

A MODEL OF BRILL-NOETHER THEORY FOR RANK TWO VECTOR BUNDLES AND ITS PETRI MAP *

TAN XIAO-JIANG[†]

Abstract. We study here the Brill-Noether theory for rank two vector bundles. First we construct a parameter space H_d for all base point free rank two vector bundles of degree d which generated by its sections. Then for each $E \in H_d$, we define a $2g \times d$ matrix W_E for which we call it the Brill-Noether matrix of E , it shares the same properties as the Brill-Noether matrix W_D for effective divisor D . By using W_E , the Brill-Noether variety $C_{2,d}^r = \{E \in H_d \mid \dim H^0(C, E) \geq r+1\}$ could be given by $C_{2,d}^r = \{E \in H_d \mid \text{rank}(W_E) \leq d-r+1\}$, so $C_{2,d}^r$ is a determinant variety, we get its expected dimension is $4(g-1) + 1 - (r+1)(2(g-1) - d + r + 1) + 2r + 1$. On the other hand, by using W_E , we define the Petri map to be $P : H^0(C, K(-E)) \otimes \text{Im}\{H^0(C, E) \mapsto H^0(C, [D])\} \mapsto H^0(C, K[D](-E))$, we show that $C_{2,d}^r$ has the expected dimension if and only if the Petri map is injective.

1. Introduction . Let C be a smooth irreducible complex projective curve of genus g (C a Compact Riemann surface), L a line bundle on C . We also use L to denote the sheaf of holomorphic sections of L . The Brill-Noether theory for line bundles is to study those bundles L for which both $H^0(C, L)$ and $H^1(C, L)$ are non-zero (L is then called special line bundle).

Let C_d be the d -fold symmetric product of C , C_d is a d -dimensional complex manifold. It is the space of all effective divisors of degree d . Since each line bundle L with $H^0(C, L) \neq 0$ is defined by an effective divisor, so C_d could be considered as a parameter space for all line bundles L with $\text{deg}(L) = d$ and $H^0(C, L) \neq 0$.

Define on C_d the Brill-Noether variety C_d^r to be

$$C_d^r = \{D \in C_d \mid \dim H^0(C, [D]) \geq r+1\}.$$

Where $[D]$ is the line bundle defined by divisor D .

C_d^r could be considered as a parameter space for line bundles L with $\text{deg}(L) = d$ and $\dim H^0(C, L) \geq r+1$. The key tool to study C_d^r is the Brill-Noether matrix.

Let $D = n_1 p_1 + \dots + n_k p_k$ be a given effective divisor with $d = \text{deg}(D) = n_1 + \dots + n_k$. For $i = 1, \dots, k$, let z_i be a local coordinate at p_i with $z_i(p_i) = 0$. Let $\{w_1, \dots, w_g\}$ be a linear basis of the space of all holomorphic forms on C , for each i assume at p_i , $w_t(z_i) = f_{ti}(z_i) dz_i$ for $t = 1, \dots, g$, let W_D be the matrix of the restrictions of $\{w_1, \dots, w_g\}$ on D , that is

$$W_D = \begin{bmatrix} w_1 |_D \\ w_2 |_D \\ \vdots \\ w_g |_D \end{bmatrix} = \begin{bmatrix} f_{11}(p_1) & \cdots & \frac{1}{(n_1-1)!} f_{11}^{(n_1-1)}(p_1) & f_{12}(p_2) & \cdots & \frac{1}{(n_2-1)!} f_{12}^{(n_2-1)}(p_2) & \cdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ f_{g1}(p_1) & \cdots & \frac{1}{(n_k-1)!} f_{g1}^{(n_1-1)}(p_1) & f_{g2}(p_2) & \cdots & \frac{1}{(n_2-1)!} f_{g2}^{(n_2-1)}(p_2) & \cdots \end{bmatrix}.$$

*Received February 9, 2003; accepted for publication July 5, 2003. Project supported by National Natural Science Foundation of China.

[†]School of Mathematical Sciences, Peking University, Beijing 100871, P.R.China
(tan@math.pku.edu.cn).

For a collection of Laurent tails $\mu = \{\mu_i = \sum_{k=-n_i}^{-1} b_{ik} z_i^k\}$, we denote it as a d-dimensional vector

$$\mu = (b_{1-1}, b_{1-2}, \dots, b_{1-n_1}, b_{2-1}, \dots, b_{2-n_2}, \dots) \in C^d.$$

Then μ is the Laurent of a global meromorphic function if and only if $W_D \cdot \mu^t = 0$. From this one can get Riemann-Roch theorem easily.

The matrix W_D is called the Brill-Noether matrix of D.

Now let $[D]|_D$ be the skyscraper sheaf of the restriction of $[D]$ on D, then what we have above could be represented as

$$Ker(W_D) = \{\mu \mid \mu \in C^d, W_D \cdot \mu^t = 0\} \cong Im\{H^0(X, [D]) \mapsto H^0(X, [D]|_D)\} \quad (*)$$

and in particular, we get

$$dimH^0(X, [D]) = deg(D) - rank(W_D) + 1. \quad (**).$$

so C_d^r could be defined by

$$C_d^r = \{D \in C^d \mid Rank(W_D) \leq d - r\}.$$

It is a subvariety of C_d which locally is defined by the simultaneously vanishing of all $(d - r + 1) \times (d - r + 1)$ minors of W_D (Ref [ACGH] p159).

Now let $M(m, n) = M$ be the variety of all $m \times n$ complex matrices, and for $0 \leq k \leq \min\{m, n\}$, denote by $M_k(m, n) = M_k$ the locus of matrices of rank at most k, that is

$$M_k = \{E \in M(m, n) \mid Rank(E) \leq k\}.$$

M_k is an irreducible subvariety of $M(m, n)$, and $codim(M_k) = (n - k)(m - k)$ (Ref [ACGH] p67).

