

QUADRATIC SHEAVES AND SELF-LINKAGE*

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0. Introduction and first results. The present paper is devoted to the proof of a structure theorem for self-linked pure subschemes $C \subseteq \mathbb{P}_k^n$ of codimension 2 over a field k of characteristic $p \neq 2$.

We use the symbol C because the classical case to be studied was the one of reduced curves in \mathbb{P}_k^3 . In this case, one can describe the notion of self-linkage in non-technical terms, saying that C is self-linked if and only if there are surfaces F and G such that their complete intersection is the curve C counted with multiplicity 2. This is a special case of the notion of linkage (C is linked to C' if $C \cup C'$ is the complete intersection of two surfaces F and G), classically introduced by R. Apery, F. Gaeta (see [Ap], [Gae]) and later deeply investigated by the algebraic point of view by Ch. Peskine and L. Szpiro and by P. Rao (see [P-S], [Rao1]). The special case of self-linkage was however studied before, in the work of E. Togliatti (see [To1], [To2]), and later D. Gallarati (see [Gal]), in the form of the theory of contact between surfaces.

In [Ca1] the theory of contact was related to a new theory, of the so called even sets of nodes, and later Rao used these ideas to obtain a structure theorem for projectively Cohen–Macaulay self-linked subschemes of codimension 2 in projective spaces (see [Rao2]).

Recently, Walter’s structure theorem (see [Wa]) for subcanonical subschemes of codimension 3 opened the way to solving some old conjectures about even sets of nodes and contact of hypersurfaces (see [C-C]).

A basic ingredient was the algebraic concept of quadratic sheaves, generalizing to a greater extent the geometric notion of contact and even sets. This notion was applied in [C-C] to the classification of even sets of nodes on a surface $F \subseteq \mathbb{P}_k^3$ for low values of the degree d .

On the other hand, let $F \subseteq \mathbb{P}_k^3$ be a surface whose only singularities are an even set of nodes Δ . Then there is a curve C on F passing through the points of Δ , and a surface G such that $F \cap G = 2C$ as cycles (see [Ca1], [Gal]). With this in mind it is therefore only natural to apply the structure theorem for quadratic sheaves in order to obtain a structure theorem for self-linkage.

This is done in the present paper, where we generalize the previously cited result of P. Rao to arbitrary pure subschemes of \mathbb{P}_k^n of codimension 2, (see also the survey [Ca2] of the second author for a preliminary version of these results).

MAIN THEOREM. *Let k be a field of characteristic $p \neq 2$ and $C \subseteq \mathbb{P}_k^n$ be a pure subscheme of codimension 2 which is self-linked through hypersurfaces $F := \{f = 0\}$ and $G := \{g = 0\}$ of respective degrees d, m . Let \mathfrak{S}_C be its sheaf of ideals and set $\mathcal{F}_F := \mathfrak{S}_C/f\mathcal{O}_{\mathbb{P}_k^n}(-d)$, $\mathcal{F}_G := \mathfrak{S}_C/g\mathcal{O}_{\mathbb{P}_k^n}(-m)$.*

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Assume moreover that if $n \equiv 1 \pmod 4$ and $d + m - n - 1 = 2\varrho$, then the following two equivalent congruences hold:

$$\begin{aligned} \chi(\mathcal{O}_{\mathbb{P}_k^n}(\varrho)) - \chi(\mathcal{O}_{\mathbb{P}_k^n}(\varrho - d)) - \chi(\mathcal{O}_C(\varrho)) &\equiv 0 \pmod 2, \\ \chi(\mathcal{O}_{\mathbb{P}_k^n}(\varrho)) - \chi(\mathcal{O}_{\mathbb{P}_k^n}(\varrho - m)) - \chi(\mathcal{O}_C(\varrho)) &\equiv 0 \pmod 2. \end{aligned}$$

Then there exist a locally free $\mathcal{O}_{\mathbb{P}_k^n}$ -sheaf \mathcal{E} , a symmetric map $\alpha: \check{\mathcal{E}}(-d - m) \rightarrow \mathcal{E}$ and a resolution

$$(1) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}_k^n}(-d) \oplus \check{\mathcal{E}}(-d - m) \xrightarrow{\begin{pmatrix} \gamma & t\lambda \\ \lambda & \alpha \end{pmatrix}} \mathcal{O}_{\mathbb{P}_k^n}(-m) \oplus \mathcal{E} \longrightarrow \mathcal{F}_F \longrightarrow 0$$

inducing exact sequences

$$(2) \quad 0 \longrightarrow \check{\mathcal{E}}(-d - m) \xrightarrow{\begin{pmatrix} t\lambda \\ \alpha \end{pmatrix}} \mathcal{O}_{\mathbb{P}_k^n}(-m) \oplus \mathcal{E} \longrightarrow \mathfrak{S}_C \longrightarrow 0,$$

$$(3) \quad 0 \longrightarrow \check{\mathcal{E}}(-d - m) \xrightarrow{\alpha} \mathcal{E} \longrightarrow \mathcal{F}_G \longrightarrow 0.$$

Conversely, given a subscheme C of codimension 2, assume that there does exist a sequence (2) with the above property of α being symmetric and $\deg(\det(\alpha)) = m$. Then C is self-linked through the hypersurfaces F and G of respective equations $f := \det \begin{pmatrix} 0 & t\lambda \\ \lambda & \alpha \end{pmatrix}$ and $g := \det(\alpha)$. \square

It was conjectured by Ph. Ellia that if moreover C is subcanonical then C is the complete intersection of two hypersurfaces (see [E–B]). We have been trying to deduce this statement from our main theorem. In the meantime, Ellia’s conjecture has been proven in [Ar] if $n \geq 4$, and in [F–L–K] when $n \geq 3$ and $k \cong \mathbb{C}$.

Moreover, in section 3 we discuss the problem whether the self-linkage ideal (f, g) is uniquely determined once the generator g of higher degree is fixed, and prove that the answer is positive under the condition that the subscheme be locally Gorenstein.

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First results. For the reader’s benefit we recall the following definitions and results proved from [C–C]. From now on we always assume that k is a field of characteristic $p \neq 2$.

