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GROMOV-WITTEN INVARIANTS OF THE HILBERT SCHEME OF 3-POINTS ON $\mathbb{P}^2$ *

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Abstract. Using obstruction bundles, composition law and localization formula, we compute certain 3-point genus-0 Gromov-Witten invariants of the Hilbert scheme of 3-points on the complex projective plane. Our results partially verify Ruan's conjecture about quantum corrections for this Hilbert scheme.

1. Introduction. Motivated by the pioneering work of Nakajima and Grojnowski [Nak, Gro], there have been intensive studies of the cohomology ring structure of the Hilbert schemes of points on a smooth algebraic surface (e.g. [Leh, L-S, LQW1, LQW2, LQW3, Q-W, Go2]). While our understanding of this ordinary cohomology ring structure has deepened rapidly, the quantum cohomology ring structure of these Hilbert schemes remains to be a mystery. A limited progress to the quantum cohomology ring structure has been made in [L-Q] where certain 1-point genus-0 Gromov-Witten invariants of these Hilbert schemes have been determined. These 1-point invariants come from the contributions of curves contracted by the Hilbert-Chow map from the Hilbert schemes to the symmetric products of the surface.

In this paper, we study 3-point genus-0 Gromov-Witten invariants of the Hilbert scheme $(\mathbb{P}^2)^{[3]}$ of 3-points on the complex projective plane $\mathbb{P}^2$. Again, we are primarily interested in those invariants which come from the contributions of curves contracted by the Hilbert-Chow map (2.8). These curves are homologous to $d\beta_3$ for some positive integer d, where $\beta_3 \subset (\mathbb{P}^2)^{[3]}$ is the rational curve defined by

$$\beta_3 = \{\xi + x_2 \,|\, \ell(\xi) = 2,\ \mathrm{Supp}(\xi) = x_1\}$$

with x_1 and x_2 being two fixed distinct points of the projective plane $X = \mathbb{P}^2$.

To state our main results, we introduce some notations. Let $H^*(X^{[3]})$ and $H_*(X^{[3]})$ be the cohomology and homology of $X^{[3]}$ with $\mathbb{C}$-coefficients. For $i = 2, 4, 6, 8, 10$, a linear basis $\mathfrak{B}_i$ of $H_i(X^{[3]})$ in terms of the Heisenberg operators introduced in [Nak, Gro] can be determined (see Lemma 2.3 and Definition 2.4 for details). For $\alpha_1, \ldots, \alpha_k \in H^*(X^{[3]})$, we use $\langle \alpha_1, \ldots, \alpha_k \rangle_{0,d}$ to stand for the k-point genus-0 Gromov-Witten invariant $\langle \alpha_1, \ldots, \alpha_k \rangle_{0,d\beta_3}$. Now the 3-point genus-0 Gromov-Witten invariants $\langle \alpha_1, \alpha_2, \alpha_3 \rangle_{0,d}$ of $X^{[3]}$ are reduced either to the 2-point invariants $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2) \rangle_{0,d}$ with $A_1 \in \mathfrak{B}_6$ and $A_2 \in \mathfrak{B}_8$, or to the 3-point invariants $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2), \mathrm{PD}(A_3) \rangle_{0,d}$ with $A_1, A_2, A_3 \in \mathfrak{B}_8$. Here PD denotes the Poincaré duality. Our main results are the following.

THEOREM 1.1. *Let $X = \mathbb{P}^2$, and $\mathfrak{B}_6$ and $\mathfrak{B}_8$ be defined in Definition 2.4. Let $d \geq 1$, $A_1 \in \mathfrak{B}_6$ and $A_2 \in \mathfrak{B}_8$. Let x, ℓ be a point and a line in X respectively. Then, $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2) \rangle_{0,d}$ is zero unless the pair (A_1, A_2) is one of the following:*

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(i) $(\mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(x)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle)$
(ii) $(\mathfrak{a}_{-2}(\ell)\mathfrak{a}_{-1}(\ell)|0\rangle, \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle)$
(iii) $(\mathfrak{a}_{-3}(\ell)|0\rangle, \mathfrak{a}_{-3}(X)|0\rangle)$.
Moreover, $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2)\rangle_{0,d} = 12/d$ *in cases (i) and (ii).*

Theorem 1.2. *Let* $X = \mathbb{P}^2$, *and* $\mathfrak{B}_8$ *be defined in Definition 2.4. Let* $\ell \subset X$ *be a line. Let* $d \geq 1$, $f(d) = d\,\langle \mathrm{PD}(\mathfrak{a}_{-3}(\ell)|0\rangle), \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)\rangle_{0,d}$, *and* $A_1, A_2, A_3 \in \mathfrak{B}_8$. *Then, the* 3*-point genus-*0 *Gromov-Witten invariant* $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2), \mathrm{PD}(A_3)\rangle_{0,d}$ *is zero unless the unordered triple* (A_1, A_2, A_3) *is one of the following:*
(i) $(\mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle, \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle)$
(ii) $(\mathfrak{a}_{-3}(X)|0\rangle, \mathfrak{a}_{-3}(X)|0\rangle, \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle)$
(iii) $(\mathfrak{a}_{-3}(X)|0\rangle, \mathfrak{a}_{-3}(X)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle)$
(iv) $(\mathfrak{a}_{-3}(X)|0\rangle, \mathfrak{a}_{-3}(X)|0\rangle, \mathfrak{a}_{-3}(X)|0\rangle)$.
Moreover, $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2), \mathrm{PD}(A_3)\rangle_{0,d} = -24$ *for case (i); for cases (ii) and (iii),* $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2), \mathrm{PD}(A_3)\rangle_{0,d} = -2f(d)$*; for case (iv),*

$$\begin{aligned} &\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2), \mathrm{PD}(A_3)\rangle_{0,d} \\ &= -162 - 15f(d) + 6\sum_{0<d_1<d} f(d_1) + \frac{1}{3}\sum_{0<d_1<d} f(d_1)f(d-d_1). \end{aligned}$$

These two theorems are proved by using obstruction bundles and composition laws in Sect. 3, which generalizes the earlier methods in [L-Q]. In view of our theorems, to compute all the 3-point invariants $\langle \alpha_1, \alpha_2, \alpha_3\rangle_{0,d}$ of $X^{[3]}$, it remains to determine the 2-point invariant $\langle \mathrm{PD}(\mathfrak{a}_{-3}(\ell)|0\rangle), \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)\rangle_{0,d}$. In Sect.4, using the standard $(\mathbb{C}^*)^2$-action on $X = \mathbb{P}^2$ and the virtual localization formula from [G-P], we reduce the computation of $\langle \mathrm{PD}(\mathfrak{a}_{-3}(\ell)|0\rangle), \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)\rangle_{0,d}$ to a summation over stable graphs. Even though we could not simplify this summation for a general d, we are able to calculate the summation for $d \leq 4$ by employing Mathematica. This enables us to prove the following.

Proposition 1.3. *Let* $X = \mathbb{P}^2$, *and* $\ell \subset X$ *be a line. Then, the* 2*-point genus-*0 *Gromov-Witten invariant* $\langle \mathrm{PD}(\mathfrak{a}_{-3}(\ell)|0\rangle), \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)\rangle_{0,d}$ *is equal to* -27, $27/2$, 18 *and* $27/4$ *when d is equal to* 1, 2, 3 *and* 4 *respectively.*

One of our motivations for this present work is to verify Ruan's conjecture in [Ru2] about the quantum corrections for crepant resolutions of orbifolds. The symmetric products of a smooth projective surface are global orbifolds. The Hilbert-Chow map (2.8) presents the Hilbert schemes of points on a smooth projective surface as crepant resolutions of the symmetric products of the surface. For the Hilbert scheme $(\mathbb{P}^2)^{[3]}$, our results enable us to verify Ruan's conjecture for those quantum corrections not involving $\langle \mathrm{PD}(\mathfrak{a}_{-3}(\ell)|0\rangle), \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)\rangle_{0,d}$. Since the verification involves only straight-forward computations, we omit the details.

Finally, we remark that our methods can be extended in several directions. First of all, they can be used to compute many 3-point Gromov-Witten invariants of the Hilbert scheme $(\mathbb{P}^2)^{[n]}$ for a general n. Secondly, our methods of proving Theorem 1.1 and Theorem 1.2 can be easily modified to work for an arbitrary simply connected projective surface X. In addition, the ideas of proving Proposition 1.3 can be applied to other toric surfaces. We leave the details to the interested readers.

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2. Preliminaries.

2.1. Stable maps and Gromov-Witten invariants. Let Y be a smooth projective variety. A k-pointed *stable map* to Y consists of a complete nodal curve C with k distinct ordered smooth points $p_1, \ldots, p_k$ and a morphism $\mu : C \to Y$ such that the data $(\mu, C, p_1, \ldots, p_k)$ has only finitely many automorphisms. In this case, the stable map is denoted by $[\mu : (C; p_1, \ldots, p_k) \to Y]$. For a fixed homology class $\beta \in H_2(Y; \mathbb{Z})$, let $\overline{\mathfrak{M}}_{g,k}(Y, \beta)$ be the stack parameterizing all the stable maps $[\mu : (C; p_1, \ldots, p_k) \to Y]$ such that $\mu_*[C] = \beta$ and the arithmetic genus of C is g. It is known [F-P, LT1, LT2, B-F] that $\overline{\mathfrak{M}}_{g,k}(Y, \beta)$ is a complete Deligne-Mumford stack with a projective moduli space. Moreover, it has a virtual fundamental class $[\overline{\mathfrak{M}}_{g,k}(Y, \beta)]^{\rm vir} \in A_{\mathfrak{d}}(\overline{\mathfrak{M}}_{g,k}(Y, \beta))$ where

$$\mathfrak{d} = -(K_Y \cdot \beta) + (\dim(Y) - 3)(1 - g) + k \tag{2.1}$$

is the expected complex dimension of $\overline{\mathfrak{M}}_{g,k}(Y, \beta)$, and $A_{\mathfrak{d}}(\overline{\mathfrak{M}}_{g,k}(Y, \beta))$ is the Chow group of $\mathfrak{d}$-dimensional cycles in the stack $\overline{\mathfrak{M}}_{g,k}(Y, \beta)$. The evaluation map

$$ev_k \colon \overline{\mathfrak{M}}_{g,k}(Y, \beta) \to Y^k \tag{2.2}$$

is defined by $ev_k([\mu : (C; p_1, \ldots, p_k) \to Y]) = (\mu(p_1), \ldots, \mu(p_k))$.

The Gromov-Witten invariants are defined by using the virtual fundamental class $[\overline{\mathfrak{M}}_{g,k}(Y, \beta)]^{\rm vir}$. Recall that an element $\alpha \in H^*(Y) \overset{\rm def}{=} \bigoplus_{j=0}^{2\dim_{\mathbb{C}}(Y)} H^j(Y)$ is *homogeneous* if $\alpha \in H^j(Y)$ for some j; in this case, we take $|\alpha| = j$. Let $\alpha_1, \ldots, \alpha_k \in H^*(Y)$ such that every α_i is homogeneous and

$$\sum_{i=1}^{k} |\alpha_i| = 2\mathfrak{d}. \tag{2.3}$$

Then, we have the k-point Gromov-Witten invariant defined by:

$$\langle \alpha_1, \ldots, \alpha_k \rangle_{g,\beta} = \int_{[\overline{\mathfrak{M}}_{g,k}(Y,\beta)]^{\rm vir}} ev_k^*(\alpha_1 \otimes \ldots \otimes \alpha_k). \tag{2.4}$$

Next, we summarize certain properties concerning the virtual fundamental class. To begin with, we recall that *the excess dimension* is the difference between the dimension of $\overline{\mathfrak{M}}_{g,k}(Y, \beta)$ and the expected dimension $\mathfrak{d}$ in (2.1). Let T_Y stand for the tangent bundle of Y. For $0 \le i < k$, we shall use

$$f_{k,i} : \overline{\mathfrak{M}}_{g,k}(Y, \beta) \to \overline{\mathfrak{M}}_{g,i}(Y, \beta) \tag{2.5}$$

to stand for the forgetful map obtained by forgetting the last $(k - i)$ marked points and contracting all the unstable components. It is known that $f_{k,i}$ is flat when $\beta \neq 0$ and $0 \le i < k$. The following can be found in [LT1, Beh, Get, C-K, LiJ].

Proposition 2.1. *Let $\beta \in H_2(Y; \mathbb{Z})$ and $\beta \neq 0$. Let e be the excess dimension of $\overline{\mathfrak{M}}_{g,k}(Y, \beta)$, and $\mathfrak{M} \subset \overline{\mathfrak{M}}_{g,k}(Y, \beta)$ be a closed substack. Then,*

(i) $[\overline{\mathfrak{M}}_{g,k}(Y, \beta)]^{\rm vir} = (f_{k,0})^*[\overline{\mathfrak{M}}_{g,0}(Y, \beta)]^{\rm vir}$*;*

(ii) $[\overline{\mathfrak{M}}_{g,k}(Y, \beta)]^{\rm vir}|_{\mathfrak{M}} = c_e((R^1(f_{k+1,k})_*(ev_{k+1})^*T_Y)|_{\mathfrak{M}})$ *if there exists an open substack $\mathfrak{U}$ of $\overline{\mathfrak{M}}_{g,k}(Y, \beta)$ such that $\mathfrak{M} \subset \mathfrak{U}$ (i.e, $\mathfrak{U}$ is an open neighborhood of $\mathfrak{M}$) and $(R^1(f_{k+1,k})_*(ev_{k+1})^*T_Y)|_{\mathfrak{U}}$ is a rank-e locally free sheaf over $\mathfrak{U}$.*

We also need one formula for $g = 0$ known as the composition law. Let $\{\Delta_a\}$ be a basis of $H^*(Y)$, and $\{\Delta^a\}$ be the basis of $H^*(Y)$ dual to $\{\Delta_a\}$ with respect to the intersection pairing of Y. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in H^*(Y)$ be classes of even degrees. Then the combination of (3.3) and (3.6) in [K-M] says that

$$\begin{aligned}&\langle \alpha_1\alpha_2, \alpha_3, \alpha_4\rangle_{0,\beta} + \langle \alpha_1, \alpha_2, \alpha_3\alpha_4\rangle_{0,\beta}\\ &\quad + \sum_{\beta_1+\beta_2=\beta,\, \beta_1,\beta_2\neq 0} \sum_a \langle \alpha_1, \alpha_2, \Delta_a\rangle_{0,\beta_1} \langle \Delta^a, \alpha_3, \alpha_4\rangle_{0,\beta_2}\\ &= \langle \alpha_1\alpha_3, \alpha_2, \alpha_4\rangle_{0,\beta} + \langle \alpha_1, \alpha_3, \alpha_2\alpha_4\rangle_{0,\beta}\\ &\quad + \sum_{\beta_1+\beta_2=\beta,\, \beta_1,\beta_2\neq 0} \sum_a \langle \alpha_1, \alpha_3, \Delta_a\rangle_{0,\beta_1} \langle \Delta^a, \alpha_2, \alpha_4\rangle_{0,\beta_2}. \end{aligned} \tag{2.6}$$

2.2. Basic facts about the Hilbert scheme of points on a surface. Let X be a simply connected smooth projective surface, and $X^{[n]}$ be the Hilbert scheme of points in X. An element in $X^{[n]}$ is represented by a length-n 0-dimensional closed subscheme ξ of X. For $\xi \in X^{[n]}$, let I_ξ be the corresponding sheaf of ideals. In $X^{[n]} \times X$, we have the universal codimension-2 subscheme:

$$\mathcal{Z}_n = \{(\xi, x) \subset X^{[n]} \times X \mid x \in \mathrm{Supp}(\xi)\} \subset X^{[n]} \times X. \tag{2.7}$$

Let $X^{(n)}$ be the n-th symmetric product of X. We have the Hilbert-Chow map:

$$\rho : X^{[n]} \to X^{(n)}. \tag{2.8}$$

For a subset $Y \subset X$, we define the subset $M_n(Y)$ in the Hilbert scheme $X^{[n]}$:

$$M_n(Y) = \{\xi \in X^{[n]} | \mathrm{Supp}(\xi) \text{ is a point in } Y\} \subset X^{[n]}. \tag{2.9}$$

In particular, for a fixed point $x \in X$, $M_n(x)$ is just the punctual Hilbert scheme of points on X at x. It is known that the punctual Hilbert schemes $M_n(x)$ are isomorphic for all the surfaces X and all the points $x \in X$.

