

## MODULI SPACES OF $SL(r)$ -BUNDLES ON SINGULAR IRREDUCIBLE CURVES\*

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**Introduction.** One of the problems in moduli theory, motivated by physics, is to study the degeneration of moduli spaces of semistable  $G$ -bundles on curves of genus  $g \geq 2$ . When a smooth curve  $Y$  specializes to a stable curve  $X$ , one expects that the moduli space of semistable  $G$ -bundles on  $Y$  specializes to a (nice) moduli space of generalized semistable  $G$ -torsors on  $X$ . It is well known ([Si]) that for any flat family  $\mathcal{C} \rightarrow S$  of stable curves there is a family  $\mathcal{U}(r, d)_S \rightarrow S$  of moduli spaces  $\mathcal{U}_{\mathcal{C}_s}(r, d)$  of (s-equivalence classes of) semistable torsion free sheaves of rank  $r$  and degree  $d$  on curves  $\mathcal{C}_s$  ( $s \in S$ ). If we fix a suitable representation  $G \rightarrow GL(r)$ , one would like to define a moduli space  $\mathcal{U}_X(G)$  of suitable  $G$ -sheaves on  $X$  with at least a morphism  $\mathcal{U}_X(G) \rightarrow \mathcal{U}_X(r, d)$ . Moreover, it should behave well under specialization, i.e. if a smooth curve  $Y$  specializes to  $X$ , then the moduli space of  $G$ -bundles on  $Y$  specializes to  $\mathcal{U}_X(G)$ . By my knowledge, the problem is almost completely open except for special case like  $G = SO(r)$  or  $G = Sp(r)$  ([Fa1], [Fa2]), where one has a generalisation of  $G$ -torsors which extends the case  $G = GL(r)$ . It is open even for  $G = SL(r)$  (See [Fa1], [Fa2] for the introduction).

In this paper, we will consider the case  $G = SL(r)$  and  $X$  being irreducible (the case of a reducible curve with one node was studied in [Su2]). For any projective curve  $X$ , we will use  $\mathcal{U}_X(r, d)$  to denote the moduli space of semistable torsion free sheaves of rank  $r$  and degree  $d$  on  $X$ . If  $X_\eta$  is a smooth curve and  $L_\eta$  is a line bundle of degree  $d$  on  $X_\eta$ , we use  $\mathcal{U}_{X_\eta}(r, L_\eta)$  to denote the moduli space of semistable vector bundles of rank  $r$  with fixed determinant  $L_\eta$  on  $X_\eta$ , which is a closed subvariety of  $\mathcal{U}_{X_\eta}(r, d)$ . It is known that when  $X_\eta$  specializes to  $X$  the moduli space  $\mathcal{U}_{X_\eta}(r, d)$  specializes to  $\mathcal{U}_X(r, d)$ . It is natural to expect that if  $L_\eta$  specializes to a torsion free sheaf  $L$  on  $X$  then  $\mathcal{U}_{X_\eta}(r, L_\eta)$  specializes to a closed subscheme  $\mathcal{U}_X(r, L) \subset \mathcal{U}_X(r, d)$  (or a scheme with a morphism to  $\mathcal{U}_X(r, d)$ ). It is important that we should look for an intrinsic  $\mathcal{U}_X(r, L)$  (i.e. independent of  $X_\eta$ ) which should not be *too bad* and should represent a *moduli problem*.

Let  $S = \text{Spec}(A)$  where  $A$  is a discrete valuation ring, let  $\mathcal{C} \rightarrow S$  be a proper flat family of curves with closed fibre  $\mathcal{C}_0 \cong X$  and smooth generic fibre  $\mathcal{C}_\eta$ . Then we have a  $S$ -flat scheme  $\mathcal{U}(r, d)_S \rightarrow S$  with generic fibre  $\mathcal{U}_{\mathcal{C}_\eta}(r, d)$  and closed fibre being  $\mathcal{U}_X(r, d)$ . For any line bundle  $\mathcal{L}_\eta$  of degree  $d$  on  $\mathcal{C}_\eta$ , there is a unique extension  $\mathcal{L}$  on  $\mathcal{C}$  such that  $\mathcal{L}|_{\mathcal{C}_0} := L$  is torsion free of degree  $d$  (since  $X$  is irreducible). Then  $\mathcal{U}_{\mathcal{C}_\eta}(r, \mathcal{L}_\eta) \subset \mathcal{U}(r, d)_S$  is an irreducible, reduced, locally closed subscheme. Let

$$f : \mathcal{U}(r, \mathcal{L})_S := \overline{\mathcal{U}_{\mathcal{C}_\eta}(r, \mathcal{L}_\eta)} \subset \mathcal{U}(r, d)_S \rightarrow S$$

be the Zariski closure of  $\mathcal{U}_{\mathcal{C}_\eta}(r, \mathcal{L}_\eta)$  in  $\mathcal{U}(r, d)_S$ . Then  $f : \mathcal{U}(r, \mathcal{L})_S \rightarrow S$  is flat and projective, but there is no reason that its closed fibre  $f^{-1}(0)$  (even its support  $f^{-1}(0)_{\text{red}}$ )

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is independent of the family  $\mathcal{C} \rightarrow S$  and  $\mathcal{L}_\eta$ . However, there are conjectures ([NS]) that  $f^{-1}(0)$  is intrinsic for irreducible curves  $X$  with only one node. To state them, we introduce the notation for any stable irreducible curves. Let  $X$  be an irreducible stable curve with  $\delta$  nodes  $\{x_1, \dots, x_\delta\}$ , and  $L$  a torsion free sheaf of rank one and degree  $d$  on  $X$ . A torsion free sheaf  $F$  of rank  $r$  and degree  $d$  on  $X$  is called with a determinant  $L$  if there exists a morphism  $(\wedge^r F) \rightarrow L$  which is an isomorphism outside the nodes of  $X$ . The subset  $\mathcal{U}_X(r, L) \subset \mathcal{U}_X(r, d)$  consists of  $s$ -equivalence classes  $[F] \in \mathcal{U}_X(r, d)$  such that  $[F]$  contains a sheaf with a fixed determinant  $L$ . Then D.S. Nagaraj and C.S. Seshadri made the following conjectures (See Conjecture (a) and (b) at page 136 of [NS]):

(1) If  $L$  is a line bundle on  $X$  and  $\mathcal{U}_X(r, L)^0 \subset \mathcal{U}_X(r, L)$  is the subset of locally free sheaves, then  $\mathcal{U}_X(r, L)$  is the closure of  $\mathcal{U}_X(r, L)^0$  in  $\mathcal{U}_X(r, d)$ .

(2) Let  $\mathcal{L}_\eta$  (resp.  $L$ ) be a line bundle (resp. torsion free sheaf of rank one) of degree  $d$  on smooth curve  $Y$  (resp.  $X$ ). Assume that  $\mathcal{L}_\eta$  specializes to  $L$  as  $Y$  specializes to  $X$ . Then  $\mathcal{U}_X(r, L)$  is the specialization of  $\mathcal{U}_Y(r, \mathcal{L}_\eta)$ .

We answer (1) completely. In fact, even if  $L$  is not locally free (thus  $\mathcal{U}_X(r, L)$  contains no locally free sheaf), we prove that torsion free sheaves of type 1 (See Section 1) are dense in  $\mathcal{U}_X(r, L)$ .

**THEOREM 1.** *Let  $L$  be a torsion free sheaf of rank 1 and degree  $d$ . Define*

$$\mathcal{U}_X(r, L)^0 = \{F \in \mathcal{U}_X(r, L) \mid (\wedge^r F) \cong L\}$$

*which coincides with the subset of locally free sheaves when  $L$  is locally free. Then*

(1)  $\mathcal{U}_X(r, L)$  is the closure of  $\mathcal{U}_X(r, L)^0$ . If  $L$  is not locally free,  $\mathcal{U}_X(r, L)^0$  is the subset of torsion free sheaves of type 1.

(2) There is a canonical scheme structure on  $\mathcal{U}_X(r, L)^0$ , which is reduced when  $L$  is locally free, such that when smooth curve  $\mathcal{C}_\eta$  specializes to  $X$  and  $\mathcal{L}_\eta$  specializes to  $L$  on  $X$ , the specialization  $f^{-1}(0)$  of  $\mathcal{U}_{\mathcal{C}_\eta}(r, \mathcal{L}_\eta)$  contains a dense open subscheme which is isomorphic to  $\mathcal{U}_X(r, L)^0$ . In particular,

$$f^{-1}(0)_{\text{red}} \cong \mathcal{U}_X(r, L).$$

If the specialization  $f^{-1}(0)$  has no *embedded point*, then our theorem also proved Conjecture (2). Unfortunately,  $\mathcal{U}_X(r, L)$  seems not represent a nice moduli functor, we can not say anything about the scheme structure of  $\mathcal{U}_X(r, L)$ . To remedy this, we consider the specialization of  $\mathcal{U}_{\mathcal{C}_\eta}(r, \mathcal{L}_\eta)$  in the so called generalized Gieseker space  $G(r, d)$  (See [NSe]). Let  $X$  be an irreducible stable curve with only one node  $p_0$  and  $L$  be a line bundle of degree  $d$  on  $X$ . Then, when  $r = 2$ , we show that there is a Cohen-Macaulay closed subscheme  $G(r, L) \subset G(r, d)$  of pure dimension  $(r^2 - 1)(g - 1)$ , which represents a nice moduli functor (See Definition 3.2). Moreover,  $G(r, L)$  satisfies the requirements in (2) for specializations. It is known ([NSe] that there is a canonical birational morphism  $\theta : G(r, d) \rightarrow \mathcal{U}_X(r, d)$ . We prove in Lemma 3.4 that the *set-theoretic image* of  $G(r, L)$  is  $\mathcal{U}_X(r, L)$ . Thus we can endow  $\mathcal{U}_X(r, L)$  a scheme structure by the *scheme-theoretic image* of  $G(r, L)$ . Then we have

**THEOREM 2.** *Let  $X$  be an irreducible curve of genus  $g \geq 2$  with only one node  $p_0$ . Let  $L$  be a line bundle of degree  $d$  on  $X$ . Then, when  $r = 2$  and  $(2, d) = 1$ , we have*

(1) *There is a Cohen-Macaulay projective scheme  $G(2, L)$  of pure dimension*

$3(g - 1)$ , which represents a moduli functor.

