

PERIODS OF STREBEL DIFFERENTIALS AND ALGEBRAIC CURVES DEFINED OVER THE FIELD OF ALGEBRAIC NUMBERS *

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Abstract. In [8] we have shown that if a compact Riemann surface admits a Strebel differential with rational periods, then the Riemann surface is the complex model of an algebraic curve defined over the field of algebraic numbers. We will show in this article that even if all geometric data are defined over \mathbb{Q} , the Strebel differential can still have a transcendental period. We construct a Strebel differential q on an arbitrary complete nonsingular algebraic curve C defined over $\overline{\mathbb{Q}}$ such that (i) all poles of q are \mathbb{Q} -rational points of C ; (ii) the residue of \sqrt{q} at each pole is a positive integer; and (iii) q has a transcendental period.

1. Introduction. The periodic function $e^{i\pi\theta}$ has the following remarkable property: for every rational number θ , $e^{i\pi\theta}$ gives an algebraic number, while for every non-rational algebraic number θ , it gives a transcendental number. It follows that if $e^{i\pi\theta}$ is algebraic and θ is not rational, then θ is transcendental. This somewhat reciprocal algebraicity-transcendence relation between the periods of a function and its values is commonly seen among the periodic functions appearing in algebraic geometry such as modular functions and theta functions [1].

The purpose of this paper is to examine a similar relation between the periods of Strebel differentials on a complete nonsingular algebraic curve and its field of definition. In an earlier paper [8], we have established that if a compact Riemann surface C admits a Strebel differential q such that the length of every *critical horizontal trajectory* is rational, then C is the complex model of an algebraic curve defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers. (We refer to [8] for the definitions of the notions used in this article.) Let us call the length of a critical horizontal trajectory a *period* of q . Thus the rationality of the periods makes the field of definition of C algebraic. Following the analogy of the relation between the periods and the values of theta functions, one can ask questions about the transcendence of the periods of a Strebel differential and the field of definition of an algebraic curve.

Let Γ be a *ribbon graph*, that is, a graph with a prescribed cyclic order of half-edges at each vertex. A ribbon graph is a *metric ribbon graph* if a positive real number, conventionally referred to as the *length*, is assigned to each edge. From every metric ribbon graph Γ , one can construct a compact Riemann surface C and a *Strebel differential* q on it in a unique manner. The number of poles of q is equal to the number of *boundary circuits* of Γ , and the residue of \sqrt{q} at each pole is the total length of the boundary circuit corresponding to the pole.

Conversely, starting with a compact Riemann surface C with $n > 0$ marked points and an n -tuple of positive real numbers, one constructs a unique metric ribbon graph as the union of the critical horizontal trajectories of the Strebel differential q on C . The integral of \sqrt{q} between zeros of q along a critical horizontal trajectory is called a *period* of q . The periods of the Strebel differential determine a metric on the ribbon graph.

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Harer [4] used this idea to establish an orbifold isomorphism

$$(1) \quad \coprod_{\Gamma} \frac{\mathbb{R}_+^{e(\Gamma)}}{\text{Aut}_{\partial}(\Gamma)} \xrightarrow{\sim} \mathfrak{M}_{g,n} \times \mathbb{R}_+^n,$$

where Γ runs over all ribbon graphs with valence of each vertex no less than 3 and with Euler characteristic $2 - 2g - n$, $e(\Gamma)$ is the number of edges of Γ , $\text{Aut}_{\partial}(\Gamma)$ is the group of ribbon graph automorphisms of Γ that fix its boundary circuits, and $\mathfrak{M}_{g,n}$ is the moduli space of smooth compact complex algebraic curves of genus g with n ordered marked points. This isomorphism plays a key role in many recent papers (cf. [3], [5], [6], [7], [9], [12]). For an explicit construction of this isomorphism, we refer to [8].

The set of *rational points*

$$\coprod_{\Gamma} \frac{\mathbb{Q}_+^{e(\Gamma)}}{\text{Aut}_{\partial}(\Gamma)} \subset \coprod_{\Gamma} \frac{\mathbb{R}_+^{e(\Gamma)}}{\text{Aut}_{\partial}(\Gamma)}$$

is well defined (although not as an orbifold over \mathbb{Q}) because a ribbon graph automorphism acts as a permutation of edges. Let $\mathfrak{M}_{g,n}(\overline{\mathbb{Q}})$ denote the moduli space of n -pointed complete nonsingular algebraic curves defined over $\overline{\mathbb{Q}}$. Comparing Belyi's theorem [2], Grothendieck's idea of *dessins d'enfants* [10], and Strebel theory [11], we have established in [8] that there is a natural injective map j :

$$(2) \quad \begin{array}{ccc} \coprod_{\Gamma} \frac{\mathbb{R}_+^{e(\Gamma)}}{\text{Aut}_{\partial}(\Gamma)} & \xrightarrow{\sim} & \mathfrak{M}_{g,n} \times \mathbb{R}_+^n \\ \cup & & \cup \\ \coprod_{\Gamma} \frac{\mathbb{Q}_+^{e(\Gamma)}}{\text{Aut}_{\partial}(\Gamma)} & \xrightarrow{j} & \mathfrak{M}_{g,n}(\overline{\mathbb{Q}}) \times \mathbb{Q}_+^n. \end{array}$$

