$\int_{\Lambda_i} |C_i \cdot \frac{1}{t}\eta_i|^2 = +\infty$$

and

$$\int_{\Lambda_i} < C_i \cdot \frac{1}{t}\eta_i, df_i >= 0.$$

Thus $\omega \notin \mathcal{H}^1(\bar{L})$. So we can write $\omega|_{\bar{L}-K(2r_0)} = df$, for some harmonic function f on $\bar{L} - K(2r_0)$. Then we can write $\omega = (\omega - d(\rho f)) + d(\rho f)$, where function ρ with takes values between 0 and 1 and such that

$$\rho(x) = 1, \text{ for } x \in L - K(2r_0) \quad \text{and} \quad \rho(x) = 0, \text{ for } x \in K(r_0).$$

Thus, $\theta = \omega - d(\rho f)$ has compact support $K(2r_0)$. But from the Hodge projection π, we can write $\theta = \pi\theta + d\varphi$ for some function φ with $\int |d\varphi|^2 < +\infty$. So we have $\omega = \pi\theta + d\varphi + d(\rho f) = \pi\theta + d(\varphi + \rho f)$ with $\int |d(\varphi + \rho f)|^2 < +\infty$ and thus $d(\varphi + \rho f) \in \mathcal{H}_0^1(L)$. So $[\pi]\theta = [\omega]$. □

THEOREM 14. *Let L be an SL submanifold which is strongly AC to an SL cone, then dim $\mathcal{H}^1(L) = dim H_c^1(L)$.*

Proof. From lemma 11, 12 and 13, we have dim $\mathcal{H}^1(\bar{L}) = \dim H_c^1(L)$. Combined with $\mathcal{H}^1(L) \cong \mathcal{H}^1(\bar{L})$, we prove the Lemma. □

4. Proof of the main result.

LEMMA 15. *Let $F = (f_1, f_2, f_3) : \mathbb{C}^3 \to \mathbb{R}^3$ be a homogeneous SL fibration so that its central fiber L_0 is a regular cone. Then every connected component L of regular fiber is diffeomorphic to $\mathbb{R} \times T^2, \mathbb{R}^2 \times S^1$ or $\mathbb{R}^3$.*

Proof. Let

$$g_i = \frac{f_i^2}{1 + f_i^2}, \text{ for } i = 1, 2, 3,$$

then

$$\begin{aligned} \{g_i, g_j\} &=< J\text{grad } g_i, \text{ grad } g_j > \\ &= \frac{2f_i f_j}{(1 + f_i^2)^2(1 + f_j^2)^2} < J\text{grad } f_i, \text{grad } f_j >= 0. \end{aligned}$$

So $G = (g_1, g_2, g_3) : \mathbb{C}^3 \to \mathbb{R}^3$ defines a 3-degree of freedom Liouville integrable Hamiltonian system. Because f_i are homogeneous polynomials, Hamiltonian vector fields X_{g_i} are bounded on $\mathbb{C}^3$. So from a theorem in [10, Cor 2, p.17] solutions of Cauchy problem: $\frac{dz}{dt} = X_{g_i}, z(t_0) = z_0$ are complete, i.e., X_{g_i} are complete. Thus by the generalized Liouville theorem, every connected component of regular fiber $G^{-1}(v_1, v_2, v_3)$ is differential homeomorphic to $\mathbb{R} \times T^2, \mathbb{R}^2 \times S^1, \mathbb{R}^3$ or T^3. Now every connected component L of regular fiber $F^{-1}(s_1, s_2, s_3)$ is a connected component of regular fiber $G^{-1}(v_1, v_2, v_3)$, where $v_i = \frac{s_i^2}{1+s_i^2}$. So L is differential homeomorphic to $\mathbb{R} \times T^2, \mathbb{R}^2 \times S^1, \mathbb{R}^3$ or T^3. But from Example in section 2, we know that L is strongly AC to SL cone and is not compact, so L is not diffeomorphic to T^3. □