Let $\xi \in X^{[n-k]}$ and $\eta \in X^{[k]}$. If $\mathrm{Supp}(\xi) \cap \mathrm{Supp}(\eta) = \emptyset$, then we use $\xi + \eta$ to represent the closed subscheme $\xi \cup \eta$ in $X^{[n]}$. Similarly, given a subvariety Y of $X^{[n-k]}$ and a point $\eta \in X^{[k]}$ such that $\left(\bigcup_{\xi \in Y} \mathrm{Supp}(\xi)\right) \cap \mathrm{Supp}(\eta) = \emptyset$, we use $Y + \eta$ to represent the subvariety in $X^{[n]}$ consisting of all the points $\xi + \eta$ with $\xi \in Y$.

Next, we review some results on homology groups of the Hilbert scheme $X^{[n]}$ due to Göttsche [Go1], Grojnowski [Gro], and Nakajima [Nak]. Their results say that the space $\mathbb{H} \overset{\text{def}}{=} \bigoplus_{n=0}^{\infty} \bigoplus_{k=0}^{4n} H_k(X^{[n]})$ is an irreducible highest weight representation of the Heisenberg algebra generated by $\mathfrak{a}_{-n}(a), n \in \mathbb{Z}, a \in H_*(X) \overset{\text{def}}{=} \bigoplus_{k=0}^{4} H_k(X)$. Moreover, $|0\rangle \overset{\text{def}}{=} 1 \in H_0(X^{[0]}; \mathbb{C}) = \mathbb{C}$ is a highest weight vector. It follows that the space $\mathbb{H}$ is a linear span of elements of the form $\mathfrak{a}_{-n_1}(a_1) \ldots \mathfrak{a}_{-n_k}(a_k)|0\rangle$ where $k \geq 0$, $n_1, \ldots, n_k > 0$, and $a_1, \ldots, a_k \in H_*(X)$. The geometric interpretation of $\mathfrak{a}_{-n_1}(a_1) \ldots \mathfrak{a}_{-n_k}(a_k)|0\rangle$ for homogeneous classes $a_1, \ldots, a_k \in H_*(X)$ can be understood as follows. For $i = 1, \ldots, k$, let $a_i \in H_{|a_i|}(X)$ be represented by a cycle X_i such that $X_1, \ldots, X_k$ are in general position. Then,

$$\mathfrak{a}_{-n_1}(a_1) \ldots \mathfrak{a}_{-n_k}(a_k)|0\rangle \in H_m(X^{[n]}) \tag{2.10}$$

where $n = \sum_{i=1}^{k} n_i$ and $m = \sum_{i=1}^{k}(2n_i - 2 + |a_i|)$. Up to a scalar, $\mathfrak{a}_{-n_1}(a_1)\ldots\mathfrak{a}_{-n_k}(a_k)|0\rangle$ is represented by the closure of the real-$\sum_{i=1}^{k}(2n_i - 2 + |a_i|)$-dimensional subset:

$$\{\xi_1 + \ldots + \xi_k \in X^{[n]} | \xi_i \in M_{n_i}(X_i), \mathrm{Supp}(\xi_i) \cap \mathrm{Supp}(\xi_j) = \emptyset \text{ for } i \neq j\} \quad (2.11)$$

where $M_{n_i}(X_i)$ is the subset of $X^{[n_i]}$ defined by (2.9).

Definition 2.2. *Let $x \in X$, and C be a real-2-dimensional submanifolds of X. Then, we define $\beta_n = \mathfrak{a}_{-2}(x)\mathfrak{a}_{-1}(x)^{n-2}|0\rangle$, $\beta_C = \mathfrak{a}_{-1}(C)\mathfrak{a}_{-1}(x)^{n-1}|0\rangle$, and*

$$B_n = \frac{1}{(n-2)!}\mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(X)^{n-2}|0\rangle, \quad D_C = \frac{1}{(n-1)!}\mathfrak{a}_{-1}(C)\mathfrak{a}_{-1}(X)^{n-1}|0\rangle.$$

Lemma 2.3. *Let x and ℓ be a point and a line in $X = \mathbb{P}^2$ respectively. Then,*

(i) *a basis of $H_2(X^{[3]};\mathbb{Z})$ consists of β_3 and β_ℓ;*
(ii) *a basis of $H_4(X^{[3]})$ consists of the five homology classes $\mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(x)^2|0\rangle$, $\mathfrak{a}_{-2}(\ell)\mathfrak{a}_{-1}(x)|0\rangle$, $\mathfrak{a}_{-1}(\ell)^2\mathfrak{a}_{-1}(x)|0\rangle$, $\mathfrak{a}_{-1}(\ell)\mathfrak{a}_{-2}(x)|0\rangle$, and $\mathfrak{a}_{-3}(x)|0\rangle$;*
(iii) *a basis of $H_6(X^{[3]})$ consists of the classes $\mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(x)|0\rangle$, $\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(x)|0\rangle$, $\mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)\mathfrak{a}_{-1}(x)|0\rangle$, $\mathfrak{a}_{-3}(\ell)|0\rangle$, $\mathfrak{a}_{-2}(\ell)\mathfrak{a}_{-1}(\ell)|0\rangle$, and $\mathfrak{a}_{-1}(\ell)^3|0\rangle$;*
(iv) *a basis of $H_8(X^{[3]})$ consists of the five classes $\mathfrak{a}_{-3}(X)|0\rangle$, $\mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle$, $\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle$, $\mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)^2|0\rangle$, and $\mathfrak{a}_{-1}(X)^2\mathfrak{a}_{-1}(x)|0\rangle$;*
(v) *a basis of $H_{10}(X^{[3]};\mathbb{Z})$ consists of the divisors B_3 and D_ℓ.*

Proof. The proof of (i) and (v) was contained in the proof of the Theorem 4.1 in [LQZ], while the rest statements follow by exploiting (2.10). □

Definition 2.4. *For $X = \mathbb{P}^2$ and $i = 2, 4, 6, 8$ and 10, let $\mathfrak{B}_i$ stand for the linear basis of the homology group $H_i(X^{[3]})$ given in Lemma 2.3.*

Fix $p \in X^{[3]}$. Then a basis $\{\Delta_a\}$ of $H^*(X^{[3]})$ is given by the Poincaré duals of

$$[p],\ \mathfrak{B}_i\,(i = 2, 4, 6, 8, 10),\ [X^{[3]}] \quad (2.12)$$

where $[p] = \mathfrak{a}_{-1}(x)^3|0\rangle \in H_0(X^{[3]})$ and $[X^{[3]}] = 1/6\ \mathfrak{a}_{-1}(X)^3|0\rangle \in H_{12}(X^{[3]})$ are the homology classes corresponding to p and $X^{[3]}$ respectively.

The following is the main result proved in [L-Q].

Lemma 2.5. *Let $d \geq 1$, and x and ℓ be a point and a line in $X = \mathbb{P}^2$ respectively.*

(i) *If α stands for the Poincaré duals of the homology classes $\mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(x)^2|0\rangle$, $\mathfrak{a}_{-1}(\ell)^2\mathfrak{a}_{-1}(x)|0\rangle$, $\mathfrak{a}_{-1}(\ell)\mathfrak{a}_{-2}(x)|0\rangle$, and $\mathfrak{a}_{-3}(x)|0\rangle$, then $\langle\alpha\rangle_{0,d\beta_n} = 0$.*

(ii) *If α is the Poincaré dual of $\mathfrak{a}_{-2}(\ell)\mathfrak{a}_{-1}(x)|0\rangle$, then $\langle\alpha\rangle_{0,d\beta_n} = 2(K_X \cdot \ell)/d^2$.*

2.3. Curves from the punctual Hilbert scheme.

Lemma 2.6. *Fix $n \geq 2$. Let $\mathrm{Hilb}^n(\mathbb{C}^2, 0)$ be the punctual Hilbert scheme of points on $\mathbb{C}^2$ at the origin, and u, v be the coordinates of $\mathbb{C}^2$. Then, $H_2(\mathrm{Hilb}^n(\mathbb{C}^2, 0);\mathbb{Z}) \cong \mathbb{Z}$. Moreover, a generator of $H_2(\mathrm{Hilb}^n(\mathbb{C}^2, 0);\mathbb{Z})$ is given by*

$$\sigma_n = \{(\lambda u + \mu v^{n-1}, u^2, uv, v^n) \mid \lambda, \mu \in \mathbb{C} \text{ with } |\lambda| + |\mu| \neq 0\}. \quad (2.13)$$

Proof. The first statement was proved in [E-S]. To prove the second statement, following [E-S], take a $\mathbb{C}^*$-action on $\mathbb{C}^2$ given by $t \cdot (u, v) = (t^{-\alpha}u, t^{-\beta}v)$ with $\beta \gg \alpha$.

For $\xi \in \mathrm{Hilb}^n(\mathbb{C}^2;0)$, we use the ideal $I_\xi \subset \mathbb{C}[u,v]$ to represent ξ. Then the $\mathbb{C}^*$-invariant ideal in $\mathbb{C}[u,v]$ corresponding to a generator σ_n of $H_2(\mathrm{Hilb}^n(\mathbb{C}^2,0);\mathbb{Z})$ is (v^{n-1}, uv, u^2). Therefore σ_n is the closure of the cell

$$\begin{aligned}&\{I \in \mathbb{C}[u,v] \mid \ell(\mathbb{C}[u,v]/I) = n, \quad \lim_{t\to 0}(t\cdot I) = (v^{n-1}, uv, u^2)\}\\ = &\{(v^{n-1}+au, uv, u^2) \mid a \in \mathbb{C}\} \cong \mathbb{C}.\end{aligned}$$

Finally, notice that if $a \neq 0$, then $(v^{n-1}+au, uv, u^2) = (v^{n-1}+au, v^n)$. So letting $a \to \infty$, we see that the ideal (u, v^n) is also contained in σ_n. Thus,

$$\sigma_n = \{(v^{n-1}+au, uv, u^2) \mid a \in \mathbb{C}\} \cup \{(u, v^n)\}$$

which is the same as $\{(\lambda u + \mu v^{n-1}, u^2, uv, v^n) \mid \lambda, \mu \in \mathbb{C} \text{ with } |\lambda| + |\mu| \neq 0\}$. □

Let $R = \mathcal{O}_{\mathbb{C}^2,0}$ be the local ring of $\mathbb{C}^2$ at the origin, and $\mathfrak{m} = (u,v)$ be the maximal ideal of R. Let $\eta \in \mathrm{Hilb}^n(\mathbb{C}^2,0)$. It is known that there exists an embedding

$$\tau : \mathrm{Hilb}^n(\mathbb{C}^2,0) \to \mathrm{Grass}(R/\mathfrak{m}^n, n)$$

where $R/\mathfrak{m}^n$ is considered as a $\mathbb{C}$-vector space of dimension $\binom{n+1}{2}$, and τ maps an element $\eta \in \mathrm{Hilb}^n(\mathbb{C}^2,0)$ to the n-dimensional quotient of $R/\mathfrak{m}^n$ in the exact sequence

$$0 \to I_{\eta,0}/\mathfrak{m}^n \to R/\mathfrak{m}^n \to R/I_{\eta,0} = \mathcal{O}_{\eta,0} \to 0.$$

Let $\mathfrak{p} : \mathbb{G} \to \mathbb{P}^{N-1}$ be the Plücker embedding where $N = \binom{n+1}{2}\left(\binom{n+1}{2} - n\right)$.

Lemma 2.7. *Identify $M_n(x)$ with $\mathrm{Hilb}^n(\mathbb{C}^2,0)$, and regard σ_n as a curve in $M_n(x) \subset X^{[n]}$. Then as a curve in $X^{[n]}$, σ_n is homologous to β_n.*

Proof. According to the results in Sect. 3 of [LQZ], it suffices to show that the image $(\mathfrak{p}\circ\tau)(\sigma_n)$ is a line. Fix a basis for the $\mathbb{C}$-vector space $R/\mathfrak{m}^n$:

$$\overline{1}, \overline{u}, \overline{u}^2, \overline{u}\,\overline{v}, \overline{u}^3, \overline{u}^2\,\overline{v}, \overline{u}\,\overline{v}^2, \ldots, \overline{u}^{n-1}, \overline{u}^{n-2}\overline{v}, \ldots, \overline{u}\,\overline{v}^{n-2}, \overline{v}, \ldots, \overline{v}^{n-1}.$$

Note the special ordering of this basis. Recall from (2.13) that for any $\eta \in \sigma_n \subset \mathrm{Hilb}^n(\mathbb{C}^2,0)$, $I_{\eta,0} = (\lambda u + \mu v^{n-1}, u^2, uv, v^n)$ for some $\lambda, \mu \in \mathbb{C}$ with $|\lambda| + |\mu| \neq 0$. So a basis for the subspace $I_{\eta,0}/\mathfrak{m}^n \subset R/\mathfrak{m}^n$ can be chosen as

$$\lambda\overline{u} + \mu\overline{v}^{n-1}, \overline{u}^2, \overline{u}\,\overline{v}, \overline{u}^3, \overline{u}^2\overline{v}, \overline{u}\,\overline{v}^2, \ldots, \overline{u}^{n-1}, \overline{u}^{n-2}\overline{v}, \ldots, \overline{u}\,\overline{v}^{n-2},$$

and the matrix representation of $I_{\eta,0}/\mathfrak{m}^n$ is given by the $\binom{n}{2} \times \binom{n+1}{2}$-matrix:

$$\begin{bmatrix} 0 & \lambda & 0 & \ldots & 0 & 0 & \ldots & 0 & \mu \\ 0 & 0 & 1 & \ldots & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 & \ldots & 0 & 0 \end{bmatrix}. \tag{2.14}$$

Thus, $(\mathfrak{p}\circ\tau)(\eta) = [0, \ldots, 0, \lambda, 0, \ldots, 0, \mu, 0, \ldots, 0]$ where the positions of λ and μ are independent of $\eta \in \sigma_n$. So the image $(\mathfrak{p}\circ\tau)(\sigma_n)$ is a line. □

Note that the flat limits of the elements $(\lambda u + v, v^n)$, $\lambda \in \mathbb{C}^*$ in $\mathrm{Hilb}^n(\mathbb{C}^2,0)$ as $\lambda \to 0$ and $\lambda \to \infty$ are equal to (v, u^n) and (u, v^n) respectively. So in the punctual Hilbert scheme $\mathrm{Hilb}^n(\mathbb{C}^2,0)$, we have the projective curve:

$$\tilde{\sigma}_n = \{(\lambda u + v, v^n) \mid \lambda \in \mathbb{C}^*\} \cup \{(v, u^n),\ (u, v^n)\}. \tag{2.15}$$

Lemma 2.8. *As a curve in* $X^{[n]}$, $\tilde{\sigma}_n$ *is homologous to* $\binom{n}{2}\sigma_n$.

Proof. It suffices to show that $\tilde{\sigma}_n \sim \binom{n}{2}\sigma_n$ in $H_2(\mathrm{Hilb}^n(\mathbb{C}^2,0);\mathbb{Z})$. By (2.15), if $\eta \in \tilde{\sigma}_n - \{(v,u^n),\ (u,v^n)\}$, then a basis for the subspace $I_{\eta,0}/\mathfrak{m}^n \subset R/\mathfrak{m}^n$ is

$$\lambda\overline{u}+\overline{v}, \lambda\overline{u}^2+\overline{u}\,\overline{v}, \lambda\overline{u}\,\overline{v}+\overline{v}^2, \ldots,$$
$$\lambda\overline{u}^{n-1}+\overline{u}^{n-2}\overline{v}, \lambda\overline{u}^{n-2}\overline{v}+\overline{u}^{n-3}\overline{v}^2, \ldots, \lambda\overline{u}\,\overline{v}^{n-2}+\overline{v}^{n-1}.$$

As in the proof of Lemma 2.7, we see that the degree of $(\mathfrak{p}\circ\tau)(\tilde{\sigma}_n - \{(v,u^n),\ (u,v^n)\})$ is $\binom{n}{2}$. So $(\mathfrak{p}\circ\tau)(\tilde{\sigma}_n)$ has degree $\binom{n}{2}$. By Lemma 2.6, there exists an integer d such that $\tilde{\sigma}_n \sim d\sigma_n$ in $H_2(\mathrm{Hilb}^n(\mathbb{C}^2,0);\mathbb{Z})$. Since $(\mathfrak{p}\circ\tau)(\sigma_n)$ is a line, $d = \binom{n}{2}$. ☐

3. 3-point genus-0 Gromov-Witten invariants of $(\mathbb{P}^2)^{[3]}$.