DEFINITION 0.1. Let X be a projective, locally Cohen–Macaulay scheme. We say that a coherent, locally Cohen–Macaulay sheaf of \mathcal{O}_X -modules \mathcal{F} is a $\delta/2$ -quadratic sheaf on X , $\delta = 0, 1$, if there exists a symmetric bilinear map

$$\mathcal{F} \times \mathcal{F} \rightarrow \mathcal{O}_X(-\delta)$$

inducing an isomorphism $\sigma: \mathcal{F}(\delta) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

REMARK 0.2. If \mathcal{F} is $\delta/2$ -quadratic, then it is reflexive since the natural map $\mathcal{F} \rightarrow \mathcal{F}^{\vee\vee}$ equals $\check{\sigma}^{-1} \circ \sigma(-\delta)$.

Assume now that $F \subseteq \mathbb{P}_k^n$ is a hypersurface of degree d and that \mathcal{F} is a $\delta/2$ -quadratic sheaf on F .

The main result of sections 1 and 2 of [C–C] concerns a characterization of quadratic sheaves on hypersurfaces in \mathbb{P}_k^n (including the needed parity condition as pointed out in theorem 9.1 of [E–P–W]).

THEOREM 0.3. *Let $F \subseteq \mathbb{P}_k^n$ be a hypersurface of degree d and let \mathcal{F} be a $\delta/2$ -quadratic sheaf on F . Then \mathcal{F} fits into an exact sequence of the form*

$$(0.3.1) \quad 0 \longrightarrow \check{\mathcal{E}}(-d - \delta) \xrightarrow{\varphi} \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow 0$$

where \mathcal{E} is a locally free $\mathcal{O}_{\mathbb{P}_k^n}$ -sheaf and φ is a symmetric map if and only if the following parity condition holds: if $n \equiv 1 \pmod{4}$ and $n + 1 - d - \delta = 2r$, then also $\chi(\mathcal{F}(-r))$ is even.

Moreover we can choose \mathcal{E} such that $H_*^j(\mathbb{P}_k^n, \mathcal{E}) = 0$ for $n > j > (n - 1)/2$. □

REMARK 0.4. Following [Ca1] we say that \mathcal{F} is split symmetric if one can choose \mathcal{E} to be a direct sum of line bundles. This is possible if and only if \mathcal{F} is arithmetically Cohen-Macaulay i.e., if and only if $H_*^i(\mathbb{P}_k^n, \mathcal{F}) = 0$ for each $i = 1, \dots, n - 2$.

1. From self-linkage to quadratic sheaves. Our first main theorem is an application of the theory of quadratic sheaves. For another application see [C-C].

DEFINITION 1.1. Let $C \subseteq \mathbb{P}_k^n$ be a pure subscheme of codimension 2 and let $\mathfrak{S}_C \subseteq \mathcal{O}_{\mathbb{P}_k^n}$ be its sheaf of ideals: C is said to be self-linked with respect to the complete intersection $X := F \cap G$ of the two hypersurfaces F, G of respective degrees d, m , if $C \subseteq X$ and one of the following equivalent conditions holds

- i) $\mathfrak{S}_X : \mathfrak{S}_C = \mathfrak{S}_C$;
- ii) $\mathfrak{S}_C / \mathfrak{S}_X = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{O}_C, \mathcal{O}_X)$.

For the above well-known equivalence see e.g. theorem 21.23 of [Ei] (see also [P-S]).

REMARK 1.2. Indeed, since \mathfrak{S}_C and \mathfrak{S}_X coincide with $\mathcal{O}_{\mathbb{P}_k^n}$ in codimension 1 and are torsion free, every $\phi \in \mathcal{H}om_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathfrak{S}_C, \mathfrak{S}_X)$ is given by a rational function, which is in turn regular by the normality of $\mathcal{O}_{\mathbb{P}_k^n}$.

Condition i) of definition 1.1 can be thus rewritten as (see [Ca2], proposition 2.6)

$$(1.2.1) \quad \mathfrak{S}_C = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathfrak{S}_C, \mathfrak{S}_X).$$

By duality for finite maps (see [Ha], exercise II.6.10),

$$(1.2.2) \quad \mathcal{H}om_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{O}_C, \mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{O}_C, \omega_X(n + 1 - m - d)) = \omega_C(n + 1 - m - d).$$

Finally, let $C, F := \{f = 0\}, G := \{g = 0\}$ be as above, and assume $m = \deg(G) \geq d = \deg(F)$. Then we can replace g by $g + af$ (a is a homogeneous polynomial of degree $m - d$), and obtain by Bertini's theorem that G is smooth outside C . Indeed, if C is reduced we can even have that G is smooth at the generic points of C , so that G is a normal, whence irreducible, hypersurface. Moreover in this case (cfr. [Ca1], proposition 2.6) condition i) is equivalent to $\text{cycle}(X) = 2C$.

DEFINITION 1.3. Let $\mathcal{F}_G := \mathfrak{S}_C / g\mathcal{O}_{\mathbb{P}_k^n}(-m), \mathcal{F}_F := \mathfrak{S}_C / f\mathcal{O}_{\mathbb{P}_k^n}(-d)$.

By the above definition it follows that both \mathcal{F}_G and \mathcal{F}_F are obviously locally Cohen-Macaulay. By remark 1.2, since

$$\mathfrak{S}_X / g\mathcal{O}_{\mathbb{P}_k^n}(-m) \cong (f\mathcal{O}_{\mathbb{P}_k^n}(-d) + g\mathcal{O}_{\mathbb{P}_k^n}(-m)) / g\mathcal{O}_{\mathbb{P}_k^n}(-m) \cong f\mathcal{O}_G(-d),$$

we have pairings

$$\mathcal{F}_G \times \mathcal{F}_G \longrightarrow \mathfrak{S}_X / g\mathcal{O}_{\mathbb{P}_k^n}(-m) \cong f\mathcal{O}_G(-d),$$

and analogously

$$\mathcal{F}_F \times \mathcal{F}_F \longrightarrow g\mathcal{O}_F(-m).$$

To verify that a twist of \mathcal{F}_G yields a quadratic sheaf on G (similarly for \mathcal{F}_F on F) we have to show that the pairing is perfect, i.e. $\mathcal{F}_G \cong \text{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{F}_G, \mathcal{O}_G(-d))$.

