

INDEX OF DIRAC OPERATOR AND SCALAR CURVATURE ALMOST NON-NEGATIVE MANIFOLDS*

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Abstract. A manifold M is called *scalar curvature almost non-negative* if for any constant $\varepsilon > 0$, there is a Riemannian metric g on M such that $s_g \cdot \text{diam}(M, g)^2 \geq -\varepsilon$ and the sectional curvature $\text{Sec}_g \leq 1$, where s_g (resp. $\text{diam}(M, g)$) is the scalar curvature (resp. diameter) of (M, g) .

Among others we prove that for a scalar curvature almost non-negative manifold M with $\hat{A}(M) \neq 0$ (resp. $\mathcal{A}(M) \in \mathbb{Z}_2$ nonzero and $\chi(M) \neq 0$ if $n = 2 \pmod{8}$), there is a constant $\varepsilon(n) > 0$ such that, if the scalar curvature $s_M \geq -\varepsilon(n)$, then

- (i) the fundamental group $\pi_1(M)$ is finite;
- (ii) M admits a real analytic Ricci flat metric g_0 such that its Riemannian universal covering \tilde{M} is isometric to the product of Ricci flat Kähler-Einstein manifolds and/or Joyce manifolds of dimension 8 with special holonomy group $\text{Spin}(7)$.

0. Introduction. Let M be a closed Spin manifold. It is well-known that M admits a metric with positive scalar curvature only if the index of the Dirac operator vanishes, by the classical Lichnerowicz formula and the Atiyah-Singer index theorem (cf. [LM]). The Gromov-Lawson conjecture, confirmed by Stolz [St], asserts the converse for simply connected Spin manifolds of dimension at least 5. There are many manifolds with non-negative scalar curvature but do not accept metrics of positive scalar curvature, e.g. torus, K_3 -surfaces, etc.

A basic theorem of Bourguignon shows that such a Spin manifold must be Ricci flat. Starting from [St], Futaki [Fu] characterized all simply connected Spin manifolds of dimension at least 5 with non-negative scalar curvature. It turns out that such a manifold either admits a metric with positive scalar curvature, or it is the product of Ricci flat Kähler-Einstein manifolds and/or Joyce manifold of dimension 8 with Spin(7)-holonomy (manifolds in the latter class are called *rigidly scalar flat*). A key-point involved is that, every harmonic spinor must be parallel and therefore, the holonomy group must be special (cf. [Fu] [Wa]).

Recall that a manifold is called *almost flat* (resp. *almost Ricci flat*) if for any positive constant $\varepsilon > 0$, there is a Riemannian metric g on M such that

$$|\text{Sec}_g \cdot \text{diam}(M, g)^2| \leq \varepsilon \quad (\text{resp. } |\text{Ric}_g \cdot \text{diam}(M, g)^2| \leq \varepsilon),$$

where Sec_g (resp. Ric_g) is the sectional (resp. Ricci) curvature.

The celebrated Gromov theorem asserts that an almost flat manifold must be an infra-nilmanifold, i.e. a finite regular cover must be a nilmanifold. By the Bochner technique it is also well-known that an almost Ricci flat manifold has the first Betti number at most n , where n is the dimension. In contrast, however, very recently Lohkamp [Lo] proved that for every compact manifold M of dimension at least 3 and for any given positive constant ε , there is a metric g so that its scalar curvature satisfies that $|s_g \cdot \text{diam}(M, g)^2| \leq \varepsilon$.

We call a manifold M is of *scalar curvature almost non-negative* if for any constant $\varepsilon > 0$, there is a Riemannian metric g on M such that $s_g \cdot \text{diam}(M, g)^2 \geq -\varepsilon$ and the sectional curvature $\text{Sec}_g \leq 1$, where s_g (resp. $\text{diam}(M, g)$) is the scalar curvature

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(resp. diameter) of (M, g) . Note that almost flat manifolds must be of scalar curvature non-negative.

The main purpose of the present paper is to characterize manifolds with almost non-negative scalar curvature. We will prove that there are indeed nontrivial topological obstructions for manifolds being of scalar curvature almost non-negative. For example, following from our theorem the connected sum, $K_3 \# S^1 \times S^3$, of a K_3 -surface with $S^1 \times S^3$, does not admit metrics of almost non-negative scalar curvature.