Let $X = \mathbb{P}^2$ and $d \ge 1$. For simplicity, we shall use $\langle \alpha_1, \ldots, \alpha_k\rangle_{0,d}$ to stand for $\langle \alpha_1, \ldots, \alpha_k\rangle_{0,d\beta_3}$. Our goal is to compute the 3-point Gromov-Witten invariants $\langle \alpha_1,\alpha_2,\alpha_3\rangle_{0,d}$ of $X^{[3]}$. Recall from Lemma 2.5 that the 1-point Gromov-Witten invariants $\langle\alpha_1\rangle_{0,d}$ of $X^{[3]}$ have been calculated. Since the expected complex dimension of the stack $\overline{\mathfrak{M}}_{0,3}(X^{[3]}, d\beta_3)$ is 6, it remains to compute the 2-point Gromov-Witten invariants $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2)\rangle_{0,d}$ when A_1 runs over the basis $\mathfrak{B}_6$ of $H_6(X^{[3]})$ in Lemma 2.3 (iii) and A_2 runs over the basis $\mathfrak{B}_8$ of $H_8(X^{[3]})$ in Lemma 2.3 (iv), and $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2), \mathrm{PD}(A_3)\rangle_{0,d}$ when A_1, A_2, A_3 run over the basis $\mathfrak{B}_8$.

3.1. $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2)\rangle_{0,d}$ with $A_1 \in \mathfrak{B}_6$ and $A_2 \in \mathfrak{B}_8$.

Lemma 3.1. *The* 2*-point Gromov-Witten invariants* $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2)\rangle_{0,d}$ *are equal to zero for the following pairs of* $(A_1, A_2) \in \mathfrak{B}_6 \times \mathfrak{B}_8$*:*

$$(\mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(x)|0\rangle, \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle),\ (\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(x)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle),$$
$$(\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(x)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)^2|0\rangle),\ (\mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)\mathfrak{a}_{-1}(x)|0\rangle, \mathfrak{a}_{-3}(X)),$$
$$(\mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)\mathfrak{a}_{-1}(x)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)^2|0\rangle),$$
$$(\mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)\mathfrak{a}_{-1}(x)|0\rangle, \mathfrak{a}_{-1}(X)^2\mathfrak{a}_{-1}(x)|0\rangle),\ (\mathfrak{a}_{-3}(\ell)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)^2|0\rangle),$$
$$(\mathfrak{a}_{-3}(\ell)|0\rangle, \mathfrak{a}_{-1}(X)^2\mathfrak{a}_{-1}(x)|0\rangle),\ (\mathfrak{a}_{-2}(\ell)\mathfrak{a}_{-1}(\ell)|0\rangle, \mathfrak{a}_{-1}(X)^2\mathfrak{a}_{-1}(x)|0\rangle),$$
$$(\mathfrak{a}_{-1}(\ell)^3|0\rangle, \mathfrak{a}_{-3}(X)),\ (\mathfrak{a}_{-1}(\ell)^3|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle),$$
$$(\mathfrak{a}_{-1}(\ell)^3|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)^2|0\rangle),\ (\mathfrak{a}_{-1}(\ell)^3|0\rangle, \mathfrak{a}_{-1}(X)^2\mathfrak{a}_{-1}(x)|0\rangle).$$

Proof. These follow from similar geometric arguments. For instance, let us show that $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2)\rangle_{0,d} = 0$ when $A_1 = \mathfrak{a}_{-1}(\ell)^3|0\rangle$ and $A_2 = \mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)^2|0\rangle$.

Choose five lines $\ell_1, \ldots, \ell_5 \subset X = \mathbb{P}^2$ in general position. By (2.11), we see that up to a scalar, A_1 is represented by the closure of the subset

$$\{x_1+x_2+x_3|\ x_1,x_2,x_3 \text{ are distinct and } x_i \in \ell_i \text{ for each } i\}. \tag{3.1}$$

Similarly, A_2 is represented by the closure of the subset

$$\{x+x_4+x_5|\ x,x_4,x_5 \text{ are distinct and } x_i \in \ell_i \text{ for each } i\}. \tag{3.2}$$

Let $\mathfrak{M}$ be the substack of $\overline{\mathfrak{M}}_{0,2}(X^{[3]}, d\beta_3)$ parametrizing all the stable maps $[\mu : (C;p_1,p_2) \to X^{[3]}]$ with $\mu(p_1) \in A_1$ and $\mu(p_2) \in A_2$. We claim that $\mathfrak{M} = \emptyset$. Indeed, assume $[\mu : (C;p_1,p_2) \to X^{[3]}]$ is an object of $\mathfrak{M}$. On one hand, by (3.1), $\rho(\mu(C)) = 2(\ell_i \cap \ell_j) + x_k$ where ρ is the Hilbert-Chow map (2.8), $\{i,j,k\}$ is a permutation of $\{1,2,3\}$, and $x_k \in \ell_k$. On the other hand, by (3.2), we obtain

$$\rho(\mu(C)) = 2(\ell_4 \cap \ell_5) + x$$

for some $x \in X$, or $\rho(\mu(C)) = 2x_i + x_j$ where $\{i, j\}$ is a permutation of $\{4, 5\}$, $x_i \in \ell_i$, and $x_j \in \ell_j$. Since the lines $\ell_1, \ldots, \ell_5 \subset X = \mathbb{P}^2$ are in general position, such $\rho(\mu(C))$ does not exist. So $\mathfrak{M} = \emptyset$. Hence $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2) \rangle_{0,d} = 0$. □

LEMMA 3.2. *The* 2*-point Gromov-Witten invariants* $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2) \rangle_{0,d}$ *are equal to zero for the following pairs of* $(A_1, A_2) \in \mathfrak{B}_6 \times \mathfrak{B}_8$*:*

$$\begin{gathered}
(\mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(x)|0\rangle, \mathfrak{a}_{-3}(X)|0\rangle),\ (\mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(x)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)^2|0\rangle),\\
(\mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(x)|0\rangle, \mathfrak{a}_{-1}(X)^2\mathfrak{a}_{-1}(x)|0\rangle),\ (\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(x)|0\rangle, \mathfrak{a}_{-3}(X)|0\rangle),\\
(\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(x)|0\rangle, \mathfrak{a}_{-1}(X)^2\mathfrak{a}_{-1}(x)|0\rangle),\ (\mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)\mathfrak{a}_{-1}(x)|0\rangle, \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle),\\
(\mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)\mathfrak{a}_{-1}(x)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle),\ (\mathfrak{a}_{-3}(\ell)|0\rangle, \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle),\\
(\mathfrak{a}_{-3}(\ell)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle),\ (\mathfrak{a}_{-2}(\ell)\mathfrak{a}_{-1}(\ell)|0\rangle, \mathfrak{a}_{-3}(X)|0\rangle),\\
(\mathfrak{a}_{-2}(\ell)\mathfrak{a}_{-1}(\ell)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)^2|0\rangle),\ (\mathfrak{a}_{-1}(\ell)^3|0\rangle, \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle).
\end{gathered}$$

Proof. These invariants are equal to certain genus-0 Gromov-Witten invariants of a K3 surface. So our lemma follows from the fact that all the genus-0 Gromov-Witten invariants of a K3 surface are equal to zero. For instance, let us show that $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2) \rangle_{0,d} = 0$ when $A_1 = \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(x)|0\rangle$ and $A_2 = \mathfrak{a}_{-3}(X)|0\rangle$.

Fix $x \in X$, and a small analytic open subset U of X such that $x \in U$. We may assume that U is independent of X. Note that for a stable map $[\mu : (C; p_1, p_2) \to X^{[3]}] \in \overline{\mathfrak{M}}_{0,2}(X^{[3]}, d\beta_3)$, either $\mu(C) \subset U^{[3]}$ or $\mu(C) \cap U^{[3]} = \emptyset$. So the analytic open substack $\mathfrak{U} \subset \overline{\mathfrak{M}}_{0,2}(X^{[3]}, d\beta_3)$ parametrizing all stable maps $[\mu\colon (C; p_1, p_2) \to X^{[3]}]$ with $\mu(C) \subset U^{[3]}$ depends only on U, and is independent of X.

Let $\mathfrak{M}$ be the substack of $\overline{\mathfrak{M}}_{0,2}(X^{[3]}, d\beta_3)$ parametrizing all the stable maps $[\mu : (C; p_1, p_2) \to X^{[3]}]$ such that $\mu(p_1) \in A_1$ and $\mu(p_2) \in A_2$. Note from the descriptions of A_1 and A_2 that if $[\mu : (C; p_1, p_2) \to X^{[3]}] \in \mathfrak{M}$, then $\mu(C) \subset M_3(x) \subset U^{[3]}$. So $\mathfrak{M} \subset \mathfrak{U}$. In fact, $\mathfrak{M}$ parametrizes all the stable maps $[\mu : (C; p_1, p_2) \to X^{[3]}] \in \mathfrak{U}$ with $\mu(C) \subset M_3(x) \subset U^{[3]}$. So $\mathfrak{M}$ is also independent of X.

In summary, we showed that $\mathfrak{M} \subset \mathfrak{U}$ where $\mathfrak{U}$ is analytic open in $\overline{\mathfrak{M}}_{0,2}(X^{[3]}, d\beta_3)$, and $\mathfrak{M}$ and $\mathfrak{U}$ are independent of X. It follows from the constructions of the virtual fundamental class (see [LT2, LT3, Ru1]) that the restriction $[\overline{\mathfrak{M}}_{0,2}(X^{[3]}, d\beta_3)]^{\mathrm{vir}}|_{\mathfrak{M}}$ is independent of the smooth surface X. So we have $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2) \rangle_{0,d} = \langle \mathrm{PD}(A_1'), \mathrm{PD}(A_2') \rangle_{0,d}$ where $A_1' = \mathfrak{a}_{-2}(X')\mathfrak{a}_{-1}(x')|0\rangle$, $A_2' = \mathfrak{a}_{-3}(X')|0\rangle$, $x' \in X'$, and X' is a K3 surface. Therefore, we conclude that $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2) \rangle_{0,d} = 0$. □

To compute other 2-point invariants $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2) \rangle_{0,d}$, we recall from [L-Q] some results concerning obstruction bundles and virtual fundamental classes. Fix $n \ge 2$. Let $B_* = \{\xi \in X^{[n]} \,|\, |\mathrm{Supp}(\xi)| = n - 1\}$ and $X^{(n)}_{s*} = \rho(B_*)$ where ρ is the Hilbert-Chow map. Let $j_2 : X^{(n)}_{s*} \to X$ be the morphism defined by sending $2x + x_3 + \ldots + x_n$ to x. For $k \ge 0$, let $\mathfrak{U}_k$ be the open substack of $\overline{\mathfrak{M}}_{0,k}(X^{[n]}, d\beta_n)$ parametrizing stable maps $[\mu\colon (C; p_1, \ldots, p_k) \to X^{[n]}]$ such that $\mu(C) \subset B_*$. For $k \ge 1$, note that $\mathfrak{U}_k = f_{k,0}^{-1}(\mathfrak{U}_0)$. Put $\tilde{ev}_k = ev_k|_{\mathfrak{U}_k}$ and $\tilde{f}_{k,0} = f_{k,0}|_{\mathfrak{U}_k}$. Then we can regard $\tilde{ev}_k$ and $\tilde{f}_{k,0}$ as morphisms from $\mathfrak{U}_k$ to $(B_*)^k$ and $\mathfrak{U}_0$ respectively. In addition, there exist morphisms ϕ and j_1 forming a commutative diagram:

$$\begin{tikzcd}
\mathfrak{U}_1 \arrow[r, "\widetilde{ev}_1"] \arrow[d, "\tilde{f}_{1,0}"] & B_* \arrow[r, phantom, "\overset{j_1}{\cong}"] \arrow[d, "\rho"] & \mathbb{P}(j_2^* T_X^*) \arrow[d, "\pi"] & \\
\mathfrak{U}_0 \arrow[r, "\phi"] & \rho(B_*) \arrow[r, phantom, "="] & X^{(n)}_{s*} \arrow[r, "j_2"] & X
\end{tikzcd} \tag{3.3}$$

where $\pi \colon \mathbb{P}(j_2^* T_X^*) \to X_{s*}^{(n)}$ is the natural projection of the $\mathbb{P}^1$-bundle. By the Lemma 3.1 in [L-Q], the restriction of $R^1(f_{1,0})_*(ev_1^* T_{X^{[n]}})$ to $\mathfrak{U}_0$ is a locally free sheaf of rank $(2d-1)$. Since the excess dimension of $\mathfrak{U}_0$ is $(2d-1)$, Proposition 2.1 implies that if $\mathfrak{M}$ is a closed substack of $\overline{\mathfrak{M}}_{0,k}(X^{[n]}, d\beta_n)$ contained in $\mathfrak{U}_k$, then

$$[\overline{\mathfrak{M}}_{0,k}(X^{[n]}, d\beta_n)]^{\text{vir}}|_{\mathfrak{M}} = \Big\{ \tilde{f}_{k,0}^* \big(c_{2d-1}(R^1(f_{1,0})_*(ev_1)^* T_{X^{[n]}})|_{f_{k,0}(\mathfrak{M})} \big) \Big\} |_{\mathfrak{M}}. \quad (3.4)$$

The following summerizes the formula (32), Lemma 3.2 and Remark 3.1 in [L-Q].

Lemma 3.3.

(i) $\mathcal{O}_{B_*}(B_*) \cong j_1^* \mathcal{O}_{\mathbb{P}(j_2^* T_X^*)}(-2)$.

(ii) *Let $\mathcal{V}$ denote the restriction of $R^1(f_{1,0})_*(ev_1)^* T_{X^{[n]}}$ to $\mathfrak{U}_0$. Then, the locally free sheaf $\mathcal{V}$ sits in the exact sequence*

$$0 \to (j_2 \circ \phi)^* \mathcal{O}_X(-K_X) \to \mathcal{V} \to \mathcal{E} \to 0$$

where $\mathcal{E} = R^1(\tilde{f}_{1,0})_*(j_1 \circ \widetilde{ev}_1)^*((j_2 \circ \pi)^* T_X \otimes \mathcal{O}_{\mathbb{P}(j_2^* T_X^*)}(-1))$.

(iii) *Over $\phi^{-1}(2x_2 + x_3 + \ldots + x_n) \cong \overline{\mathfrak{M}}_{0,0}(\mathbb{P}^1, d[\mathbb{P}^1])$ where $x_2, \ldots, x_n$ are distinct points in X, there is an isomorphism of locally free sheaves:*

$$\mathcal{E}|_{\phi^{-1}(2x_2+x_3+\ldots+x_n)} \cong R^1(f_{1,0})_*(ev_1)^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)).$$

Next, using Lemma 3.3, we compute other 2-point Gromov-Witten invariants.