(2) Let  $\mathcal{C} \rightarrow S$  be a proper family of curves over a discrete valuation ring, which has smooth generic fibre  $\mathcal{C}_\eta$  and closed fibre  $\mathcal{C}_0 \cong X$ . If there is a line bundle  $\mathcal{L}$  on  $\mathcal{C}$  such that  $\mathcal{L}|_{\mathcal{C}_0} \cong L$ . Then there exists an irreducible, reduced, Cohen-Macaulay  $S$ -projective scheme  $f : G(2, \mathcal{L})_S \rightarrow S$ , which represents a moduli functor, such that  $f^{-1}(0) \cong G(2, L)$ ,  $f^{-1}(\eta) \cong \mathcal{U}_{\mathcal{C}_\eta}(2, \mathcal{L}_\eta)$ .

(3) There exists a proper birational  $S$ -morphism  $\theta : G(2, \mathcal{L})_S \rightarrow \mathcal{U}(2, \mathcal{L})_S$  which induces a morphism  $\theta : G(2, L) \rightarrow \mathcal{U}_X(2, L)$ .

Theorem 1 is proved in Section 1. In Section 2, we introduce the objects which are used to define Gieseker moduli space. Then Theorem 2 is proved in Section 3.

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**1. Torsion-free sheaves with fixed determinant on irreducible curves.**

Let  $X$  be a stable irreducible curve of genus  $g$  with  $\delta$  nodes  $x_1, \dots, x_\delta$ . Any torsion free sheaf  $\mathcal{F}$  of rank  $r$  on  $X$  can be written into (locally at  $x_i$ )

$$\mathcal{F} \otimes \hat{\mathcal{O}}_{X, x_i} \cong \hat{\mathcal{O}}_{X, x_i}^{\oplus a_i} \oplus m_{x_i}^{\oplus (r-a_i)}.$$

We call that  $\mathcal{F}$  has type  $r - a_i$  at  $x_i$ . Let  $\mathcal{U}_X(r, d)$  be the moduli space of  $s$ -equivalence classes of semistable torsion free sheaves of rank  $r$  and degree  $d$  on  $X$ . Inspired by [NS], we make the following definition.

**DEFINITION 1.1.** Let  $L$  be a torsion free sheaf of rank one and degree  $d$  on  $X$ . A torsion free sheaf  $\mathcal{F}$  of rank  $r$  and degree  $d$  on  $X$  is called with a determinant  $L$  if there exists a non-trivial morphism  $\wedge^r \mathcal{F} \rightarrow L$  which is an isomorphism outside the nodes.

**LEMMA 1.2.** For any exact sequence  $0 \rightarrow \mathcal{F}_1 \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}_2 \rightarrow 0$  of torsion free sheaves with rank  $r_1, r, r_2$  respectively, we have a morphism

$$(\wedge^{r_1} \mathcal{F}_1) \otimes (\wedge^{r_2} \mathcal{F}_2) \rightarrow \frac{\wedge^r \mathcal{F}}{\text{torsion}},$$

which is an isomorphism outside the nodes. In particular, if a semistable sheaf  $\mathcal{F}$  has a fixed determinant  $L$ , then the associated graded torsion free sheaf  $gr(\mathcal{F})$  will also have the fixed determinant  $L$ .

*Proof.* There is a morphism  $\wedge^{r_2} \mathcal{F}_2 \rightarrow \text{Hom}(\wedge^{r_1} \mathcal{F}_1, \wedge^r \mathcal{F} / \text{torsion})$ , which locally is defined as follows: For any  $\omega \in \wedge^{r_2} \mathcal{F}_2$ , choose a preimage  $\tilde{\omega} \in \wedge^{r_2} \mathcal{F}$  with respect to  $\wedge^{r_2} \beta$ . Then the image of  $\omega$  is defined to be the morphism

$$\wedge^{r_1} \mathcal{F}_1 \rightarrow \wedge^r \mathcal{F} / \text{torsion},$$

which takes any  $f \in \wedge^{r_1} \mathcal{F}_1$  to the section  $(\wedge^{r_1} \alpha)(f) \wedge \tilde{\omega} \in \wedge^r \mathcal{F} / \text{torsion}$ , which does not depend on the choice of  $\tilde{\omega}$  since the image of  $\wedge^{r_1+1} \alpha$  is a torsion sheaf. The morphism defined above is isomorphism outside the nodes (See Lemma 1.2 of [KW]). Thus we have the desired morphism

$$(\wedge^{r_1} \mathcal{F}_1) \otimes (\wedge^{r_2} \mathcal{F}_2) \rightarrow (\wedge^{r_1} \mathcal{F}_1) \otimes \text{Hom}(\wedge^{r_1} \mathcal{F}_1, \frac{\wedge^r \mathcal{F}}{\text{torsion}}) \rightarrow \frac{\wedge^r \mathcal{F}}{\text{torsion}}.$$

DEFINITION 1.3. *The subset  $\mathcal{U}_X(r, L) \subset \mathcal{U}_X(r, d)$  and  $\mathcal{U}_X(r, L)^0 \subset \mathcal{U}_X(r, L)$  are defined to be*

$$\mathcal{U}_X(r, L) = \left\{ \begin{array}{l} s\text{-equivalence classes } [\mathcal{F}] \in \mathcal{U}_X(r, d) \text{ such that} \\ [\mathcal{F}] \text{ contains a sheaf with a fixed determinant } L \end{array} \right\}$$

$$\mathcal{U}_X(r, L)^0 = \{ [\mathcal{F}] \in \mathcal{U}_X(r, L) \mid \wedge^r \mathcal{F} \cong L \}$$

When  $L$  is a line bundle,  $\mathcal{U}_X(r, L)^0$  consists of locally free sheaves with the fixed determinant  $L$ . When  $L$  is not a line bundle,  $\mathcal{U}_X(r, L)^0$  consists of torsion free sheaves of type 1 at each node of  $X$ .

We first consider the case that  $L$  is a line bundle and  $X$  has only one node  $p_0$ . Let  $\pi : \tilde{X} \rightarrow X$  be the normalization with  $\pi^{-1}(p_0) = \{p_1, p_2\}$ . The normalization  $\phi : \mathcal{P} \rightarrow \mathcal{U}_X(r, d)$  was studied in [Su1], where  $\mathcal{P}$  is the moduli spaces of semistable generalized parabolic bundles (GPB) of degree  $d$  and rank  $r$  on  $\tilde{X}$ . A GPB of degree  $d$  and rank  $r$  on  $\tilde{X}$  is a pair  $(E, Q)$  consisting of a vector bundle  $E$  of degree  $d$  and rank  $r$  on  $\tilde{X}$  and a  $r$ -dimensional quotient  $E_{p_1} \oplus E_{p_2} \rightarrow Q$ . There is a flat morphism (See Lemma 5.7 of [Su1])

$$Det : \mathcal{P} \rightarrow J_{\tilde{X}}^d$$

sending  $(E, Q)$  to  $det(E)$ . Let  $\tilde{L} = \pi^*(L)$  and  $\mathcal{P}^{\tilde{L}} = Det^{-1}(\tilde{L})$ . Then  $\mathcal{P}^{\tilde{L}}$  is an irreducible projective variety (See the proof of Lemma 5.7 in [Su1]). Let  $\mathcal{D}_i$  ( $i = 1, 2$ ) be the divisor consisting of  $(E, Q)$  such that  $E_{p_i} \rightarrow Q$  is not an isomorphism (See [Su1] for details). Let  $\mathcal{D}_i^{\tilde{L}} = \mathcal{D}_i \cap \mathcal{P}^{\tilde{L}}$ .

LEMMA 1.4. *The set  $\mathcal{U}_X(r, L)$  is contained in the image  $\phi(\mathcal{P}^{\tilde{L}})$ . Moreover,*

$$\mathcal{U}_X(r, L) \setminus \mathcal{U}_X(r, L)^0 \subset \phi(\mathcal{D}_1^{\tilde{L}} \cap \mathcal{D}_2^{\tilde{L}}).$$

*Proof.* Let  $F \in \mathcal{U}_X(r, L)$  with  $F \otimes \hat{\mathcal{O}}_{p_0} \cong \hat{\mathcal{O}}_{p_0}^{\oplus a} \oplus m_{p_0}^{\oplus(r-a)}$ . Let  $\tilde{E} = \pi^*F/torsion$ . Then, by local computations (See, for example, Remark 2.1, 2.6 of [NS]), we have

$$(1.1) \quad 0 \rightarrow F \xrightarrow{d} \pi_*\tilde{E} \rightarrow_{p_0}\tilde{Q} \rightarrow 0$$

where  $dim(\tilde{Q}) = a$  and the quotient  $\pi_*\tilde{E} \rightarrow_{p_0}\tilde{Q}$  induces two surjective maps  $\tilde{E}_{p_i} \rightarrow \tilde{Q}$  ( $i = 1, 2$ ). Denote their kernel by  $K_i$ , we have

$$0 \rightarrow K_i \rightarrow \tilde{E}_{p_i} \rightarrow \tilde{Q} \rightarrow 0.$$

On the other hand, for  $F \in \mathcal{U}_X(r, L)$ , let  $\mathcal{Q}$  be the cokernel of  $\wedge^r F \rightarrow L$ , then

$$0 \rightarrow det(\tilde{E}) \rightarrow \tilde{L} \rightarrow \pi^*\mathcal{Q} \rightarrow 0$$

where  $\pi^*\mathcal{Q} = {}_{p_1}V_1 \oplus {}_{p_2}V_2$  and  $n_1, n_2$  is respectively the dimension of  $V_1, V_2$ . Thus  $det(\tilde{E}) = \tilde{L} \otimes \mathcal{O}_{\tilde{X}}(-n_1p_1 - n_2p_2)$  where  $n_i \geq 0$  and  $n_1 + n_2 = r - a$ .