By using the Brill-Noether matrix, locally we have a holomorphic map $BN : C_D \rightarrow M(m, n)$ with $BN(D) = W_D$ for each $D \in C_d$. C_d^r is then could be given by $C_d^r = BN^{-1}(M_{d-r})$. From the Theory of determinant variety, we get that if $C_d^r \neq \emptyset$, then $codim(C_d^r) \leq codim(M_{d-r}) = (g - (d - r))(d - (d - r))$. So if $C_d^r \neq \emptyset$, then

$$dimC_d^r \geq d - r(g - d + r) = g - (r + 1)(g - d + 1) = \rho(g, d, r) + r.$$

where $\rho(g, d, r) = g - (r + 1)(g - d + r)$ is the Brill-Noether number for line bundles. (Ref [ACGH] p215).

It was conjectured by Brill-Noether and Proved by Griffiths-Harris [GH] that for generic C, C_d^r do have the expected dimension $\rho(g, d, r) + r$.

On the other hand, by study the tangent map of $BN : C_D \rightarrow M(m, n), D \mapsto W_D$, Petri got that the variety C_d^r is smooth and has the "expected dimension" $\rho(g, d, r) + r$ at $D \in C_d^r - C_d^{r+1}$ if and only if the cup product homomorphism

$$\mu : H^0(C, [D]) \otimes H^0(C, K[-D]) \mapsto H^0(C, K)$$

is injective, where, K is the canonical line bundle of C(Ref [ACGH] p163).

The map μ is called the Petri map. Again, it was proved by Gieseker[G] that for generic C, the cup product homomorphism μ is indeed injective. This gives another prove of the result of Griffiths-Harris.

In this paper, we are trying to generalize those ideals to the study of rank two vector bundles.

First we will define a parameter space H_d for all base point free rank two vector bundles of degree d which generated by its sections(we called such vector bundles the effective vector bundles). H_d is a d-dimensional holomorphic vector bundle on C_d , so it is a 2d-dimensional complex manifold.

For each $E \in H_d$, we construct a $2g \times d$ matrix W_E for E which we call it the Brill-Noether matrix of E , it shares the same properties for E as the Brill-Noether matrix W_D for line bundle $[D]$. In particular, we have

$$\dim H^0(C, E) = d - \text{Rank}(W_E) + 2.$$

From this, the Brill-Noether variety of rank two vector bundles

$$C_{2,d}^r = \{E \in H_d \mid \dim H^0(C, E) \geq r + 1\}$$

could be given by

$$C_{2,d}^r = \{E \in H_d \mid \text{Rank}(W_E) \leq d - r + 1\}.$$

This defines $C_{2,d}^r$ as a subvariety of H_d .

Also by using W_E , locally we get a holomorphic map

$$BN : H_d \mapsto M(2d, g); BN(E) = W_E,$$

so $C_{2,d}^r = BN^{-1}(M_{d-r+1})$, and from the theory of determinant variety, we get that if $C_{2,d}^r \neq \emptyset$ then

$$\text{codim} C_{2,d}^r \leq (2g - (d - r + 1))(d - (d - r + 1))$$

so if $C_{2,d}^r \neq \emptyset$, then

$$\dim C_{2,d}^r \geq 2d - (2g - (d - r + 1))(d - (d - r + 1)) = 2d - (r + 1)(2(g - 1) - d + r + 1) =$$

$$2d - (r + 1)(2(g - 1) - d + r + 1) + 2(2(g - 1) - d + r + 1) =$$

$$4(g - 1) + 1 - (r + 1)(2(g - 1) - d + r + 1) + 2r + 1 = \rho_2(g, d, r) + 2r + 1$$

here $\rho_2(g, d, r) = 4(g - 1) + 1 - (r + 1)(2(g - 1) - d + r + 1)$ is the Brill-Noether number for rank two vector bundles.

Also, by studying the tangent map of $BN : H_d \mapsto M(2g, d)$, we generalize the Petri map to rank two vector bundles. This is for each $E \in C_{2,d}^r$, we define a cup product homomorphism

$$P : H^0(C, K(-E)) \otimes \text{Im}\{H^0(C, E) \mapsto H^0(C, [D])\} \mapsto H^0(C, K[D](-E)).$$

Here $[D] = E/I$ is the quotient bundle of E with respect to the trivial line bundle I . We call P the Petri map for rank two vector bundles, and we show that $C_{2,d}^r$ has the "expected dimension" $\rho_2(g, d, r) + 2r + 1$ if and only if the Petri map P is injective.

2. The parameter space H_d .

DEFINITION 1. A point $p \in C$ is called a base point of vector bundle E if $s(p) = 0$ for all $s \in H^0(C, E)$. E is said to be base point free if E don't have base point.

DEFINITION 2 [A]. A rank two vector bundle E is said to be generated by its sections, if E has a splitting

$$0 \mapsto L_1 \mapsto E \mapsto L_2 \mapsto 0$$

such that both $H^0(C, L_1)$ and $\text{Im}\{H^0(C, E) \mapsto H^0(C, L_2)\}$ are not zero. Where L_1 is a line sub-bundle of E , and $L_2 = E/L_1$.

The Brill-Noether theory for rank two vector bundles is to study those bundles E with both $H^0(C, E)$ and $H^1(C, E)$ are non-zero. E is then called special rank two vector bundle. If E has a base point p , then $E \otimes [-p]$ is also special and we have $\dim H^0(C, E \otimes [-p]) = \dim H^0(C, E)$, $\dim H^1(C, E \otimes [-p]) = \dim H^1(C, E) + 2$ and $\deg(E \otimes [-p]) = \deg(E) - 2$. We can reduce the degree of E . If E is not generated by its sections, since $H^0(C, E) \neq 0$, let $s \in H^0(C, E)$ with $s \neq 0$, let L_1 be the line sub-bundle of E which generated by s , $L_2 = E/L_1$. Since E is not generated by its sections, so $H^0(C, E) = H^0(C, L_1)$, the study of $H^0(C, E)$ could be reduced to the study of $H^0(C, L_1)$, that is reduced to the study of Brill-Noether for line bundles. So to study the Brill-Noether for rank two vector bundles, we can restrict ourself to the study of base point free vector bundles which generated by its sections.