Lemma 3.4. *Let $X = \mathbb{P}^2$ and $d \geq 1$. Then,*

(i) *$\langle \text{PD}(A_1), \text{PD}(A_2) \rangle_{0,d} = 0$ for the two choices of (A_1, A_2):*

$$(\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(x)|0\rangle, \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle), \ (\mathfrak{a}_{-2}(\ell)\mathfrak{a}_{-1}(\ell)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle);$$

(ii) *$\langle \text{PD}(A_1), \text{PD}(A_2) \rangle_{0,d} = -4(K_X \cdot \ell)/d$ for the two choices of (A_1, A_2):*

$$(\mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(x)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle), \ (\mathfrak{a}_{-2}(\ell)\mathfrak{a}_{-1}(\ell)|0\rangle, \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle).$$

Proof. (i) Since the proofs for the two choices of (A_1, A_2) are similar, we only prove $\langle \text{PD}(A_1), \text{PD}(A_2) \rangle_{0,d} = 0$ for $A_1 = \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(x)|0\rangle$ and $A_2 = \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle$. Fix a point x and a line ℓ in $X = \mathbb{P}^2$ such that $x \notin \ell$. By (2.11), we see that up to a scalar, A_1 is represented by the closure of the subset

$$\{x' + \xi | \ \xi \in M_2(x) \text{ and } x' \neq x\}. \quad (3.5)$$

Similarly, A_2 is represented by the closure of the subset

$$\{\xi + x_1 | \ x_1 \in \ell, \xi \in M_2(x_2) \text{ for some } x_2 \notin \ell\}. \quad (3.6)$$

Working with algebraic cycles instead of cohomology classes, we have

$$\langle \text{PD}(A_1), \text{PD}(A_2) \rangle_{0,d} = [\overline{\mathfrak{M}}_{0,2}(X^{[3]}, d\beta_3)]^{\text{vir}} \cdot ev_2^*[A_1 \times A_2]. \quad (3.7)$$

Note that $ev_2^*[A_1 \times A_2]$ is an algebraic cycle supported in $ev_2^{-1}(A_1 \times A_2)$. By (3.5) and (3.6), $ev_2^{-1}(A_1 \times A_2)$ parametrizes all the stable maps $[\mu : (C; p_1, p_2) \to X^{[3]}]$ satisfying $\rho(\mu(C)) \in 2x + \ell$. In particular, $ev_2^{-1}(A_1 \times A_2) \subset \mathfrak{U}_2$. Applying (3.4) to $\mathfrak{M} = ev_2^{-1}(A_1 \times A_2)$ and combining with Lemma 3.3 (ii), we obtain

$$\begin{aligned} [\overline{\mathfrak{M}}_{0,2}(X^{[3]}, d\beta_3)]^{\text{vir}}|_{\mathfrak{M}} &= \Big\{ \tilde{f}_{2,0}^* \big(c_{2d-1}(R^1(f_{1,0})_*(ev_1)^* T_{X^{[n]}})|_{f_{2,0}(\mathfrak{M})} \big) \Big\} |_{\mathfrak{M}} \\ &= \Big\{ \tilde{f}_{2,0}^* \big((j_2 \circ \phi)^*(-K_X) \cdot c_{2d-2}(\mathcal{E})|_{f_{2,0}(\mathfrak{M})} \big) \Big\} |_{\mathfrak{M}}. \quad (3.8) \end{aligned}$$

Now $(j_2 \circ \phi)^*(-K_X) = 3(j_2 \circ \phi)^*[\ell']$ where the line ℓ' in $X = \mathbb{P}^2$ is chosen not to contain the fixed point x. We have $(j_2 \circ \phi)^{-1}(\ell') \cap f_{2,0}(\mathfrak{M}) = \emptyset$. Therefore, $(j_2 \circ \phi)^*(-K_X)|_{f_{2,0}(\mathfrak{M})} = 0$. By (3.8), $[\overline{\mathfrak{M}}_{0,2}(X^{[3]}, d\beta_3)]^{\rm vir}|_{\mathfrak{M}} = 0$. Since $ev_2^*[A_1 \times A_2]$ is supported in $\mathfrak{M} = ev_2^{-1}(A_1 \times A_2)$, we see from (3.7) that $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2)\rangle_{0,d} = 0$.

(ii) Again, the proofs for the two choices of (A_1, A_2) are similar. So we only prove $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2)\rangle_{0,d} = -4(K_X \cdot \ell)/d$ for $A_1 = \mathfrak{a}_{-2}(\ell)\mathfrak{a}_{-1}(\ell)|0\rangle$ and $A_2 = \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle$. We follow the argument for the Lemma 3.3 (ii) in [L-Q].

Fix three lines $\ell_1, \ell_2, \ell_3 \subset X = \mathbb{P}^2$ in general position. Then A_1 is represented by the closure of the subset $\{\xi + x |\ \xi \in M_2(\ell_1), x \in \ell_2, x \notin |\mathrm{Supp}(\xi)|\}$. Similarly, A_2 is represented by the closure of the subset

$$\{\xi + x |\ \xi \in M_2(X), x \in \ell_3, x \notin |\mathrm{Supp}(\xi)|\}.$$

So $ev_2^{-1}(A_1 \times A_2)$ parametrizes all the stable maps $[\mu : (C; p_1, p_2) \to X^{[3]}]$ satisfying $\rho(\mu(C)) \in 2\ell_1 + (\ell_2 \cap \ell_3) \subset B_*$, and $ev_2^*[A_1 \times A_2]$ is a cycle in $ev_2^{-1}(A_1 \times A_2) \subset \mathfrak{U}_2$. As in (3.7) and (3.8), we see that $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2)\rangle_{0,d}$ is equal to

$$\tilde{f}_{2,0}^*\big((j_2 \circ \phi)^*(-K_X) \cdot c_{2d-2}(\mathcal{E})\big)\ ev_2^*[A_1 \times A_2].$$

Since $\tilde{f}_{2,0}^*\big((j_2 \circ \phi)^*(-K_X) \cdot c_{2d-2}(\mathcal{E})\big)$ is supported in $\mathfrak{U}_2$, recalling the definition of $\tilde{ev}_2$ from the paragraph containing (3.3), we see that $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2)\rangle_{0,d}$ equals

$$\tilde{f}_{2,0}^*\big((j_2 \circ \phi)^*(-K_X) \cdot c_{2d-2}(\mathcal{E})\big)\ \tilde{ev}_2^*\big(([A_1][B_*]) \times ([A_2][B_*])\big).$$

Now, $[A_i][B_*] = [A_i \cap B_*]c_1(\mathcal{O}_{B_*}(B_*))$. Let $\mathbb{D}$ stand for the first Chern class of the tautological line bundle over $B_* \cong \mathbb{P}(j_2^* T_X^*)$. Then we obtain from Lemma 3.3 (i) that the invariant $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2)\rangle_{0,d}$ is equal to

$$4\tilde{f}_{2,0}^*\big((j_2 \circ \phi)^*(-K_X) \cdot c_{2d-2}(\mathcal{E})\big) \cdot \tilde{ev}_2^*\big(([A_1 \cap B_*]\mathbb{D}) \times ([A_2 \cap B_*]\mathbb{D})\big). \tag{3.9}$$

Fix a line ℓ such that $\ell_1, \ell_2, \ell_3, \ell$ are in general position. We claim that

$$\tilde{f}_{2,0}^*(j_2 \circ \phi)^*[\ell] \cdot \tilde{ev}_2^*\big(([A_1 \cap B_*]\mathbb{D}) \times ([A_2 \cap B_*]\mathbb{D})\big) = [\tilde{ev}_2^{-1}(\xi_1 \times \xi_2)] \tag{3.10}$$

where ξ_1 and ξ_2 are two fixed points in $M_2(x_1) + x_2$ with $\{x_1\} = \ell_1 \cap \ell$, and $\{x_2\} = \ell_2 \cap \ell_3$. To see this, let $\tilde{e}_1$ and $\tilde{e}_2$ be the restrictions to $\mathfrak{U}_2$ of the two evaluation maps from $\overline{\mathfrak{M}}_{0,2}(X^{[3]}, d\beta_3)$ to $X^{[3]}$. We regard $\tilde{e}_1$ and $\tilde{e}_2$ as morphisms from $\mathfrak{U}_2$ to B_*. Then, $\tilde{ev}_2 = \tilde{e}_1 \times \tilde{e}_2$ and $\phi \circ \tilde{f}_{2,0} = \rho \circ \tilde{e}_1$. So

$$\begin{aligned} &\tilde{f}_{2,0}^*(j_2 \circ \phi)^*[\ell] \cdot \tilde{ev}_2^*\big(([A_1 \cap B_*]\mathbb{D}) \times ([A_2 \cap B_*]\mathbb{D})\big) \\ &= \tilde{f}_{2,0}^*(j_2 \circ \phi)^*[\ell] \cdot \tilde{e}_1^*([A_1 \cap B_*]\mathbb{D}) \cdot \tilde{e}_2^*([A_2 \cap B_*]\mathbb{D}) \\ &= \tilde{e}_1^*((j_2 \circ \rho)^*[\ell] \cdot [A_1 \cap B_*] \cdot \mathbb{D}) \cdot \tilde{e}_2^*([A_2 \cap B_*]\mathbb{D}). \end{aligned}$$

Now the cycle $(j_2 \circ \rho)^*[\ell] \cdot [A_1 \cap B_*] \cdot \mathbb{D}$ is represented by $\eta_1 + \ell_2$ where η_1 is a fixed point in $M_2(x_1)$. So $\tilde{e}_1^*((j_2 \circ \rho)^*[\ell] \cdot [A_1 \cap B_*] \cdot \mathbb{D})$ is represented by the substack $\mathfrak{M}_2$ of $\overline{\mathfrak{M}}_{0,2}(X^{[3]}, d\beta_3)$ parametrizing all the stable maps $[\mu : (C; p_1, p_2) \to X^{[3]}]$ such that $\mu(C) = M_2(x_1) + x$ for some $x \in \ell_2$ and $\mu(p_1) = \eta_1 + x$. It follows that

$$\begin{aligned} &\tilde{f}_{2,0}^*(j_2 \circ \phi)^*[\ell] \cdot \tilde{ev}_2^*\big(([A_1 \cap B_*]\mathbb{D}) \times ([A_2 \cap B_*]\mathbb{D})\big) \\ &= [\mathfrak{M}_2] \cdot \tilde{e}_2^*([A_2 \cap B_*]\mathbb{D}) = [\tilde{ev}_2^{-1}(\xi_1 \times \xi_2)] \end{aligned}$$

where $\xi_1 = \eta_1 + x_2$ and ξ_2 is a fixed point in $M_2(x_1) + x_2$. This proves (3.10).

By (3.9) and (3.10), $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2)\rangle_{0,d}$ is equal to

$$\begin{aligned} &12\tilde{f}_{2,0}^{*}\big(c_{2d-2}(\mathcal{E})\big) \cdot [\tilde{ev}_2^{-1}(\xi_1 \times \xi_2)] \\ &= -4(K_X \cdot \ell) \cdot c_{2d-2}(\mathcal{E}) \cdot (\tilde{f}_{2,0})_*[\tilde{ev}_2^{-1}(\xi_1 \times \xi_2)]. \end{aligned} \tag{3.11}$$

Note that $\tilde{ev}_2^{-1}(\xi_1 \times \xi_2)$ parametrizes all the stable maps $[\mu : (C; p_1, p_2) \to X^{[3]}]$ in $\overline{\mathfrak{M}}_{0,2}(X^{[3]}, d\beta_3)$ satisfying $\mu(p_1) = \xi_1$ and $\mu(p_2) = \xi_2$. For these stable maps, we must have $\mu(C) = M_2(x_1) + x_2$. So the restriction of $\tilde{f}_{2,0}$ to $\tilde{ev}_2^{-1}(\xi_1 \times \xi_2)$ is a degree-d^2 morphism to $\phi^{-1}(2x_1 + x_2)$. Thus, $(\tilde{f}_{2,0})_*[\tilde{ev}_2^{-1}(\xi_1 \times \xi_2)] = d^2[\phi^{-1}(2x_1 + x_2)]$. By (3.11), we obtain $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2)\rangle_{0,d} = -4(K_X \cdot \ell)d^2 \cdot c_{2d-2}(\mathcal{E}|_{\phi^{-1}(2x_1+x_2)})$. By Lemma 3.3 (iii) and the Theorem 9.2.3 in [C-K], $c_{2d-2}(\mathcal{E}|_{\phi^{-1}(2x_1+x_2)}) = 1/d^3$. Therefore, we have $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2)\rangle_{0,d} = -4(K_X \cdot \ell)/d$. □

In view of Lemma 3.1, Lemma 3.2 and Lemma 3.4, the only 2-point Gromov-Witten invariant $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2)\rangle_{0,d}$ with $A_1 \in \mathfrak{B}_6$ and $A_2 \in \mathfrak{B}_8$ that has not been computed is when $A_1 = \mathfrak{a}_{-3}(\ell)|0\rangle$ and $A_2 = \mathfrak{a}_{-3}(X)|0\rangle$. This invariant

$$\langle \mathrm{PD}(\mathfrak{a}_{-3}(\ell)|0\rangle), \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)\rangle_{0,d} \tag{3.12}$$

will be studied in Sect. 4 by using the localization formula.

We summarize the results in this subsection into a theorem.

Theorem 3.5. *Let $X = \mathbb{P}^2$, and $\mathfrak{B}_6$ and $\mathfrak{B}_8$ be defined in Definition 2.4. Let $d \geq 1$, $A_1 \in \mathfrak{B}_6$ and $A_2 \in \mathfrak{B}_8$. Let x, ℓ be a point and a line in X respectively. Then, $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2)\rangle_{0,d}$ is zero unless the pair (A_1, A_2) is one of the following:*

(i) $(\mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(x)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle)$

(ii) $(\mathfrak{a}_{-2}(\ell)\mathfrak{a}_{-1}(\ell)|0\rangle, \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle)$

(iii) $(\mathfrak{a}_{-3}(\ell)|0\rangle, \mathfrak{a}_{-3}(X)|0\rangle)$.

Moreover, $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2)\rangle_{0,d} = 12/d$ in cases (i) and (ii). □

3.2. $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2), \mathrm{PD}(A_3)\rangle_{0,d}$ with $A_1, A_2, A_3 \in \mathfrak{B}_8$.

Lemma 3.6. *The Gromov-Witten invariants $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2), \mathrm{PD}(A_3)\rangle_{0,d}$ are equal to zero for the following triples of $(A_1, A_2, A_3) \in (\mathfrak{B}_8)^3$:*

$$\begin{gathered} A_1 = \mathfrak{a}_{-3}(X)|0\rangle,\ A_2 \neq \mathfrak{a}_{-3}(X)|0\rangle,\ A_3 \neq \mathfrak{a}_{-3}(X)|0\rangle, \\ A_1 = A_2 = \mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)^2|0\rangle,\ A_3 \text{ arbitrary}, \\ A_1 = \mathfrak{a}_{-1}(X)^2\mathfrak{a}_{-1}(x)|0\rangle,\ A_2 \text{ arbitrary},\ A_3 \text{ arbitrary}, \\ (\mathfrak{a}_{-3}(X)|0\rangle, \mathfrak{a}_{-3}(X)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)^2|0\rangle), \\ A_1 = A_2 = \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle,\ A_3 \neq \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle, \\ (\mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)^2|0\rangle), \\ A_1, A_2, A_3 \in \{\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)^2|0\rangle\}. \end{gathered}$$

Proof. The arguments are similar to those for Lemma 3.1 and Lemma 3.2. □

Lemma 3.7. *Let $X = \mathbb{P}^2$, $\ell \subset X$ be a line, and $d \geq 1$. Then,*

(i) *$\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2), \mathrm{PD}(A_3)\rangle_{0,d} = 0$ for the following triple:*

$$(A_1, A_2, A_3) = (\mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle);$$

(ii) $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2), \mathrm{PD}(A_3)\rangle_{0,d} = 8(K_X \cdot \ell)$ *for the triple:*

$$(A_1, A_2, A_3) = (\mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle, \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle).$$

Proof. The arguments are similar to those for Lemma 3.4 (i) and (ii). □

According to Lemma 3.6 and Lemma 3.7, it remains to compute the invariants $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2), \mathrm{PD}(A_3)\rangle_{0,d}$ for the following 3 triples of $(A_1, A_2, A_3) \in (\mathfrak{B}_8)^3$:

$$\begin{aligned} A_1 = A_2 &= \mathfrak{a}_{-3}(X)|0\rangle, \\ A_3 &= \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle, \mathfrak{a}_{-3}(X)|0\rangle. \end{aligned}$$

In the next two lemmas, we shall calculate them in terms of (3.12). Put

$$\mathcal{E}_i = \pi_1(\pi_2^*\mathcal{O}_X(i)|_{\mathcal{O}_{\mathcal{Z}_3}}) \tag{3.13}$$

where π_1 and π_2 denote the projections of $X^{[3]} \times X$ to the two factors. It is known that $c_1(\mathcal{E}_i) = iD_\ell - B_3/2$. Using the commutation relations among standard operators on $\mathbb{H}$ (e.g. the Theorem 3.1 in [LQW4]), we obtain

$$\begin{aligned} c_1(\mathcal{E}_0)^2 &= \mathfrak{a}_{-3}(X)|0\rangle - \mathfrak{a}_{-1}(X)^2\mathfrak{a}_{-1}(x)|0\rangle \\ &\quad -\frac{1}{2}\mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)^2|0\rangle - \frac{1}{2}\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(K_X)|0\rangle. \end{aligned} \tag{3.14}$$

Lemma 3.8. *Let* $d \geq 1$ *and* $A = \mathfrak{a}_{-3}(X)|0\rangle$. *Let* w_1, w_2 *denote the two invariants* $\langle \mathrm{PD}(A), \mathrm{PD}(A), \mathrm{PD}(A_3)\rangle_{0,d}$ *for* $A_3 = \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle$, $\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle$ *respectively. Then,* $w_1 = w_2 = -2d\ \langle \mathrm{PD}(\mathfrak{a}_{-3}(\ell)|0\rangle), \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)\rangle_{0,d}$.