Let $\psi \in \text{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{F}_G, \mathcal{O}_G(-d))$, then ψ induces $\psi' : \mathfrak{S}_C \rightarrow \mathcal{O}_G(-d) \cong \mathfrak{S}_X/g\mathcal{O}_{\mathbb{P}_k^n}(-m)$. If ψ' is in the image of a $\phi \in \text{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathfrak{S}_C, \mathfrak{S}_X)$, then, by remark 1.2, $\phi \in \mathfrak{S}_C$, and clearly ϕ induces a zero ψ if and only if $\phi \in g\mathcal{O}_{\mathbb{P}_k^n}(-m)$.

Thus, we are left with the verification that

$$\text{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathfrak{S}_C, \mathfrak{S}_X) \rightarrow \text{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathfrak{S}_C, \mathfrak{S}_X/g\mathcal{O}_{\mathbb{P}_k^n}(-m))$$

is surjective. By the $\text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^i$ -exact sequence, the above surjectivity is equivalent to the injectivity of $j : \text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^1(\mathfrak{S}_C, g\mathcal{O}_{\mathbb{P}_k^n}(-m)) \rightarrow \text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^1(\mathfrak{S}_C, \mathfrak{S}_X)$.

To this purpose, consider

$$\begin{aligned} 0 \longrightarrow \text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^1(\mathcal{O}_C, \mathfrak{S}_X) &\xrightarrow{i} \text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^1(\mathcal{O}_C, \mathcal{O}_G(-d)) \longrightarrow \\ &\longrightarrow \text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^2(\mathcal{O}_C, g\mathcal{O}_{\mathbb{P}_k^n}(-m)) \xrightarrow{j'} \text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^2(\mathcal{O}_C, \mathfrak{S}_X) \end{aligned}$$

where j' corresponds to j via the natural equivalence $\text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^1(\mathfrak{S}_C, \cdot) \cong \text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^2(\mathcal{O}_C, \cdot)$. It follows that j is injective if and only if i is an isomorphism. We have the exact sequence

$$\begin{aligned} \text{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{O}_{\mathbb{P}_k^n}, \mathfrak{S}_X) &\longrightarrow \text{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathfrak{S}_C, \mathfrak{S}_X) \longrightarrow \\ &\longrightarrow \text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^1(\mathcal{O}_C, \mathfrak{S}_X) \longrightarrow \text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^1(\mathcal{O}_{\mathbb{P}_k^n}, \mathfrak{S}_X) \cong 0. \end{aligned}$$

Since $\text{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{O}_{\mathbb{P}_k^n}, \mathfrak{S}_X) \cong \mathfrak{S}_X$ and $\text{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathfrak{S}_C, \mathfrak{S}_X) \cong \mathfrak{S}_C$ then

$$\text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^1(\mathcal{O}_C, \mathfrak{S}_X) \cong \mathfrak{S}_C/\mathfrak{S}_X \cong \text{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{O}_C, \mathcal{O}_X) \cong \omega_C(n+1-m-d).$$

We also have

$$\begin{aligned} \text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^1(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}_k^n}(-d)) &\longrightarrow \text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^1(\mathcal{O}_C, \mathcal{O}_G(-d)) \longrightarrow \\ &\longrightarrow \text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^2(\mathcal{O}_C, g\mathcal{O}_{\mathbb{P}_k^n}(-m-d)) \xrightarrow{g} \text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^2(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}_k^n}(-d)). \end{aligned}$$

Since $C \subseteq G$, then the multiplication by g is zero. Moreover, $\text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^1(\mathcal{O}_C, \mathcal{O}_{\mathbb{P}_k^n}(-d)) = 0$ since C is locally Cohen–Macaulay. We obtain that

$$\text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^1(\mathcal{O}_C, \mathcal{O}_G(-d)) \cong \text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^2(\mathcal{O}_C, g\mathcal{O}_{\mathbb{P}_k^n}(-m-d)) \cong \omega_C(n+1-m-d).$$

The following easy lemma thus concludes the proof that $i : \omega_C(m+d-n-1) \rightarrow \omega_C(m+d-n-1)$ is an isomorphism.

LEMMA 1.4. *If \mathcal{H} is a coherent $\mathcal{O}_{\mathbb{P}_k^n}$ -sheaf and $\varrho : \mathcal{H} \rightarrow \mathcal{H}$ is either injective or surjective, then it is an isomorphism.*

Proof. Let ϱ be injective (resp. surjective) and $\mathcal{K} := \text{coker}(\varrho)$ (resp. $\mathcal{K} := \text{ker}(\varrho)$). By Serre’s theorem B we get $h^1(\mathbb{P}_k^n, \mathcal{H}(t)) = 0$ (resp. $h^1(\mathbb{P}_k^n, \mathcal{K}(t)) = 0$) for each t large

enough, hence $h^0(\mathbb{P}_k^n, \mathcal{K}(t)) = 0$ for t large enough, whence $\mathcal{K} = 0$ by Serre's theorem A. \square

Set $d = 2d' + \delta$, $m = 2m' + \mu$, $\delta, \mu \in \{0, 1\}$. Then $\mathcal{F}_G(d')$ is a $\delta/2$ -quadratic sheaf on G , and similarly $\mathcal{F}_F(m')$ is a $\mu/2$ -quadratic sheaf on F . According to theorem 0.3, we obtain two locally free $\mathcal{O}_{\mathbb{P}_k^n}$ -sheaves \mathcal{E}'_G and \mathcal{E}'_F provided that the respective parity conditions hold.