Let  $h : \tilde{E} \rightarrow E$  be the Hecke modifications at  $p_1$  and  $p_2$  such that  $ker(h_{p_i}) \subset K_i$  has dimension  $n_i$  for  $i = 1, 2$ . Then we have

$$(1.2) \quad 0 \rightarrow \tilde{E} \xrightarrow{h} E \rightarrow {}_{p_1}\tilde{Q}_1 \oplus {}_{p_2}\tilde{Q}_2 \rightarrow 0$$

with  $\dim(\tilde{Q}_i) = n_i$ . Thus  $\det(E) = \det(\tilde{E}) \otimes \mathcal{O}_{\tilde{X}}(n_1p_1 + n_2p_2) = \tilde{L}$  and  $\phi(E, Q) = F$  if we define  $Q$  by the exact sequence

$$(1.3) \quad 0 \rightarrow F \xrightarrow{(\pi_*h) \cdot d} \pi_*E \rightarrow {}_{p_0}Q \rightarrow 0.$$

To describe the GPB  $(E, E_{p_1} \oplus E_{p_2} \xrightarrow{q} Q \rightarrow 0)$ , note that (1.3) induces

$$F_{p_0} \xrightarrow{d_{p_0}} \tilde{E}_{p_1} \oplus \tilde{E}_{p_2} \xrightarrow{h_{p_1} \oplus h_{p_2}} E_{p_1} \oplus E_{p_2} \xrightarrow{q} Q \rightarrow 0.$$

Then  $d_{p_0}(F_{p_0}) \cap \tilde{E}_{p_i} = K_i$  by (1.1) and  $h_{p_i}(K_i) = \ker(q_i)$  by the exactness of (1.3), where  $q_i : E_{p_i} \rightarrow Q$  ( $i = 1, 2$ ) are projections induced by  $E_{p_1} \oplus E_{p_2} \xrightarrow{q} Q \rightarrow 0$ . Thus  $\dim(\ker(q_i)) = r - a - n_i$  by the construction of  $h$ .

For any  $F \in \mathcal{U}_X(r, L) \setminus \mathcal{U}_X(r, L)^0$ , the cokernel  $\mathcal{Q}$  of  $\wedge^r F \rightarrow L$  must be non-trivial. This implies that both  $V_1$  and  $V_2$  in  $\pi^*\mathcal{Q} = {}_{p_1}V_1 \oplus {}_{p_2}V_2$  are non-trivial since for any  $i = 1, 2$ , we have

$$\text{Hom}_{\mathcal{O}_{\tilde{X}}}({}_{p_i}V_i, {}_{p_i}\mathbb{C}) = \text{Hom}_{\mathcal{O}_{\tilde{X}}}(\pi^*\mathcal{Q}, {}_{p_i}\mathbb{C}) = \text{Hom}_{\mathcal{O}_X}(\mathcal{Q}, \pi_*({}_{p_i}\mathbb{C})) \neq 0.$$

Thus their dimensions  $n_1$  and  $n_2$  must be positive and  $n_1 + n_2 = r - a$ , which means that  $\ker(q_i) \neq 0$  ( $i = 1, 2$ ) and the GPB  $(E, Q)$  must be in  $\mathcal{D}_1 \cap \mathcal{D}_2$ . Thus

$$\mathcal{U}_X(r, L) \setminus \mathcal{U}_X(r, L)^0 \subset \phi(\mathcal{D}_1^{\tilde{L}} \cap \mathcal{D}_2^{\tilde{L}}).$$

*Remark 1.5.* This is also indicated in the following consideration. There is a  $\mathbb{P}^1$ -bundle  $p : \mathbb{P} \rightarrow J_{\tilde{X}}^d$  and the normalization map  $\phi_1 : \mathbb{P} \rightarrow J_X^d$ . The morphism  $\text{Det} : \mathcal{P} \rightarrow J_{\tilde{X}}^d$  can be lift to a rational morphism

$$\widetilde{\text{Det}} : \mathcal{P} \dashrightarrow \mathbb{P} \xrightarrow{\phi_1} J_X^d,$$

which is well-defined on  $\mathcal{P} \setminus \mathcal{D}_1 \cap \mathcal{D}_2$ . When  $L$  is a line bundle,  $\widetilde{\text{Det}}^{-1}(L)$  is disjoint with  $\mathcal{D}_i \setminus (\mathcal{D}_1 \cap \mathcal{D}_2)$ .

**LEMMA 1.6.** *Let  $\Lambda$  be a discrete valuation ring and  $T = \text{Spec}(\Lambda)$ . Then, for any  $F \in \mathcal{U}_X(r, L)$ , there is a  $T$ -flat sheaf  $\mathcal{F}$  on  $X \times T$  such that*

- (1)  $\mathcal{F}_t = \mathcal{F}|_{X \times \{t\}}$  is locally free for  $t \neq 0$  and  $\mathcal{F}_0 = F$ ,
- (2)  $\wedge^r(\mathcal{F}|_{X \times (T \setminus \{0\})}) = p_X^*L$ .

*In particular,  $\mathcal{U}_X(r, L)^0$  is dense in  $\mathcal{U}_X(r, L)$ .*

*Proof.* Let  $(E, Q) \in \mathcal{P}^{\tilde{L}}$  be the GPB such that  $\phi(E, Q) = F$  (Lemma 1.4). Then there exists a  $T$ -flat family of vector bundles  $\mathcal{E}$  on  $\tilde{X} \times T$  with  $\det(\mathcal{E}) = p_{\tilde{X}}^*\tilde{L}$ , and a  $T$ -flat quotient

$$\mathcal{E}_{p_1} \oplus \mathcal{E}_{p_2} \xrightarrow{q} \mathcal{Q} \rightarrow 0$$

such that  $(\mathcal{E}_0, \mathcal{Q}_0) = (\mathcal{E}, \mathcal{Q})|_{\tilde{X} \times \{0\}} = (E, Q)$ . The quotient  $\mathcal{E}_{p_1} \oplus \mathcal{E}_{p_2} \xrightarrow{q} \mathcal{Q} \rightarrow 0$  is determined by the two projections  $q_i : \mathcal{E}_{p_i} \rightarrow \mathcal{Q}$  ( $i = 1, 2$ ), which can be chosen to be isomorphisms for  $t \neq 0$  since  $\mathcal{P}^{\tilde{L}}$  is irreducible. The two maps  $q_i$  are given by two matrices

$$\begin{pmatrix} t^{a_1} & 0 & \dots & 0 \\ 0 & t^{a_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{a_r} \end{pmatrix}, \quad \begin{pmatrix} t^{b_1} & 0 & \dots & 0 \\ 0 & t^{b_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & t^{b_r} \end{pmatrix}$$

where  $0 \leq a_1 \leq a_2 \leq \dots \leq a_r$  and  $0 \leq b_1 \leq b_2 \leq \dots \leq b_r$ . When  $t = 0$ , they give the GPB  $(E, Q)$ . We recall that when  $F$  is not locally free, the numbers  $n_1$  and  $n_2$  in the proof of Lemma 1.4 are positive. Thus the two projections  $E_{p_i} \rightarrow Q$  are not isomorphism. Namely, there are  $k_1, k_2$  such that  $a_{k_1} > 0, b_{k_2} > 0$  but  $a_j = 0$  ( $j < k_1$ ) and  $b_j = 0$  ( $j < k_2$ ). It is clear now that we can change the positive numbers  $a_{k_1}, \dots, a_r, b_{k_2}, \dots, b_r$  freely such that the resulted family  $(\mathcal{E}, \mathcal{Q})$  has the property that  $(\mathcal{E}_0, \mathcal{Q}_0) = (E, Q)$ . We modify the  $T$ -flat quotient  $\mathcal{E}_{p_1} \oplus \mathcal{E}_{p_2} \xrightarrow{q} \mathcal{Q} \rightarrow 0$  by choosing  $a_{k_1}, \dots, a_r, b_{k_2}, \dots, b_r$  such that

$$\sum_{i=k_1}^r a_i - \sum_{i=k_2}^r b_i = 0.$$

Thus we get a  $T$ -flat sheaf  $\mathcal{F}$  on  $X \times T$  such that  $\mathcal{F}_0 = F$ . Moreover, on  $T \setminus \{0\}$ ,  $\mathcal{F}$  is obtained from  $\mathcal{E}|_{\tilde{X} \times (T \setminus \{0\})}$  by identifying  $\mathcal{E}_{p_1}$  and  $\mathcal{E}_{p_2}$  through the isomorphism

$$q_1 \cdot q_2^{-1} : \mathcal{E}_{p_1} \rightarrow \mathcal{E}_{p_2}.$$

$\wedge^r \mathcal{F}|_{X \times (T \setminus \{0\})}$  is obtained from  $\det(\mathcal{E})|_{\tilde{X} \times (T \setminus \{0\})} = p_{\tilde{X}}^* \tilde{L}$  by identifying  $\tilde{L}_{p_1} \otimes K(T)$  and  $\tilde{L}_{p_2} \otimes K(T)$  through the isomorphism  $\wedge^r(q_1 \cdot q_2^{-1})$ , where  $K(T)$  denote the field of rational functions on  $T$ . By the choice of  $a_{k_1}, \dots, a_r, b_{k_2}, \dots, b_r$ , we know that  $\wedge^r(q_1 \cdot q_2^{-1})$  is the identity map. Thus

$$\wedge^r \mathcal{F}|_{X \times (T \setminus \{0\})} = (p_X^* L)|_{X \times (T \setminus \{0\})}.$$

LEMMA 1.7. *For any stable irreducible curve  $X$ ,  $\mathcal{U}_X(r, L)^0$  is dense in  $\mathcal{U}_X(r, L)$ .*

*Proof.* Let  $\delta$  be the number of nodes of  $X$ , we will prove the lemma by induction to  $\delta$ . When  $\delta = 1$ , it is Lemma 1.6. Assume that the lemma is true for curves with  $\delta - 1$  nodes. Then we show that for any  $F \in \mathcal{U}_X(r, L)$  there is a  $T$ -flat sheaf  $\mathcal{F}$  on  $X \times T$ , where  $T = \text{Spec}(\Lambda)$  and  $\Lambda$  is a discrete valuation ring, such that

- (1)  $\mathcal{F}_t = \mathcal{F}|_{X \times \{t\}}$  is locally free for  $t \neq 0$  and  $\mathcal{F}_0 = F$ ,
- (2)  $\wedge^r(\mathcal{F}|_{X \times (T \setminus \{0\})}) = p_X^* L$ .