Proof. Since the arguments for w_1 and w_2 are almost the same, we only prove that $w_2 = -2d\ \langle \mathrm{PD}(\mathfrak{a}_{-3}(\ell)|0\rangle), \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)\rangle_{0,d}$. Let $c_1 = c_1(\mathcal{E}_0) = -B_3/2$ (we regard a divisor as either a homology class or a cohomology class depending on the context). Apply the composition law (2.6) to $\alpha_1 = \alpha_2 = c_1, \alpha_3 = \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle), \alpha_4 = \mathrm{PD}(\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle)$, and to the basis $\{\Delta_a\}$ of $H^*(X^{[3]})$ given by (2.12).

First of all, the left-hand-side of (2.6) is equal to

$$\begin{aligned} &\langle c_1^2, \alpha_3, \alpha_4\rangle_{0,d} + \langle c_1, c_1, \alpha_3\alpha_4\rangle_{0,d} \\ &+ \sum_{d_1+d_2=d,\, d_1,d_2>0} \sum_a \langle c_1, c_1, \Delta_a\rangle_{0,d_1} \langle \Delta^a, \alpha_3, \alpha_4\rangle_{0,d_2}. \end{aligned} \tag{3.15}$$

By (3.14) and Lemma 3.6, $\langle c_1^2, \alpha_3, \alpha_4\rangle_{0,d} = w_2$. Since the intersection number $(c_1 \cdot \beta_3)$ is equal to 1, $\langle c_1, c_1, \alpha_3\alpha_4\rangle_{0,d} = d^2 \langle \alpha_3\alpha_4\rangle_{0,d}$ and $\langle c_1, c_1, \Delta_a\rangle_{0,d_1} = d_1^2 \langle \Delta_a\rangle_{0,d_1}$. By Lemma 2.5, $\langle \Delta_a\rangle_{0,d_1} \neq 0$ only when $\Delta_a = \mathrm{PD}(\mathfrak{a}_{-2}(\ell)\mathfrak{a}_{-1}(x)|0\rangle)$. Note that $\Delta^a = -1/2\,\mathrm{PD}(\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle)$. So $\langle \Delta^a, \alpha_3, \alpha_4\rangle_{0,d_2} = 0$ by Lemma 3.6. It follows from (3.15) that the left-hand-side of (2.6) is equal to

$$w_2 + d^2 \langle \alpha_3\alpha_4\rangle_{0,d}. \tag{3.16}$$

We claim that $\langle \alpha_3\alpha_4\rangle_{0,d} = -12(K_X \cdot \ell)/d^2$. To prove this, note from (3.14) that $\mathfrak{a}_{-3}(X)|0\rangle = c_1^2 + \mathfrak{a}_{-1}(X)^2\mathfrak{a}_{-1}(x)|0\rangle + 1/2\,\mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)^2|0\rangle - 3/2\,\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle$. Choose lines ℓ', ℓ'' in $X = \mathbb{P}^2$ such that ℓ, ℓ', ℓ'' are in general position. Then, $\big(\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle\big) \cap \big(\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell')|0\rangle\big) \cap \big(\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell'')|0\rangle\big) = \emptyset$. It follows that $\big(\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle\big)^3 = 0$. In view of the linear basis in Lemma 2.3 (ii), we see that

$\big(\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle\big)^2$ is a linear combination of $\mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(x)^2|0\rangle$, $\mathfrak{a}_{-1}(\ell)^2\mathfrak{a}_{-1}(x)|0\rangle$, $\mathfrak{a}_{-1}(\ell)\mathfrak{a}_{-2}(x)|0\rangle$, and $\mathfrak{a}_{-3}(x)|0\rangle$. Hence $\langle \mathrm{PD}(\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle)\ \alpha_4\rangle_{0,d} = 0$ according to Lemma 2.5 (i), and we see that $\langle\alpha_3\alpha_4\rangle_{0,d}$ is equal to

$$\langle c_1^2\alpha_4\rangle_{0,d} + \langle \mathrm{PD}(\mathfrak{a}_{-1}(X)^2\mathfrak{a}_{-1}(x)|0\rangle)\ \alpha_4\rangle_{0,d} + \frac{1}{2}\langle \mathrm{PD}(\mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)^2|0\rangle)\ \alpha_4\rangle_{0,d}.$$

Since $(D_\ell)^2 = \mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)^2|0\rangle + 1/2\,\mathfrak{a}_{-1}(X)^2\mathfrak{a}_{-1}(x)|0\rangle$, we obtain

$$\langle\alpha_3\alpha_4\rangle_{0,d} = \langle c_1^2\alpha_4\rangle_{0,d} + \frac{1}{2}\langle D_\ell^2\alpha_4\rangle_{0,d} + \frac{3}{4}\langle \mathrm{PD}(\mathfrak{a}_{-1}(X)^2\mathfrak{a}_{-1}(x)|0\rangle)\ \alpha_4\rangle_{0,d}. \tag{3.17}$$

Since $\mathfrak{a}_{-1}(X)^2\mathfrak{a}_{-1}(x)|0\rangle \cdot \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle = 2\mathfrak{a}_{-2}(\ell)\mathfrak{a}_{-1}(x)|0\rangle$, the third term in (3.17) is equal to $3(K_X\cdot\ell)/d^2$ by Lemma 2.5 (ii). Since $D_\ell^2 \cdot \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle = \mathfrak{a}_{-2}(\ell)\mathfrak{a}_{-1}(x)|0\rangle + 4\mathfrak{a}_{-1}(\ell)\mathfrak{a}_{-2}(x)|0\rangle$, the second term in (3.17) is equal to $(K_X\cdot\ell)/d^2$ by Lemma 2.5. Using a similar argument, we see that the first term in (3.17) is equal to $-16(K_X\cdot\ell)/d^2$. Thus, $\langle\alpha_3\alpha_4\rangle_{0,d} = -12(K_X\cdot\ell)/d^2$ in view of (3.17).

Combining with (3.16), we see that the left-hand-side of (2.6) is equal to $w_2 - 12(K_X\cdot\ell)$. Similarly, the right-hand-side of (2.6) is equal to

$$-2d\ \langle \mathrm{PD}(\mathfrak{a}_{-3}(\ell)|0\rangle), \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)\rangle_{0,d} - 12(K_X\cdot\ell).$$

Hence we have $w_2 = -2d\ \langle \mathrm{PD}(\mathfrak{a}_{-3}(\ell)|0\rangle), \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)\rangle_{0,d}$. ◻

Lemma 3.9. *Let* $d\geq 1$. *Put* $f(d) = d\,\langle \mathrm{PD}(\mathfrak{a}_{-3}(\ell)|0\rangle), \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)\rangle_{0,d}$. *Let* w_3 *denote* $\langle \mathrm{PD}(A), \mathrm{PD}(A), \mathrm{PD}(A)\rangle_{0,d}$ *for* $A = \mathfrak{a}_{-3}(X)|0\rangle$. *Then* w_3 *equals*

$$\begin{gathered} -24K_X^2 - 18(K_X\cdot\ell) + 5(K_X\cdot\ell)f(d) \\ -2(K_X\cdot\ell)\sum_{0<d_1<d} f(d_1) + \frac{1}{3}\sum_{0<d_1<d} f(d_1)f(d-d_1). \end{gathered}$$

Proof. Our idea is the same as in the proof of Lemma 3.8. Let $c_1 = c_1(\mathcal{E}_0)$. Apply (2.6) to $\alpha_1 = \alpha_2 = c_1$ and $\alpha_3 = \alpha_4 = \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)$. Then, the left-hand-side of (2.6) is still of the form (3.15). By (3.14), Lemma 3.6 and Lemma 3.8, $\langle c_1^2, \alpha_3, \alpha_4\rangle_{0,d} = w_3 - (K_X\cdot\ell)/2\,w_2 = w_3 + (K_X\cdot\ell)f(d)$. Also, $\langle c_1, c_1, \alpha_3\alpha_4\rangle_{0,d} = d^2\,\langle\alpha_3\alpha_4\rangle_{0,d} = 24K_X^2 + 18(K_X\cdot\ell)$, and $\sum_a \langle c_1, c_1, \Delta_a\rangle_{0,d_1}\,\langle\Delta^a, \alpha_3, \alpha_4\rangle_{0,d_2}$ is equal to

$$\begin{aligned} &-\frac{d_1^2}{2}\ \langle \mathrm{PD}(\mathfrak{a}_{-2}(\ell)\mathfrak{a}_{-1}(x)|0\rangle)\rangle_{0,d_1}\ \langle \mathrm{PD}(\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle), \alpha_3, \alpha_4\rangle_{0,d_2} \\ &= -\frac{d_1^2}{2}\cdot\frac{2(K_X\cdot\ell)}{d_1^2}\cdot(-2f(d_2)) = 2(K_X\cdot\ell)f(d_2) \end{aligned}$$

by Lemma 2.5 (ii) and Lemma 3.8. So the left-hand-side of (2.6) is

$$w_3 + (K_X\cdot\ell)f(d) + 24K_X^2 + 18(K_X\cdot\ell) + 2(K_X\cdot\ell)\sum_{0<d_1<d} f(d_1). \tag{3.18}$$

Similarly, the right-hand-side of (2.6) is equal to

$$6(K_X\cdot\ell)f(d) + \frac{1}{3}\sum_{0<d_1<d} f(d_1)f(d-d_1). \tag{3.19}$$

Now we prove the lemma by comparing (3.18) and (3.19). ◻

The results in this subsection are summarized into a theorem.

Theorem 3.10. *Let* $X=\mathbb{P}^2$, *and* $\mathfrak{B}_8$ *be defined in Definition 2.4. Let* $\ell \subset X$ *be a line. Let* $d \ge 1$, $f(d) = d\,\langle \mathrm{PD}(\mathfrak{a}_{-3}(\ell)|0\rangle), \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)\rangle_{0,d}$, *and* $A_1, A_2, A_3 \in \mathfrak{B}_8$. *Then, the* 3*-point genus-*0 *Gromov-Witten invariant* $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2), \mathrm{PD}(A_3)\rangle_{0,d}$ *is zero unless the unordered triple* (A_1, A_2, A_3) *is one of the following:*

(i) $(\mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle, \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle)$
(ii) $(\mathfrak{a}_{-3}(X)|0\rangle, \mathfrak{a}_{-3}(X)|0\rangle, \mathfrak{a}_{-2}(X)\mathfrak{a}_{-1}(\ell)|0\rangle)$
(iii) $(\mathfrak{a}_{-3}(X)|0\rangle, \mathfrak{a}_{-3}(X)|0\rangle, \mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(\ell)|0\rangle)$
(iv) $(\mathfrak{a}_{-3}(X)|0\rangle, \mathfrak{a}_{-3}(X)|0\rangle, \mathfrak{a}_{-3}(X)|0\rangle)$.

Moreover, $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2), \mathrm{PD}(A_3)\rangle_{0,d} = -24$ *for case (i); for cases (ii) and (iii),* $\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2), \mathrm{PD}(A_3)\rangle_{0,d} = -2f(d)$*; for case (iv),*

$$\begin{aligned}&\langle \mathrm{PD}(A_1), \mathrm{PD}(A_2), \mathrm{PD}(A_3)\rangle_{0,d}\\ &= -162 - 15f(d) + 6\sum_{0<d_1<d} f(d_1) + \frac{1}{3}\sum_{0<d_1<d} f(d_1)f(d-d_1).\end{aligned}$$

□

4. Computation of $\langle \mathrm{PD}(\mathfrak{a}_{-3}(\ell)|0\rangle), \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)\rangle_{0,d}$.

In this section, we study the remaining 2-point Gromov-Witten invariant

$$\langle \mathrm{PD}(\mathfrak{a}_{-3}(\ell)|0\rangle), \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)\rangle_{0,d}$$

in (3.12). Using the standard $(\mathbb{C}^*)^2$-action on $X=\mathbb{P}^2$ and the virtual localization formula in [G-P], we reduce the computation to a summation over stable graphs. This allows us to calculate $\langle \mathrm{PD}(\mathfrak{a}_{-3}(\ell)|0\rangle), \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)\rangle_{0,d}$ for $d \le 4$.

4.1. The contracted $(\mathbb{C}^*)^2$-invariant curves in $(\mathbb{P}^2)^{[3]}$.

Let $T \subset \mathrm{SL}_3(\mathbb{C})$ be the subgroup consisting of diagonal matrices. Then $T \simeq (\mathbb{C}^*)^2$ acts on $\mathbb{P}^2$ with fixed points $P_0=(1,0,0)$, $P_1=(0,1,0)$ and $P_2=(0,0,1)$. There is an induced action of T on the Hilbert scheme $(\mathbb{P}^2)^{[3]}$ with a finite number of fixed points. The T-fixed points in $(\mathbb{P}^2)^{[3]}$ are enumerated as follows. If (u_i, v_i) are the local coordinates at the fixed point P_i, then there are three T-fixed points in $M_3(P_i) \subset (\mathbb{P}^2)^{[3]}$ corresponding to the partitions (3), $(2,1)$ and $(1,1,1)$ of 3. The corresponding ideals are (u_i^3, v_i), (u_i^2, u_iv_i, v_i^2) and (u_i, v_i^3). Also for each ordered pair of points (P_i, P_j) with $i \ne j$, we have two fixed points $R_{i,j}^{(1)} = \xi_{i,1} + P_j$ and $R_{i,j}^{(2)} = \xi_{i,2} + P_j$ in $(\mathbb{P}^2)^{[3]}$, where $\xi_{i,1}, \xi_{i,2} \in M_2(P_i)$ correspond to the ideals $(u_i, v_i^2), (u_i^2, v_i)$ respectively. Finally, $P_0+P_1+P_2$ is also a T-fixed point in $(\mathbb{P}^2)^{[3]}$.

Next, we start enumerating T-invariant curves. Observe that a T-invariant curve is the closure of a 1-dimensional T-orbit. Thus, a T-invariant curve is the T-orbit of a point in a fixed component of a 1-parameter subgroup of T corresponding to the kernel of the T-action along the curve. In particular a T-invariant curve is a smooth rational curve, and must contain exactly two fixed points.

We are only interested in T-invariant curves that are contracted under the Hilbert-Chow morphism $(\mathbb{P}^2)^{[3]} \to (\mathbb{P}^2)^{(3)}$. Such curves must be entirely contained in $M_3(P_i)$ for some i, or in $M_2(P_i)+P_j$ for some $i \ne j$. Since $M_2(P_i) \simeq \mathbb{P}^1$, we immediately obtained six T-invariant curves $C_{i,j} \overset{\mathrm{def}}{=} M_2(P_i)+P_j$, with $1 \le i,j \le 3$ and $i \ne j$, contracted by the Hilbert-Chow morphism $(\mathbb{P}^2)^{[3]} \to (\mathbb{P}^2)^{(3)}$.