Let M be a Spin manifold. Let S^+ (resp. S^-) be the Spinor bundles on M . Let $\mathcal{D}^+ : \Gamma(S^+) \rightarrow \Gamma(S^-)$ denote the Dirac operator. The Atiyah-Singer index theorem asserts that $\text{ind}(\mathcal{D}^+) = \hat{A}(M)$ is a Spin cobordism invariant and defines a ring homomorphism

$$\mathcal{A} : \Omega_*^{Spin} \rightarrow KO^{-*}(pt),$$

which is non-zero only if the dimension is $0 \pmod{4}$ or $1, 2 \pmod{8}$. In particular, if $n = 0 \pmod{4}$, $\mathcal{A}(M) = \hat{A}(M)$ is the \hat{A} -genus of M .

Let $\mathcal{M}(n, d)$ denote the set of all closed Riemannian n -manifolds such that the sectional $K_M \leq 1$, and $\text{diam} M \leq d$.

A manifold $M \in \mathcal{M}(n, d)$ is called *non-collapsing* if there is no sequence of metrics of bounded curvature collapsing to a lower dimensional space in the Gromov-Hausdorff topology.

THEOREM A. *Let $M \in \mathcal{M}(n, d)$ be a closed non-collapsing Spin manifold such that $\mathcal{A}(M) \neq 0$. There is a constant $\varepsilon(n) > 0$ such that if the scalar curvature $s_M \geq -\varepsilon(n)$, then M admits a Ricci flat real analytic Riemannian metric g_0 with restricted holonomy group $\text{Hol}^0(M, g_0)$ a product whose irreducible components are $SU(m)$, $Sp(m)$, G_2 or $Spin(7)$.*

By the local formula for \hat{A} -genus (Euler characteristic), $M \in \mathcal{M}(n, d)$ is non-collapsing if $\hat{A}(M) \neq 0$ (resp. the Euler characteristic $\chi(M) \neq 0$). Therefore Theorem A together with the Cheeger-Gromoll splitting theorem [CGr] implies immediately

COROLLARY 0.1. *Let $M \in \mathcal{M}(n, d)$ be a closed Spin manifold. Suppose that $\hat{A}(M) \neq 0$ (resp. $\mathcal{A}(M) \neq 0$ and $\chi(M) \neq 0$ if $n = 2 \pmod{8}$). There is a constant $\varepsilon(n) > 0$ such that, if the scalar curvature $s_M \geq -\varepsilon(n)$, then*

- (i) *the fundamental group $\pi_1(M)$ is finite;*
- (ii) *M admits a real analytic Ricci flat metric g_0 such that its Riemannian universal covering \tilde{M} is isometric to the product of Ricci flat Kähler-Einstein manifolds and/or Joyce manifolds of dimension 8 with special holonomy group $Spin(7)$.*

The above theorem together with the Stolz theorem [St] clearly implies

COROLLARY 0.2. *Let $M \in \mathcal{M}(n, d)$ be a closed non-collapsing Spin manifold. If M is simply connected and $n \geq 5$. There is a constant $\varepsilon(n) > 0$ such that, if the scalar curvature $s_M \geq -\varepsilon$, then either M admits a metric of positive scalar curvature or admits a Ricci flat metric.*

The proof of the Theorem A together with Corollaries 3.1 and 3.2 in [Wa] imply the following two corollaries

COROLLARY 0.3. *Let $M \in \mathcal{M}(4, D)$ be a closed Spin manifold so that $\hat{A}(M) \neq 0$. There is a constant $\varepsilon > 0$ such that, if the scalar curvature $s_M \geq -\varepsilon$, then M is bi-*

Lipschitz isometric to a Calabi-Yau K_3 -surface.