For  $F \in \mathcal{U}_X(r, L)$ , we can assume that  $F$  is not locally free. Let  $p_0 \in X$  be a node at where  $F$  is not locally free. Let  $\pi : \tilde{X} \rightarrow X$  be the partial normalization at  $p_0$  and  $\pi^{-1}(p_0) = \{p_1, p_2\}$ . Let  $\tilde{L} = \pi^* L$  and  $\tilde{E} = \pi^* F / \text{torsion}$ , then by the same arguments of Lemma 1.4

$$0 \rightarrow F \xrightarrow{d} \pi_* \tilde{E} \rightarrow_{p_0} \tilde{Q} \rightarrow 0.$$

Note that  $\wedge^r \tilde{E} = \pi^*(\wedge^r F) / (\text{torsion at } \{p_1, p_2\})$  and the cokernel of  $\wedge^r \tilde{E} \rightarrow \tilde{L}$  at  $\{p_1, p_2\}$  is  $_{p_1} \mathbb{C}^{n_1} \oplus_{p_2} \mathbb{C}^{n_2}$ , we have the morphism

$$\wedge^r \tilde{E} \rightarrow \tilde{L} \otimes \mathcal{O}_{\tilde{X}}(-n_1 p_1 - n_2 p_2)$$

which is an isomorphism outside the nodes of  $\tilde{X}$ . As the same with proof of Lemma 1.4, we have the Hecke modification  $E$  of  $\tilde{E}$  at  $p_1$  and  $p_2$  such that

$$0 \rightarrow \tilde{E} \xrightarrow{h} E \rightarrow_{p_1} \tilde{Q}_1 \oplus_{p_2} \tilde{Q}_2 \rightarrow 0$$

with  $\dim(\tilde{Q}_i) = n_i$ . Thus  $\wedge^r E \cong (\wedge^r \tilde{E}) \otimes \mathcal{O}_{\tilde{X}}(n_1 p_1 + n_2 p_2) \rightarrow \tilde{L}$  and the generalized parabolic sheaf (GPS)  $(E, Q)$  defines  $F$  by the exact sequence

$$0 \rightarrow F \xrightarrow{(\pi_* h) \cdot d} \pi_* E \rightarrow_{p_0} Q \rightarrow 0,$$

where  $Q$  is defined by requiring above sequence exact. The two projections  $E_{p_i} \rightarrow Q$  ( $i = 1, 2$ ) are not isomorphism, thus, by choosing suitable bases of  $E_{p_1}$  and  $Q$ , they are given by matrices

$$P_1 = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}, \quad P_2 = A \cdot \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \cdot B$$

where  $A, B$  are invertable  $r \times r$  matrices and  $\mathrm{rank}(P_i) = r_i < r$  ( $i = 1, 2$ ). Since  $E \in \mathcal{U}_{\tilde{X}}(r, \tilde{L})$ , by the assumption, there is a  $T$ -flat sheaf  $\mathcal{E}$  on  $\tilde{X} \times T$  such that  $\mathcal{E}_0 := \mathcal{E}|_{\tilde{X} \times \{0\}} = E$  and  $\mathcal{E}|_{\tilde{X} \times (T \setminus \{0\})}$  locally free with determinant  $p_{\tilde{X}}^*(\tilde{L})$ . Define the morphisms  $q_i : \mathcal{E}_{p_i} := \mathcal{E}|_{\{p_i\} \times T} \rightarrow Q \otimes \mathcal{O}_T$  ( $i = 1, 2$ ) by using matrices

$$Q_1 = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \\ 0 & \dots & 1 & 0 \dots & 0 \\ 0 & \dots & 0 & t^{a_{r_1+1}} \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \\ 0 & \dots & 0 & \dots & c \cdot t^{a_r} \end{pmatrix}, \quad Q_2 = A \cdot \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \\ 0 & \dots & 1 & 0 \dots & 0 \\ 0 & \dots & 0 & t^{b_{r_2+1}} \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \\ 0 & \dots & 0 & \dots & t^{b_r} \end{pmatrix} \cdot B$$

where  $t$  is the local parameter of  $\Lambda$ ,  $a_{r_1+1}, \dots, a_r, b_{r_2+1}, \dots, b_r$  are positive integers satisfying  $a_{r_1+1} + \dots + a_r = b_{r_2+1} + \dots + b_r$ , and  $c$  is any constant. Then these morphisms  $q_i$  ( $i = 1, 2$ ) define a family  $(\mathcal{E}, Q \otimes \mathcal{O}_T)$  of GPS, which induces a  $T$ -flat sheaf  $\mathcal{F}$  on  $X \times T$  such that  $\mathcal{F}_0 = F$  and  $\mathcal{F}_t$  ( $t \neq 0$ ) are locally free. The determinant  $\det(\mathcal{F}|_{X \times T^0})$ , where  $T^0 = T \setminus \{0\}$ , is defined by the sheaf  $(\det(\mathcal{E}|_{\tilde{X} \times T^0})) = p_{\tilde{X}}^*(\tilde{L})$  through the isomorphism

$$\det(q_2^{-1} \cdot q_1) : (\det(\mathcal{E}|_{\tilde{X} \times T^0})_{p_1}) = (\wedge^r \mathcal{E}_{p_1})|_{T^0} \rightarrow (\wedge^r \mathcal{E}_{p_2})|_{T^0} = (\det(\mathcal{E}|_{\tilde{X} \times T^0})_{p_2}),$$

which is a scale product by  $\det(Q_2^{-1} \cdot Q_1) = \det(AB)^{-1} \cdot c$ . Thus we can choose suitable constant  $c$  such that  $\det(\mathcal{F}|_{X \times T^0}) = p_X^*(L)$ . We are done.

**LEMMA 1.8.** *When  $L$  is not locally free,  $\mathcal{U}_X(r, L)^0$  consists of torsion free sheaves of type 1 at each node of  $X$ , which is dense in  $\mathcal{U}_X(r, L)$ .*

*Proof.* The proof follows the same idea. For simplicity, we assume that  $X$  has only one node  $p_0$ . Let  $F$  be a torsion free sheaf of rank  $r$  and degree  $d$  on  $X$  with type  $t(F) \geq 1$  at  $p_0$ . Then

$$\deg(\wedge^r F / \text{torsion}) = d - t(F) + 1.$$

Thus  $F \in \mathcal{U}_X(r, L)^0$  if and only if  $t(F) = 1$ .

For any  $F \in \mathcal{U}_X(r, L)$  of type  $t(F) > 1$ , let  $\tilde{E} = \pi^* F / \text{torsion}$ , then

$$0 \rightarrow F \xrightarrow{d} \pi_* \tilde{E} \rightarrow_{p_0} \tilde{Q} \rightarrow 0$$

where  $\dim(\tilde{Q}) = r - t(F)$ . Let  $\tilde{L} = \pi^*L/\text{torsion}$ , then  $\deg(\tilde{L}) = d - 1$  and  $L = \pi_*\tilde{L}$ . The condition  $F \in \mathcal{U}_X(r, L)$  implies  $\det(\tilde{E}) = \tilde{L}(-n_1p_1 - n_2p_2)$  where  $n_i \geq 0$  and  $n_1 + n_2 = t(F) - 1$ . As in the proof of Lemma 1.4, let  $h : \tilde{E} \rightarrow E$  be the Hecke modifications at  $p_1$  and  $p_2$  such that  $\dim(\ker(h_{p_1})) = n_1 + 1$  and  $\dim(\ker(h_{p_2})) = n_2$ . Then we have  $\det(E) = \det(\tilde{E}) \otimes \mathcal{O}_{\tilde{X}}((n_1 + 1)p_1 + n_2p_2) = \tilde{L}(p_1)$ , and there is an GPB  $(E, E_{p_1} \oplus E_{p_2} \xrightarrow{q} Q \rightarrow 0)$  such that  $\phi(E, Q) = F$ , where  $q_i : E_{p_i} \rightarrow Q$  ( $i = 1, 2$ ) satisfy  $\dim(\ker(q_1)) = t(F) - n_1 - 1$  and  $\dim(\ker(q_2)) = t(F) - n_2$ . The two projections  $q_i : E_{p_i} \rightarrow Q$  ( $i = 1, 2$ ) are given by matrices

$$P_1 = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix}, \quad P_2 = A \cdot \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \cdot B$$

where  $\text{rank}(P_1) = r - t(F) + n_1 + 1$ ,  $\text{rank}(P_2) = r - t(F) + n_2$ . Let  $T = \text{Spec}(\mathbb{C}[t])$  and  $\mathcal{E} = p_{\tilde{X}}^*E$ . Choose deformations  $P_i(t)$  of  $P_i$  ( $i = 1, 2$ ) as following

$$\begin{pmatrix} 1 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & & & & \\ 0 & \dots & 1 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & t & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & t & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & t \end{pmatrix}, \quad A \cdot \begin{pmatrix} 1 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & & & & \\ 0 & \dots & 1 & 0 & \dots & 0 & 0 \\ 0 & \dots & 0 & t & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & t & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \cdot B$$

where the number of  $t$  in  $P_2(t)$  is  $t(F) - n_2 - 1$ . Then we get a family  $(\mathcal{E}, Q \otimes \mathcal{O}_T)$  of GPB on  $\tilde{X} \times T$ , which induces a  $T$ -flat sheaf  $\mathcal{F}$  on  $X \times T$  such that  $\mathcal{F}_0 = F$  and  $\mathcal{F}_t$  ( $t \neq 0$ ) are torsion free of type 1. To see that  $\wedge^r \mathcal{F}_t \cong L$  ( $t \neq 0$ ), we note that  $\det(\mathcal{E}) = p_{\tilde{X}}^*\tilde{L}(p_1)$  and  $L$  is determined by the GPB

$$(\det(E) = \tilde{L}(p_1), G \subset \det(E)_{p_1} \oplus \det(E)_{p_2})$$

where  $G$  is the graph of zero map  $\det(E)_{p_2} \rightarrow \det(E)_{p_1}$ . Thus we have a non-trivial morphism  $\wedge^r \mathcal{F}_t \rightarrow L$ , which must be an isomorphism when  $t \neq 0$ .