We now analyze T-invariant curves in $M_3(P_i)$, by using a tangent space analysis. Suppose that $(s,t)(u_i,v_i) = (\lambda_i(s,t)u_i, \mu_i(s,t)v_i)$ where λ_i and μ_i are independent characters of T. Let $Q_{i,0}, Q_{i,1}, Q_{i,2} \in M_3(P_i)$ be the three T-fixed points corresponding to the ideals $(u_i^2, u_i v_i, v_i^2), (u_i^3, v_i), (u_i, v_i^3)$ respectively. For simplicity, denote the tangent space of $(\mathbb{P}^2)^{[3]}$ at the point $Q_{i,j}$ by $T_{Q_{i,j}}$. By [E-S], we have the following decompositions for the tangent spaces as a representation of T:

$$T_{Q_{i,0}} = 2\lambda_i^{-1} + 2\mu_i^{-1} + \lambda_i^{-2}\mu_i + \lambda_i\mu_i^{-2} \tag{4.1}$$

$$T_{Q_{i,1}} = \lambda_i^{-1}\mu_i^2 + \lambda_i^{-1}\mu_i + \lambda_i^{-1} + \mu_i^{-3} + \mu_i^{-2} + \mu_i^{-1} \tag{4.2}$$

$$T_{Q_{i,2}} = \lambda_i^{-3} + \lambda_i^{-2} + \lambda_i^{-1} + \lambda_i^2\mu_i^{-1} + \lambda_i\mu_i^{-1} + \mu_i^{-1}. \tag{4.3}$$

The kernel of each character appearing in equations (4.1), (4.2), (4.3) determines 1-parameter subgroup whose fixed locus contains T-invariant curves. Since we are interested only in T-invariant curves contained in $M_3(P_i)$, we need only to analyze characters of the form $\lambda_i^k\mu_i^\ell$ with $k\ell \neq 0$. (The kernel of a character λ_i^k or μ_i^ℓ will have fixed locus that moves out of the punctual Hilbert scheme.)

Looking at $T_{Q_{i,0}}$ we see that the character $\lambda_i\mu_i^{-2}$ has multiplicity one. This means that its kernel has one-dimensional fixed component containing the point $Q_{i,0}$. Now the character $\lambda_i^{-1}\mu_i^2$ in $T_{Q_{i,1}}$ has the same kernel as the character $\lambda_i\mu_i^{-2}$ in $T_{Q_{i,0}}$. So there is a unique T-invariant curve, denoted by $C_{0,1}^{(i)}$, which contains $Q_{i,0}$ and $Q_{i,1}$, and is the fixed locus of $\ker(\lambda_i\mu_i^{-2})$. Similar analysis shows that there are two other T-invariant curves $C_{0,2}^{(i)}$ and $C_{1,2}^{(i)}$ in $M_3(P_i)$; namely, $C_{0,2}^{(i)}$ through $Q_{i,0}$ and $Q_{i,2}$ which is the fixed locus of $\ker(\lambda_i^{-2}\mu_i)$, while $C_{1,2}^{(i)}$ through $Q_{i,1}$ and $Q_{i,2}$ which is the fixed locus of $\ker(\lambda_i^{-1}\mu_i)$. This analysis partially proves the following.

Lemma 4.1. *There are 15 T-invariant curves contracted under the Hilbert-Chow morphism $(\mathbb{P}^2)^{[3]} \to (\mathbb{P}^2)^{(3)}$. They are described as follows:*

(i) *the six curves $C_{i,j} = M_2(P_i) + P_j$ where $1 \le i, j \le 3$ and $i \neq j$;*

(ii) *the nine curves $C_{k,\ell}^{(i)} \subset M_3(P_i)$ where $1 \le i \le 3$ and $0 \le k < \ell \le 2$.*

Furthermore, $C_{1,2}^{(i)} \sim 3\beta_3$ and $C_{0,1}^{(i)} \sim C_{0,2}^{(i)} \sim \beta_3$ for every i.

Proof. It remains to prove the last sentence. Identify $M_3(P_i)$ with the punctual Hilbert scheme $\mathrm{Hilb}^3(\mathbb{C}^2, 0)$. By (2.15), $C_{1,2}^{(i)} = \tilde{\sigma}_3$. It follows from Lemma 2.8 that $C_{1,2}^{(i)} \sim 3\beta_3$. Similarly, we see from (2.13) and Lemma 2.7 that $C_{0,1}^{(i)} \sim C_{0,2}^{(i)} \sim \beta_3$. □

Next, we compute the equivariant first Chern classes of the restrictions of the tautological bundles (3.13) to the T-fixed points in $(\mathbb{P}^2)^{[3]}$. Let $w_i = c_1(\lambda_i)$ and $z_i = c_1(\mu_i)$ in the equivariant Chow group $A_*^T(pt)$. If we put $(w_0, z_0) = (w, z)$, then $(w_1, z_1) = (-w, -w + z)$ and $(w_2, z_2) = (-z, -z + w)$.

Lemma 4.2. *Let $g_0 = 0$, $g_1 = w$, and $g_2 = z$. There are T-linearizations on $\mathcal{E}_0$ and $\mathcal{E}_1$ such that $c_1(\mathcal{E}_0|_{R_{i,j}^{(1)}}) = z_i$, $c_1(\mathcal{E}_0|_{R_{i,j}^{(2)}}) = w_i$, $c_1(\mathcal{E}_0|_{Q_{i,0}}) = z_i + w_i$, $c_1(\mathcal{E}_0|_{Q_{i,1}}) = 3z_i$, $c_1(\mathcal{E}_0|_{Q_{i,2}}) = 3w_i$ and $c_1(\mathcal{E}_1|_{R_{i,j}^{(1)}}) = 2g_i + g_j + z_i$, $c_1(\mathcal{E}_1|_{R_{i,j}^{(2)}}) = 2g_i + g_j + w_i$, $c_1(\mathcal{E}_1|_{Q_{i,0}}) = 3g_i + z_i + w_i$, $c_1(\mathcal{E}_1|_{Q_{i,1}}) = 3g_i + 3z_i$, $c_1(\mathcal{E}_1|_{Q_{i,2}}) = 3g_i + 3w_i$.*

Proof. The proofs of these conclusions are similar. For instance, let us prove $c_1(\mathcal{E}_1|_{R_{i,j}^{(2)}}) = 2g_i + g_j + w_i$. Note that the fiber $\mathcal{E}_1|_{R_{i,j}^{(2)}}$ is canonically identified with $\mathcal{O}_X(1) \otimes \mathcal{O}_X/I_{R_{i,j}^{(2)}}$. Since $R_{i,j}^{(2)} = \xi_{i,2} + P_j$, $\mathcal{E}_1|_{R_{i,j}^{(2)}}$ is canonically identified with $\big(\mathcal{O}_X(1) \otimes \mathcal{O}_X/I_{\xi_{i,2}}\big) \oplus \big(\mathcal{O}_X(1) \otimes \mathcal{O}_X/I_{P_j}\big)$. Therefore,

$$c_1(\mathcal{E}_1|_{R_{i,j}^{(2)}}) = 2c_1(\mathcal{O}_X(1)|_{P_i}) + c_1(\mathcal{O}_X/I_{\xi_{i,2}}) + c_1(\mathcal{O}_X(1)|_{P_j}).$$

Since $\mathcal{O}_X(1)|_{P_i} \cong (\mathbb{C} \oplus \mathbb{C})/(\mathbb{C}P_i)$, we have $c_1(\mathcal{O}_X(1)|_{P_i}) = g_i$. Using $c_1(\mathcal{O}_X/I_{\xi_{i,2}}) = c_1(\lambda_i) = w_i$, we conclude that $c_1(\mathcal{E}_1|_{R^{(2)}_{i,j}}) = 2g_i + g_j + w_i$. □

4.2. The Euler characteristic for a covering. An important step in computing the virtual Euler class of the T-fixed locus $\mathfrak{M}_{0,2}((\mathbb{P}^2)^{[3]}, d\beta_3)^T$ is to compute (as a representation) $\chi(f^*T_{(\mathbb{P}^2)^{[3]}})$ where $f : \mathbb{P}^1 \to (\mathbb{P}^2)^{[3]}$ is a degree-d morphism such that the image is one of the 15 T-invariant curves in Lemma 4.1 and f is totally ramified at the two T-fixed points in $f(\mathbb{P}^1)$.

4.2.1. Degree-d coverings of $C^{(i)}_{k,\ell}$. Observe that if $\mathbb{P}^1 \to (\mathbb{P}^2)^{[3]}$ is a degree-d T-equivariant morphism with image $C^{(i)}_{k,\ell}$, then the characters of T-action on $\mathbb{P}^1$ are (using multiplicative notation) $\alpha^{1/d}, \beta^{1/d}$ where α, β are the characters of the T-action on the image curve $C^{(i)}_{k,\ell}$. Let $S_{i,k}$ and $S_{i,\ell}$ be the two fixed points of the action on $\mathbb{P}^1$ denoted so that the image of $S_{i,k}$ is $Q_{i,k}$ and the image of $S_{i,\ell}$ is $Q_{i,\ell}$. If V is a T-equivariant vector bundle on $\mathbb{P}^1$, then the localization theorem for equivariant K-theory says that

$$\chi(V) = \frac{V|_{S_{i,k}}}{1 - T^*_{\mathbb{P}^1}|_{S_{i,k}}} + \frac{V|_{S_{i,\ell}}}{1 - T^*_{\mathbb{P}^1}|_{S_{i,\ell}}} \tag{4.4}$$

where $T^*_{\mathbb{P}^1}$ is the cotangent bundle of $\mathbb{P}^1$. Since $T^*_{\mathbb{P}^1}|_{S_{i,k}} \cong T^*_{C^{(i)}_{k,\ell}}|_{Q_{i,k}}$, we can use formulas (4.1), (4.2), (4.3) to determine $\chi(f^*T_{(\mathbb{P}^2)^{[3]}})$.

First of all, let $f(\mathbb{P}^1) = C^{(i)}_{0,1}$. The curve $C^{(i)}_{0,1}$ is a component of the fixed locus of $\ker(\lambda_i\mu_i^{-2})$. Thus, reading off (4.1) and (4.2), we see that $T_{C^{(i)}_{0,1}}|_{Q_{i,0}} = \lambda_i\mu_i^{-2}$ and $T_{C^{(i)}_{0,1}}|_{Q_{i,1}} = \lambda_i^{-1}\mu_i^2$. Thus $T_{\mathbb{P}^1}|_{S_{i,0}} = \gamma_i\theta_i^{-2}$ and $T_{\mathbb{P}^1}|_{S_{i,1}} = \gamma_i^{-1}\theta_i^2$ where $\gamma_i^d = \lambda_i$ and $\theta_i^d = \mu_i$. Substituting (4.1) and (4.2) into the localization formula (4.4) yields

$$\begin{aligned}\chi(f^*T_{(\mathbb{P}^2)^{[3]}}) = {} & \frac{\lambda_i\mu_i^{-2} + \mu_i^{-1} + \lambda_i^{-1} + \lambda_i^{-2}\mu_i + \lambda_i^{-1} + \mu_i^{-1}}{1 - \gamma_i^{-1}\theta_i^2} \\ & + \frac{\lambda_i^{-1}\mu_i^2 + \lambda_i^{-1}\mu_i + \lambda_i^{-1} + \mu_i^{-3} + \mu_i^{-2} + \mu_i^{-1}}{1 - \gamma_i\theta_i^{-2}}.\end{aligned}$$

Since $1/(1 - \gamma_i^{-1}\theta_i^2) = -\gamma_i\theta_i^{-2}/(1 - \gamma_i\theta_i^{-2})$, the right hand side can be rewritten as

$$\begin{aligned}\frac{1}{1 - \gamma_i\theta_i^{-2}} \big[& (\lambda_i^{-1}\mu_i^2 + \lambda_i^{-1}\mu_i + \lambda_i^{-1} + \mu_i^{-3} + \mu_i^{-2} + \mu_i^{-1}) - \gamma_i\theta_i^{-2}\big((\lambda_i^2\mu_i^{-4})(\lambda_i^{-1}\mu_i^2) \\ & + (\lambda_i\mu_i^{-2})\lambda_i^{-1}\mu_i + \lambda_i^{-1} + (\lambda_i^{-2}\mu_i^4)\mu_i^{-3} + (\lambda_i^{-1}\mu_i^2)\mu_i^{-2} + \mu_i^{-1}\big)\big].\end{aligned}$$

Using $\lambda_i = \gamma_i^d$ and $\mu_i = \theta_i^d$, we conclude that $\chi(f^*T_{(\mathbb{P}^2)^{[3]}})$ is equal to

$$\begin{aligned}& \lambda_i^{-1}\mu_i^2 \sum_{m=0}^{2d} (\gamma_i\theta_i^{-2})^m + \lambda_i^{-1}\mu_i \sum_{m=0}^{d} (\gamma_i\theta_i^{-2})^m + \lambda_i^{-1} \\ & - \mu_i^{-3}(\gamma_i\theta_i^{-2})^{-2d+1} \sum_{m=0}^{2d-2} (\gamma_i\theta_i^{-2})^m - \mu_i^{-2}(\gamma_i\theta_i^{-2})^{-d+1} \sum_{m=0}^{d-2} (\gamma_i\theta_i^{-2})^m + \mu_i^{-1}.\end{aligned}$$

To simplify this further, set $\Theta^{(i)}_{0,1} = \sum_{m=1}^{d-1} (\gamma_i\theta_i^{-2})^m = \sum_{m=1}^{d-1} (\lambda_i\mu_i^{-2})^{m/d}$ (with the understanding that $\Theta_{0,1}^{(i)} = 0$ when $d = 1$). Then we see that $\chi(f^*T_{(\mathbb{P}^2)^{[3]}})$ equals

$$(1 + \lambda_i^{-1}\mu_i^2 + \lambda_i\mu_i^{-2} + \lambda_i^{-1}\mu_i + \mu_i^{-1} + \lambda_i^{-1} + \mu_i^{-1} - \lambda_i^{-1}\mu_i^{-1})$$
$$+ (\lambda_i^{-1}\mu_i^2 + 1 + \lambda_i^{-1}\mu_i - \lambda_i^{-2}\mu_i - \lambda_i^{-1}\mu_i^{-1} - \lambda_i^{-1})\Theta_{0,1}^{(i)}. \tag{4.5}$$

By symmetry, if $f(\mathbb{P}^1) = C_{0,2}^{(i)}$, then $\chi(f^*T_{(\mathbb{P}^2)^{[3]}})$ is equal to

$$(1 + \mu_i^{-1}\lambda_i^2 + \mu_i\lambda_i^{-2} + \mu_i^{-1}\lambda_i + \lambda_i^{-1} + \mu_i^{-1} + \lambda_i^{-1} - \mu_i^{-1}\lambda_i^{-1})$$
$$+ (\mu_i^{-1}\lambda_i^2 + 1 + \mu_i^{-1}\lambda_i - \mu_i^{-2}\lambda_i - \mu_i^{-1}\lambda_i^{-1} - \mu_i^{-1})\Theta_{0,2}^{(i)} \tag{4.6}$$

where $\Theta_{0,2}^{(i)} = \sum_{m=1}^{d-1}(\mu_i\lambda_i^{-2})^{m/d}$, and as above $\Theta_{0,2}^{(i)} = 0$ if $d = 1$.