It is convenient to rewrite such conditions. If $n \equiv 1 \pmod 4$ and $d+m-n-1 = 2\varrho$, then we want that $\chi(\mathcal{F}_F(\varrho)) \equiv \chi(\mathcal{F}_G(\varrho)) \equiv 0 \pmod 2$. By two obvious exact sequences the above conditions are equivalent to the two congruences

$$(1.5) \quad \begin{aligned} \chi(\mathcal{O}_{\mathbb{P}_k^n}(\varrho)) - \chi(\mathcal{O}_{\mathbb{P}_k^n}(\varrho - d)) - \chi(\mathcal{O}_C(\varrho)) &\equiv 0 \pmod 2, \\ \chi(\mathcal{O}_{\mathbb{P}_k^n}(\varrho)) - \chi(\mathcal{O}_{\mathbb{P}_k^n}(\varrho - m)) - \chi(\mathcal{O}_C(\varrho)) &\equiv 0 \pmod 2. \end{aligned}$$

In turn the two congruences above are equivalent each other. In fact, it suffices to show that

$$\chi(\mathcal{O}_{\mathbb{P}_k^n}(\varrho - d)) + \chi(\mathcal{O}_{\mathbb{P}_k^n}(\varrho - m)) = 0.$$

To this purpose recall that

$$\chi(\mathcal{O}_{\mathbb{P}_k^n}(h)) = \frac{(h+n)\dots(h+1)}{n!}.$$

In our case $n = 4a + 1$ and ϱ is even. Since, by the linkage condition, dm is even too, the parity of $d + m - n - 1$ yields $d = 2d'$, $m = 2m'$, hence

$$\chi(\mathcal{O}_{\mathbb{P}_k^n}(\varrho - d)) = \frac{(2a+i)\dots(-2a+i)}{(4a+1)!} = -\frac{(2a-i)\dots(-2a-i)}{(4a+1)!} = \chi(\mathcal{O}_{\mathbb{P}_k^n}(\varrho - m)),$$

where $i = m' - d'$.

We define $\mathcal{E}_G := \mathcal{E}'_G(-d')$ and $\mathcal{E}_F := \mathcal{E}'_F(-m')$. Then theorem 0.3 rewrites as follows.

PROPOSITION 1.6. *Assume that if $n \equiv 1 \pmod 4$ and $n + 1 - d - m = 2\varrho$ then the two equivalent congruences (1.5) above hold.*

Then there exist two locally free $\mathcal{O}_{\mathbb{P}_k^n}$ -sheaves \mathcal{E}_G and \mathcal{E}_F , such that $H_^i(\mathbb{P}_k^n, \mathcal{E}_G) = H_*^i(\mathbb{P}_k^n, \mathcal{E}_F) = 0$ for $n > i > (n - 1)/2$, fitting into exact sequences*

$$(1.6.1) \quad \begin{aligned} 0 &\longrightarrow \check{\mathcal{E}}_F(-d-m) \xrightarrow{\alpha_F} \mathcal{E}_F \longrightarrow \mathcal{F}_F \longrightarrow 0, \\ 0 &\longrightarrow \check{\mathcal{E}}_G(-d-m) \xrightarrow{\alpha_G} \mathcal{E}_G \longrightarrow \mathcal{F}_G \longrightarrow 0, \end{aligned}$$

where ${}^t\alpha_F = \alpha_F$, ${}^t\alpha_G = \alpha_G$. \square

In the above proposition and in what follows, the superscript t denotes the dual morphism twisted by $-d - m$.

REMARK 1.7. Let $\ell \subseteq \mathbb{P}_k^n$ be a line disjoint from C . Then the restrictions of the sequences (1.6.1) to ℓ are still exact. On the other hand $\mathcal{F}_F|_\ell \cong \mathcal{O}_{F \cap \ell}$ and $\mathcal{F}_G|_\ell \cong \mathcal{O}_{G \cap \ell}$, hence

$$c_1(\mathcal{E}_F) = -\frac{1}{2} \text{rk}(\mathcal{E}_F)(d+m) + \frac{d}{2}, \quad c_1(\mathcal{E}_G) = -\frac{1}{2} \text{rk}(\mathcal{E}_G)(d+m) + \frac{m}{2}.$$

2. From quadratic sheaves to self-linkage. From now on we will assume that the parity condition holds.

Let $\mathcal{O}_{\mathbb{P}_k^n}(-d) \xrightarrow{f} \mathcal{O}_{\mathbb{P}_k^n}$ be the multiplication by f . Then it induces a map $\vartheta: \mathcal{O}_{\mathbb{P}_k^n}(-d) \rightarrow \mathcal{F}_G$. Since we have a chain of homomorphisms

$$(2.1) \quad \begin{aligned} \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{O}_{\mathbb{P}_k^n}(-d), \mathcal{E}_G) &\longrightarrow \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{O}_{\mathbb{P}_k^n}(-d), \mathcal{F}_G) \longrightarrow \\ &\xrightarrow{\vartheta} \mathrm{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^1(\mathcal{O}_{\mathbb{P}_k^n}(-d), \check{\mathcal{E}}_G(-d-m)) \cong \\ &\cong H^1(\mathbb{P}_k^n, \check{\mathcal{E}}_G(-m)) \cong H^{n-1}(\mathbb{P}_k^n, \mathcal{E}_G(m+d-n-1)) = 0 \end{aligned}$$

we can lift ϑ to a map $\lambda: \mathcal{O}_{\mathbb{P}_k^n}(-d) \rightarrow \mathcal{E}_G$.

Arguing as in (2.1) we obtain that the natural map

$$\mathrm{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{E}_G, \mathfrak{S}_C) \rightarrow \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{E}_G, \mathcal{F}_G)$$

is surjective whence we infer that the surjection $\mathcal{E}_G \rightarrow \mathcal{F}_G$ can be lifted to $\nu: \mathcal{E}_G \rightarrow \mathfrak{S}_C$. Notice that the map $\nu \circ \lambda$ is given by $\tilde{f} \in H^0(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d))$ congruent $f \pmod{g}$.

Let $r := \mathrm{rk}(\mathcal{E}_G)$. We have a map

$$\Lambda^{r-1}(\alpha_G): \Lambda^{r-1}(\check{\mathcal{E}}_G(-d-m)) \rightarrow \Lambda^{r-1}\mathcal{E}_G.$$

Since $\Lambda^{r-1}\check{\mathcal{E}}_G \cong \mathcal{E}_G \otimes \det(\mathcal{E}_G)^{-1}$ and $\Lambda^{r-1}\mathcal{E}_G \cong \check{\mathcal{E}}_G \otimes \det(\mathcal{E}_G)$, twisting by $\det(\mathcal{E}_G)^{-1}(-d)$ and taking remark 1.7 into account, we obtain a map $\alpha_G^{adj}: \mathcal{E}_G \rightarrow \check{\mathcal{E}}_G(-d)$ such that $\alpha_G \circ \alpha_G^{adj}(-m): \mathcal{E}_G(-m) \rightarrow \mathcal{E}_G$ is the multiplication by g and $\alpha_G^{adj}(d) = \check{\alpha}_G^{adj}$ since α_G , hence α_G^{adj} , is symmetric.