LEMMA 1. If E is a base point free rank two vector bundle which generated by its sections, then the trivial line bundle I is a line sub-bundle of E .

Proof. This is a special case of Lemma 1.1 of [TE].

Let E be a base point free rank two vector bundles which generated by its sections, assume $\deg(E) = d$, by our Lemma, I is a line sub-bundle of E , so E has a splitting

$$0 \mapsto I \mapsto E \mapsto L \mapsto 0$$

where $L = E/I$. Since E is generated by its sections, we have $\text{Im}\{H^0(C, E) \mapsto H^0(C, L)\} \neq 0$. Choose $s \in \text{Im}\{H^0(C, E) \mapsto H^0(C, L)\}$ with $s \neq 0$, let $D = \text{div}(s)$, then $D \geq 0$, and $L = [D]$. E is then an extension of $[D]$ by I , it is determined by an element $e \in H^1(C, [-D])$. Since $s \in H^0(C, [D])$ can be lift to a section of E , we get in particular that $s \cdot e = 0$, and from sequence

$$0 \mapsto [-D] \mapsto \overset{s}{\cdot} I \mapsto I|_D \mapsto 0 \tag{***}$$

we get an exact sequence

$$0 \mapsto H^0(C, [-D]) \mapsto H^0(C, I) \mapsto H^0(C, I|_D) \mapsto H^1(C, [-D]) \mapsto \dots$$

$s \cdot e = 0$ if and only if $e \in \text{Im}\{H^0(C, I|_D) \mapsto H^1(C, [-D])\}$. Let e be the image of some $f \in H^0(C, I|_D)$, f is then determined uniquely up to a constant. So from E we get a triple $\{I, D, f\}$.

Conversely, if we have a triple $\{I, D, f\}$, where D is an effective divisor of degree d , and $f \in H^0(C, I|_D)$, then let $e \in H^1(C, [-D])$ be the image of f in the map $H^0(C, I|_D) \mapsto H^1(C, [-D])$ which induced from sequence $(***)$, let E be the extension of $[D]$ by I which determined by e , then E has a splitting $0 \mapsto I \mapsto E \mapsto [D] \mapsto 0$, and $s \in \text{Im}\{H^0(C, E) \mapsto H^0(C, [D])\}$, where s is the canonical section of D ($s \in H^0(C, [D])$, with $\text{div}(s) = D$). We get a base point free rank two vector bundle E of degree d which generated by its sections.

So to give a base point free rank two vector bundle of degree d which generated by its sections will be the same as to give a triple $\{I, D, f\}$, here $D \in C_d$ and $f \in H^0(C, [D]|_D)$, or the same the set of all base point free rank two vector bundle of degree d which generated by its sections could be represented by the set of all triples $\{I, D, f\}$. We will denote this as $E = \{I, D, f\}$.

Now let H_d be the vector bundle on C_d which for each $D \in C_d$, $H_d|_D = H^0(C, I|_D)$, by using local coordinate, it is easy to see that H_d is a holomorphic vector bundle of dimension d on C_d .

Each point of H_d could be represented as a triple $E = \{I, D, f\}$, and each triple $E = \{I, D, f\}$ could be represented as a point in H_d , so H_d could be considered as a parameter space for the set of all base point free rank two vector bundles of degree d which generated by its sections.

3. Brill-Noether matrix for $E = \{I, D, f\}$. Let L be a line bundle, $D = n_1p_1 + \dots + n_kp_k \geq 0$ be a given effective divisor of degree d . For $i = 1, \dots, k$, let z_i be a local coordinate at p_i with $z_i(p_i) = 0$. Then each $f \in H^0(C, L|_D)$ could be represented as a set of polynomials $f = \{f_i(z_i)\}_{i=1}^k$, where $f_i(z_i) = a_0^i + a_1^iz_i + \dots + a_{n_i-1}^iz_i^{n_i-1}$ is a polynomial of z_i of degree less than n_i . So f could also be denoted as a d -dimensional vector $f = (a_0^1, a_1^1, \dots, a_{n_1-1}^1; a_0^2, a_1^2, \dots, a_{n_2-1}^2; \dots)$. This gives $H^0(C, L|_D) \cong C^d$, where $d = \text{deg}(d)$.

DEFINITION 3. Let L_1, L_2 be two line bundles, $D = n_1p_1 + \dots + n_kp_k \geq 0$ be a given effective divisor. For $f = \{f_i(z_i)\}_{i=1}^k \in H^0(C, L_1|_D)$ and $g = \{g_i(z_i)\}_{i=1}^k \in H^0(C, L_2|_D)$, we define $f * g \in H^0(C, L_1 \otimes L_2|_D)$ to be

$$f * g = \{f_i(z_i)g_i(z_i)(\text{mod}(z_i^{n_i}))\}_{i=1}^k.$$

LEMMA 2. $f * g = g * f$, and $(f * g) * h = f * (g * h)$.

Proof. Trivial.

LEMMA 3. For $E = \{I, D, f\}$, a section $s \in H^0(C, [D])$ could be lift to be a section of $H^0(C, E)$ (which means $s \in \text{Im}\{H^0(C, E) \mapsto H^0(C, [D])\}$), if and only if

$$s|_D * f \in \text{Im}\{H^0(C, [D]) \mapsto H^0(C, [D]|_D)\}.$$

Proof. See [T].

Now let (w_1, \dots, w_g) be a linear basis of $H^0(C, K)$ of the space of all holomorphic forms on C . then for effective divisor D , the Brill-Noether matrix W_D for D could be defined by

$$W_D = \begin{bmatrix} w_1|_D \\ w_2|_D \\ \vdots \\ w_g|_D \end{bmatrix}.$$

An element $t \in H^0(C, [D]|_D)$ is in the image of map $H^0(C, [D]) \mapsto H^0(C, [D]|_D)$, if and only if

$$W_D * t = \begin{bmatrix} w_1|_D * t \\ w_2|_D * t \\ \vdots \\ w_g|_D * t \end{bmatrix} = 0.$$

That is $\text{Im}\{H^0(C, [D]) \mapsto H^0(C, [D]|_D)\} = \text{Ker}\{W_D\}$.