Next, let $f(\mathbb{P}^1) = C_{1,2}^{(i)}$. Then $T_{C_{1,2}^{(i)}}|_{Q_{i,1}} = \lambda_i^{-1}\mu_i$ and $T_{C_{1,2}^{(i)}}|_{Q_{i,2}} = \lambda_i\mu_i^{-1}$. Thus $T_{\mathbb{P}^1}|_{S_{i,1}} = \gamma_i^{-1}\theta_i$ and $T_{\mathbb{P}^1}|_{S_{i,2}} = \gamma_i\theta_i^{-1}$. By (4.4), (4.2) and (4.3), $\chi(f^*T_{(\mathbb{P}^2)^{[3]}})$ equals

$$\frac{1}{1 - \gamma_i\theta_i^{-1}}[(\lambda_i^{-1}\mu_i^2 + \lambda_i^{-1}\mu_i + \lambda_i^{-1} + \mu_i^{-3} + \mu_i^{-2} + \mu_i^{-1})$$
$$- \gamma_i\theta_i^{-1}(\lambda_i^2\mu_i^{-1} + \lambda_i\mu_i^{-1} + \mu_i^{-1} + \lambda_i^{-3} + \lambda_i^{-2} + \lambda_i^{-1})]$$

As above, the numerator is divisible by $(1 - \gamma_i\theta_i^{-1})$, and $\chi(f^*T_{(\mathbb{P}^2)^{[3]}})$ is equal to

$$\lambda_i^{-1}\sum_{s=0}^{2}\mu_i^s\sum_{m=0}^{(s+1)d}(\gamma_i\theta_i^{-1})^m - \sum_{s=1}^{3}\lambda_i^{-s}\sum_{m=1}^{sd-1}(\gamma_i\theta_i^{-1})^m.$$

Let $\Theta_{1,2}^{(i)} = \sum_{m=1}^{d-1}(\lambda_i\mu_i^{-1})^{m/d}$ with $\Theta_{1,2}^{(i)} = 0$ when $d = 1$. Then $\chi(f^*T_{(\mathbb{P}^2)^{[3]}})$ equals

$$\lambda_i^{-1}(1 + \Theta_{1,2}^{(i)}) + \mu_i^{-1} + \lambda_i^{-1}\mu_i(1 + \theta_{1,2}^{(i)}) + (1 + \theta_{1,2}^{(i)}) + \lambda_i\mu_i^{-1}$$
$$\lambda_i^{-1}\mu_i^2(1 + \Theta_{1,2}^{(i)}) + \mu_i(1 + \Theta_{1,2}^{(i)}) + \lambda_i(1 + \Theta_{1,2}^{(i)}) + \lambda_i^2\mu_i^{-1}$$
$$- \lambda_i^{-1}\Theta_{1,2}^{(i)} - (\lambda_i^{-2}\Theta_{1,2}^{(i)} + \lambda_i^{-1}\mu_i^{-1}(1 + \Theta_{1,2}^{(i)}))$$
$$- (\lambda_i^{-3}\Theta_{1,2} + \lambda_i^{-2}\mu_i^{-1}(1 + \Theta_{1,2}^{(i)}) + \lambda_i^{-1}\mu_i^{-2}(1 + \Theta_{1,2}^{(i)})).$$

Rearranging the terms, we conclude that $\chi(f^*T_{(\mathbb{P}^2)^{[3]}})$ is equal to

$$(\lambda_i^{-1} + \mu_i^{-1} + \lambda_i^{-1}\mu_i + 1 + \lambda_i\mu_i^{-1} + \lambda_i^{-1}\mu_i^2 + \mu_i + \lambda_i + \lambda_i^2\mu_i^{-1} \tag{4.7}$$
$$- \lambda_i^{-1}\mu_i^{-1} - \lambda_i^{-2}\mu_i^{-1} - \lambda_i^{-1}\mu_i^{-2})$$
$$+ (1 + \lambda_i^{-1}\mu_i^2 + \mu_i + \lambda_i + \lambda_i^{-1}\mu_i - \lambda_i^{-2} - \lambda_i^{-1}\mu_i^{-1} - \lambda_i^{-3} - \lambda_i^{-2}\mu_i^{-1} - \lambda_i^{-1}\mu_i^{-2})\Theta_{1,2}^{(i)}.$$

4.2.2. Degree-d coverings of $C_{i,j}$. Consider maps $f : \mathbb{P}^1 \to (\mathbb{P}^2)^{[3]}$ which are degree-d and have image $C_{i,j}$. To compute $\chi(f^*T_{(\mathbb{P}^2)^{[3]}})$, we recall from subsection 4.1 that the T-fixed points on $C_{i,j}$ are $R_{i,j}^{(1)}$ and $R_{i,j}^{(2)}$. Using the results in [E-S], we have the following decompositions for the tangent spaces of $(\mathbb{P}^2)^{[3]}$ at $R_{i,j}^{(1)}$ and $R_{i,j}^{(2)}$ as representations of T:

$$T_{R_{i,j}^{(1)}} = \lambda_i^{-1}\mu_i + \lambda_i^{-1} + \mu_i^{-2} + \mu_i^{-1} + \lambda_j^{-1} + \mu_j^{-1}, \tag{4.8}$$

$$T_{R_{i,j}^{(2)}} = \lambda_i^{-2} + \lambda_i^{-1} + \lambda_i\mu_i^{-1} + \mu_i^{-1} + \lambda_j^{-1} + \mu_j^{-1}. \tag{4.9}$$

Also, $T_{C_{i,j}}|_{R_{i,j}^{(1)}} = \lambda_i^{-1}\mu_i$ and $T_{C_{i,j}}|_{R_{i,j}^{(2)}} = \lambda_i\mu_i^{-1}$. By (4.4), $\chi(f^*T_{(\mathbb{P}^2)^{[3]}})$ equals

$$\frac{\lambda_i^{-1}\mu_i + \lambda_i^{-1} + \mu_i^{-2} + \mu_i^{-1} + \lambda_j^{-1} + \mu_j^{-1}}{1-\gamma_i\theta_i^{-1}} + \frac{\lambda_i^{-2} + \lambda_i^{-1} + \lambda_i\mu_i^{-1} + \mu_i^{-1} + \lambda_j^{-1} + \mu_j^{-1}}{1-\gamma_i^{-1}\theta_i}.$$

So we obtain the following formula for $\chi(f^*T_{(\mathbb{P}^2)^{[3]}})$:

$$\begin{aligned}(1 + \lambda_i^{-1}\mu_i + \lambda_i\mu_i^{-1} + \lambda_i^{-1} + \mu_i^{-1} + \lambda_j^{-1} + \mu_j^{-1} - \lambda_i^{-1}\mu_i^{-1})\\ + (1 + \lambda_i^{-1}\mu_i - \lambda_i^{-2} - \lambda_i^{-1}\mu_i^{-1})\Theta_{1,2}^{(i)}.\end{aligned} \tag{4.10}$$

4.3. T-invariant stable maps, stable graphs and localizations. Let $X = \mathbb{P}^2$. Note that if $[f : (C; p_1, p_2) \to X^{[3]}] \in \overline{\mathfrak{M}}_{0,2}(X^{[3]}, d\beta_3)$ is T-invariant and if $\mathbb{P}^1$ is an irreducible component of C with nonconstant $f|_{\mathbb{P}^1}$, then $f(\mathbb{P}^1)$ is one of the 15 T-invariant curves in Lemma 4.1. The restriction $f|_{\mathbb{P}^1}$ is ramified at exactly two points with ramification index $\deg(f|_{\mathbb{P}^1})$. Since $f|_{\mathbb{P}^1}$ is ramified at every special point, $\mathbb{P}^1$ contains at most two special points. Moreover, f maps the contracted components and the special points (i.e., marked points, nodal points and ramification points) of C into the T-fixed point set $(X^{[3]})^T$.

Following the book [C-K], to each T-invariant stable map $[f : (C; p_1, p_2) \to X^{[3]}] \in \overline{\mathfrak{M}}_{0,2}(X^{[3]}, d\beta_3)$, we can associate a marked graph Γ called *a stable graph of genus-*0. The graph Γ has one vertex for each connected component of $f^{-1}((X^{[3]})^T)$. It has one edge e for each non-contracted component $C_e \simeq \mathbb{P}^1$, whose two vertices correspond to the connected components of $f^{-1}((X^{[3]})^T)$ containing the two ramification points in the component C_e. The edge e is marked with the degree $d_e \overset{\text{def}}{=} \deg(f|_{C_e})$. Note that the morphism f defines a labeling map $\mathfrak{L}$ from the vertices of Γ to $(X^{[3]})^T$. Finally, a vertex is marked with $\{1\}$ (respectively, $\{2\}$, or $\{1, 2\}$) if the connected component of $f^{-1}((X^{[3]})^T)$ corresponding to the vertex contains the marked point p_1 (respectively, p_2, or both p_1 and p_2).

To a stable graph Γ, we introduce the following notation (cf. [C-K]). Recall that a flag F is a pair (v, e) consisting of an edge e and a vertex v of e. For a flag $F = (v, e)$, define $i(F) = \mathfrak{L}(v)$. Let $S(v)$ be the number of markings of v, and $val(v)$ be the valance of v (i.e., the number of edges e such that v is a vertex of e). Let $n(F) = n(v) = val(v) + S(v)$. If $val(v) = 1$, let $F(v)$ be the single flag containing v; if $val(v) = 2$, let $F_1(v)$ and $F_2(v)$ denote the two flags containing v.

Now the connected components of $\overline{\mathfrak{M}}_{0,2}(X^{[3]}, d\beta_3)^T$ are enumerated by stable graphs corresponding to stable maps whose images are unions of the 15 T-invariant curves in Lemma 4.1 and whose contracted components and special points are mapped into $(X^{[3]})^T$. We use Γ to denote these stable graphs, and use $\mathfrak{M}_\Gamma$ to denote the corresponding connected components of $\overline{\mathfrak{M}}_{0,2}(X^{[3]}, d\beta_3)^T$. If Γ is a stable graph, let $M_\Gamma = \prod_{n(v)\ge 3} \overline{M}_{0,n(v)}$ where $\overline{M}_{0,n(v)}$ is the (fine) moduli space of $n(v)$-pointed stable rational curves. As discussed in [C-K], there is a finite map $M_\Gamma \to \mathfrak{M}_\Gamma$ such that $\mathfrak{M}_\Gamma = M_\Gamma/\mathbf{A}_\Gamma$ where $\mathbf{A}_\Gamma$ fits in the exact sequence

$$0 \to \prod_e \mathbb{Z}/d_e\mathbb{Z} \to \mathbf{A}_\Gamma \to \mathrm{Aut}(\Gamma) \to 0.$$

Since a stable curve is connected, we see from the description of the T-invariant curves in Lemma 4.1 that a summation over all the stable graphs Γ breaks up as

$$\sum_\Gamma = \sum_{1\le i\ne j\le 3} \sum_{\Gamma\in\mathcal{S}_{d,i,j}} + \sum_{i=1}^{3} \sum_{\Gamma\in\mathcal{T}_{d,i}} \tag{4.11}$$

where $\mathcal{S}_{d,i,j}$ is the set of all stable graphs Γ such that $f(C) = C_{i,j}$ for every $[f : (C; p_1, p_2) \to X^{[3]}] \in \mathfrak{M}_\Gamma$, and $\mathcal{T}_{d,i}$ is the set of all stable graphs Γ such that $f(C) \subset C_{0,1}^{(i)} \cup C_{0,2}^{(i)} \cup C_{1,2}^{(i)}$ for every $[f : (C; p_1, p_2) \to X^{[3]}] \in \mathfrak{M}_\Gamma$.

Our goal of this section is to study $\langle \mathrm{PD}(\mathfrak{a}_{-3}(\ell)|0\rangle), \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)\rangle_{0,d}$. To apply the localization formula more effectively, we rewrite this 2-point invariant by using the Chern classes of tautological bundles over $X^{[3]} = (\mathbb{P}^2)^{[3]}$ defined in (3.13). Let

$$A = (c_1(\mathcal{E}_1) - c_1(\mathcal{E}_0))c_1(\mathcal{E}_0)^2 \quad \text{and} \quad B = c_1(\mathcal{E}_0)^2.$$

Intersecting (3.14) with $D_\ell = c_1(\mathcal{E}_1) - c_1(\mathcal{E}_0)$, we see that A is equal to

$$\begin{aligned} &3\mathfrak{a}_{-3}(\ell)|0\rangle - 3\mathfrak{a}_{-1}(X)\mathfrak{a}_{-1}(\ell)\mathfrak{a}_{-1}(x)|0\rangle \\ &- \frac{1}{2}\mathfrak{a}_{-1}(\ell)^3|0\rangle + 3\mathfrak{a}_{-1}(X)\mathfrak{a}_{-2}(x)|0\rangle + \frac{3}{2}\mathfrak{a}_{-2}(\ell)\mathfrak{a}_{-1}(\ell)|0\rangle. \end{aligned}$$

By Lemma 3.1, Lemma 3.2 and Lemma 3.4 (i), we obtain

$$\langle A, B\rangle_{0,d} = 3\ \langle \mathrm{PD}(\mathfrak{a}_{-3}(\ell)|0\rangle), \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)\rangle_{0,d} \tag{4.12}$$

where for notational simplicity, we make no distinction between the algebraic cycles A, B and their corresponding cohomology classes.

By the virtual localization formula of [G-P], we have

$$\langle A, B\rangle_{0,d} = \int_{[\overline{\mathfrak{M}}_{0,2}(X^{[3]}, d\beta_3)]^{\mathrm{vir}}} ev_2^*(A \otimes B) = \sum_\Gamma \frac{1}{|\mathbf{A}_\Gamma|} \int_{[M_\Gamma]^{\mathrm{vir}}} \frac{(A\otimes B)_\Gamma}{e(N_\Gamma^{\mathrm{vir}})}. \tag{4.13}$$

Here $[M_\Gamma]^{\mathrm{vir}}$ is the pullback of $[\mathfrak{M}_\Gamma]^{\mathrm{vir}}$ to M_Γ via the finite map $M_\Gamma \to \mathfrak{M}_\Gamma$. Likewise, $(A \otimes B)_\Gamma$ is the pullback of $ev_2^*(A \otimes B)|_{\mathfrak{M}_\Gamma}$ to M_Γ, and $e(N_\Gamma^{\mathrm{vir}})$ is the pullback of the Euler class of the moving part N_Γ^{vir} of the tangent-obstruction complex.

Let Γ be a stable graph such that the labeling $\mathfrak{L}$ maps the marked vertices of Γ to the same point in $(X^{[3]})^T$. Then we have $(A \otimes B)_\Gamma = (1_X \otimes AB)_\Gamma$ where $1_X \in H^0(X)$ is the fundamental cohomology class. By the fundamental class axiom, $\langle 1_X, AB\rangle_{0,d} = 0$. Thus in view of (4.13) and (4.11), we obtain

$$\begin{aligned} &\langle A, B\rangle_{0,d} = \langle A, B\rangle_{0,d} - \langle 1_X, AB\rangle_{0,d} \\ &= \sum_\Gamma \int_{[M_\Gamma]^{\mathrm{vir}}} \frac{(A\otimes B)_\Gamma - (1_X \otimes AB)_\Gamma}{|\mathbf{A}_\Gamma|\ e(N_\Gamma^{\mathrm{vir}})} = \sum_{1\le i\ne j\le 3}\ \sum_{\Gamma\in\mathcal{S}'_{d,i,j}} + \sum_{i=1}^{3}\ \sum_{\Gamma\in\mathcal{T}'_{d,i}} \end{aligned} \tag{4.14}$$

where the three prime signs indicate that we only sum over stable graphs Γ such that the two marked vertices of Γ have distinct labels in $(X^{[3]})^T$. In other words, putting $S'_{d,i,j} = \sum_{\Gamma\in\mathcal{S}'_{d,i,j}}$ and $T'_{d,i} = \sum_{\Gamma\in\mathcal{T}'_{d,i}}$, we have

$$\langle A, B\rangle_{0,d}\ =\ \sum_{1\le i\ne j\le 3} S'_{d,i,j}\ +\ \sum_{i=1}^{3} T'_{d,i}. \tag{4.15}$$

4.4. Computation of $S'_{d,i,j}$. Let $\mathcal{S}''_{d,i,j} = \mathcal{S}'_{d,i,j}/\sim$ where $\Gamma_1 \sim \Gamma_2$ if Γ_1 and Γ_2 are identical except that the vertex which is marked with $\{1\}$ (respectively, with $\{2\}$) in Γ_1 is marked with $\{2\}$ (respectively, with $\{1\}$) in Γ_2. Then each graph Γ in $\mathcal{S}''_{d,i,j}$ gives rise to two graphs Γ_1, Γ_2 in $\mathcal{S}'_{d,i,j}$. However, there is no ambiguity to define

$$e_{d,i,j} = \sum_{\Gamma\in\mathcal{S}''_{d,i,j}} \int_{[M_{\Gamma_1}]^{\mathrm{vir}}} \frac{1}{|\mathbf{A}_{\Gamma_1}|\ e(N_{\Gamma_1}^{\mathrm{vir}})}. \tag{4.16}$$