Next we will prove that  $\mathcal{U}_X(r, L)$  is the underlying scheme of specialization of the moduli spaces of semistable bundles with fixed determinant. This in particular implies that  $\mathcal{U}_X(r, L) \subset \mathcal{U}_X(r, d)$  is a closed subset. Let  $S = \text{Spec}(R)$  and  $R$  be a discrete valuation ring. Let  $\mathcal{X} \rightarrow S$  be a flat proper family of curves with smooth generic fibre and closed fibre  $\mathcal{X}_0 = X$ . Let  $\mathcal{L}$  be a relative torsion free sheaf on  $\mathcal{X}$  of rank one and (relative) degree  $d$  such that  $\mathcal{L}|_X = L$ . It is well known that there exists a moduli scheme  $f : \mathcal{U}(r, d)_S \rightarrow S$  such that for any  $s \in S$  the fibre  $f^{-1}(s)$  is the moduli space of semistable torsion free sheaves of rank  $r$  and degree  $d$  on  $\mathcal{X}_s$  (where  $\mathcal{X}_s$  denote the fibre of  $\mathcal{X} \rightarrow S$  at  $s$ ). Since  $\mathcal{X}$  is smooth over  $S^0 = S \setminus \{0\}$ , there is a family  $\mathcal{U}(r, \mathcal{L}|_{S^0})_{S^0} \rightarrow S^0$  of moduli spaces of semistable bundles with fixed determinant  $\mathcal{L}|_{\mathcal{X}_s}$  on  $\mathcal{X}_s$  ( $s \in S^0$ ). We have

$$\mathcal{U}(r, \mathcal{L}|_{S^0})_{S^0} \subset \mathcal{U}(r, d)_S.$$

Let  $Z$  be the Zariski closure of  $\mathcal{U}(r, \mathcal{L}|_{S^0})_{S^0}$  inside  $\mathcal{U}(r, d)_S$ . We get a flat family

$$f : Z \rightarrow S$$

of projective schemes. For any  $0 \neq s \in S$ , the fibre  $Z_s$  is the moduli space of semistable bundles on  $\mathcal{X}_s$  with fixed determinant  $\mathcal{L}|_{\mathcal{X}_s}$ .

LEMMA 1.9. *The fibre  $Z_0$  of  $f : Z \rightarrow S$  at  $s = 0$  is contained in  $\mathcal{U}_X(r, L)$  as a set.*

*Proof.* We can assume that for any  $[F] \in Z_0$  there is a discrete valuation ring  $A$  and  $T = \mathrm{Spec}(A) \rightarrow S$  such that there is a  $T$ -flat family of torsion free sheaf  $\mathcal{F}$  on  $\mathcal{X}_T = \mathcal{X} \times_S T \rightarrow T$ , so that

$$\wedge^r \mathcal{F}_\eta \cong \mathcal{L}_\eta, \quad \mathcal{F}|_X \cong F.$$

By Proposition 5.3 of [Se] and its proof (see [Se], it deals with one node curve, but generalization to our case is straightforward since its proof is completely local), there is a birational morphism  $\sigma : \Gamma \rightarrow \mathcal{X}_T$  and a vector bundle  $\mathcal{E}$  on  $\Gamma$  such that  $\sigma_* \mathcal{E} = \mathcal{F}$ . Moreover, the morphism  $\sigma$  is an isomorphism over  $\mathcal{X}_T \setminus \{x_1, \dots, x_k\}$ . Since  $(\wedge^r \mathcal{E})|_{\Gamma_\eta} \cong (\sigma^* \mathcal{L})^{\vee\vee}|_{\Gamma_\eta}$ , note that  $(\wedge^r \mathcal{E})^{-1} \otimes (\sigma^* \mathcal{L})^{\vee\vee}$  is torsion free (thus  $T$ -flat), we can extend the isomorphism into a morphism  $\wedge^r \mathcal{E} \rightarrow (\sigma^* \mathcal{L})^{\vee\vee}$ . Since  $\sigma_*$  and  $\sigma^*$  are adjoint functors,  $\sigma_* \mathcal{O}_\Gamma = \mathcal{O}_{\mathcal{X}_T}$ , we have  $\sigma_*((\sigma^* \mathcal{N})^\vee) = \mathcal{N}^\vee$  for any coherent sheaf  $\mathcal{N}$ . Then, by using  $\sigma^*(\mathcal{N}^\vee) = \sigma^* \sigma_*((\sigma^* \mathcal{N})^\vee) \rightarrow (\sigma^* \mathcal{N})^\vee$ , we have a canonical morphism  $\sigma_*((\sigma^* \mathcal{N})^{\vee\vee}) \rightarrow \mathcal{N}^{\vee\vee}$ . In particular, there is a canonical morphism

$$\sigma_*((\sigma^* \mathcal{L})^{\vee\vee}) \rightarrow \mathcal{L}^{\vee\vee} \cong \mathcal{L}$$

which induce a morphism  $\vartheta : \wedge^r \mathcal{F} = \wedge^r(\sigma_* \mathcal{E}) \rightarrow \sigma_* \wedge^r \mathcal{E} \rightarrow \mathcal{L}$ . Modified by some power of the maximal ideal of  $A$ , we can assume the morphism  $\vartheta$  being nontrivial on  $X$ , which means that  $\vartheta$  is an isomorphism on  $\mathcal{X}_T \setminus \{x_1, \dots, x_k\}$  since  $X$  is irreducible. Thus  $[F] \in \mathcal{U}_X(r, L)$ .

THEOREM 1.10.  *$\mathcal{U}_X(r, L)$  is the closure of  $\mathcal{U}_X(r, L)^0$  in  $\mathcal{U}_X(r, d)$ . When smooth curve  $\mathcal{X}_s$  specializes to  $\mathcal{X}_0 = X$  and  $\mathcal{L}_s$  specializes to  $L$ , the moduli spaces  $\mathcal{U}_{\mathcal{X}_s}(r, \mathcal{L}_s)$  of semistable bundles of rank  $r$  with fixed determinant  $\mathcal{L}_s$  on  $\mathcal{X}_s$  specializes to an irreducible scheme  $Z_0$  with  $(Z_0)_{\mathrm{red}} \cong \mathcal{U}_X(r, L)$ .*

*Proof.* Let  $\mathcal{U}(r, d)_S^0 \subset \mathcal{U}(r, d)_S$  be the open subscheme of torsion free sheaves of type at most 1. Then there is a well-defined  $S$ -morphism (taking determinant  $\det(\bullet) = \wedge^r(\bullet)$ )

$$\det : \mathcal{U}(r, d)_S^0 \rightarrow \mathcal{U}(1, d)_S.$$

The given family of torsion free sheaves  $\mathcal{L}$  on  $\mathcal{X}$  of rank one and degree  $d$  gives a  $S$ -point  $[\mathcal{L}] \in \mathcal{U}(1, d)_S$ . It is clear that

$$Z^0 := (\det)^{-1}([\mathcal{L}]) \subset Z$$

and the fibre of  $f|_{Z^0} : Z^0 \rightarrow S$  at  $s = 0$  is irreducible with support  $\mathcal{U}_X(r, L)^0$  (it is also reduced when  $L$  is a line bundle). Thus  $f^{-1}(0) = Z_0$  contains the closure  $\overline{\mathcal{U}_X(r, L)^0}$  of  $\mathcal{U}_X(r, L)^0$  in  $\mathcal{U}_X(r, d)$ . On the other hand, by Lemma 1.9, Lemma 1.8 and Lemma 1.7, we have

$$\overline{\mathcal{U}_X(r, L)^0} \subset (Z_0)_{\mathrm{red}} \subset \mathcal{U}_X(r, L) \subset \overline{\mathcal{U}_X(r, L)^0}.$$

Hence  $\mathcal{U}_X(r, L) = \overline{\mathcal{U}_X(r, L)^0} = (Z_0)_{\text{red}}$ . In particular, the fibre of  $f : Z \rightarrow S$  at  $s = 0$  is irreducible.

**2. Stability and Gieseker functor.** Let  $X$  be a stable curve with  $\delta$  nodes  $\{x_1, \dots, x_\delta\}$ . Any semistable curve with stable model  $X$  can be obtained from  $X$  by destabilizing the nodes  $x_i$  with chains  $R_i$  ( $i = 1, \dots, \delta$ ) of projective lines. It will be denoted as  $X_{\vec{n}}$ , where  $\vec{n} = (n_1, \dots, n_\delta)$  and  $n_i$  is the length of  $R_i$  (See [NSE] for the example of  $\delta = 1$ ). Then  $X_{\vec{n}}$  are the curves which are semi-stably equivalent to  $X$ , we use  $\pi : X_{\vec{n}} \rightarrow X$  to denote the canonical morphism contracting  $R_1, \dots, R_\delta$  to  $x_1, \dots, x_\delta$  respectively. A vector bundle  $E$  of rank  $r$  on a chain  $R = \cup C_i$  of projective lines is called *positive* if  $a_{ij} \geq 0$  in the decomposition  $E|_{C_i} = \oplus_{j=1}^r \mathcal{O}(a_{ij})$  for all  $i$  and  $j$ . A *positive*  $E$  is called *strictly positive* if for each  $C_i$  there is at least one  $a_{ij} > 0$ .  $E$  is called *standard* (resp. *strictly standard*) if it is positive (resp. strictly positive) and  $a_{ij} \leq 1$  for all  $i$  and  $j$  (See [NSE], [Se]).

For any semistable curve  $X_{\vec{n}} = \cup X_{\vec{n}}^k$  of genus  $g \geq 2$ , let  $\omega_{X_{\vec{n}}}$  be its canonical bundle and

$$\lambda_k = \frac{\text{deg}(\omega_{X_{\vec{n}}}|_{X_{\vec{n}}^k})}{2g - 2},$$

it is easy to see that  $\lambda_k = 0$  if and only if the irreducible component  $X_{\vec{n}}^k$  is a component of the chains of projective lines.

DEFINITION 2.1. A sheaf  $E$  of constant rank  $r$  on  $X_{\vec{n}}$  is called (semi)stable, if for every subsheaf  $F \subset E$ , we have

$$\chi(F) < (\leq) \frac{\chi(E)}{r} \cdot r(F) \quad \text{when } r(F) \neq 0, r,$$

$$\chi(F) \leq 0 \quad \text{when } r(F) = 0, \text{ and } \chi(F) < \chi(E) \text{ when } r(F) = r, F \neq E,$$

where, for any sheaf  $F$ , the rank  $r(F)$  is defined to be  $\sum \lambda_k \cdot \text{rank}(F|_{X_{\vec{n}}^k})$ .

Let  $C = X_{\vec{n}}$  and  $C_0 = X_{(0, n_2, \dots, n_\delta)}$  (namly,  $C_0$  is obtained from  $C$  by contracting the chain  $R_1 = \bigcup_{k=1}^{n_1} \mathbb{P}_k^1$  of projective lines  $\mathbb{P}_k^1 = \mathbb{P}^1$ ).

LEMMA 2.2. Let  $\pi : C \rightarrow C_0$  be the canonical morphism, let  $E$  be a torsion free sheaf that is locally free on  $R_1$ . If  $E|_{R_1}$  is positive and  $\pi_*E$  is stable (semistable) on  $C_0$ , then  $E$  is stable (semistable) on  $C$ . In particular, a vector bundle on  $X_{\vec{n}}$  is stable (semistable) if  $E|_{R_i}$  ( $1 \leq i \leq \delta$ ) are positive and  $\pi_*E$  is stable (semistable) on  $X$ , where  $\pi : X_{\vec{n}} \rightarrow X$  is the canonical morphism contracting  $R_1, \dots, R_\delta$  to  $x_1, \dots, x_\delta$ .