Now we can prove the following

PROPOSITION 16. *Let $F = (f_1, f_2, f_3) : \mathbb{C}^3 \to \mathbb{R}^3$ be a homogeneous SL-fibration so that its central fiber L_0 is a regular cone. We let $n_i = \deg f_i$, so arranged that $n_1 \leq n_2 \leq n_3$. Then we must have $(n_1, n_2, n_3) = (2, 2, 3)$.*

Proof. By observation in section 1, we have $\deg f_k \geq 2$ for all k. Let θ_i denote the deformation 1-form associated to the family $L_{s_i} = \{f_i = s_i; f_j = 0 \text{ for } j \neq i\}$. Because we assume that f_i are irreducible, θ_1, θ_2 and θ_3 are linearly independent at any point $p \in L_0^*$. Note that dt is 1-form in L_0^*. Now we prove the proposition by studying case by case:

Case 1: *$\deg F = (2, 2, n_3)$ with $n_3 \neq 3$.*

In this case, θ_1 and θ_2 have order 1. By Lemma 4, we can let $\theta_1 = \frac{1}{t}\eta_1$ and $\theta_2 = \frac{1}{t}\eta_2$, where η_1 and η_2 are harmonic 1−forms on $\Sigma = L_0 \cap S^5$. Now if $n_3 = 2$, we can also let $\theta_3 = \frac{1}{t}\eta_3$, where η_3 is the harmonic form on Σ. Thus $< \theta_i, dt >= 0$ for $i = 1, 2, 3$ and θ_1, θ_2 and θ_3 are linearly dependent on L_0^*. This is impossible. If $n_3 \geq 4$, by Lemma 4 we can let $\theta_3 = d(t^{-\mu_i}\phi_i)$, where ϕ_i is an eigenfunction of Σ which is not constant. Since Σ is compact, we know that there is a point p of Σ such that $\phi_i(p) = 0$. So $\theta_3(p)$ doesn't contain dt as component at point p and thus θ_1, θ_2 and θ_3 are linearly dependent at point p.

Case 2: $\deg F = (2, 3, n_3)$.

In this case. the deformation 1−form θ_2 of L_{s_2} has order 2. Hence $\theta_2 = C_2 d(\frac{1}{t})$ and therefore n_3 can not be 3. The 1−form associated to the family L_{s_3} can be written $\theta_3 = d(t^{-\mu_i}\phi_i)$, where ϕ_i is a non-constant eigenfunction on Σ. Because Σ is compact, then there is a point $m \in \Sigma$ such that ϕ_i attains maximum at point m. So $\theta_3(m) = \phi_i(m)dt^{-\mu_i}$, and $\theta_2(m)$ and $\theta_3(m)$ is linearly dependent at point m. This is a contradiction.

Case 3: $\deg F = (n_1, n_2, n_3)$ *with $n_3 \geq n_2 \geq 4$.*

If there exists such F, then there is a point $s_0 = (s_{10}, s_{20}, s_{30}) \in \mathbb{R}^3$, such that $L_{s_0} = F^{-1}(s_{10}, s_{20}, s_{30})$ is a smooth SL submanifold. We have proven that L_{s_0} is strongly AC to an SL cone L_0 in the example before Lemma 8. Let L be a connected component of L_{s_0}. By Theorem 14, we have $\dim \mathcal{H}^1(L) = \dim H_c^1(L)$. By Poincare Lemma and Lemma 15, we have dim $H_c^1(L) \leq 1$. Thus $\dim \mathcal{H}^1(L) \leq 1$.

Because L is smooth, we can get three deformation 1-forms θ_1, θ_2 and θ_3. Certainly these forms are harmonic on L. But we assume that $n_3 \geq n_2 \geq 4$, so θ_2 and θ_3 has order at least 3. Now by the homogeneous, we can get $\theta_2, \theta_3 \in \mathcal{H}^1(L)$, which contradicts to $\dim \mathcal{H}^1(L) \leq 1$. □

In order to discuss the case of $(n_1, n_2, n_3) = (2, 2, 3)$, we need the following Lemma. In the following we denote by $diag(a_1, a_2, \cdots, a_n)$ the $n \times n$ diagonal matrix with diagonal entries $a_1, a_2, \cdots, a_n$.