Now for $E = \{I, D, f\}$, we define its Brill-Noether matrix W_E to be

$$W_E = \begin{bmatrix} w_1|_D \\ w_2|_D \\ \vdots \\ w_g|_D \\ w_1|_D * f \\ w_2|_D * f \\ \vdots \\ w_g|_D * f \end{bmatrix} = \begin{bmatrix} W_D \\ W_D * f \end{bmatrix}.$$

THEOREM 1. $\text{Ker}\{W_E\} = \{v \in C^d \mid W_E \cdot v = 0\} \cong \text{Im}\{H^0(C, E) \mapsto H^0(C, [D]) \mapsto H^0(C, [D]|_D)\}$.

Proof. By $H^0(C, [D] |_D) \cong C^d$, each $v \in C^d$ could be identified to an element $v \in H^0(C, [D] |_D)$, let W_D be the Brill-Noether matrix for D , then $W_D \cdot v = W_D * v$, and $(W_D * f) * v = W_D * (f * v)$. So $W_E \cdot v = 0$ if and only if $W_D * v = 0$ and $W_D * (f * v) = 0$. From $W_D * v = 0$, we get that $v \in Im\{H^0(C, [D]) \mapsto H^0(C, [D] |_D)\}$. Let it be the image of some $s \in H^0(C, [D])$, this is $v = s |_D$. Then from $(W_D * f) * v = 0$, we get $(W_D * f) * s |_D = W_D * (f * s |_D) = 0$. That means $f * s |_D \in Im\{H^0(C, [D]) \mapsto H^0(C, [D] |_D)\}$. By our Lemma 3, s is then can be lift to a section of E .

Conversely, if $v \in Im\{H^0(C, E) \mapsto H^0(C, [D]) \mapsto H^0(C, [D] |_D)\}$, let it be the image of some $s \in H^0(C, [D])$, so $W_D \cdot v = 0$, and since s can be lift to a section of E , by our Lemma 3, $f * v \in Im\{H^0(C, [D]) \mapsto H^0(C, [D] |_D)\}$, so $W_D * f * v = 0$, we get $W_E \cdot v = 0$. This completes the proof.

Now from the exact sequence

$$0 \mapsto I \mapsto E \mapsto [D] \mapsto 0$$

we get exact sequence

$$0 \mapsto H^0(C, I) \mapsto H^0(C, E) \mapsto H^0(C, [D]) \mapsto H^1(C, I) \mapsto \dots$$

Since $dimH^0(C, I) = 1$, so

$$\begin{aligned} dimH^0(C, E) &= dimIm\{H^0(C, E) \mapsto H^0(C, [D])\} + 1 = \\ dimIm\{H^0(C, E) \mapsto H^0(C, [D]) \mapsto H^0(C, [D] |_D)\} + 2 &= \\ dimKer(W_E) + 2 &= d - rank(W_E) + 2. \end{aligned}$$

That is

THEOREM 2. Let $E = \{I, D, F\}$ and W_E be its Brill-Noether matrix, then we have $Ker(W_E) \cong Im\{H^0(C, E) \mapsto H^0(C, [D]) \mapsto H^0(C, [D] |_D)\}$, and in particular $dimH^0(C, E) = d - rank(W_E) + 2$.

Now we define the Brill-Noether variety $C_{2,d}^r$ for rank two vector bundles to be

$$C_{2,d}^r = \{E \in H_d \mid dimH^0(C, E) \geq r + 1\}.$$

By Theorem 2, $C_{2,d}^r$ could also be given by

$$C_{2,d}^r = \{E \in H_d \mid rank(W_E) \leq d - r + 1\}.$$

This gives $C_{2,d}^r$ as a subvariety of H_d which $C_{2,d}^r$ is defined locally by the simultaneously vanishing of all $(d - r + 2) \times (d - r + 2)$ minors of W_E .

By using the Brill-Noether matrix W_E , locally, we get a holomorphic map $BN : H_d \mapsto M(2g, d)$ with $BN(E) = W_E$ for each $E \in H_d$, where $M(2g, d)$ is the variety of all $2g \times d$ complex matrices. Let

$$M_{d-r+1} = \{E \in M(2g, d) \mid rank(E) \leq d - r + 1\}.$$

Then M_{d-r+1} is a subvariety of $M(2g, d)$, and $codim(M_{d-r+1}) = (2g - (d - r + 1)) \times (d - (d - r + 1))$. By definition, we have $C_{2,d}^r = BN^{-1}(M_{d-r+1})$. So from the Theory of determinant variety, we get that if $C_{2,d}^r \neq \emptyset$, then

$$codimC_{2,d}^r \leq (2g - (d - r + 1)) \times (d - (d - r + 1)).$$

This is

$$dimC_{2,d}^r \geq 2d - (2g - (d - r + 1)) \times (d - (d - r + 1)) =$$

$$4(g - 1) + 1 - (r + 1)(2(g - 1) - d + r + 1) + 2r + 1 = \rho_2(g, d, r) + 2r + 1.$$

Here $\rho_2(g, d, r) = 4(g - 1) + 1 - (r + 1)(2(g - 1) - d + r + 1)$ is the Brill-Noether number for rank two vector bundles. Same as the case of line bundles, we get that the expected dimension of $C_{2,d}^r$ is $\rho_2(g, d, r) + 2r + 1$, this is

THEOREM 3. If $C_{2,d}^r \neq \emptyset$, then each component of $C_{2,d}^r$ will have dimension at least $\rho_2(g, d, r) + 2r + 1$.

4. The Petri map. Since $C_{2,d}^r = BN^{-1}(M_{d-r+1})$, to get the dimension of $C_{2,d}^r$, analogous to the case of line bundles, we should consider the tangent map

$$BN^* : T_E \mapsto T_{BN(E)}$$

for each $E = \{I, f, D\} \in H_d$. Here T_E and $T_{BN(E)}$ are the tangent space of E and $BN(E)$ in H_d and $M(2g, d)$.