COROLLARY 0.4. *Let $M \in \mathcal{M}(8, D)$ be a closed Spin manifold. Assume that $\hat{A}(M) \neq 0$. There is a constant $\varepsilon > 0$ such that, if the scalar curvature $s_M \geq -\varepsilon$, then M is simply connected.*

Observe that the \hat{A} -genus of a K_3 -surface $\hat{A}(K_3) \neq 0$. Corollary 0.1 implies easily that the connected sum $K_3 \# S^1 \times S^3$ does not admit an almost non-negative scalar curvature metric. This also answers in negative the following question:

PROBLEM 0.5. *Let M (resp. N) be a manifold with positive (resp. non-negative) scalar curvature. Does the connected sum $M \# N$ admit a metric with almost non-negative scalar curvature?*

This is maybe interesting by comparing with the amazing discovery of Schoen-Yau [SY], that the connected sum of any two n -manifolds ($n \geq 3$) with positive scalar curvature admits a metric with positive scalar curvature.

Some remarks on the above results are in order.

REMARK 0.6. The Seiberg-Witten theory [Wi] implies that 4-manifolds with non-trivial Seiberg-Witten invariant do not accept any metric with positive scalar curvature. Furthermore, Witten ([Wi]) proved that a scalar curvature non-negative 4-manifold with non-trivial Seiberg-Witten invariant is hyper-Kähler, if $b_2^+ \geq 2$. Seiberg-Witten theory may be used to study scalar curvature almost non-negative 4-manifolds in the spirit of the presented paper.

REMARK 0.7. Theorem A should be compared with a recent result in [Lot] where sectional curvature is assumed being almost non-negative.

REMARK 0.8. Theorem A holds identically, if the upper bound for the sectional curvature is replaced by a upper bound for the Ricci curvature together with a positive lower bound for the conjugate radii.

Now let us start to sketch the idea in the proof of Theorem A.

Suppose Theorem A is false, then there is a sequence of n -dimensional Spin manifolds (M_i, g_i) so that $\mathcal{A}(M_i) \neq 0$, the scalar curvatures $s_{g_i} \geq -1/i$, the diameter $\text{diam } M_i \leq D$ and $\text{Sec}_{g_i} \leq 1$ for some positive constant D . By the Gromov precompactness theorem and the Cheeger-Gromov theorem, passing to a subsequence if necessary, we may assume that (M_i, g_i) converges in $L^{2,p}$ -class to a manifold (X, g_∞) (since M_i are non-collapsing.) Let $\phi_i \in \Gamma(S_i^+)$ be a harmonic spinor with unit L^2 -norm. Using the Lichnerowicz formula and the scalar curvature bound we prove that ϕ_i converges to a parallel harmonic spinor ϕ_∞ with nontrivial norm. Therefore ϕ_i is almost parallel for i large, which implies that the Ricci curvature of (M_i, g_i) converges in L^p -class to zero. By regularity of Einstein equation we know that there is a real analytic Ricci flat metric on X . A contradiction.

1. Preliminaries. In this section we give some necessary preliminary results needed in next sections.

a). Harmonic coordinates

A local coordinate (h^1, \dots, h^n) is *harmonic* if each component is a harmonic function, i.e., $\Delta h^i = 0$ for $i = 1, \dots, n$, where Δ is the Laplacian operator. In a harmonic

coordinate, the Ricci curvature of the metric tensor g satisfies the equation

$$(\text{Ric}_g)_{ij} = -\frac{1}{2} \Delta g_{ij} + Q(g, \partial g).$$

here $g_{ij} = g(\frac{\partial}{\partial h^i}, \frac{\partial}{\partial h^j})$ and $Q(\cdot, \cdot)$ is a quadratic form of its variables (c.f. [Pe]).

The existence of harmonic coordinate ball of uniform radius is studied in [JK] [An], in terms of various curvature bound. We quote the following

THEOREM 1.1 [AN]. *Let (M, g) be a Riemannian n -manifold (not necessarily complete) such that*

$$(1.4.1) \quad |\text{Ric}_M| \leq \lambda, \quad \text{inj } B(x, \frac{1}{2} \cdot \text{dist}(x, \partial M)) \geq i_0(x) > 0$$