Lemma 17. *Let $\widetilde{A} = diag(1, -k, 0, 1, -k, 0)$ with $k > 0$ and let matrix $\widetilde{B}$ be symmetric such that (1) $\widetilde{A}E\widetilde{B} = \widetilde{B}E\widetilde{A}$, where matrix E has the form*

$$E = \begin{pmatrix} 0 & -I_{n\times n} \\ I_{n\times n} & 0 \end{pmatrix}$$

and (2) there is a $s_0 \in R$ such that $exp(E\widetilde{B}s_0) = I$, then there is a symplectic matrix Q such that $Q^T\widetilde{A}Q = \widetilde{A}$ and $Q^T\widetilde{B}Q$ has the form $diag(\alpha, \beta, \gamma, \alpha, \beta, \gamma)$.

Proof. From $\widetilde{A}E\widetilde{B} = \widetilde{B}E\widetilde{A}$, we find that $\widetilde{B}$ has the following form:

$$\widetilde{B} = \begin{pmatrix} a & c & 0 & 0 & e & 0 \\ c & b & 0 & e & 0 & 0 \\ 0 & 0 & b_{33} & 0 & 0 & b_{36} \\ 0 & e & 0 & a & -c & 0 \\ e & 0 & 0 & -c & b & 0 \\ 0 & 0 & b_{63} & 0 & 0 & b_{66} \end{pmatrix}$$

and furthermore,

$$c = e = 0 \quad \text{when} \quad k \neq 1. \tag{4.1}$$

Let

$$B_1 = \begin{pmatrix} a & c & 0 & e \\ c & b & e & 0 \\ 0 & e & a & -c \\ e & 0 & -c & b \end{pmatrix}, B_2 = \begin{pmatrix} b_{33} & b_{36} \\ b_{36} & b_{66} \end{pmatrix}.$$

From $\exp(E\widetilde{B}s_0) = I$, we can obtain $\exp(EB_1s_0) = I$ and $\exp(EB_2s_0) = I$. Now we first consider the matrix B_2. Let λ be an eigenvalue of EB_2 with the corresponding eigenvector ξ, (i.e., $(EB_2)\xi = \lambda\xi$,) then $\exp(EB_2s)\xi = e^{\lambda s}\xi$. Since $\exp(EB_2s_0) = I$, we have $e^{\lambda s_0} = 1$. Thus λ must be $\pm\mu i$ for $\mu \in \mathbb{R}$. If $b_{33} = 0$, then $b_{36} = 0$. Thus B_2 has the diagnol form and we abe done. If $b_{33} \neq 0$, without loss of generality, we can assume $b_{33} > 0$. Take $w = (\frac{b_{33}b_{66} - b_{36}^2}{b_{33}^2})^{\frac{1}{4}}$ and cosider the symplectic matrix

$$Q_1 = \begin{pmatrix} w & -\frac{b_{36}}{b_{33}}\frac{1}{w} \\ 0 & \frac{1}{w} \end{pmatrix},$$

then

$$Q_1^T B_2 Q_1 = \begin{pmatrix} \sqrt{b_{33}b_{66} - b_{36}^2} & 0 \\ 0 & \sqrt{b_{33}b_{66} - b_{36}^2} \end{pmatrix}.$$

Next we consider the matrix B_1. From $\exp(EB_1s_0) = I$, the eigenvalues of EB_1 are $\pm\nu i$ for $\nu \in \mathbb{R}$. But by direct calculation, the eigenvalues of EB_1 are

$$\lambda = \frac{\pm(a-b)i \pm \sqrt{4(e^2+c^2) - (a+b)^2}}{2}.$$

So we obtain

$$4(e^2 + c^2) \leq (a+b)^2. \tag{4.2}$$

If $a + b = 0$, then $e = c = 0$ and B_1 is of the diagonal form. So without loss of generality, we can assume $a+b > 0$. From (4.2), we have $(a+b)^2 - 4c^2 \geq 0$. If $c = 0$, we take $u = 0$. If $c \neq 0$, we take $u = \frac{-1}{2c}[(a+b) - \sqrt{(a+b)^2 - 4c^2}]$. Notice that $1 - u^2 \geq 0$. If $1 - u^2 = 0$, then $a + b = \pm 2c$ and $e = 0$. It is easy to prove that in this case there isn't any s_0 such that $\exp(EB_1 s_0) = I$. Thus $1 - u^2 > 0$. So we can take the symplectic matrix