Now let $E = \{I, D, f\}$, then

$$BN(E) = W_E = \begin{bmatrix} W_D \\ W_D * f \end{bmatrix}.$$

Since for each $D \in C_d$, the tangent space of C_d at D is $T_D = H^0(C, [D] |_D)$ (Ref [ACGH] P160), so by definition we get that the tangent space of H_d at E is $T_E = H^0(C, [D] |_D) \oplus H^0(C, I |_D)$.

Now let $t = (-v, u) \in T_E = H^0(C, [D] |_D) \oplus H^0(C, I |_D)$, then by direct calculation, we have

$$BN^*(t) = \begin{bmatrix} \dot{W}_D * (-v) \\ \dot{W}_D * (-v) * f + W_D * u \end{bmatrix}.$$

Where \dot{W}_D means the differential of W_D with respect to the local coordinates, and $\dot{f} = I$.

To get the dimension of $C_{2,d}^r$, we need to get the dimension of the space $V = \{t \in T_E | BN^*(t) \in T_{BN(E)}(M_{d-r+1})\}$. But from the theory of determinant variety(Ref [ACGH] p69), we know that $BN^*(t) \in T_{BN(E)}(M_{d-r+1})$ if and only if $Ker(W_E) \cdot BN^*(t) \subset Im(W_E) = C^{2g} \cdot W_E$. Here $Ker(W_E) = \{(b, e) = (b_1, \dots, b_g; e_1, \dots, e_g) \in C^d | (b, e)W_E = 0\}$.

Now let $(b, e) = (b_1, \dots, b_g; e_1, \dots, e_g) \in Ker(W_E)$, this is $(b, e) \cdot W_E = b \cdot W_D + e \cdot W_D * f = 0$. Choose an open cover $\{U_\alpha\}_{\alpha=1}^k$ of C , let $s = \{s_\alpha\}_{\alpha=1}^k \in H^0(C, [D])$ be the canonical section of $[D]$, this is $s \in H^0(C, [D])$ and $div(s) = D$. for the linear basis $\{w_1, \dots, w_g\}$ of the holomorphic forms, let w_i be given with respect to the open cover by $w_i = \{w_{\alpha i}\}$, let $bw = b_1w_1 + \dots + b_gw_g = \{b_1w_{\alpha 1} + \dots + b_gw_{\alpha g}\} = \{bw_\alpha\} \in H^0(C, K)$, and $ew = e_1w_1 + \dots + e_gw_g = \{e_1w_{\alpha 1} + \dots + e_gw_{\alpha g}\} = \{ew_\alpha\} \in H^0(C, K)$, let $f = \{f_\alpha\}$ be a given representation for $f \in H^0(C, I |_D)$, where f_α is a holomorphic function on U_α .

LEMMA 4. $(b, e) \in Ker(W_E)$ if and only if

$$F = \{F_\alpha = \begin{bmatrix} e \cdot w_\alpha \\ -(b \cdot w_\alpha + e \cdot w_\alpha * f_\alpha) / s_\alpha \end{bmatrix}\} \in H^0(C, K(-E)).$$

Here $(-E)$ is the dual vector bundle of E .

Proof. For later using and also for making our notations easy to understand, we will give a proof of this Lemma in detail, and we will also use the proof to give a proof of Riemann-Roch Theorem for rank two vector bundles.

Let $\{U_\alpha\}_{\alpha=1}^k$ be the open cover of C . Then on $U_\alpha \cap U_\beta$, the transition matrix of $E = \{I, f, D\}$ can be given by

$$E_{\alpha\beta} = \begin{bmatrix} 1 & (f_\alpha - f_\beta) / s_\beta \\ 0 & s_\alpha / s_\beta \end{bmatrix}$$

where $e = \{e_{\alpha\beta} = (f_\alpha - f_\beta)/s_\beta\}$ is a representation of $e \in H^1(C, [-D])$.

From $E_{\alpha\beta}$, and by the definition of dual vector bundle, the transition matrix of $K(-E)$ can be given on $U_\alpha \cap U_\beta$ by

$$(K(-E))_{\alpha\beta} = \begin{bmatrix} k_{\alpha\beta} & 0 \\ -k_{\alpha\beta}(f_\alpha - f_\beta)/s_\beta & k_{\alpha\beta}s_\beta/s_\alpha \end{bmatrix}$$

where $\{k_{\alpha\beta}\}$ is the transition function of the canonical line bundle K .

By definition, $K(-E)$ is an extension of K by $K[-D]$, which determined also by $f \in H^0(C, I|_D)$.

Now let $(b, e) \in Ker(W_E)$, that is $b \cdot W_D + e \cdot W_D * f = 0$, let $ew = e_1w_1 + \dots + e_gw_g \in H^0(C, K)$, $bw = b_1w_1 + \dots + b_gw_g \in H^0(C, K)$, then $b \cdot W_D + e \cdot W_D * f = 0$ means $ew|_D * f = -bw|_D$, by our Lemma 3(also Ref [T]), that means, ew can be lift to a section of $K(-E)$ and

$$F = \{F_\alpha = \begin{bmatrix} e \cdot w_\alpha \\ -(b \cdot w_\alpha + e \cdot w_\alpha * f_\alpha)/s_\alpha \end{bmatrix}\} \in H^0(C, K(-E)).$$

is one of the lift. This can also be proved by direct computation that $F_\alpha = K(-E)_{\alpha\beta} \cdot F_\beta$.