Then, given any $C > 1$ and $\alpha \in (0, 1)$, there is an $\varepsilon_0 = \varepsilon_0(\lambda, C, n, \alpha)$ with the following property: given any $x \in M$, there is a harmonic coordinate system $U = \{u_i\}_1^n$ defined on $B(x, \varepsilon(x)) \subset M$ such that if $g_{ij} = g(\nabla u_i, \nabla u_j)$, then $g_{ij}(x) = \delta_{ij}$ and

$$(1.4.2) \quad C^{-1} \cdot I \leq g(y) \leq C \cdot I, \quad (\text{as bilinear forms})$$

$$(1.4.3) \quad \varepsilon(x)^{1+\alpha} \|g_{ij}(y)\|_{C^{1,\alpha}} \leq C$$

for all $y \in B(x, \varepsilon(x))$ (resp. $\frac{\varepsilon(x)^{2-\frac{n}{p}}}{\sqrt{\text{vol } B(x, \varepsilon(x))}} \|g_{ij}\|_{L^{2,p}(B(x, \varepsilon(x)))} \leq C$), where

$$(1.4.4) \quad \frac{\varepsilon(x)}{i_0(x)} \geq \varepsilon_0 \cdot \frac{\text{dist}(x, \partial M)}{\text{diam}M} > 0$$

where $B(x, r)$ is the metric ball with radius r , I the identity matrix and inj is the injectivity radius.

b). Compactness theorems

THEOREM 1.2 [CG]. *Let (M_i, g_i) be a sequence of compact Riemannian manifolds whose sectional curvature, diameter, and injectivity radius satisfy*

$$\lambda \leq \text{Sec}_{g_i} \leq \Lambda, \quad \text{diam} \leq d, \quad i_M > i_0,$$

where the constants are independent of i . Then, replacing M_i by a subsequence if necessary, M_i converges to a metric space X , such that

- (i) X is a differentiable manifold;
- (ii) there is a diffeomorphism $f_i : X \rightarrow M_i$ for all sufficiently large i ;
- (iii) the pullback metrics $f_i^*(g_i)$ converges in $C^{1,\alpha}$ -class to a $C^{1,\alpha}$ (resp. $L^{2,p}$) Riemannian metric g_∞ in X , for any prescribed real number $\alpha \in (0, 1)$ (resp. positive integer $p > 1$).

For the sake of simplicity, in the rest of the paper we fix the real number $\alpha \in (0, 1)$ (resp. the integer $p > n$).

For a sequence (M_i, g_i) as above, the limit metric g_∞ is not necessarily C^2 . However, if the L^p -Ricci curvature (for p large) of g_∞ vanishes identically, then by the elliptic regularity of the Einstein equation one may conclude the smoothness of g_∞ (indeed real analytic). That is

THEOREM 1.3 [AN]. *Let (M_i, g_i) be a sequence as in Theorem 1.2. If the L^p -Ricci curvature (for some large p) of g_∞ vanishes identically. Then (M_i, g_i) has a subsequence converging to a C^∞ Riemannian manifold X , such that*

- (i) *there is a diffeomorphism $f_i : X \rightarrow M_i$ for all sufficiently large i ;*
- (ii) *the pullback metrics $f_i^*(g_i)$ converges in C^∞ -class to a real analytic Riemannian metric g_∞ .*

2. Convergence of harmonic spinors. Let M be a closed Spin Riemannian manifold. Let S^+ (resp. S^-) be the positive (resp. negative) spinor bundle. Let $D : \Gamma(S^+) \rightarrow \Gamma(S^-)$ be the Dirac operator. The classical Lichnerowicz formula reads

$$(2.1) \quad D^2 = \nabla^* \nabla + \frac{1}{4}s$$

where s is the scalar curvature of M .

Let M_i be a sequence of closed Spin Riemannian n -manifolds such that

$$\text{diam}(M_i)^2 \cdot s_i \geq -\varepsilon_i ; \text{Sec}_{g_i} \leq 1$$

where s_{g_i} is the scalar curvature of M_i . By rescaling we may assume that $\text{diam}(M_i) \leq 1$ and $s_{g_i} \geq -\varepsilon_i$. Note that the sectional curvature $\text{Sec}_{g_i} \geq -\lambda(n)$.

Let $\phi_i \in \Gamma(S_i^+)$ denote the harmonic spinors (i.e. $D_i \phi_i = 0$.) By normalizing we may assume that

$$(2.2) \quad \frac{1}{\text{vol}(M_i)} \int_{M_i} |\phi_i|^4 = 1$$

LEMMA 2.3. *Let ϕ_i be as above. Then*

$$|\phi_i|_{L^\infty} \leq C$$

where C is a constant depending only on n , ε_i and $\lambda(n)$.