$$Q_2 = \frac{1}{\sqrt{1-u^2}} \begin{pmatrix} 1 & u & 0 & 0 \\ u & 1 & 0 & 0 \\ 0 & 0 & 1 & -u \\ 0 & 0 & -u & 1 \end{pmatrix}.$$

One easily checks that

$$B_3 = Q_2^T B_1 Q_2 = \begin{pmatrix} a_1 & 0 & 0 & e \\ 0 & b_1 & e & 0 \\ 0 & e & a_1 & 0 \\ e & 0 & 0 & b_1 \end{pmatrix},$$

where $a_1 = \frac{1}{1-u^2}(bu^2 + 2cu + a)$ and $b_1 = \frac{1}{1-u^2}(au^2 + 2cu + b)$. One verifies

$$(a_1 + b_1)^2 = (a+b)^2 - 4c^2. \tag{4.3}$$

Now from (4.2) and (4.3) we have $(a_1 + b_1)^2 - 4e^2 \geq 0$. If $e = 0$, we take $v = 0$. If $e \neq 0$, we take $v = \frac{-1}{2e}(a_1 + b_1 - \sqrt{(a_1+b_1)^2 - 4e^2})$. Again we have $1 - v^2 \geq 0$.

If $1 - v^2 = 0$, then $a_1 + b_1 = \pm 2e$. If we take the symplectic matrix

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

then we have

$$R^T B_3 R = \begin{pmatrix} a_1 & e & 0 & 0 \\ e & b_1 & 0 & 0 \\ 0 & 0 & a_1 & -e \\ 0 & 0 & -e & b_1 \end{pmatrix}.$$

As before, we can prove that $1 - v^2 = 0$ is impossible.

Now we are reduced to the case $1 - v^2 > 0$. We take symplectic matrix

$$Q_3 = \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & 0 & 0 & v \\ 0 & 1 & v & 0 \\ 0 & v & 1 & 0 \\ v & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$B_4 = Q_3^T B_3 Q_3 = \begin{pmatrix} a_2 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & b_2 \end{pmatrix}.$$

We let

$$Q = \frac{1}{\sqrt{1-u^2}}\frac{1}{\sqrt{1-v^2}}\begin{pmatrix} 1 & u & 0 & uv & v & 0 \\ u & 1 & 0 & v & uv & 0 \\ 0 & 0 & w & 0 & 0 & -\frac{b_{36}}{b_{33}}\frac{1}{w} \\ -uv & v & 0 & 1 & -u & 0 \\ v & -uv & 0 & -u & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{w} \end{pmatrix}$$

where u, v and w are taken as before. Then we can check that $Q^T\widetilde{B}Q$ has diagonal form. If $k = 1$, we have $Q^T\widetilde{A}Q = \widetilde{A}$. If $k \neq 1$, from (4.1) we can take $u = v = 0$. Thus $Q^T\widetilde{A}Q = \widetilde{A}$. This proves the Lemma. □

Now we discuss the case of $(2, 2, 3)$.

THEOREM 18. *Let $F = (f_1, f_2, f_3) : \mathbb{C}^3 \to \mathbb{R}^3$ be a homogeneous SL-fibration with $(n_1, n_2, n_3) = (2, 2, 3)$ so that its central fiber L_0 is a regular cone. Then there is a unitary matrix S so that if we make the Darboux coordinates change*

$$(p_1, p_2, p_3, q_1, q_2, q_3)^T = S^{-1}(x_1, x_2, x_3, y_1, y_2, y_3)^T$$

and let $w_k = p_k + iq_k$. Then (f_1, f_2, f_3) is linearly equivalent to

$$\widetilde{f_1} = |w_1|^2 - |w_2|^2, \quad \widetilde{f_2} = |w_1|^2 - |w_3|^2 \quad \text{and} \quad \widetilde{f_3} = \mathrm{Im}(w_1w_2w_3).$$