Conversely, let

$$F = \{F_\alpha = \begin{bmatrix} e \cdot w_\alpha \\ v_\alpha \end{bmatrix}\} \in H^0(C, K(-E)).$$

then $ew = e_1w_1 + \dots + e_gw_g = \{ew_\alpha = e_1w_1|_{U_\alpha} + \dots + e_gw_g|_{U_\alpha}\}$, is a section of K , here $e = (e_1, \dots, e_g)$, and F is a lift of ew . $ew \in H^0(C, K)$ can be lift to a section of $H^0(C, K(-E))$, by our Lemma 3, there exists an $bw = b_1w_1 + \dots + b_gw_g \in H^0(C, K)$, such that $ew|_D * f = -bw|_D$, or the same, $ew|_D * f + bw|_D = 0$, that is $(b, e) \cdot W_E = 0$, so $(b, e) \in Ker(W_E)$.

Now if $ew = 0$, that is $e = 0$, then $F = \{F_\alpha = \begin{bmatrix} 0 \\ v_\alpha \end{bmatrix}\} \in H^0(C, K(-E))$ means $v = \{v_\alpha\} \in H^0(C, K \otimes [-D])$, but we know that $H^0(C, K \otimes [-D]) = \{w \in H^0(C, K) \mid w|_D = 0\}$. Assume $v = b_1w_1 + \dots + b_gw_g = bw$, here $b = (b_1, \dots, b_g)$, then $bw|_D = 0$ means $bW_D = 0$, so $(b, 0)W_E = 0$, this is $(b, 0) \in Ker(W_E)$. That completes the proof.

From the proof, we get

COROLLARY 1. $H^0(C, K(-E)) \cong Ker(W_E)$, and in particular

$$dimH^0(C, K(-E)) = 2g - rank(W_E).$$

But from the definition of W_E , we know that

$$dimH^0(C, E) = d - rank(W_E) + 2.$$

We get the Riemann-Roch Theorem for base point free rank two vector bundle which generated by its sections:

RIEMANN-ROCH THEOREM. If E is a base point free rank two vector bundle which generated by its sections, then

$$dimH^0(C, E) - dimH^0(C, K(-E)) = deg(E) - 2(g - 1).$$

Same as the case of line bundles, Riemann-Roch Theorem for all rank two vector bundles could be derived easily from this, we will not give it here.

Now, let $t \in T_E$, to get the dimension of $C_{2,d}^r$, we need to get the dimension of space $V = \{t \in T_E \mid BN^*(t) \in T_{BN(E)}(M_{d-r+1})\}$. So we need to know under what condition $BN^*(t) \in T_{BN(E)}(M_{d-r+1})$. From the theory of determinant variety, we know this same that

$$(b, e)BN^*(t) = (b, e) \left[\begin{array}{c} \dot{W}_D * (-v) \\ \dot{W}_D * (-v) * f + W_D * u \end{array} \right] \in Im(W_E).$$

for all $(b, e) \in Ker(W_E)$. For this, we will first define a short exact sequence of sheaves.

Let V be a vector bundle on C , we will use V itself to denote the sheaf of holomorphic sections of V . For $E = \{I, f, D\}$, let $\{U_\alpha\}_{\alpha=1}^k$ be the given open cover of C , and $s = \{s_\alpha\}_{\alpha=1}^k \in H^0(C, [D])$ be the canonical section of $[D]$, this is $s \in H^0(C, [D])$ and $div(s) = D$. Let $f = \{f_\alpha\}$ be a given representation for $f \in H^0(C, I|_D)$, where f_α is a holomorphic function on U_α . Then by using the transition matrix $E_{\alpha\beta}$ given in the proof of Lemma 4, one can check directly that

$$F = \{F_\alpha = \begin{bmatrix} f_\alpha \\ s_\alpha \end{bmatrix}\} \in H^0(C, E).$$

is the lift of the canonical section s . Now let $P_1 : K(-E) \mapsto K$ be the projective map which induced from sequence $0 \mapsto K[-D] \mapsto K \otimes [-E] \mapsto K \mapsto 0$, then from F and P_1 , we define a map of sheaves $K(-E) \mapsto K \oplus K$ by

$$x \mapsto (P_1(x), -(x, F))$$

here $x \in K(-E)$, and $(,) : K(-E) \otimes E \mapsto K$ is the duality map. We also define a map of sheaves $K \oplus K \mapsto K|_D$ to be $(s, t) \mapsto (s|_D * f + t|_D)$ for $(s, t) \in K \oplus K$.

Locally, let $\{U_\alpha\}$ be the given open cover of C , if $\begin{bmatrix} a \\ b \end{bmatrix} \in K(-E)|_{U_\alpha}$, then $K(-E) \mapsto K$ is defined by $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto (a, -af_\alpha - bs_\alpha)$, and the map $K \oplus K \mapsto K|_D$ could be given by $(c, d) \mapsto (c|_D * f + d|_D)$.

LEMMA 5. The sequence $0 \mapsto K(-E) \mapsto K \oplus K \mapsto K|_D \mapsto 0$ is a short exact sequence of sheaves on C .

Proof. We will use the local representation to give the proof.

If $\begin{bmatrix} a \\ b \end{bmatrix} \in K(-E)$, and $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto (a, (-af - bs)) = 0$, then $a = 0$, and since $s \neq 0$ so $bs = 0$ means $b = 0$, the map $K(-E) \mapsto K \oplus K$ is injective.

If $(c, d) \in K \oplus K$, and $(c, d) \mapsto (c|_D * f + d|_D) = 0$, we then get $c|_D * f = -d|_D$, by our Lemma 3, c can be lift locally to section of $K(-E)$ and same as Lemma 4, $\begin{bmatrix} c \\ -(cf + d)/s \end{bmatrix} \in K(-E)$ is one of the lift. But $\begin{bmatrix} c \\ -(cf + d)/s \end{bmatrix} \mapsto (c, -cf + (cf + d)) = (c, d)$. This shows that the sequence is exact at $K \oplus K$.

Also it is easy to see that the map $K \oplus K \mapsto K|_D$ is an onto map. This completes the proof.