We can then define

$$\mu := \check{\lambda}(-d) \circ \alpha_G^{adj}: \mathcal{E}_G \rightarrow \mathcal{O}_{\mathbb{P}_k^n}.$$

PROPOSITION 2.2. \mathfrak{S}_C coincides with the sheaf of ideals of $\mathcal{O}_{\mathbb{P}_k^n}$ generated by g and $\mathrm{im}(\mu)$.

Proof. Let us consider the second sequence in (1.6.1). We have a chain map

$$(2.2.1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \check{\mathcal{E}}_G(-d-m) & \xrightarrow{\alpha_G} & \mathcal{E}_G & \longrightarrow & \mathcal{F}_G \longrightarrow 0 \\ & & \downarrow \beta & & \downarrow \nu & & \downarrow id \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}_k^n}(-m) & \xrightarrow{g} & \mathfrak{S}_C & \longrightarrow & \mathcal{F}_G \longrightarrow 0 \end{array}$$

where β is induced by the restriction of ν .

By the mapping cone construction we obtain a resolution

$$0 \longrightarrow \check{\mathcal{E}}_G(-d-m) \xrightarrow{s} \mathcal{O}_{\mathbb{P}_k^n}(-m) \oplus \mathcal{E}_G \longrightarrow \mathfrak{S}_C \longrightarrow 0$$

where s has components β, α_G . Recall that $\alpha_G(m) \circ \alpha_G^{adj}$ is the multiplication by g whence $(\nu \circ \alpha_G)(m) \circ \alpha_G^{adj} = g\nu$, since $(\nu \circ g)(m) = g\nu$.

Diagram (2.2.1) yields $g\beta = \nu \circ \alpha_G$. Composing on the right with $\alpha_G^{adj} \circ \lambda$, we obtain

$$g\beta(m) \circ \alpha_G^{adj} \circ \lambda = g\nu \circ \lambda: \mathcal{O}_{\mathbb{P}_k^n}(-d) \rightarrow \mathfrak{S}_C(m) \subseteq \mathcal{O}_{\mathbb{P}_k^n}(m).$$

Since g is obviously a non-zero divisor in \mathfrak{S}_C , then $\nu \circ \lambda = \beta(m) \circ \alpha_G^{adj} \circ \lambda: \mathcal{O}_{\mathbb{P}_k^n}(-d) \rightarrow \mathcal{O}_{\mathbb{P}_k^n}$ and since $\nu \circ \lambda$ is given by $\tilde{f} \equiv f \pmod{g}$ the same is true for $\beta(m) \circ \alpha_G^{adj} \circ \lambda$.

It follows from the above identities that $s^t\mu = (\nu \circ \lambda(-m), g\lambda(-m))$ is represented by the product matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} \tilde{f} \\ g \end{pmatrix}.$$

Thus we obtain the following commutative diagram

$$(2.2.2) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}_k^n}(-d-m) & \xrightarrow{\begin{pmatrix} \tilde{f} \\ g \end{pmatrix}} & \mathcal{O}_{\mathbb{P}_k^n}(-m) \oplus \mathcal{O}_{\mathbb{P}_k^n}(-d) & \longrightarrow & \mathfrak{S}_X \longrightarrow 0 \\ & & \downarrow \scriptstyle t\mu & & \downarrow \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} & & \downarrow \\ 0 & \longrightarrow & \check{\mathcal{E}}_G(-d-m) & \xrightarrow{s} & \mathcal{O}_{\mathbb{P}_k^n}(-m) \oplus \mathcal{E}_G & \longrightarrow & \mathfrak{S}_C \longrightarrow 0 \end{array}$$

whose right column is the inclusion.

The mapping cone of (2.2.2) is a resolution of $\mathfrak{S}_C/\mathfrak{S}_X \cong \omega_C(n+1-d-m)$ (see (1.2.2)). Therefore the dual of the mapping cone of diagram (2.2.2) yields a resolution of \mathfrak{S}_C (see proposition 2.5 of [P-S]) and \mathfrak{S}_C coincides with the sheaf of ideals locally generated by the maximal minors of

$$\begin{pmatrix} \beta & 1 & 0 \\ \alpha_G & 0 & \lambda \end{pmatrix}$$

which is the ideal locally generated by the maximal minors of (α_G, λ) . Our statement follows then from the very definition of μ and the identity $\det(\alpha_G) = g$. \square

Since $\mu \circ \alpha_G = g({}^t\lambda)$, then μ induces an endomorphism ψ of \mathcal{F}_G fitting into the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \check{\mathcal{E}}_G(-d-m) & \xrightarrow{\alpha_G} & \mathcal{E}_G & \longrightarrow & \mathcal{F}_G \longrightarrow 0 \\ & & \downarrow \scriptstyle t\lambda & & \downarrow \scriptstyle \mu & & \downarrow \scriptstyle \psi \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}_k^n}(-m) & \xrightarrow{g} & \mathfrak{S}_C & \longrightarrow & \mathcal{F}_G \longrightarrow 0. \end{array}$$

Since $\mathcal{F}_G := \mathfrak{S}_C/g\mathcal{O}_{\mathbb{P}_k^n}(-m)$, proposition 2.2 implies the surjectivity of ψ . Moreover ψ is also injective by lemma 1.4 above. We can replace the given surjection $\pi: \mathcal{E}_G \rightarrow \mathcal{F}_G$ with $\psi \circ \pi$, whence we may also replace ν by μ in the arguments of proposition 2.2, thus obtaining the following

PROPOSITION 2.3. *There exists $\gamma \in H^0(\mathbb{P}_k^n, \mathcal{O}_{\mathbb{P}_k^n}(d-m))$ such that*

$$f = \det \begin{pmatrix} \gamma & {}^t\lambda \\ \lambda & \alpha_G \end{pmatrix}.$$