Proof. Let $L_0 = F^{-1}(0)$ and let $x_0 \in L_0$. Let Σ be a connected component of $L_0 \cap S^5$ containing x_0. Let $f_1(x) = x^TAx$ and $f_2(x) = x^TBx$, where $x^T = (x_1, x_2, x_3, y_1, y_2, y_3)$. Because A and B are symmetric matrices, $EA, EB \in sp(6, \mathbb{R})$, the later is the Lie algebra of symplectic group $Sp(6, \mathbb{R})$. So

$$G = \{\exp(EAt), \exp(EBs)|t, s \in \mathbb{R}\} \subset Sp(6, \mathbb{R})$$

is a Lie subgroup of the symplectic group. From $\{f_1, f_2\} = 0$, we know that $AEB = BEA$ and thus $(EA)(EB) = (EB)(EA)$. Thus G is a commutative Lie subgroup of $Sp(6, R)$. On $\mathbb{R}^6$ we define the distribution

$$D = \{(EA)x, (EB)x|x \in \mathbb{R}^6\}.$$

Then the distribution D is completely integrable because

$$[(EA)x, (EB)x] = \overline{\nabla}_{(EA)x}(EB)x - \overline{\nabla}_{(EB)x}(EA)x = (EB)(EA)x - (EA)(EB)x = 0.$$

Thus the orbit $G \cdot x_0$ is the maximal connected integral submanifold of D through x_0.

On the other hand, we will prove Σ is also the maximal connected integral submanifold of D through x_0. Let W_3 be the normal deformation vector field of L_0 associated to the family $L_{s_i} = \{f_1 = f_2 = 0, f_3 = s_3\}$. As in the proof of Lemma 5, we have

$$< W_3, \mathrm{grad} f_1 >=< W_3, \mathrm{grad} f_2 >= 0. \tag{4.4}$$

Because $\deg f_3 = 3$, from Lemma 4, we have $\theta_3 = W_3 \rfloor \omega_0 = C_3t^{-2}dt$, where C_3 is a constant on Σ. Thus $W_3(x) = C_3|x|^{-3}Jx$ for any $x \in \Sigma$. So from (4.4) we have equations

$$< x, (EA)x >=< x, (EB)x >= 0. \tag{4.5}$$

On the other hand $J\text{grad}f_1 = 2EAx$ and $J\text{grad}f_2 = 2EBx$ are vector fields on L_0^*. Then (4.5) says that $(EA)x$ and $(EB)x$ are 2 linearly independent vector fields on Σ. So Σ is also the integral submanifold of D through x_0. Thus we have $\Sigma = G \cdot x_0$.

On Σ, there are 2 commuting linearly independent vector fields $(EA)x$ and $(EB)x$. From [1, Lemma 2, p.274] and its proof, we know that Σ is a 2-torus and that $\exp(EAt) \cdot x_0$ is a circle on Σ. Let t_0 be the first t such that $\exp(EAt_0)x_0 = x_0$. Thus for any $x = \exp(EAt)\exp(EBs) \cdot x_0 \in \Sigma$, we have $\exp(EAt_0)x = x$, namely, $(\exp(EAt_0) - I)x = 0$. But Σ is full as the submanifold of $\mathbb{R}^6$ as we observed in section 1, so we must have $\exp(EAt_0) = I$ and $\exp(EAt)$ is a circle on G. For the same reason, $\exp(EBs)$ is a circle on G. We let s_0 be the first s such that $\exp(EBs_0) = I$.