From this short exact sequence, we get a long exact sequence

$$0 \mapsto H^0(C, K(-E)) \mapsto H^0(C, K \oplus K) \mapsto H^0(C, K|_D) \mapsto H^1(C, K(-E)) \mapsto \dots$$

$a \in H^0(C, K|_D)$ is in the image of map $H^0(C, K \oplus K) = H^0(C, K) \oplus H^0(C, K) \mapsto H^0(C, K|_D)$ if and only if $\delta(a) = 0$, here $\delta : H^0(C, K|_D) \mapsto H^1(C, K(-E))$ is the co-boundary map. But from Serrra duality, we know that for $\delta(a) \in H^1(C, K(-E))$, $\delta(a) = 0$ if and only if for any $f \in H^0(C, E)$, we have $(\delta(a), f) = 0$. Here $(,) : H^1(C, K(-E)) \otimes H^0(C, E) \mapsto H^1(C, K)$ is the duality map.

Now assume, for open cover $\{U_\alpha\}$, a is given by $a = \{a_\alpha\}$, where $a_\alpha \in H^0(U_\alpha, K|_{U_\alpha})$ and $a_\alpha|_{D \cap U_\alpha} = a|_{D \cap U_\alpha}$. Then by direct calculation, we get $\delta(a) \in H^1(C, K(-E))$, could be represented as

$$\delta(a) = \left\{ \begin{bmatrix} 0 \\ k_{\alpha\beta}(-a_\alpha + a_\beta)/s_\alpha \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ \tilde{\delta}(a) \end{bmatrix},$$

where $\tilde{\delta} : H^0(C, K|_D) \mapsto H^1(C, K[-D])$ is the co-boundary map from the following sequence

$$0 \mapsto H^0(C, K[-D]) \mapsto^s H^0(C, K) \mapsto H^0(C, K|_D) \mapsto H^1(C, K[-D]) \mapsto \cdots.$$

So for any $f = \begin{bmatrix} y_\alpha \\ x_\alpha \end{bmatrix} \in H^0(C, E)$, the dual map could be given by

$$\begin{aligned} (\delta(a), f) &= \left(\left\{ \begin{bmatrix} 0 \\ (-a_\alpha + a_\beta)/s_\alpha \end{bmatrix} \right\}, \left\{ \begin{bmatrix} y_\alpha \\ x_\alpha \end{bmatrix} \right\} \right) \\ &= \{(-a_\alpha + a_\beta)/s_\alpha\} \cdot x_\alpha = (\tilde{\delta}(a), \{x_\alpha\}). \end{aligned}$$

but $\delta(a) = 0$ if and only if $(\delta(a), f) = 0$ for all $f \in H^0(C, E)$, from what we get above, this is same that $\delta(a) = 0$ if and only if for any $x = \{x_\alpha\} \in \text{Im}\{H^0(C, E) \mapsto H^0(C, [D])\}$, $(\tilde{\delta}(a), x) = 0$. We get the following Lemma.

LEMMA 6. For $a \in H^0(C, K|_D)$, $\delta(a) \in H^1(C, K(-E))$, with $\delta(a) = 0$ if and only if for any $x = \{x_\alpha\} \in \text{Im}\{H^0(C, E) \mapsto H^0(C, [D])\}$, $(\tilde{\delta}(a), x) = 0$.

Now go back to the tangent map of $BN : H_d \mapsto M(2g, d)$.

For $E = \{I, f, D\} \in C_{2,d}^r$, we know

$$BN(E) = W_E = \begin{bmatrix} W_D \\ W_D * f \end{bmatrix},$$

if $t = (u, -v) \in T_E = H^0(C, [D]|_D) \oplus H^0(C, I|_D)$, then

$$BN^*(t) = \begin{bmatrix} \dot{W}_D * (-v) \\ \dot{W}_D * (-v) * f + W_D * u \end{bmatrix}.$$

But we know that $BN^*(t) \in T_{BN(E)}(M_{d-r+1})$ if and only if $\text{Ker}(W_E) \cdot BN^*(t) \in \text{Im}(W_E)$. Since $\text{Im}(W_E) = C^{2g} \cdot W_E = \{(c, d) \begin{bmatrix} W_D \\ W_D * f \end{bmatrix} \mid (c, d) \in C^{2g}\}$. If we identify $C^{2g} = C^g \oplus C^g \cong H^0(C, K) \oplus H^0(C, K)$, then we get

$$\text{Im}(W_E) = \text{Im}\{H^0(C, K) \oplus H^0(C, K) \mapsto H^0(C, K|_D)\}.$$

Where the map $H^0(C, K) \oplus H^0(C, K) \mapsto H^0(C, K|_D)$ is induced from above exact sequence.

From this we get $BN^*(t) \in T_{BN(E)}(M_{d-r+1})$ if and only if for any $(b, e) \in \text{Ker}(W_E)$, $(b, e)BN^*(t) \in \text{Im}(W_E)$. This is $\delta((b, e)BN^*(t)) = 0$. By Lemma 6, we get

LEMMA 7. let $t \in T_E$, then $BN^*(t) \in T_{BN(E)}(M_{d-r+1})$ if and only if for any $(b, e) \in \text{Ker}(W_E)$, we have $(\tilde{\delta}((b, e)BN^*(t)), x) = 0$ for all $x \in \text{Im}\{H^0(C, E) \mapsto H^0(C, [D])\}$.