By the definition of $\mathcal{S}_{d,i,j}$, $f(C) = C_{i,j}$ for every stable map $[f : (C; p_1, p_2) \to X^{[3]}]$ in $\mathfrak{M}_{\Gamma_1}$ or $\mathfrak{M}_{\Gamma_2}$. Recall that $R_{i,j}^{(1)}$ and $R_{i,j}^{(2)}$ are the two T-fixed points in $C_{i,j}$. So

$$\int_{[M_{\Gamma_1}]^{\rm vir}} \frac{(A \otimes B)_{\Gamma_1} - (1_X \otimes AB)_{\Gamma_1}}{|\mathbf{A}_{\Gamma_1}| \; e(N_{\Gamma_1}^{\rm vir})} + \int_{[M_{\Gamma_2}]^{\rm vir}} \frac{(A \otimes B)_{\Gamma_2} - (1_X \otimes AB)_{\Gamma_2}}{|\mathbf{A}_{\Gamma_2}| \; e(N_{\Gamma_2}^{\rm vir})}$$
$$= -(A|_{R_{i,j}^{(1)}} - A|_{R_{i,j}^{(2)}})(B|_{R_{i,j}^{(1)}} - B|_{R_{i,j}^{(2)}}) \cdot \int_{[M_{\Gamma_1}]^{\rm vir}} \frac{1}{|\mathbf{A}_{\Gamma_1}| \; e(N_{\Gamma_1}^{\rm vir})}.$$

Combining this with Lemma 4.2 and (4.16), we conclude that

$$S'_{d,i,j} = -(2g_i + g_j)(w_i^2 - z_i^2)^2 \; e_{d,i,j}. \tag{4.17}$$

To compute $e_{d,i,j}$, we calculate the contribution from a graph Γ_1 by considering the restriction of the tangent-obstruction complex on $\overline{\mathfrak{M}}_{0,2}(X^{[3]}, d\beta_3)$ to $\mathfrak{M}_{\Gamma_1}$. Following [G-P], the fibers of its cohomology sheaves, $\mathcal{T}^1$ and $\mathcal{T}^2$, at a point associated to a stable map $[f : (C; p_1, p_2) \to X^{[3]}]$ fit into the exact sequence

$$\begin{array}{rlclclcl} 0 & \to & \mathrm{Ext}^0(\Omega_C(p_1+p_2), \mathcal{O}_C) & \to & H^0(C, f^*T_{X^{[3]}}) & \to & \mathcal{T}^1 & \\ & \to & \mathrm{Ext}^1(\Omega_C(p_1+p_2), \mathcal{O}_C) & \to & H^1(C, f^*T_{X^{[3]}}) & \to & \mathcal{T}^2 & \to \quad 0. \end{array}$$

To obtain the contribution of the moving parts of each term in the sequence, we use an analysis similar to that carried out for $\mathbb{P}^r$ in [G-P]. As was the case for $\mathbb{P}^r$, the fixed part $\mathcal{T}^{2,\mathrm{f}}$ vanishes. So the fixed stack is smooth with tangent bundle $\mathcal{T}^{1,\mathrm{f}}$. In particular $[\mathfrak{M}_{\Gamma_1}]^{\rm vir} = [\mathfrak{M}_{\Gamma_1}]$. As a result, denoting the contributions from the edges, vertices and flags of the graph Γ_1 by $e_{\Gamma_1}^{\rm e}, e_{\Gamma_1}^{\rm v}, e_{\Gamma_1}^{\rm F}$ respectively, we obtain

$$e(N_{\Gamma_1}^{\rm vir}) = e_{\Gamma_1}^{\rm e} \cdot e_{\Gamma_1}^{\rm v} \cdot e_{\Gamma_1}^{\rm F}. \tag{4.18}$$

First of all, we have $e_{\Gamma_1}^{\rm e} = \prod_e e(\chi(((f|_{C_e})^* T_{X^{[3]}})^{\rm m}))$ where $((f|_{C_e})^* T_{X^{[3]}})^{\rm m}$ denotes the moving part in $(f|_{C_e})^* T_{X^{[3]}}$. It follows from (4.10) that

$$e_{\Gamma_1}^{\rm e} = \prod_e \frac{(-1)^{d_e - 1}((d_e-1)!)^2 w_i w_j z_i z_j (w_i - z_i)^2}{(w_i + z_i) P(1 + \frac{2d_e w_i}{-w_i + z_i}, d_e - 1) P(1 - \frac{d_e(w_i + z_i)}{w_i - z_i}, d_e - 1)} \tag{4.19}$$

where $P(a, n)$ denotes the polynomial $a(a+1)\ldots(a+n-1)$.

Now the contributions of vertices and flags are given by

$$e_{\Gamma_1}^{\rm v} = \prod_v e(T_{\mathfrak{L}(v)}) \cdot \prod_{val(v)=n(v)=2} (\omega_{F_1(v)} + \omega_{F_2(v)}) \cdot \prod_{val(v)=n(v)=1} \omega_{F(v)}^{-1} \tag{4.20}$$
$$e_{\Gamma_1}^{\rm F} = \prod_{n(F) \geq 3} (\omega_F - e_F) \cdot \prod_F e(T_{i(F)})^{-1} \tag{4.21}$$

where for a flag $F = (v, e)$, we put $\omega_F = e(T_{i(F)} C_{i,j})/d_e$, and define e_F to be the first Chern class of the bundle on M_Γ whose fiber is the cotangent space of the component associated to v at the point corresponding to the flag F (c.f. [C-K, p.285]). Note that $T_{i(F)} = T_{\mathfrak{L}(v)}$ has been computed in (4.8) and (4.9). Thus, $\omega_F = (-w_i + z_i)/d_e$ if $i(F) = R_{i,j}^{(1)}$, and $\omega_F = (w_i - z_i)/d_e$ if $i(F) = R_{i,j}^{(2)}$.

4.5. Computation of $T'_{d,i}$. Recall from (4.14) and (4.11) that $\mathcal{T}'_{d,i}$ is the set of all stable graphs Γ such that $f(C) \subset C^{(i)}_{0,1} \cup C^{(i)}_{0,2} \cup C^{(i)}_{1,2}$ for every $[f : (C; p_1, p_2) \to X^{[3]}] \in \mathfrak{M}_\Gamma$, and that the marked vertices of Γ have distinct labels in $(X^{[3]})^T$. The T-fixed points in $C^{(i)}_{0,1} \cup C^{(i)}_{0,2} \cup C^{(i)}_{1,2}$ are $Q_{i,0}, Q_{i,1}, Q_{i,2}$. For $0 \le j < k \le 2$, let $\mathcal{T}'_{d,i,j,k}$ be the subset of $\mathcal{T}'_{d,i}$ consisting of all $\Gamma \in \mathcal{T}'_{d,i}$ such that the labeling $\mathfrak{L}$ maps the marked vertices of Γ to $\{Q_{i,j}, Q_{i,k}\}$. Then, $\mathcal{T}'_{d,i,0,1}$, $\mathcal{T}'_{d,i,0,2}$ and $\mathcal{T}'_{d,i,1,2}$ form a partition of $\mathcal{T}'_{d,i}$. So

$$\sum_{\Gamma \in \mathcal{T}'_{d,i}} = \sum_{\Gamma \in \mathcal{T}'_{d,i,0,1}} + \sum_{\Gamma \in \mathcal{T}'_{d,i,0,2}} + \sum_{\Gamma \in \mathcal{T}'_{d,i,1,2}} . \tag{4.22}$$

Put $\mathcal{T}''_{d,i,j,k} = \mathcal{T}'_{d,i,j,k}/\sim$ where the relation $\sim$ is defined the same way as in the first paragraph of subsection 4.4. As in (4.17) and (4.16), we get

$$\sum_{\Gamma \in \mathcal{T}'_{d,i,j,k}} \int_{[M_\Gamma]^{\mathrm{vir}}} \frac{(A \otimes B)_\Gamma - (1_X \otimes AB)_\Gamma}{|\mathbf{A}_\Gamma| \, e(N^{\mathrm{vir}}_\Gamma)} = \gamma_{i,j,k} \cdot f_{d,i,j,k} \tag{4.23}$$

where $\gamma_{i,j,k} = -(A|_{Q_{i,j}} - A|_{Q_{i,k}})(B|_{Q_{i,j}} - B|_{Q_{i,k}})$ and

$$f_{d,i,j,k} = \sum_{\Gamma \in \mathcal{T}''_{d,i,j,k}} \int_{[M_{\Gamma_1}]^{\mathrm{vir}}} \frac{1}{|\mathbf{A}_{\Gamma_1}| \, e(N^{\mathrm{vir}}_{\Gamma_1})}. \tag{4.24}$$

By Lemma 4.2, we have $\gamma_{i,0,1} = -3g_i(w_i^2 + 2w_iz_i - 8z_i^2)^2$, $\gamma_{i,0,2} = -3g_i(-8w_i^2 + 2w_iz_i + z_i^2)^2$ and $\gamma_{i,1,2} = -243g_i(w_i^2 - z_i^2)^2$. Combining (4.22) and (4.23) yields

$$\begin{aligned} T'_{d,i} &= \sum_{\Gamma \in \mathcal{T}'_{d,i}} \int_{[\mathfrak{M}_\Gamma]^{\mathrm{vir}}} \frac{(A \otimes B)_\Gamma - (1_X \otimes AB)_\Gamma}{e(N^{\mathrm{vir}}_\Gamma)} \\ &= \gamma_{i,0,1} \cdot f_{d,i,0,1} + \gamma_{i,0,2} \cdot f_{d,i,0,2} + \gamma_{i,1,2} \cdot f_{d,i,1,2}. \end{aligned} \tag{4.25}$$

The $f_{d,i,j,k}$ can be calculated via graph sums in a manner similar to the calculation of the $e_{d,i,j}$ in subsection 4.4. Note that if $f_{d,i,0,1}$ is written as a function of the variables w_i and z_i, then $f_{d,i,0,2}$ can be obtained from $f_{d,i,0,1}$ by switching w_i and z_i. Also, for an edge e of a stable graph Γ and for $0 \le j < k \le 2$, define $e \in [Q_{i,j}Q_{i,k}]$ if the labeling $\mathfrak{L}$ of T maps the two vertices of e to the set $\{Q_{i,j}, Q_{i,k}\}$. By Lemma 4.1, the curves $C^{(i)}_{0,1}$, $C^{(i)}_{0,2}$ and $C^{(i)}_{1,2}$ are homologous to β_3, β_3 and $3\beta_3$ respectively. Therefore, for each stable graph Γ, the edges e satisfy

$$\sum_{e \in [Q_{i,0}Q_{i,1}]} d_e + \sum_{e \in [Q_{i,0}Q_{i,2}]} d_e + \sum_{e \in [Q_{i,1}Q_{i,2}]} 3d_e = d. \tag{4.26}$$

4.6. Cases when $1 \le d \le 4$. When the degree d is small, we can use Mathematica and the setups of subsections 4.4 and 4.5 to make explicite computations. We now do this for $1 \le d \le 4$.

When $1 \le d \le 4$, we have verified via Mathematica that

$$e_{d,i,j} = \frac{w_i + z_i}{dw_iw_j(w_i - z_i)^2 z_iz_j} \quad \text{and} \quad S'_{d,i,j} = \frac{(2g_i + g_j)(w_i + z_i)^3}{dw_iw_jz_iz_j}. \tag{4.27}$$

Unfortunately, we are not able to prove this formula for general d.

Also, for $1 \le d \le 4$, the functions $f_{d,i,0,1}$ are given by

$$f_{1,i,0,1} = \frac{w_i + z_i}{w_i(w_i - 2z_i)^2(w_i - z_i)z_i^2} \tag{4.28}$$

$$\begin{aligned} f_{2,i,0,1} &= \frac{2w_i^2 + 7w_iz_i + 5z_i^2}{2w_i(w_i - 2z_i)^2(w_i - z_i)(2w_i - z_i)z_i^2} \\ &= \frac{1}{2}f_{1,i,0,1} + \frac{3(w_i + z_i)}{w_i(w_i - 2z_i)^2(w_i - z_i)(2w_i - z_i)z_i} \end{aligned} \tag{4.29}$$

$$\begin{aligned} f_{3,i,0,1} &= \frac{2(w_i + z_i)(w_i + 4z_i)}{3w_i(w_i - 2z_i)^2(w_i - z_i)(2w_i - z_i)z_i^2} \\ &= \frac{1}{3}f_{1,i,0,1} + \frac{3(w_i + z_i)}{w_i(w_i - 2z_i)^2(w_i - z_i)(2w_i - z_i)z_i} \end{aligned} \tag{4.30}$$

$$\begin{aligned} f_{4,i,0,1} &= \frac{2w_i^2 + 7w_iz_i + 5z_i^2}{4w_i(w_i - 2z_i)^2(w_i - z_i)(2w_i - z_i)z_i^2} \\ &= \frac{1}{4}f_{1,i,0,1} + \frac{3(w_i + z_i)}{2w_i(w_i - 2z_i)^2(w_i - z_i)(2w_i - z_i)z_i} \end{aligned} \tag{4.31}$$

Recall that if we regard $f_{d,i,0,1}$ as a function of z_i and w_i, then $f_{d,i,0,2}$ can be obtained from $f_{d,i,0,1}$ by switching z_i and w_i. So $f_{d,i,0,2}$ is known for $1 \le d \le 4$. Furthermore,

$$f_{1,i,1,2} = 0 \tag{4.32}$$

$$f_{2,i,1,2} = \frac{w_i + z_i}{w_i(w_i - 2z_i)(w_i - z_i)^2(2w_i - z_i)z_i} \tag{4.33}$$

$$f_{3,i,1,2} = \frac{w_i + z_i}{w_i(w_i - 2z_i)(w_i - z_i)^2(2w_i - z_i)z_i} \tag{4.34}$$

$$f_{4,i,1,2} = \frac{w_i + z_i}{2w_i(w_i - 2z_i)(w_i - z_i)^2(2w_i - z_i)z_i}. \tag{4.35}$$

Combining formulas (4.28)-(4.35) with (4.25), we conclude that

$$T'_{1,i} = \frac{-3g_i(w_i^3 - 6w_i^2z_i - 6w_iz_i^2 + z_i^3)}{w_i^2z_i^2} \tag{4.36}$$

$$T'_{2,i} = \frac{-3g_i(w_i^3 + 12w_i^2z_i + 12w_iz_i^2 + z_i^3)}{2w_i^2z_i^2} = \frac{1}{2}T'_{1,i} - \frac{27g_i(w_i + z_i)}{w_iz_i} \tag{4.37}$$

$$T'_{3,i} = \frac{-3g_i(w_i^3 + 21w_i^2z_i + 21w_iz_i^2 + z_i^3)}{3w_i^2z_i^2} = \frac{1}{3}T'_{1,i} - \frac{27g_i(w_i + z_i)}{w_iz_i} \tag{4.38}$$

$$T'_{4,i} = \frac{-3g_i(w_i^3 + 12w_i^2z_i + 12w_iz_i^2 + z_i^3)}{4w_i^2z_i^2} = \frac{1}{4}T'_{1,i} - \frac{27g_i(w_i + z_i)}{2w_iz_i}. \tag{4.39}$$

In view of formulas (4.15), (4.27) and (4.36)-(4.39), we obtain

$$\langle A, B\rangle_{0,1} = -81 \tag{4.40}$$

$$\langle A, B\rangle_{0,2} = -\frac{81}{2} + 81 = \frac{81}{2} \tag{4.41}$$

$$\langle A, B\rangle_{0,3} = -\frac{81}{3} + 81 = 54 \tag{4.42}$$

$$\langle A, B\rangle_{0,4} = -\frac{81}{4} + \frac{81}{2} = \frac{81}{4}. \tag{4.43}$$

Proposition 4.3. *Let* $X = \mathbb{P}^2$*, and* $\ell \subset X$ *be a line. Then, the* 2*-point genus-*0 *Gromov-Witten invariant* $\langle \mathrm{PD}(\mathfrak{a}_{-3}(\ell)|0\rangle), \mathrm{PD}(\mathfrak{a}_{-3}(X)|0\rangle)\rangle_{0,d}$ *is equal to* -27, $27/2$, 18 *and* $27/4$ *when* d *is equal to* 1, 2, 3 *and* 4 *respectively.*

Proof. Follows immediately from (4.12) and (4.40)-(4.43). □

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