*Proof.* Let  $C = \tilde{C}_0 \cup R_1$  and  $\tilde{C}_0 \cap R_1 = \{p_1, p_2\}$ , where  $\pi : \tilde{C}_0 \rightarrow C_0$  is the partial normalization of  $C_0$  at  $x_1$ . Let  $\tilde{E} = E|_{\tilde{C}_0}$ ,  $E' = E|_{R_1}$ . Then we have exact sequence

$$(2.1) \quad 0 \rightarrow E'(-p_1 - p_2) \rightarrow E \rightarrow \tilde{E} \rightarrow 0.$$

If  $E|_{R_1}$  is positive and  $\pi_*E$  stable (semistable), then  $\pi_*E'(-p_1 - p_2) = 0$ . For any  $E_1 \subset E$ , consider the sequence (2.1), let  $\tilde{E}_1 \subset \tilde{E}$  be the image of  $E_1$  in  $\tilde{E}$  and  $K \subset E'(-p_1 - p_2)$  be the kernel of  $E_1 \rightarrow \tilde{E}_1$ , then we have

$$0 \rightarrow \pi_*E_1 \rightarrow \pi_*\tilde{E}_1 \rightarrow R^1\pi_*K = {}_{x_1}H^1(K),$$

and  $\chi(E_1) = \chi(\tilde{E}_1) + \chi(K) = \chi(\pi_*\tilde{E}_1) - h^1(K) \leq \chi(\pi_*E_1)$ . Since  $r(E_1) = r(\pi_*E_1)$ ,

$$\chi(E_1) - \frac{\chi(E)}{r}r(E_1) \leq \chi(\pi_*E_1) - \frac{\chi(\pi_*E)}{r}r(\pi_*E_1).$$

Thus we will be done if we can check that  $\chi(E_1) < \chi(E)$  when  $r(E_1) = r(E)$  and  $E_1 \neq E$ . In this case, the quotient  $E_2 = E/E_1$  is torsion outside the chains  $\{R_i\}$ . If  $E_2|_R = 0$ , where  $R = \cup R_i$ , then  $E_2$  is a nontrivial torsion and we are done. If  $E_2|_R \neq 0$ , then  $\chi(E_2) \geq \chi(E_2|_R)$ . Since  $E|_R$  is positive and the surjective map

$$E|_R = \bigoplus_{j=1}^r \mathcal{L}_j \rightarrow E_2|_R \rightarrow 0,$$

we have  $H^1(E_2|_R) = 0$  and there is at least one line bundle  $\mathcal{L}_j$  such that  $\mathcal{L}_j \hookrightarrow E_2|_R$  on a sub-chain. Thus  $\chi(E_2) \geq \chi(E_2|_R) = h^0(E_2|_R) > 0$  and  $\chi(E_1) < \chi(E)$ .

*Remark 2.3.* It is easy to show that if  $E$  is semistable on  $X_{\bar{n}}$ , then  $E$  is standard on the chains and  $\pi_*E$  is torsion free. It is expected that (semi)stability of  $E$  also implies the (semi)stability of  $\pi_*E$ .

DEFINITION 2.4. Let  $\mathcal{C} \rightarrow S$  be a flat family of stable curves of genus  $g \geq 2$ . The associated functor  $\mathcal{G}_S$  (called the Gieseker functor) is defined as follows:

$$\mathcal{G}_S : \{S\text{-schemes}\} \rightarrow \{\text{sets}\},$$

where  $\mathcal{G}_S(T) =$  set of closed subschemes  $\Delta \subset \mathcal{C} \times_S T \times_S Gr(m, r)$  such that

(1) the induced projection map  $\Delta \rightarrow T \times_S Gr(m, r)$  over  $T$  is a closed embedding over  $T$ . Let  $\mathcal{E}$  denote the rank  $r$  vector bundle on  $\Delta$  which is induced by the tautological rank  $r$  quotient bundle on  $Gr(m, r)$ .

(2) the projection  $\Delta \rightarrow T$  is a flat family of semistable curves and the projection  $\Delta \rightarrow \mathcal{C} \times_T T$  over  $T$  is the canonical morphism  $\pi : \Delta \rightarrow \mathcal{C} \times_S T$  contracting the chains of projective lines.

(3) the vector bundles  $\mathcal{E}_t = \mathcal{E}|_{\Delta_t}$  on  $\Delta_t$  ( $t \in T$ ) are of rank  $r$  and degree  $d = m + r(g - 1)$ . The quotients  $\mathcal{O}_{\Delta_t}^m \rightarrow \mathcal{E}_t$  induce isomorphisms

$$H^0(\mathcal{O}_{\Delta_t}^m) \cong H^0(\mathcal{E}_t).$$

LEMMA 2.5 ([GI],[NSE],[SE]). The functor  $\mathcal{G}_S$  is represented by a  $PGL(m)$ -stable open subscheme  $\mathcal{Y} \rightarrow S$  of the Hilbert scheme. The fibres  $\mathcal{Y}_s$  ( $s \in S$ ) are reduced, and the singularities of  $\mathcal{Y}_s$  are products of normal crossings. A point  $y \in \mathcal{Y}_s$  is smooth if and only if the corresponding curve  $\Delta_y$  is a stable curve, namely all chains in  $\Delta_y$  are of length 0.

Let  $Quot$  be the Quot-scheme of rank  $r$  and degree  $d$  quotients of  $\mathcal{O}_{\mathcal{C}}^m$  on  $\mathcal{C} \rightarrow S$  (we choose the canonical polarization on any flat family  $\mathcal{C} \rightarrow S$  of stable curves of genus  $g \geq 2$ ). There is a universal quotient

$$\mathcal{O}_{\mathcal{C} \times_S Quot}^m \rightarrow \mathcal{F} \rightarrow 0$$

on  $\mathcal{C} \times_S Quot \rightarrow Quot$ . Let  $\mathcal{R} \subset Quot$  be the  $PGL(m)$ -stable open subscheme consisting of  $q \in Quot$  such that the quotient map  $\mathcal{O}_{\mathcal{C} \times_S \{q\}}^m \rightarrow \mathcal{F}_q \rightarrow 0$  induces an isomorphism  $H^0(\mathcal{O}_{\mathcal{C} \times_S \{q\}}^m) \cong H^0(\mathcal{F}_q)$  (thus  $H^1(\mathcal{F}_q) = 0$ ). We can assume that  $d$  is large enough so that all semistable torsion free sheaves of rank  $r$  and degree  $d$

on  $\mathcal{C} \rightarrow S$  can be realized as points of  $\mathcal{R}$ . Let  $\mathcal{R}^s$  ( $\mathcal{R}^{ss}$ ) be the open set of stable (semistable) quotients, and let  $\mathcal{W}$  be the closure of  $\mathcal{R}^{ss}$  in  $Quot$ . Then there is an ample  $PGL(m)$ -line bundle  $\mathcal{O}_{\mathcal{W}}(1)$  on  $\mathcal{W}$  such that  $\mathcal{R}^s$  (resp.  $\mathcal{R}^{ss}$ ) is precisely the set of GIT stable (resp. GIT semistable) points. The moduli scheme  $\mathcal{U}(r, d) \rightarrow S$  is the GIT quotient of  $\mathcal{R}^{ss} \rightarrow S$ .

Let  $\Delta \subset \mathcal{C} \times_S \mathcal{Y} \times_S Gr(m, r)$  be the universal object of  $\mathcal{G}_S(\mathcal{Y})$ , and

$$\mathcal{O}_{\Delta}^m \rightarrow \mathcal{E} \rightarrow 0$$

be the induced quotient on  $\Delta$  by the universal quotient on Grassmannian over  $\mathcal{Y}$ . Then there is a commutative diagram over  $S$

$$\begin{array}{ccc} \Delta & \xrightarrow{\pi} & \mathcal{C} \times_S \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{Y} & \xlongequal{\quad} & \mathcal{Y} \end{array}$$

LEMMA 2.6. *If  $S$  is a smooth scheme, then  $\pi_*\mathcal{O}_{\Delta} = \mathcal{O}_{\mathcal{C} \times_S \mathcal{Y}}$  and there is a birational  $S$ -morphism*

$$\theta : \mathcal{Y} \rightarrow \mathcal{R}$$

such that pullback of the universal quotient  $\mathcal{O}_{\mathcal{C} \times_S \mathcal{R}}^m \rightarrow \mathcal{F} \rightarrow 0$  (by  $id \times \theta$ ) is

$$\mathcal{O}_{\mathcal{C} \times_S \mathcal{Y}}^m \rightarrow \pi_*\mathcal{E} \rightarrow 0.$$

*Proof.* Similar with Proposition 6 and Proposition 9 of [NSe] (See also [Se]).

LEMMA 2.7. *Let  $\mathcal{Y}^s = \theta^{-1}(\mathcal{R}^s)$  and  $\mathcal{Y}^0 = \theta^{-1}(\mathcal{R}^{ss})$ . Then*

$$\theta : \mathcal{Y}^s \rightarrow \mathcal{R}^s, \quad \theta : \mathcal{Y}^0 \rightarrow \mathcal{R}^{ss}$$

are proper birational morphisms.

*Proof.* The proof in [NSe] and [Se] for irreducible one node curves is completely local. Thus can be generalied to general stable curves.