But by direct calculation, we get

$$\begin{aligned} (b, e)BN^*(t) &= (b, e) \begin{bmatrix} \dot{W}_D * (-u) \\ \dot{W}_D * (-u) * f + W_D * v \end{bmatrix} = b\dot{W}_D * u + e\dot{W}_D * u * f + eW_D * v \\ &= \begin{bmatrix} eW_D \\ -(b\dot{W}_D + e\dot{W}_D * f) \end{bmatrix} * \begin{bmatrix} u \\ -v \end{bmatrix}. \end{aligned}$$

Notice that by using local coordinate, it is easy to see that

$$\begin{bmatrix} eW_D \\ -(b\dot{W}_D + e\dot{W}_D * f) \end{bmatrix} = \begin{bmatrix} eW_D \\ -(bW_D + eW_D * f)/s \end{bmatrix}.$$

Since $(b, e) \in \text{Ker}(W_E)$, by Lemma 4, we get

$$\begin{bmatrix} eW_D \\ -(bW_D + eW_D * f)/s \end{bmatrix} \in \text{Im}\{H^0(C, K(-E)) \mapsto H^0(C, K(-E)|_D)\}.$$

let it be the image of some $F \in H^0(C, K(-E))$. Now notice that $E|_D = I|_D \oplus [D]|_D = T_E$ and $K(-E)|_D = K|_D \oplus K[D]|_D = (I|_D \oplus [D]|_D)^* = T_E^*$ then follow the proof of Lemma 1.5 p162 [ACGH] step by step, for $x \in \text{Im}\{H^0(C, E) \mapsto H^0(C, [D])\}$, we have

$$(\tilde{\delta}((b, e)BN^*(t)), x) = (\tilde{\delta}(F * t), x) = (\delta_1(t), (F \otimes x)) = (t, (F \otimes x)|_D)$$

Where $\delta_1 : (I|_D \oplus [D]|_D) \mapsto H^1(C, E[-D])$ is the co-boundary map follow from sequence $0 \mapsto E[-D] \mapsto^s E \mapsto E|_D \mapsto 0$. So $t \in V = \{t \in T_E \mid BN^*(t) \in T_{BN(E)}(M_{d-r+1})\}$ if and only if for any $F \in H^0(C, K(-E))$ and $x \in \text{Im}\{H^0(C, E) \mapsto H^0(C, [D])\}$, we have $(t, (F \otimes x)|_D) = 0$. We get

LEMMA 8. $t \in V = \{t \in T_E \mid BN^*(t) \in T_{BN(E)}(M_{d-r+1})\}$, if and only if

$$t \in \{\text{Im}\{H^0(C, K(-E)) \otimes \text{Im}\{H^0(C, E) \mapsto H^0(C, [D])\} \mapsto H^0(C, K(-E)[D]) \mapsto H^0(C, K(-E)[D]|_D)\}^\perp.$$

Now assume $E \in C_{2,d}^r - C_{2,d}^{r+1}$, From what we get above, the expected dimension of $C_{2,d}^r$ at E could be given by

$$\begin{aligned} \dim(C_{2,d}^r) &= \dim(V) = \\ &= 2d - \dim\{\text{Im}\{H^0(C, K(-E)) \otimes \text{Im}\{H^0(C, E) \mapsto H^0(C, [D])\} \mapsto H^0(C, K(-E)[D]) \mapsto H^0(C, K(-E)[D]|_D)\} \\ &= 2d - (2(g-1) - d + r + 1)r + 2(g-1) - d + r + 1 + \dim W. \end{aligned}$$

where $(2(g-1) - d + r + 1)r = \dim[H^0(C, K(-E)) \otimes \text{Im}\{H^0(C, E) \mapsto H^0(C, [D])\}] = \dim H^0(C, K(-E)) \times \dim \text{Im}\{H^0(C, E) \mapsto H^0(C, [D])\}$, and $2(g-1) - d + r + 1 = \dim \text{Ker}\{H^0(C, K(-E)[D]) \mapsto H^0(C, K(-E)[D]|_D)\}$, $W = \text{Ker}\{H^0(C, K(-E)) \otimes \text{Im}\{H^0(C, E) \mapsto H^0(C, [D])\} \mapsto H^0(C, K(-E)[D])\}$.

We then get

$$\begin{aligned} \dim(C_{2,d}^r) &= 4(g-1) + 1 - (r+1)(2(g-1) - d + r + 1) + 2r + 1 + \dim W \\ &= \rho(2, d, r) + 2r + 1 + \dim W. \end{aligned}$$

THEOREM 3. $C_{2,d}^r$ has the expected dimension $\rho(2, d, r) + 2r + 1$ at $E \in C_{2,d}^r - C_{2,d}^{r+1}$, if and only if for all $E \in C_{2,d}^r$, $W = \{0\}$.

This is the same that $C_{2,d}^r$ has the expected dimension $\rho(2, d, r) + 2r + 1$, if and only if for all $E \in C_{2,d}^r$, the map

$$H^0(C, K(-E)) \otimes \text{Im}\{H^0(C, E) \mapsto H^0(C, [D])\} \mapsto H^0(C, K(-E)[D])$$

is injective.

Compare with the case of line bundles, we then called the map

$$H^0(C, K(-E)) \otimes \text{Im}\{H^0(C, E) \mapsto H^0(C, [D])\} \mapsto H^0(C, K(-E)[D])$$

the Petri map for rank two vector bundles. We have

THEOREM 4. $C_{2,d}^r$ has the expected dimension $\rho(2, d, r) + 2r + 1$, if and only if for all $E \in C_{2,d}^r$, the Petri map is injective.

This is a generalization of Lemma 1.6 of [ACGH] P163.

REFERENCES

- [ACGH] ARBARELLO, E., CORNABA, M., GRIFFITHS, P., HARRIS, J., *Geometry of Algebraic curves*, Vol I, Springer-Verlag, N.Y. 1984.
- [A] ATIYAH, M.F., *Vector Bundles on Elliptic Curves*, Proc London Math Soc, 7 (1957), pp. 414–452.
- [G] GIESEKER, D., *Stable curves and special divisors*, Invent Math., 66 (1982), pp. 25–275.
- [GH] GRIFFITHS, P., HARRIS, J., *The dimension of the variety of special linear systems on a general curve*, Duke Math. J, 47 (1980), pp. 233–272.
- [LN] LANGE, H., NARASIMHAN, M.S., *Maximal Sub-bundles of Rank Two Bundles on Curves*, Math Ann, 266 (1983), pp. 55–72.
- [T] TAN, X.J., *Some results on the existence of rank two special stable vector bundles*, Manuscripta Math, 75 (1992), pp. 365–373.
- [TE] TEIXIDOR I BIGAS, M., *On the Gieseker-Petri map for rank two vector bundles*, Manuscripta Math, 75 (1992), pp. 375–382.