Now by Williammson's theorem [9], we can reduce A to normal forms by means of a real symplectic transformation. In [1, Appendix 6] we can find the list of normal forms. From $\exp(EAt_0) = I$, eigenvalues of EA are of the form 0 or $\pm\mu i$ for $\mu \in R$. Thus we only need to discuss the case with eigenvalues 0 or $\pm\mu i$. In other words, we need to check which symmetric matrices C with eigenvalues 0 or $\pm\mu i$ satisfy the equation $\exp(ECt_0) = I$. After that we can find a symplectic matrix P_1 such that

$$A' = P_1^T A P_1 = \text{diag}(\pm\mu_1^2, \pm\mu_2^2, \pm\mu_3^2, \pm\delta_1, \pm\delta_2, \pm\delta_3),$$

where $\pm\mu_j i (j = 1, 2, 3)$ are eigenvalues of EA and $\delta_j = 1$ if $\mu_j \neq 0; \delta_j = 0$ if $\mu_j = 0$. Certainly there is another symplectic matrix P_2 such that

$$\widetilde{A} = (P_1P_2)^T A(P_1P_2) = diag(r_1, r_2, r_3, r_1, r_2, r_3),$$

where $r_j = \pm\mu_j (j = 1, 2, 3)$.

Without loss of generality, we can assume that $\det(EA) = 0$ and $\pm i$ are eigenvalues of EA. This is because if not, we can take $A_1 = (uA - vB)$ for $u, v \in \mathbb{R}$ such that $\det(EA_1) = 0$ and $\pm i$ are eigenvalues of (EA_1). If we take $\bar{f}_1 = uf_1 - vf_2 = us_1 - vs_2, \bar{f}_2 = f_2 = s_2$ and $\bar{f}_3 = f_3 = s_3$, then $\bar{F} = (\bar{f}_1, \bar{f}_2, \bar{f}_3)$ also defines homogeneous SL fibration. This SL fibration has the same geometric structure as the SL fibration F. This is the meaning of "linearly equivalent" in the theorem.

Let $P = P_1P_2$, then $\widetilde{A} = P^T AP = diag(1, -k, 0, 1, -k, 0)$. We note that $k > 0$. Because if $k \leq 0$, then $C(\Sigma) = \{0\}$. Let $\widetilde{B} = P^T BP$. First we have $\exp(E\widetilde{B}s_0) = I$. On the other hand, from $AEB = BEA$, we have

$$\widetilde{A}E\widetilde{B} = (P^T AP)E(P^T BP) = P^T AEBP = P^T BEAP = \widetilde{B}E\widetilde{A}.$$

So from Lemma 17, there is a symplectic matrix Q such that $Q^T\widetilde{A}Q = \widetilde{A}$ and $Q^T\widetilde{B}Q = diag(\alpha, \beta, \gamma, \alpha, \beta, \gamma)$.

Thus if we take $S = PQ$, then $S^T AS = diag(1, -k, 0, 1, -k, 0)$ and $S^T BS = diag(\alpha, \beta, \gamma, \alpha, \beta, \gamma)$. If we take suitable linearly transformation, we can take $S^T BS = diag(1, 0, -l, 1, 0, -l)$ with $l > 0$.

Now if we take the Darboux coordinates

$$(p_1, p_2, p_3, q_1, q_2, q_3)^T = S^{-1}(x_1, x_2, x_3, y_1, y_2, y_3)^T,$$

we have proven that

$$\widetilde{f}_1 = (p_1^2 + q_1^2) - k(p_2^2 + q_2^2)$$
$$\widetilde{f}_2 = (p_1^2 + q_1^2) - l(p_3^2 + q_3^2).$$

Because Poisson bracket is preserved by the symplectic transformation, we still have $\{\widetilde{f}_i, \widetilde{f}_j\} = 0$ at the Darboux coordinates (p_j, q_j). From $\{\widetilde{f}_j, \widetilde{f}_3\} = 0$ for $j = 1, 2$, we have

$$p_1 \frac{\partial \widetilde{f}_3}{\partial q_1} - q_1 \frac{\partial \widetilde{f}_3}{\partial p_1} = k(p_2 \frac{\partial \widetilde{f}_3}{\partial q_2} - q_2 \frac{\partial \widetilde{f}_3}{\partial p_2}) = l(p_3 \frac{\partial \widetilde{f}_3}{\partial q_3} - q_3 \frac{\partial \widetilde{f}_3}{\partial p_3}).$$