Proof. Note that

$$\Delta |\phi_i|^2 = 2 \langle \nabla_i^* \nabla_i \phi_i, \phi_i \rangle - 2 |\nabla_i \phi_i|^2$$

By the Lichnerowicz formula we get that

$$0 = \nabla_i^* \nabla_i \phi_i + \frac{1}{4} s_{g_i} \phi_i$$

Therefore

$$\Delta |\phi_i|^2 \leq -\frac{1}{2} s_{g_i} |\phi_i|^2 \leq \frac{1}{2} \varepsilon_i |\phi_i|^2$$

By [Ga] Proposition 3.2 it follows that

$$|\phi_i|_{L^\infty}^2 \leq C_1 \frac{(\int_{M_i} |\phi_i|^4)^{\frac{1}{2}}}{\sqrt{\text{vol}(M_i)}} = C_1$$

The desired result follows. \square

Let $2r$ denote a uniform lower bound for the conjugate radii of M_i . By Theorem 1.1, there are radii r balls, $B_i^1(r), \dots, B_i^m(r)$, in the tangent spaces $T_{p_k^i} M_i$, where $p_1^i, \dots, p_m^i \in M_i$ are m points so that the exponential maps $\exp_{p_k^i} : B_i^k(r) \subset T_{p_k^i} M_i \rightarrow M_i$ are embeddings and $\exp_{p_1^i}(B_i^1(r)), \dots, \exp_{p_m^i}(B_i^m(r))$ is a harmonic coordinate

covering of M_i . Fixing k , consider the pullback metric $\exp_{p_k^i}^*(g_i)$ on $B_i^k(r)$. By Theorem 1.2 there is an integer N such that for all $i > N$, there is a diffeomorphism $f_i^k : B^k(r) := B_N^k(r) \rightarrow B_i^k(r)$ so that the pullback metric tensors $(f_i^k)^*(\exp_{p_k^i}^*(g_i))$ converge in $L^{2,p}$ -class (cf. [Ya])

Let $\hat{\phi}_i = (f_i^k)^*(\exp_{p_k^i}^*(\phi_i)) \in \Gamma(\hat{S}^+)$ be the pullback spinors, where \hat{S}^+ is a spinor bundle of $B^k(r)$, which is isomorphic to the pullback spinor bundles \hat{S}_i^\pm on $B^k(r)$ for all i sufficiently large (passing to a subsequence if necessary.)

LEMMA 2.4. *Let M_i be as above. If $\lim_i \varepsilon_i = 0$, then a subsequence of $\hat{\phi}_i$ above converges in $L^{2,p}$ -class (for any integer $p > 1$) to a non-trivial parallel harmonic spinor $\hat{\phi}$ with respect to the limit metric g_∞ .*

Proof. Recall that all $\hat{\phi}_i$ satisfy

$$(2.4.1) \quad \nabla_i^* \nabla_i \hat{\phi}_i + \frac{1}{4} s_{g_i} \hat{\phi}_i = 0$$

By Theorem 1.2 it follows that in the harmonic coordinate ball $B^k(r)$ and a local frame of sections of \hat{S}^+ (resp. \hat{S}^-), (2.4.1) gives a second order uniformly elliptic equation, such that

(2.4.2) the coefficients of the second order term are uniformly $C^{1,\alpha}$ - (resp. $L^{2,p}$ -) bounded;

(2.4.3) the coefficients of the first order terms (also the quadratic terms) are uniformly $C^{0,\alpha}$ - (resp. $L^{1,p}$ -) bounded;

(2.4.4) the coefficients of the zero order terms are uniformly L^p -bounded;

By [ADN] we know that $\|\hat{\phi}_i\|_{L^{2,p}} \leq C \|\hat{\phi}_i\|_{L^\infty}$, where C is a universal constant. Therefore, by Lemma 2.3 $\hat{\phi}_i$ contains a convergence subsequence in $L^{2,p'}$ - (resp. $C^{1,\alpha}$ -) topology for any $p' < p$, noting that there is a Sobolev embedding $L^{2,p} \subset C^{1,1-\frac{2}{p}}$.