Proof. Recall that we are now assuming $\nu = \mu$, whence by the very definition of μ , $\mu \circ \lambda \equiv f \pmod{g}$. Since

$$\mu \circ \lambda = \det \begin{pmatrix} 0 & {}^t\lambda \\ \lambda & \alpha_G \end{pmatrix},$$

then we obtain the existence of a γ such that

$$f = \det \begin{pmatrix} 0 & {}^t\lambda \\ \lambda & \alpha_G \end{pmatrix} + \gamma g = \det \begin{pmatrix} 0 & {}^t\lambda \\ \lambda & \alpha_G \end{pmatrix} + \det \begin{pmatrix} \gamma & {}^t\lambda \\ 0 & \alpha_G \end{pmatrix} = \det \begin{pmatrix} \gamma & {}^t\lambda \\ \lambda & \alpha_G \end{pmatrix}. \quad \square$$

Since $f \neq 0$ we get a coherent sheaf \mathcal{F} supported on F and an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_k^n}(-d) \oplus \check{\mathcal{E}}_G(-d-m) \xrightarrow{\begin{pmatrix} \gamma & t\lambda \\ \lambda & \alpha_G \end{pmatrix}} \mathcal{O}_{\mathbb{P}_k^n}(-m) \oplus \mathcal{E}_G \longrightarrow \mathcal{F} \longrightarrow 0.$$

It follows that $\mathcal{F}(m')$ is $\mu/2$ -quadratic and we can easily construct the following exact sequence of the vertical complexes C_i 's

$$\begin{array}{ccccccc} & & C_1 & & C_2 & & C_3 \\ 0 & \longrightarrow & \check{\mathcal{E}}_G(-d-m) & \longrightarrow & \mathcal{O}_{\mathbb{P}_k^n}(-m) \oplus \mathcal{E}_G & \longrightarrow & \mathfrak{S}_C \longrightarrow 0 \\ & & \downarrow \begin{pmatrix} 0 \\ id \end{pmatrix} & & \downarrow id & & \downarrow \eta \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}_k^n}(-d) \oplus \check{\mathcal{E}}_G(-d-m) & \longrightarrow & \mathcal{O}_{\mathbb{P}_k^n}(-m) \oplus \mathcal{E}_G & \longrightarrow & \mathcal{F} \longrightarrow 0. \end{array}$$

The associated long exact sequence gives $\mathcal{O}_{\mathbb{P}_k^n}(-d) \cong H_1(C_1) \cong H_0(C_3) \cong \ker(\eta)$. On the other hand $f \in \ker(\eta)$, thus $\mathcal{F} \cong \mathcal{F}_F$.

If we set $\mathcal{E} := \mathcal{E}_G$ and $\alpha := \alpha_G$, the above discussion proves the “only if” part of the statement of the following main theorem.

THEOREM 2.4. *Let k be a field of characteristic $p \neq 2$ and $C \subseteq \mathbb{P}_k^n$ be a pure subscheme of codimension 2 which is self-linked through hypersurfaces $F := \{f = 0\}$ and $G := \{g = 0\}$ of respective degrees d, m . Let \mathfrak{S}_C be its sheaf of ideals and set $\mathcal{F}_F := \mathfrak{S}_C/f\mathcal{O}_{\mathbb{P}_k^n}(-d)$, $\mathcal{F}_G := \mathfrak{S}_C/g\mathcal{O}_{\mathbb{P}_k^n}(-m)$.*

Assume that if $n \equiv 1 \pmod 4$ and $d + m - n - 1 = 2\rho$, then the following two equivalent congruences hold:

$$\begin{aligned} \chi(\mathcal{O}_{\mathbb{P}_k^n}(\rho)) - \chi(\mathcal{O}_{\mathbb{P}_k^n}(\rho - d)) - \chi(\mathcal{O}_C(\rho)) &\equiv 0 \pmod 2, \\ \chi(\mathcal{O}_{\mathbb{P}_k^n}(\rho)) - \chi(\mathcal{O}_{\mathbb{P}_k^n}(\rho - m)) - \chi(\mathcal{O}_C(\rho)) &\equiv 0 \pmod 2. \end{aligned}$$

Then there exist a locally free $\mathcal{O}_{\mathbb{P}_k^n}$ -sheaf \mathcal{E} , a symmetric map $\alpha: \check{\mathcal{E}}(-d-m) \rightarrow \mathcal{E}$ and a resolution

$$(2.4.1) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}_k^n}(-d) \oplus \check{\mathcal{E}}(-d-m) \xrightarrow{\begin{pmatrix} \gamma & t\lambda \\ \lambda & \alpha \end{pmatrix}} \mathcal{O}_{\mathbb{P}_k^n}(-m) \oplus \mathcal{E} \longrightarrow \mathcal{F}_F \longrightarrow 0$$

inducing exact sequences

$$(2.4.2) \quad 0 \longrightarrow \check{\mathcal{E}}(-d-m) \xrightarrow{\begin{pmatrix} t\lambda \\ \alpha \end{pmatrix}} \mathcal{O}_{\mathbb{P}_k^n}(-m) \oplus \mathcal{E} \longrightarrow \mathfrak{S}_C \longrightarrow 0,$$

$$(2.4.3) \quad 0 \longrightarrow \check{\mathcal{E}}(-d-m) \xrightarrow{\alpha} \mathcal{E} \longrightarrow \mathcal{F}_G \longrightarrow 0.$$

Conversely, given a subscheme C of codimension 2, assume that there does exist a sequence (2.4.2) with the above property of α being symmetric and $\deg(\det(\alpha)) = m$. Then C is self-linked through the hypersurfaces F and G of respective equations $f := \det \begin{pmatrix} 0 & t\lambda \\ \lambda & \alpha \end{pmatrix}$ and $g := \det(\alpha)$.