Belyi's theorem shows that *every* complete nonsingular algebraic curve over $\overline{\mathbb{Q}}$ is constructed as the image of the map j if we do not specify the number n of marked points. However, in the light of the geometric $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ actions [10], j cannot be surjective. Indeed, we shall prove the following:

THEOREM 1. *Let C be an arbitrary complete nonsingular algebraic curve defined over $\overline{\mathbb{Q}}$. Then there is a Strebel differential q on the complex model of C such that*

1. *every pole of q is a $\overline{\mathbb{Q}}$ -rational point of C ;*
2. *the residue of \sqrt{q} at each pole is a positive integer; and*
3. *q has a transcendental period.*

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2. Construction of the Strebel Differential. Let us start by constructing a simple example on \mathbb{P}^1 . We wish to exhibit a Strebel differential q_c on \mathbb{P}^1 that has poles at $0, 1, \infty, c$ and c^2 , such that $\sqrt{q_c}$ has residues $2, 2, 2, 4$, and 2 , resp., at these poles, with $c \in \overline{\mathbb{Q}} \setminus \{0, 1\}$ a constant to be determined later. We define two rational maps f and g in order to construct certain Strebel differentials on \mathbb{P}^1 . First we choose

$$(3) \quad f : \mathbb{P}^1 \ni x \longmapsto y = \frac{1}{(1-c)^2} \frac{(x-c)^2}{x} \in \mathbb{P}^1.$$

Since

$$f'(x) = \frac{1}{(1-c)^2} \frac{x^2 - c^2}{x^2},$$

f is ramified at $x = \pm c$. We note that

$$f(0) = f(\infty) = \infty, \quad f(1) = f(c^2) = 1, \quad f(c) = 0, \quad f(-c) = -\frac{4c}{(1-c)^2}.$$

The other rational map is

$$(4) \quad g : \mathbb{P}^1 \ni y \mapsto \zeta = \frac{4(y^2 - y + 1)^3}{27y^2(1-y)^2} \in \mathbb{P}^1.$$

Consider the meromorphic quadratic differential

$$q_0 = \frac{1}{4\pi^2} \frac{d\zeta^2}{\zeta(1-\zeta)}$$

on \mathbb{P}^1 , and put

$$(5) \quad q_1 = g^*(q_0) = -\frac{1}{\pi^2} \frac{y^2 - y + 1}{y^2(1-y)^2} dy^2.$$

It has quadratic poles at 0, 1, and ∞ , and simple zeros at $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The residue of $\sqrt{q_1}$ at each pole is 2, and the three periods of q_1 are all 1. Let $\Gamma_y = g^{-1}([0, 1])$. It has been shown in [8] that Γ_y is the ribbon graph consisting of the set of critical trajectories of the Strebel differential q_1 .

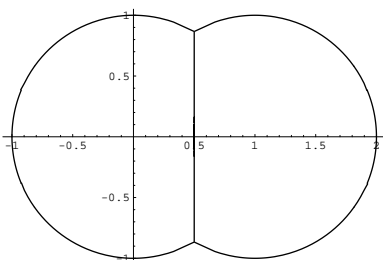


FIG. 1. Graph Γ_y , the inverse image of $[0, 1]$ via $\zeta = \frac{4(y^2 - y + 1)^3}{27y^2(1-y)^2}$.

Next, let s be a real algebraic number such that $0 < s < \frac{\sqrt{3}}{2}$, choose the constant c so that

$$(6) \quad f(-c) = -\frac{4c}{(1-c)^2} = \frac{1}{2} + is,$$

and define $q_c = f^*(q_1)$. Since the two critical values of the double-sheeted holomorphic covering map f are $f(c) = 0$ and $f(-c)$, one sees that the Strebel differential q_c on \mathbb{P}^1 has quadratic poles at 0, 1, ∞ , c and c^2 with residues of $\sqrt{q_c}$ at these poles 2, 2, 2, 4, and 2, resp., and a new double zero at $-c$. Let $\Gamma_c = f^{-1}(\Gamma_y)$. Because of the choice of c in Eq. 6, Γ_c is the ribbon graph consisting of the critical trajectories for

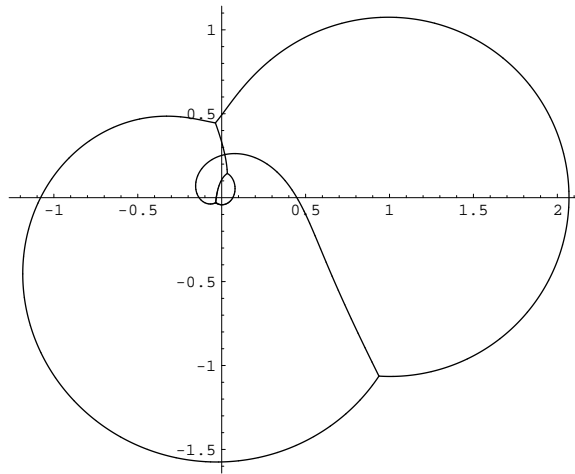


FIG. 2. Graph Γ_c for $s = \frac{5\sqrt{3}}{11}$.

the Strebel differential q_c . It has four tri-valent vertices at $f^{-1}(\frac{1}{2} \pm i\frac{\sqrt{3}}{2})$ and a unique 4-valent vertex at $-c = f^{-1}(\frac{1}{2} + is)$.