Using above equations, by observation, $\widetilde{f}_3$ can not contain the following items:

$$p_i^2 q_i, p_i q_i^2, p_i^3, q_i^3, p_i^2 p_j, p_i^2 q_j, p_i q_j^2, q_i q_j^2, p_i p_j q_i, p_i q_i q_j (i \neq j).$$

So $\widetilde{f}_3$ only contains following items:

$$p_1 p_2 p_3, p_1 p_2 q_3, p_1 q_2 p_3, p_1 q_2 q_3, q_1 p_2 p_3, q_1 p_2 q_3, q_1 q_2 p_3, q_1 q_2 q_3.$$

Using $\{\widetilde{f}_i, \widetilde{f}_j\} = 0$, a straight forward computation shows that $k = l = 1$ and

$$\widetilde{f}_3 = a(p_1 p_2 p_3 - p_1 q_2 q_3 - q_1 p_2 q_3 - q_1 q_2 p_3) + b(p_1 p_2 q_3 + p_1 q_2 p_3 + q_1 p_2 p_3 - q_1 q_2 q_3)$$

$$= a\mathrm{Re}(p_1 + iq_1)(p_2 + iq_2)(p_3 + iq_3) + b\mathrm{I}m(p_1 + iq_1)(p_2 + iq_2)(p_3 + iq_3),$$

where a and b are constants. So if let $\sin\theta = \frac{a}{\sqrt{a^2+b^2}}$, $\cos\theta = \frac{b}{\sqrt{a^2+b^2}}$, then

$$\widetilde{f}_3 = \sqrt{a^2 + b^2}\mathrm{I}m[e^{i\theta}(p_1 + iq_1)(p_2 + iq_2)(p_3 + iq_3)].$$

So we can assume that

$$\widetilde{f}_3 = \mathrm{I}m(p_1 + iq_1)(p_2 + iq_2)(p_3 + iq_3) = p_1 p_2 q_3 + p_1 q_2 p_3 + q_1 p_2 p_3 - q_1 q_2 q_3$$

by some unitary translation and linear translation.

Now we must prove $S \in O(6, \mathbb{R})$. Let

$$U = diag(1, -1, 0, 1, -1, 0)$$

and let

$$V = diag(1, 0, -1, 1, 0, -1).$$

We have proven that $S^T AS = U$ and $S^T BS = V$. Then

$$\begin{aligned} S^{-1}\exp(EAt)S &= \exp(S^{-1}EASt) = \exp(ES^T ASt) \\ &= \exp(EUt) = diag(e^{it}, e^{-it}, 1) \end{aligned}$$

and

$$S^{-1}\exp(EBs)S = \exp(EVs) = \mathrm{diag}(e^{is}, 1, e^{-is}).$$

So we have $S^{-1}GS = T^2 = \{diag(e^{it_1}, e^{it_2}, e^{-i(t_1+t_2)}) | t_1, t_2 \in R\}$ or $G = STS^{-1}$. Let $C(\Sigma') = \{f_1(p, q) = f_2(p, q) = f_3(p, q) = 0\}$ and $p_0 = \frac{1}{\sqrt{3}}(1, 1, 1, 0, 0, 0)^T \in C(\Sigma')$, then there is a point $x_0 \in \Sigma$ such that $S^{-1}x_0 = cp_0$, where c is a constant. From

$G \cdot x_0 = \Sigma \subset S^5(1)$, we have $< gx_0, gx_0 >= 1$ for any $g \in G$. So for any $\tau \in T^2$, we have

$$< S\tau S^{-1}x_0, S\tau S^{-1}x_0 >= c^2 < S\tau p_0, S\tau p_0 >= 1,$$

or

$$(\tau p_0)^T(S^TS)(\tau p_0) = c^{-2}. \tag{4.6}$$

Let $\tau = \mathrm{diag}(e^{it_1}, e^{it_2}, e^{-i(t_1+t_2)})$ and let $u = \cos t_1$,$v = \cos t_2$. Let $S^TS = (m_{ij})$ and let