Proof. There remains to prove the converse assertion. Let $P \in \mathbb{P}_k^n$ and consider in $\mathcal{O}_{\mathbb{P}_k^n, P}$ the maximal minors f_i of the matrix $(\lambda \alpha)$ obtained by deleting the i^{th} column: in particular $g = f_1$. Recall that $\mathfrak{S}_{C,P} = (f_1, \dots, f_{r+1})$ in $\mathcal{O}_{\mathbb{P}_k^n, P}$, hence for each pair of indices $i, j = 2, \dots, r+1$ we have

$$u_{i,j}f + v_{i,j}g = f_i f_j$$

for suitable $u_{i,j}, v_{i,j} \in \mathcal{O}_{\mathbb{P}_k^n, P}$ (the determinantal identity (1.2) of [Ca1] with $k = j = 1$).

It follows that neither f nor g are identically zero, else C would have a codimension one component. The same identity shows that $\mathfrak{S}_C^2 \subseteq \mathfrak{S}_X$. Moreover it is always true that $\mathfrak{S}_X \subseteq \mathfrak{S}_C$, thus F and G have no common components.

We now prove that $\mathfrak{S}_X : \mathfrak{S}_C = \mathfrak{S}_C$. To this purpose we first check that $\mathcal{F} := \text{coker}(\alpha) \cong \mathfrak{S}_C/g\mathcal{O}_{\mathbb{P}_k^n}(-m)$. Indeed the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{\mathcal{E}}_G(-d-m) & \longrightarrow & \mathcal{O}_{\mathbb{P}_k^n}(-m) \oplus \mathcal{E}_G & \longrightarrow & \mathfrak{S}_C \longrightarrow 0 \\ & & \downarrow id & & \downarrow (0 \ id) & & \\ 0 & \longrightarrow & \tilde{\mathcal{E}}_G(-d-m) & \longrightarrow & \mathcal{E}_G & \longrightarrow & \mathcal{F} \longrightarrow 0. \end{array}$$

induces a surjection $\mathfrak{S}_C \rightarrow \mathcal{F}$. Arguing as above we obtain that its kernel is $g\mathcal{O}_{\mathbb{P}_k^n}(-m)$.

The map α induces an isomorphism $\mathcal{F} \cong \text{Hom}_{\mathcal{O}_{\mathbb{P}_k^n}}(\mathcal{F}, \mathcal{O}_G(-d))$, since $\text{deg}(f) = d$. We have $h \in \mathfrak{S}_{X,P} : \mathfrak{S}_{C,P}$ if and only if $h\mathfrak{S}_{C,P} \subseteq \mathfrak{S}_{X,P}$.

Taking residue classes mod g , this is equivalent to $h\mathcal{F}_P \subseteq \mathfrak{S}_{X,P}/g\mathcal{O}_{\mathbb{P}_k^n, P}(-m) \cong \mathcal{O}_{G,P}(-d)$, since $g \in \mathfrak{S}_X$, hence to the fact that $h \text{ mod } g$ is in $\text{Hom}_{\mathcal{O}_{\mathbb{P}_k^n, P}}(\mathcal{F}_P, \mathcal{O}_{G,P}(-d)) \cong \mathcal{F}_P$, i.e. $h \in \mathfrak{S}_{C,P}$. \square

3. Self-linkage of generically Gorenstein subschemes.

In this section we shall inspect more deeply the case when C is a generically Gorenstein pure subscheme of codimension 2 of \mathbb{P}_k^n which is self-linked through two hypersurfaces $F := \{f = 0\}$ and $G := \{g = 0\}$ of degrees $\text{deg}(F) =: d \leq m := \text{deg}(G)$.

The first hypothesis implies that C is also generically locally complete intersection since its codimension is 2. Owing to the isomorphism (1.2.2) we have

$$\mathfrak{S}_C/\mathfrak{S}_X \cong \omega_C(n+1-m-d)$$

whence at each generic point $P \in C$ the sheaf $\omega_{C,P}$ is invertible and a lift of a generator yields y such that $\mathfrak{S}_{C,P} = (f, g, y)\mathcal{O}_{\mathbb{P}_k^n, P}$. Since C is generically complete intersection and $\mathfrak{S}_X = (f, g)$ then \mathfrak{S}_C is either (f, y) or (g, y) at P . In the first case we have locally $g = af + by$. By changing globally g with $g + cf$ for a suitable c we can assume that a is invertible at each generic point of C , hence at each generic point $P \in C$ we may assume that we are in the case $\mathfrak{S}_C = (g, y)$ holds generically.

Let now $H := \{h = 0\}$ be another hypersurface such that C is also self-linked through H and G . Recall that we defined $X := F \cap G$. If we set $\overline{X} := H \cap G$, then the following proposition holds.

PROPOSITION 3.1. *Let C be a generically Gorenstein pure subscheme of codimension 2 of \mathbb{P}_k^n which is self-linked through the two complete intersections $X = F \cap G$ and $\overline{X} = H \cap G$. If $\text{deg}(F) \leq \text{deg}(G)$ then $X = \overline{X}$*

Proof. At each generic point $P \in C$ we have $\mathfrak{S}_C = (y, g)$ and $\mathfrak{S}_X = (f, g)$. By the factoriality of $\mathcal{O}_{\mathbb{P}_k^n, P}$ and since (y, g) is a system of parameters for the regular local ring $\mathcal{O}_{\mathbb{P}_k^n, P}$, the condition $\mathfrak{S}_X : \mathfrak{S}_C = \mathfrak{S}_C$ amounts to the identity $f = gz - y^2$ up to units.

Moreover C is self-linked with respect to both X and \overline{X} , whence in $\mathcal{O}_{\mathbb{P}_k^n, P}$

$$(f, g) : (y, g) = (y, g) = (h, g) : (y, g).$$

It follows that $y^2 \in (h, g)$, hence $(f, g) = (y^2, g) \subseteq (h, g)$ in $\mathcal{O}_{\mathbb{P}_k^n, P}$. Since such an inclusion holds at each generic point $P \in C$ then $(f, g) \subseteq (h, g)$. Changing the roles of f and h we obtain $(h, g) \subseteq (f, g)$, hence equality must hold. \square

REMARK 3.2. The condition $\deg(F) \leq \deg(G)$ is necessary as shows the following easy example. Let C be the origin in the affine plane with coordinates h, g . Let $f = h^2 - g^2$. Then $(f, g) \neq (f, h)$.