Let L denote the edge of Γ_c that connects the vertex $-c$ and one of the tri-valent vertices. From Eq. 5 one sees that the period of q_c corresponding to the edge L is equal to $\frac{1}{2} \pm \ell(s)$ depending on the choice of the tri-valent vertex, where

$$\begin{aligned}
 \ell(s) &= \int_{1/2}^{1/2+is} \sqrt{-\frac{1}{\pi^2} \frac{y^2 - y + 1}{y^2(1-y)^2}} dy \\
 (7) \quad &= \frac{1}{\pi} \int_0^s \frac{\sqrt{3/4 - t^2}}{1/4 + t^2} dt \\
 &= \frac{1}{\pi} \arcsin \left(\frac{2s(9 + 4s^2)}{3\sqrt{3}(1 + 4s^2)} \right).
 \end{aligned}$$

In Section 3, we shall show that $\ell(s)$ is transcendental at $s = r\sqrt{3}$ for every rational number r such that $0 < r < \frac{1}{2}$.

The above construction immediately gives the construction of the Strebel differential q for the general case of Theorem 1. Let C be an arbitrary complete nonsingular algebraic curve defined over $\overline{\mathbb{Q}}$. Belyi [2] has shown that there is a holomorphic map

$$(8) \quad \beta : C \longrightarrow \mathbb{P}^1$$

that is ramified only over 0, 1 and ∞ . The map β is called a *Belyi map*. Without loss of generality, we can assume that the ramification degrees over the point $0 \in \mathbb{P}^1$ are no less than 3 and the ramification degrees over $1 \in \mathbb{P}^1$ are always 2. Define $q = \beta^*(q_c)$ and $\Gamma = \beta^{-1}(\Gamma_c)$. Then q is the Strebel differential on C with poles at $\beta^{-1}(0)$, $\beta^{-1}(1)$, $\beta^{-1}(\infty)$, $\beta^{-1}(c)$, and $\beta^{-1}(c^2)$. The residue of \sqrt{q} at each of these poles is 2 times the ramification degree of β at each ramification point, 4 at each inverse image of c , and 2 at each inverse image of c^2 . Certainly the period of q corresponding to any edge of Γ that is an inverse image of L is a transcendental number if we chose $s = r\sqrt{3}$.

3. Transcendence of the Period. Let us now show that $\ell(s)$ of Eq. 7 is transcendental for every $s = r\sqrt{3}$, where r is a rational number in between 0 and 1/2.

Since

$$\sin(\pi\ell(s)) = \frac{2s(9 + 4s^2)}{3\sqrt{3}(1 + 4s^2)} = \frac{2r(3 + 4r^2)}{(1 + 12r^2)} \in \mathbb{Q} \setminus \{1/2\},$$

the claim follows from

PROPOSITION 2. *Let $0 < \ell < 1/2$ be such that $a = \sin(\pi\ell)$ is rational but not equal to $\frac{1}{2}$. Then ℓ is transcendental.*

Proof. Let

$$b = -a + i\sqrt{1 - a^2} = e^{i\pi(\ell+1/2)}.$$

This is a solution of the quadratic equation $x^2 + 2ax + 1 = 0$. Let $a = m/n$ be an irreducible fraction. Then $nx^2 + 2mx + n$ is primitive and irreducible if n is odd, and if $n = 2k$, then $kx^2 + mx + k$ is primitive and irreducible.

Suppose that $\ell \in \mathbb{Q}$. Then b is a solution of the primitive equation $x^N - 1 = 0$ for some integer N . We note that the primitive minimal polynomial of b divides $x^N - 1$. Therefore, if n is odd, then $n = 1$, and hence $a = m \geq 1$, which is a contradiction. If n is even, then $k = 1$ and the only possibility is $a = \frac{1}{2}$. Thus ℓ is not rational. It cannot be algebraic because b is algebraic. This completes the proof. \square

It would be desirable to establish that if a compact Riemann surface C admits a Strebel differential q whose periods are algebraic but not rational, then the geometric data $(C, (p_1, \dots, p_n))$, where (p_1, \dots, p_n) are the poles of q , cannot be defined over $\overline{\mathbb{Q}}$. In our simple example of Section 2, if the period corresponding to the edge L is taken to be algebraic but not rational, then the ramification points c and c^2 become transcendental. However, we do not have any general theorem in this direction.

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