Next we prove that the limit $\hat{\phi}$ is parallel with respect to the metric g_∞ . By (2.2) $\hat{\phi}$ is not zero and the desired result follows.

Integrating (2.4.1) we get

$$0 = \frac{1}{\text{vol } M_i} \int_{M_i} |\nabla_i \phi_i|^2 + \frac{1}{4 \text{vol } M_i} \int_{M_i} s_{g_i} |\phi_i|^2$$

Since $\varepsilon_i \rightarrow 0$, the second term

$$\liminf_{i \rightarrow \infty} \frac{1}{4 \text{vol } M_i} \int_{M_i} s_{g_i} |\phi_i|^2 = 0$$

Hence, passing to a subsequence we may assume that

$$(*) \quad \lim_{i \rightarrow \infty} \frac{1}{\text{vol } M_i} \int_{M_i} |\nabla_i \phi_i|^2 = 0$$

We now use a trick in [Ya]. Suppose that $\hat{\phi}$ is not parallel with respect to g_∞ . Set $\hat{a}_i = |\hat{\nabla}_i \hat{\phi}_i|^2$, $\hat{a} = |\hat{\nabla}_\infty \hat{\phi}|^2$. Then there is a point $x \in B^k(r)$ for some k such that $\hat{a} > b > 0$ on the radius δ ball $B_\delta(x, (B^k(r), \hat{g}_\infty))$ for some positive constants b and δ . Thus $\hat{a}_i > \frac{b}{2}$ on $B_\delta(x, (B^k(r), \hat{g}_i))$ for i large. Set $a_i = |\nabla_i \phi_i|^2$. By the Bishop-Gromov

volume comparison theorem we get

$$1 - \frac{\text{vol} \{a_i \leq \frac{b}{2}\}}{\text{vol}(M_i)} \geq \frac{\text{vol} B_\delta(q_i, M_i)}{\text{vol}(M_i)} \geq \frac{\alpha(\delta)}{\alpha(D)}$$

where $q_i = \exp_{p_k^i} \circ f_i^k(x)$ and $\alpha(r)$ is the volume function of radius r ball in the hyperbolic space of curvature $-\lambda(n)$. Therefore

$$\frac{\text{vol} \{a_i < \frac{b}{2}\}}{\text{vol}(M_i)} \leq 1 - \frac{\alpha(\delta)}{\alpha(D)}$$

for sufficiently large i . On the other hand, by (*) above, for any fixed $b > 0$, it holds that

$$\lim_{i \rightarrow \infty} \frac{\text{vol} \{a_i < \frac{b}{2}\}}{\text{vol}(M_i)} = 1$$

A contradiction. \square

3. Proof of Theorem A. Let M be a Spin manifold, and $\phi \in \Gamma(S^+)$ be a spinor. Using the first Bianchi identity we get

$$(3.1) \quad \sum_j e_j (\nabla_{e_i} \nabla_{e_j} - \nabla_{e_j} \nabla_{e_i} - \nabla_{[e_i, e_j]}) \phi = -\frac{1}{4} \sum_{j,k,l} e_j R_{ijkl} e_k e_l \phi = -\frac{1}{2} \sum_l \text{Ric}(e_i, e_l) e_l \phi$$

for any i , where e_1, \dots, e_n is an orthonormal basis.

Proof of Theorem A. Suppose not. Then there is a sequence of Riemannian manifolds (M_i, g_i) with $\mathcal{A}(M_i) \neq 0$ so that

$$\text{diam}(M_i) \leq 1, s_{g_i} \geq -i^{-1}, \text{Sec}_{g_i} \leq 1$$

but no one of M_i admits the desired metrics.

By the Atiyah-Singer index theorem, the index $\text{Ind}(D_i) \neq 0$. Therefore there is a nonzero harmonic spinor $\phi_i \in \ker D_i$, where D_i is the Dirac operator. By normalizing we may assume (2.2) for all ϕ_i .