In order to clarify the role of the hypothesis that C is generically Gorenstein in proposition 3.1, in the remaining part of this section we shall give an example where:

- a) C is a pure subscheme of codimension 2 of \mathbb{P}_k^n , which is not generically Gorenstein;
- b) C is self-linked through $X := F \cap G$, with $F := \{f = 0\}$ and $G := \{g = 0\}$ of degrees $\deg(F) =: d \leq m := \deg(G)$;
- c) for each choice of the second generator $\widehat{g} := af + g$ of \mathfrak{S}_X , there exists h with $\deg(h) = \deg(f)$ such that C is self-linked through $\overline{X} := H \cap \widehat{G}$, where $\widehat{G} := \{\widehat{g} = 0\}$ and $H := \{h = 0\}$,
- d) but $\overline{X} \neq X$.

EXAMPLE 3.3. Let $n \geq 2$, x and y independent linear forms in \mathbb{P}_k^n and $C \subseteq \mathbb{P}_k^n$ the subscheme associated to the ideal (x^2, xy, y^2) . Notice that C is not generically Gorenstein.

CLAIM 3.3.1. *Let C be as above. Assume that C is self-linked through two hypersurfaces $F := \{f = 0\}$ and $G := \{g = 0\}$, with $\deg(F) =: d \leq m := \deg(G)$. Then $d = 2$, $m = 3$ and there exist two other linear forms x', y' on \mathbb{P}_k^n such that $(x, y) = (x', y')$ and*

- i) either $f = x'^2$ and $g = y'^3 \pmod f$,
- ii) or $f = x'y'$ and $g = x'^3 - y'^3 \pmod f$.

Proof. Since $\deg(C) = 3$ then $\deg(F \cap G) = 6$ hence $d = 2$ and $m = 3$. There exists a Hilbert–Burch resolution of \mathfrak{S}_C

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_k^n}(-3)^2 \xrightarrow{A} \mathcal{O}_{\mathbb{P}_k^n}(-2)^3 \xrightarrow{B} \mathfrak{S}_C \longrightarrow 0$$

where

$$A := \begin{pmatrix} y & 0 \\ -x & y \\ 0 & -x \end{pmatrix}, \quad B := (x^2, xy, y^2).$$

Then the module $\omega_C(n - 4) := \text{Ext}_{\mathcal{O}_{\mathbb{P}_k^n}}^1(\mathfrak{S}_C, \mathcal{O}_{\mathbb{P}_k^n}(-5))$ has a dual resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}_k^n}(-5) \xrightarrow{tB} \mathcal{O}_{\mathbb{P}_k^n}(-3)^3 \xrightarrow{tA} \mathcal{O}_{\mathbb{P}_k^n}(-2)^2 \longrightarrow \omega_C(n - 4) \longrightarrow 0.$$

In particular $\omega_C(n - 4)$ has two generators e_1, e_2 subject to the relations $ye_1 = xe_2 = xe_1 - ye_2 = 0$. On the other hand we have an isomorphism $\mathfrak{S}_C/\mathfrak{S}_X \cong \omega_C(n - 4)$.

Therefore $\mathfrak{S}_C/\mathfrak{S}_X$ has two generators, which lie in degree 2, satisfying the above relations.

Up to a linear change $(x, y) \rightarrow (x', y')$ of generators we can assume $f = x'^2$ or $f = x'(x' + cy)$.

We consider then the two possible cases.

$c = 0$, whence $f = x'^2$. Hence, modulo f , we can find $y' = ay + bx'$ such that either $g = x'y'^2$ or $g = y'^3$. The first case is impossible since (f, g) must be a regular sequence. In the second case let $e_1 = y'^2$, $e_2 = x'y'$ mod \mathfrak{S}_X . We obtain therefore a self-linkage of C .

$c \neq 0$ whence we may set $y' = x' + cy$ and therefore $f = x'y'$. Hence, modulo f , $g = ax'^3 + by'^3$ where $ab \neq 0$, thus we can assume $g = x'^3 - y'^3$. In this case $e_1 = x'^2$, $e_2 = y'^2$ mod \mathfrak{S}_X . \square

CLAIM 3.3.2. *For each linear form $L \in k[x, y]$ there exist linear forms $\xi, \eta, M \in k[x, y]$ such that*

$$y^3 + Lx^2 = \eta^3 + M\xi^2$$

where x and ξ (resp. ξ and η) are linearly independent.

Proof. Let $L := L_0x + L_1y$.

Assume that $L_0 \neq 0$. Then we may set

$$\eta := R' \left(x + \frac{L_1}{3L_0}y \right), \quad \xi = y, \quad M := -\frac{L_1^2}{3L_0}x + \left(1 - \frac{L_1^3}{27L_0^2} \right) y,$$

where R' is a fixed cube root of L_0 .

Next consider the case $L_0 = L_1 = 0$. Then we may take $\eta = y$, ξ arbitrary and $M = 0$.

Finally let $L_0 = 0$ and $L_1 \neq 0$. Let a be such that

$$(3.3.2.1) \quad (3a^2 - L_1)^2 - 12a^4 = 0.$$

Notice that $a \neq 0$. It follows that $3ay^2 + 3a^2xy + a^3x^2 - L_1xy$ is the square of a linear form $\xi := R''(6ay + (3a^2 - L_1)x)$, where R'' is a fixed square root of $(12a)^{-1}$. Setting $\eta := y + ax$ and $M := -x$, then $y^3 + Lx^2 = \eta^3 + M\xi^2$. Notice that x and ξ are independent, else $a = 0$ which is not a root of equation (3.3.2.1). On the other hand also ξ and η are independent for $L_1 \neq -3a^2$, in which case substituting in (3.3.2.1) we would obtain $24a^4 = 0$, a contradiction. \square

Set now $f := x^2$, $g := y^3$, $\widehat{g} := y^3 + Lx^2$ and $h := \xi^2$. Since ξ and η are independent $(h, \widehat{g}) = (\eta^3, \xi^2)$ so that C is self-linked through $H := \{h = 0\}$ and $\widehat{G} := \{\widehat{g} = 0\}$, but $f \notin (h, g)$, since x and ξ are independent.

Thus we have checked that our example satisfies conditions a), b), c) and d) above.

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