There is a  $PGL(m)$ -equivariant factorisation (See [NSe], [Se], [Sch])

$$\begin{array}{ccccc} \mathcal{Y}^s & \xrightarrow{\iota} & \mathcal{Y}^0 & \xrightarrow{\iota} & \mathcal{H} \\ \theta \downarrow & & \theta \downarrow & & \lambda \downarrow \\ \mathcal{R}^s & \xrightarrow{\iota} & \mathcal{R}^{ss} & \xrightarrow{\iota} & \mathcal{W} \end{array}$$

and linearisation  $\mathcal{O}_{\mathcal{H}}(1)$ , where  $\iota$  is open embedding. Let  $L_a = \lambda^*(\mathcal{O}_{\mathcal{W}}(a)) \otimes \mathcal{O}_{\mathcal{H}}(1)$ . Then, for  $a$  large enough, the set  $\mathcal{H}(L_a)^{ss}$  ( $\mathcal{H}(L_a)^s$ ) of GIT-semistable (stable) points satisfies: (i)  $\mathcal{H}(L_a)^{ss} \subset \lambda^{-1}(\mathcal{R}^{ss})$ , (ii)  $\mathcal{H}(L_a)^s = \lambda^{-1}(\mathcal{R}^s)$ . By Lemma 2.7,  $\theta$  is proper, we have  $\lambda^{-1}(\mathcal{R}^{ss}) = \mathcal{Y}^0$  and  $\lambda^{-1}(\mathcal{R}^s) = \mathcal{Y}^s$ . Thus

$$\mathcal{H}(L_a)^s = \mathcal{Y}^s = \theta^{-1}(\mathcal{R}^s), \quad \mathcal{H}(L_a)^{ss} \subset \mathcal{Y}^0 = \theta^{-1}(\mathcal{R}^{ss}).$$

NOTATION 2.8.  $\mathcal{G}(r, d)_S = \mathcal{H}(L_a)^{ss} // PGL(m)$  is called (according to [NSE]) the generalized Gieseker semistable moduli space (or Gieseker space for simplicity). It is intrinsic by recent work [Sch].

Let  $y = (\Delta_y, \mathcal{O}_{\Delta_y}^m \rightarrow \mathcal{E}_y \rightarrow 0) \in \mathcal{Y}^0$ . Obviously, for  $y \in \mathcal{H}(L_a)^{ss} \setminus \mathcal{H}(L_a)^s$ , we have to add extra conditions besides the semistability of  $\pi_*\mathcal{E}_y$ . Alexander Schmitt ([Sch]) recently figure out a sheaf theoretic condition  $(H_3)$  (See Definition 2.2.10 in [Sch]) for  $\pi_*\mathcal{E}_y$ , which is a sufficient and necessary condition for  $y \in \mathcal{H}(L_a)^{ss}$ . The pair  $(C, E)$  of a semistable curve  $C$  with a vector bundle  $E$  is called  $H$ -(semi)stable (See [Sch]) if  $E$  is *strictly positive* on the chains of projective lines, and the direct image (on stable model of  $C$ )  $\pi_*E$  is semistable satisfying the condition  $(H_3)$ .

THEOREM 2.9. The projective  $S$ -scheme  $\mathcal{G}(r, d)_S \rightarrow S$  universally corepresents the moduli functor  $\mathcal{G}(r, d)_S^\sharp : \{S\text{-schemes}\} \rightarrow \{\text{sets}\}$ ,

$$\mathcal{G}(r, d)_S^\sharp(T) = \left\{ \begin{array}{l} \text{Equivalence classes of pairs } (\Delta_T, \mathcal{E}_T), \text{ where } \Delta_T \rightarrow T \\ \text{is a flat family of semistable curves with stable model} \\ \mathcal{C} \times_S T \rightarrow T \text{ and } \mathcal{E}_T \text{ is an } T\text{-flat sheaf such that for} \\ \text{any } t \in T, (\mathcal{E}_T)|_{\Delta_t} \text{ is } H\text{-(semi)stable vector bundle of} \\ \text{rank } r \text{ and degree } d. \end{array} \right.$$

We call that  $(\Delta_T, \mathcal{E}_T)$  is equivalent to  $(\Delta'_T, \mathcal{E}'_T)$  if there is an  $T$ -automorphism  $g : \Delta_T \rightarrow \Delta'_T$ , which is identity outside the chains, such that  $\mathcal{E}_T$  and  $g^*\mathcal{E}'_T$  are fibrewisely isomorphic.

**3. A Gieseker type degeneration for rank two.** Let  $\mathcal{C} \rightarrow S$  be a flat family of irreducible stable curves and  $\mathcal{L}$  be a line bundle on  $\mathcal{C}$  of relative degree  $d$ . We simply call the families in  $\mathcal{G}(r, d)_S^\sharp(T)$ , the families of semistable Gieseker bundles parametrized by  $T$ .

DEFINITION 3.1. The subfunctor  $\mathcal{G}_{\mathcal{L}} : \{S\text{-schemes}\} \rightarrow \{\text{sets}\}$  of  $\mathcal{G}$  is defined to be

$$\mathcal{G}_{\mathcal{L}}(T) = \left\{ \begin{array}{l} \Delta \in \mathcal{G}(T) \text{ such that for any } t \in T \text{ there is} \\ \text{a morphism } \det(\mathcal{E}|_{\Delta_t}) \rightarrow \pi^*\mathcal{L}_t \text{ on } \Delta_t \text{ which} \\ \text{is an isomorphism outside the chain of } \mathbb{P}^1\text{'s} \end{array} \right.$$

DEFINITION 3.2. The moduli functor  $\mathcal{G}(r, \mathcal{L})_S^\sharp$  of semistable Gieseker bundles with a fixed determinant is defined to be

$$\mathcal{G}(r, \mathcal{L})_S^\sharp(T) = \left\{ \begin{array}{l} (\Delta_T, \mathcal{E}_T) \in \mathcal{G}(r, d)_S^\sharp(T) \text{ such that for any } t \in T \\ \text{there exists a morphism } \det(\mathcal{E}_T|_{\Delta_t}) \rightarrow \pi^*\mathcal{L}_t \text{ on } \Delta_t \\ \text{which is an isomorphism outside the chain of } \mathbb{P}^1\text{'s} \end{array} \right.$$

When  $S = \text{Spec}(\mathbb{C})$ , the above defined functor is denoted by  $\mathcal{G}(r, L)^\sharp$ .

Let  $S = \text{Spec}(D)$  where  $D$  is a discrete valuation ring. Let  $\mathcal{C} \rightarrow S$  be a family of curves with smooth generic fibre and closed fibre  $\mathcal{C}_0 = X$ . Assume that  $X$  is irreducible with only one node  $p_0$ . Then we have the following result that is similar with Lemma 1.19 of [Vi].

LEMMA 3.3. *When  $r = 2$ , the moduli functor  $\mathcal{G}(r, \mathcal{L})_S^\sharp$  is a closed subfunctor of  $\mathcal{G}(r, d)_S^\sharp$ . More precisely, for any family  $(\Delta_T, \mathcal{E}_T) \in \mathcal{G}(r, d)_S^\sharp(T)$ , there exists a closed subscheme  $T' \subset T$  such that a morphism  $T_1 \rightarrow T$  of schemes factors through  $T_1 \rightarrow T' \hookrightarrow T$  if and only if*

$$(\Delta_T \times_T T_1, pr_1^* \mathcal{E}_T) \in \mathcal{G}(r, \mathcal{L})_S^\sharp(T_1).$$

Similarly,  $\mathcal{G}_\mathcal{C}$  is a closed subfunctor of  $\mathcal{G}$ .

*Proof.* Let  $\pi : \Delta_T \rightarrow \mathcal{C} \times_S T$  be the birational morphism contracting the chain of rational curves and  $\mathcal{L}_T$  be the pullback  $\pi^* \mathcal{L}$  to  $\Delta_T$ . Let  $f : \Delta_T \rightarrow T$  be the family of semistable curves (thus  $f_*(\mathcal{O}_{\Delta_T}) = \mathcal{O}_T$ ). Then the condition that defines the subfunctor is equivalent to the existence of a global section of  $det(\mathcal{E}_T|_{\Delta_t})^{-1} \otimes \pi^* \mathcal{L}_t$  which is nonzero outside the chain  $R_t \subset \Delta_t$  of  $\mathbb{P}^1$ s. But, in the case  $r = 2$ , any non-trivial section of  $det(\mathcal{E}_T|_{\Delta_t})^{-1} \otimes \pi^* \mathcal{L}_t$  is automatically nonzero outside the chain  $R_t \subset \Delta_t$  of  $\mathbb{P}^1$ s. There is a complex

$$(3.1) \quad \mathcal{K}_T^\bullet : \mathcal{K}_T^0 \xrightarrow{\delta_T} \mathcal{K}_T^1$$

of locally free sheaves on  $T$  such that for any base change  $T_1 \rightarrow T$  the pullback of  $\mathcal{K}_T^\bullet$  to  $T_1$  computes the direct image of  $det(\mathcal{E}_{T_1})^{-1} \otimes \mathcal{L}_{T_1}$  (which equals to the kernel of  $\delta_{T_1} : \mathcal{K}_{T_1}^0 \rightarrow \mathcal{K}_{T_1}^1$ ). There is a canonical closed subscheme  $T' \subset T$  (defined locally by some minors of  $\delta_T$ ) where  $\delta_T$  is not injective.

For simplicity, we assume that  $r$  and  $d$  are coprime  $(r, d) = 1$ . In this case, the functor  $\mathcal{G}(r, d)_S^\sharp$  is representable by an irreducible Cohen-Macaulay  $S$ -scheme  $\mathcal{G}(r, d)_S \rightarrow S$  (See [NSe]), whose fibres are reduced, irreducible projective schemes with at most normal crossing singularities. Moreover, there is a canonical proper birational  $S$ -morphism

$$(3.2) \quad \theta : \mathcal{G}(r, d)_S \rightarrow \mathcal{U}(r, d)_S,$$

where  $\mathcal{U}(r, d)_S \rightarrow S$  is the family (associated to  $\mathcal{C} \rightarrow S$ ) of moduli spaces of semistable torsion free sheaves with rank  $r$  and degree  $d$ .

For general  $r$ , let  $\mathcal{G}(r, \mathcal{L})_S \subset \mathcal{G}(r, d)_S$  be the subset of Gieseker bundles satisfying the conditions of functor, then we can show it being a closed subset. But we do not know how to define the correct *subscheme* structure on it.