Applying (3.1) to ϕ_k and integrating both sides we get

$$(3.2) \quad \lim_{k \rightarrow \infty} \frac{1}{\text{vol} M_k} \int_{M_k} \left| \sum_l \text{Ric}(e_i, e_l) e_l \phi_k \right|^p = 0$$

by Lemmas 2.3 and 2.4.

Since $\hat{\phi}$ is parallel with respect to g_∞ and

$$\frac{1}{\text{vol} M_k} \int_{M_k} |\phi_k|^4 = 1,$$

the C^0 -norm

$$|\phi_k|_{C^0} \geq \frac{1}{2}$$

for k large. Therefore (3.2) implies that

$$\lim_{k \rightarrow \infty} \frac{1}{\text{vol} M_k} \int_{M_k} \left| \sum_l \text{Ric}(e_i, e_l) e_l \right|^p = 0$$

for any i and any prescribed integer $p > 1$. This proves that the limit metric g_∞ on X is L^p -Ricci flat. By Theorem 1.3 this implies that g_∞ is a smooth Ricci flat metric.

By Lemma 2.4 the limit of ϕ_i is a parallel spinor ϕ_∞ with respect to g_∞ . By [Fu] we know that the holonomy of g_∞ must be one of the desired types. \square

REFERENCES

- [ADN] S. AGMON; A. DOUGLIS; L. NIRENBERG, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions II*, Commun. on Pure. Appl. Math., 17 (1964), pp. 35-92.
- [An] M. ANDERSON, *Convergence and rigidity of manifolds under Ricci curvature bounds*, Invent. Math., 102 (1990), pp. 429-445.
- [CFG] J. CHEEGER; K. FUKAYA; M. GROMOV, *Nilpotent structures and invariant metrics on collapsed manifolds*, J.A.M.S, 5 (1992), pp. 327-372.
- [CGr] J. CHEEGER; D. GROMOLL, *The splitting theorem for manifolds of nonnegative Ricci curvature.*, Jour. Diff. Geom., 6 (1972), pp. 119-128.
- [Fu] A. FUTAKI, *Scalar-flat closed manifolds not admitting positive scalar curvature metrics*, Invent. Math., 112 (1993), pp. 23-29.
- [Ga] S. GALLOT, *A Sobolev inequality and some geometric applications*, Spectra of Riemannian manifolds, Kaigai, Tokyo (1983), pp. 45-55.
- [GLP] M. GROMOV, J. LAFONTAINE; P.PANSU, *Structures metriques pour les varietes riemanniennes*, CedecFernand Paris (1981).
- [JK] J. JOST; H. KARCHER, *Geometrische methoden zur Gewinnung von a-priori- Schranke für harmonische Abbildungen*, Manuscripta. Math., 40 (1982), pp. 27-77.
- [Li] P. LI, *On the Sobolev constant and the p -spectrum of a compact Riemannian manifold*, Ann. Sci. École Norm. Sup., 13 (1980), pp. 451-468.
- [Lic] P. LICHNEROWICZ, *Théorie globale des connexions et des groupes d'holonomie*, Dunod, Paris (1955).
- [Lo] J. LOHKAMP, *Scalar curvature and hammocks*, Math. Ann., 313 (1999), pp. 385-407.
- [Lot] J. LOTT, *\hat{A} -genus and collapsing*, J. Geom. Anal., 10 (2000), pp. 529-543.
- [LM] B. LAWSON AND S. MICHELSON, *Spin Geometry*, Princeton Univ Press (1988).
- [Pe] P. PETERSEN, *Riemannian geometry*, GTM, Springer-Verlag (1997).
- [SY] R. SCHOEN; S. T. YAU, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math., 28 (1979), pp. 159-183.
- [St] S. STOLZ, *Simply connected manifolds of positive scalar curvature*, Ann. of Math. (2), 136 (1992), pp. 511-540.
- [Wa] M. Y. WANG, *On non-simply connected manifolds with non-trivial parallel spinors*, Ann. Global. Anal. Geom. (1995), pp. 31-42.
- [Wi] E. WITTEN, *Monopoles and four-manifolds*, Math. Res. Lett., 1 (1994), pp. 769-796.
- [Ya] T. YAMAGUCHI, *Manifolds of almost nonnegative Ricci curvature*, Jour. Diff. Geom., 28 (1988), pp. 157-167.