$$\begin{aligned} h(u,v) = {} & m_{33} + m_{44} + m_{55} - 2m_{46}v - 2m_{56}u + 2m_{12}uv \\ & + 2(m_{13} + m_{46})u^2v + 2(m_{25} + m_{56})uv^2 + (m_{11} - m_{44})u^2 \\ & + (m_{22} - m_{55})v^2 + (m_{33} - m_{66})(2u^2v^2 - u^2 - v^2) \\ & + \{-2m_{35} + 2(m_{14} + m_{36})u + 2m_{24}v + 2(m_{34} - m_{16})uv \\ & \quad +2(m_{35} - m_{26})v^2 - 4m_{36}uv^2\}\sqrt{1-u^2} \\ & + \{-2m_{34} + 2(m_{25} + m_{36})v + 2m_{15}u + 2(m_{34} - m_{16})u^2 \\ & \quad +2(m_{35} - m_{26})uv - 4m_{36}u^2v\}\sqrt{1-v^2} \\ & + \{2m_{45} - 2(m_{13} + m_{46})u - 2(m_{23} + m_{56})v \\ & \quad -2(m_{33} - m_{66})uv\}\sqrt{1-u^2}\sqrt{1-v^2} \end{aligned}$$

Then (4.6) can be rewritten

$$h(u,v) \equiv 3c^{-2} \tag{4.7}$$

for any $-1 \leq u, v \leq 1$. So we have $\frac{\partial^l h}{\partial u^n \partial v^{l-n}} = 0$. By direct calculation, we can get

$$\begin{aligned} \frac{\partial^6 h}{\partial u^3 \partial v^3}|_{u=v=0} &= m_{33} - m_{66} = 0 \\ \frac{\partial^6 h}{\partial u^3 \partial v^3}|_{u=0} &= -18(m_{13} + m_{46})v(1-v^2)^{-\frac{5}{2}} = 0 \\ \frac{\partial^6 h}{\partial u^3 \partial v^3}|_{v=0} &= -18(m_{23} + m_{56})u(1-u^2)^{-\frac{5}{3}} = 0 \\ \frac{\partial^6 h}{\partial u^3 \partial v^3} &= 18m_{45}uv(1-u^2)^{-\frac{5}{2}}(1-v^2)^{-\frac{5}{2}} = 0 \end{aligned}$$

From above equations, we can obtain $m_{45} = 0$, $m_{33} = m_{66}$, $m_{13} = -m_{46}$ and $m_{23} = -m_{56}$. Thus h can be write in the following form:

$$\begin{aligned} h(u,v) = {} & m_{33} + m_{44} + m_{55} + 2m_{13}v + 2m_{23}u + 2m_{12}uv \\ & + (m_{11} - m_{44})u^2 + (m_{22} - m_{55})v^2 \\ & + \{-2m_{35} + 2(m_{14} + m_{36})u + 2m_{24}v + 2(m_{34} - m_{16})uv \\ & \quad +2(m_{35} - m_{26})v^2 - 4m_{36}uv^2\}\sqrt{1-u^2} \\ & + \{-2m_{34} + 2(m_{25} + m_{36})v + 2m_{15}u + 2(m_{34} - m_{16})u^2 \\ & \quad +2(m_{35} - m_{26})uv - 4m_{36}u^2v\}\sqrt{1-v^2} \end{aligned}$$

Using the same method, at last, we can obtain

$$S^T S = \text{diag}(m_{11}, m_{22}, m_{33}, m_{11}, m_{22}, m_{33}) \tag{4.8}$$

and

$$m_{11} + m_{22} + m_{33} = 3c^{-2}.$$

But S is the symplectic matrix, so S^{-1} and S^T are symplectic matrices too. Thus we have

$$S^T SES^T S = S^T ES = E \tag{4.9}$$

Now from (4.8) and (4.9), we easily can get $m_{11} = m_{22} = m_{33} = 1$. From above discussion, we have $S^T S = I$ and $S \in O(6, \mathbb{R})$. Thus we have proven $S \in U(3)$ and this completes the proof of theorem 18. □

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