LEMMA 3.4.  $\mathcal{G}(r, \mathcal{L})_S \subset \mathcal{G}(r, d)_S$  is a closed subset of  $\mathcal{G}(r, d)_S$ . In fact, for the closed fibre  $\mathcal{C}_0 = X$ , we have

$$(3.3) \quad \mathcal{G}(r, \mathcal{L})_S^\sharp(\{0\}) = \theta^{-1}(\mathcal{U}_X(r, \mathcal{L}_0)).$$

*Proof.* It is enough to prove (3.3). For any  $(\Delta, E) \in \mathcal{G}(r, d)_S^\sharp(\{0\})$ , let

$$\pi : \Delta \rightarrow X$$

be the morphism contracting the chain  $R$  of  $\mathbb{P}^1$ s. Then, by definition of  $\theta$ ,

$$\theta((\Delta, E)) = \pi_*(E) := F \in \mathcal{U}_X(r, d).$$

Note that  $F$  has type of  $t(F) = \deg(E|_R)$  (See [NSe]), then  $\pi_*(det(E))$  has torsion of dimension  $t(F) - 1$  supported at the node  $p_0 = \pi(R)$ . There is a natural morphism

$$\wedge^r F = \wedge^r(\pi_* E) \rightarrow \pi_*(\wedge^r E) = \pi_*(det(E)),$$

which is an isomorphism outside  $p_0$ . Thus we have an isomorphism

$$\wedge^r F/torsion \cong \pi_* det(E)/torsion$$

since  $\deg(\wedge^r F/torsion) = \deg(\pi_* det(E)/torsion) = d - t(F) + 1$ . By using this isomorphism, it is clear that

$$(\Delta, E) \in \mathcal{G}(r, \mathcal{L})_S^\sharp(\{0\}) \iff \theta((\Delta, E)) \in \mathcal{U}_X(r, \mathcal{L}_0).$$

$\mathcal{G}(2, \mathcal{L})_S$  is in fact a degeneracy loci of a map of vector bundles. To study it, we recall some standard results (See [FP] for example). Let  $\varphi : F \rightarrow E$  be a morphism of vector bundles on a variety  $M$  with  $rk(F) = m$  and  $rk(E) = n$ . The closed subsets of  $M$

$$D_r(\varphi) = \{x \in M \mid \text{rank}(\varphi_x) \leq r\}$$

are the so called degeneracy locus of  $\varphi$ . We collect the results into

LEMMA 3.5. *The codimension of each irreducible component of  $D_r(\varphi)$  is at most  $(n - r)(m - r)$ . If  $M$  is Cohen-Macaulay and the codimension of each irreducible of  $D_r(\varphi)$  equals to  $(n - r)(m - r)$ , then  $D_r(\varphi)$  is Cohen-Macaulay.*

In (3.1),  $rk(\mathcal{K}_T^1) - rk(\mathcal{K}_T^0) = g - 1$  since  $det(\mathcal{E}_T)^{-1} \otimes \mathcal{L}_T$  has relative degree 0. Then one sees that

$$T' = D_{k_0}(\delta_T), \quad k_0 = rk(\mathcal{K}_T^0) - 1.$$

In what follows, we will use  $\text{Codim}(\bullet)$  to denote: codimension of each irreducible component of  $\bullet$ . Thus  $\text{Codim}(T') \leq g$ , and it is Cohen-Macaulay if

$$\text{Codim}(T') = g.$$

In particular, let  $X$  be the singular fibre of  $\mathcal{C} \rightarrow S$  and  $L = \mathcal{L}|_X$ . The closed fibre  $G(r, d)$  of  $\mathcal{G}(r, d)_S \rightarrow S$  is the so called generalized Gieseker moduli space (associated to  $X$ ) of [NSE], which has normal crossing singularities. The closed fibre of  $T' \rightarrow S$ , denoted by  $G(r, L)$ , is the degeneracy loci

$$D_{k_0}(\delta_{T_0}) \subset T_0 \subset G(r, d)$$

of  $\delta_{T_0} : \mathcal{K}_{T_0}^0 \rightarrow \mathcal{K}_{T_0}^1$ , where  $T_0$  is the closed fibre of  $T \rightarrow S$ . Thus

$$\text{Codim}(G(r, L)) \leq g$$

and  $G(r, L)$  is Cohen-Macaulay if  $\text{Codim}(G(r, L)) = g$ . When  $r = 2$ ,  $G(r, L) \subset G(r, d)$  is a closed subscheme that represents a moduli functor (See Theorem 3.7 for definition).

LEMMA 3.6. *When  $r = 2$ ,  $\text{Codim}(G(r, L)) = g$ . In particular,  $\mathcal{G}(r, \mathcal{L})_S \subset \mathcal{G}(r, d)_S$  is an irreducible, reduced, Cohen-Macaulay subscheme of codimension  $g$ .*

*Proof.* Assume that  $\text{Codim}(G(r, L)) = g$ . Note that there is a unique irreducible component of  $\mathcal{G}(r, \mathcal{L})_S$  with codimension  $g$  dominates  $S$  since  $\mathcal{C} \rightarrow S$  has smooth generic fibre. Thus other irreducible components (if any) of  $\mathcal{G}(r, \mathcal{L})_S$  will fall in  $G(r, L)$  and their codimension in  $G(r, d)$  are at most  $g - 1$  since  $\mathcal{G}(r, d)_S \rightarrow S$  is flat over  $S$ . This contradicts  $\text{Codim}(G(r, L)) = g$ . Hence  $\mathcal{G}(r, \mathcal{L})_S \subset \mathcal{G}(r, d)_S$  is an irreducible, Cohen-Macaulay subscheme of codimension  $g$ . It has to be reduced since it is Cohen-Macaulay and has a reduced open subscheme.

Now we prove that  $\text{Codim}(G(r, L)) = g$  in  $G(r, d)$ . Let  $J_X^0$  be the Jacobian of line bundles of degree 0 on  $X$ . Consider a morphism

$$\phi : G(r, L) \times J_X^0 \rightarrow G(r, d)$$

that sends any  $\{(\Delta, E), \mathcal{N}\} \in G(r, L) \times J_X^0$  to  $(\Delta, E \otimes \pi^* \mathcal{N}) \in G(r, d)$ , where  $\pi : \Delta \rightarrow X$  is the morphism contracting the chain  $R$  of  $\mathbb{P}^1$ s. We claim that

$$\dim \phi^{-1}((\Delta, E_0)) \leq 1, \quad \text{for any } (\Delta, E_0) \in G(r, d).$$

Let  $\sigma : J_X^0 \rightarrow J_{\tilde{X}}^0$  be the morphism induced by pulling back line bundles on  $X$  to its normalization  $\tilde{X}$ . The fibres of  $\sigma$  are of dimension 1. On the other hand, it is easy to see that the projection  $G(r, L) \times J_X^0 \rightarrow J_X^0$  induces an injective morphism

$$\rho : \phi^{-1}((\Delta, E_0)) \rightarrow J_X^0.$$

To prove the claim, it is enough to show that the image  $\text{Im}(\rho)$  falls in a finite number of fibres of  $\sigma$ . Note that, for any  $\{(\Delta, E), \mathcal{N}\} \in \phi^{-1}((\Delta, E_0))$ , we have

$$\det(E) \otimes \pi^*(\mathcal{N}^{\otimes r}) = \det(E_0)$$

on  $\Delta$ . Recall that, by definition of  $G(r, L)$ , there is a morphism  $\det(E) \rightarrow \pi^* L$  which is an isomorphism outside the chain  $R$  of  $\mathbb{P}^1$ s. We have

$$\det(E)|_{\tilde{X}} = \pi^* L|_{\tilde{X}}(-n_1 p_1 - n_2 p_2) = \tilde{L}(-n_1 p_1 - n_2 p_2),$$

where  $\tilde{L}$  is the pullback of  $L$  to  $\tilde{X}$ ,  $n_1, n_2$  are nonnegative integers such that

$$n_1 + n_2 = \deg(E_0|_R) = t(F_0), \quad F_0 := \pi_*(E_0).$$

Thus  $\sigma \circ \rho(\{(\Delta, E), \mathcal{N}\}) = \sigma(\mathcal{N}) = \tilde{\mathcal{N}} \in J_{\tilde{X}}^0$  falls in the set

$$\{\tilde{\mathcal{N}} \in J_{\tilde{X}}^0 \mid \tilde{\mathcal{N}}^{\otimes r} = \det(E_0)|_{\tilde{X}} \otimes \tilde{L}^{-1}(n_1 p_1 + n_2 p_2)\},$$

which is clearly a finite set. This proves that fibres of  $\phi$  are at most dimension 1.

There is a unique irreducible component  $G(r, L)^0$  of  $G(r, L)$  containing  $\Delta \cong X$ , which has codimension  $g$ . For any other irreducible component (if any), say  $G(r, L)^+$ , all of  $\Delta$ s in  $G(r, L)^+$  must have chain (with positive length) of  $\mathbb{P}^1$ s. Then the image  $\phi(G(r, L)^+ \times J_X^0)$  has to fall in a subvariety of  $G(r, d)$ , which has codimension at least 1. Thus  $\dim(G(r, L)^+ \times J_X^0) \leq \dim G(r, d)$ , that is,

$$\text{Codim}(G(r, L)^+) \geq g.$$

By Lemma 3.5,  $G(r, L)$  is Cohen-Macaulay of pure codimension  $g$ .

**THEOREM 3.7.** *Let  $X$  be an irreducible curve of genus  $g \geq 2$  with only one node  $p_0$ . Let  $L$  be a line bundle of degree  $d$  on  $X$ . Then, when  $r = 2$  and  $(2, d) = 1$ , we have*

- (1) *There is a Cohen-Macaulay projective scheme  $G(r, L)$  of pure dimension  $(r^2 - 1)(g - 1)$ , which represents the moduli functor*

$$\mathcal{G}(r, L)^\sharp : (\mathbb{C} - \text{schemes}) \rightarrow (\text{sets})$$

*which is defined in Definition 3.2.*

- (2) *Let  $\mathcal{C} \rightarrow S$  be a proper family of curves over a discrete valuation ring, which has smooth generic fibre  $\mathcal{C}_\eta$  and closed fibre  $\mathcal{C}_0 \cong X$ . If there is a line bundle  $\mathcal{L}$  on*

$\mathcal{C}$  such that  $\mathcal{L}|_{\mathcal{C}_0} \cong L$ . Then there exists an irreducible, reduced, Cohen-Macaulay  $S$ -projective scheme  $f : G(r, \mathcal{L})_S \rightarrow S$  such that

$$f^{-1}(0) \cong G(r, L), \quad f^{-1}(\eta) \cong \mathcal{U}_{\mathcal{C}_\eta}(r, \mathcal{L}_\eta).$$

Moreover  $G(r, \mathcal{L})_S$  represents the moduli functor  $\mathcal{G}(r, \mathcal{L})_S^\sharp$  in Definition 3.2.

(3) There exists a proper birational  $S$ -morphism  $\theta : G(r, \mathcal{L})_S \rightarrow \mathcal{U}(r, \mathcal{L})_S$  which induces a morphism  $\theta : G(r, L) \rightarrow \mathcal{U}_X(r, L)$ .

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