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ARMAND BOREL

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The foundations of the modern theory of linear algebraic groups were laid in Armand Borel's paper "Groupes linéaires algébriques", published in 1956 [Œ 39]. Below I shall review, more or less chronologically, his publications on the theory of algebraic groups proper. These are relatively few in number. The more numerous publications about applications of the theory, for example to arithmetic groups and automorphic forms, fall outside the scope of this segment of the article. Some of the publications about applications are discussed in Arthur's segment below.

Linear Algebraic Groups. A *linear algebraic group* over the field of complex numbers $\mathbb{C}$ is a subgroup G of a group $GL_n(\mathbb{C})$ of invertible $n \times n$-matrices whose elements $g = (g_{ij})_{1 \leq i,j \leq n}$ are precisely the solutions of a set $P_a(g_{ij}) = 0 (1 \leq a \leq N)$ of polynomial equations in the matrix entries. A linear algebraic group G is an affine algebraic variety, and thus the machinery of algebraic geometry can be used. Topological notions in G will be relative to the Zariski topology.[1]

Examples of algebraic groups (the adjective "linear" will be dropped) are the group of diagonal matrices (which is abelian), the group of upper triangular matrices (which is solvable), and the complex orthogonal group, defined by quadratic equations. Of course, $GL_n(\mathbb{C})$ is also an example. An algebraic group isomorphic to a group of diagonal matrices is called a *torus*.

If the P_a have coefficients in a subfield F of $\mathbb{C}$ (e.g., $F = \mathbb{Q}$), then G is said to be *defined over* F . In all this $\mathbb{C}$ may be replaced by an algebraically closed field k (of arbitrary characteristic) and F by a subfield of k. (If F is nonperfect,[2] some care has to be taken.) If G is defined over F, one has the group $G(F)$ of points of G with coordinates in F, known as the group of F*-rational points*.

When Borel took up the subject, the notion of algebraic group had already been around for some time, and a restricted assortment of results was known. Borel's book [2] contains an excellent review of the history of the theory of algebraic groups. I mention only a few points.

Algebraic groups appeared in the nineteenth century. In the 1880s É. Picard tried to develop a Galois theory for complex linear differential equations with polynomial coefficients. He had the insight that such an equation has a Galois group that is an algebraic group. Somewhat later L. Maurer established some general properties of linear algebraic groups. In the background of his work are Lie's theory of transformation groups and the theory of invariants.

The more general notion of algebraic group variety—an algebraic variety with a compatible group structure (similar to the notion of a Lie group)—also appeared in the nineteenth century. Examples are elliptic curves, which are projective curves with a structure of abelian group.

Around 1950 basic general facts about algebraic groups over arbitrary fields were being developed, for example by A. Weil, who required foundational material for

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[1] A Zariski closed set is the zero set of a family of polynomial functions.

[2] The field F of characteristic p is perfect if either $p = 0$ or if every element of F is a p^{th} power.

his work on abelian varieties over an arbitrary field. He gave a construction of the quotient of an algebraic group variety by a closed subgroup, as an algebraic variety. (In contrast to the case of Lie groups, this construction is rather delicate). It was known also that the quotient of a (linear) algebraic group by a closed normal subgroup is again an algebraic group.

Pertaining to the theory of algebraic groups is work of E. R. Kolchin in 1948, motivated by Ritt's algebraic version of Picard's work on linear differential equations. Kolchin showed that a connected solvable algebraic group can be put in upper triangular form over any algebraically closed field. This is a global analog of an old theorem of Lie about solvable Lie algebras.

At about the same time Chevalley began a study of linear algebraic groups in characteristic zero, partly inspired by Maurer's work. Lie algebras played an important role.

"Groupes Linéaires Algébriques". Borel started work on this paper in Chicago in 1954–55. In the paper he gives a systematic exposition of the theory, using methods of algebraic geometry. Perhaps he was influenced by the work of his teacher Heinz Hopf, who had introduced global geometric methods in the theory of compact Lie groups, thus circumventing the use of Lie algebras.

The paper starts with a discussion of elementary matters. One elementary but important ingredient of the theory is the Jordan decomposition. Let g be an element of the algebraic group $G \subset GL_n(k)$. Then there exist unique elements $g_s, g_u \in G$ such that $g = g_s g_u = g_u g_s$, g_s is semisimple (diagonalizable), and g_u is unipotent (all its eigenvalues are 1). If $G = GL_n(k)$, this follows from the Jordan normal form theorem. Moreover, the decomposition is intrinsic, i.e., independent of the particular imbedding $G \subset GL_n(k)$. The result was more or less in the literature, but Borel gave it its final form.

The paper then discusses solvable groups. It is shown that a connected solvable algebraic group G is a semidirect product $G = T.G_u$, where G_u is the set of unipotent elements of G, which is a connected normal closed unipotent[3] subgroup, and where T is a maximal torus of G. Moreover, any two maximal tori of G are conjugate.

Borel Subgroups. The heart of the paper is the study of Borel subgroups, as they are called now. A *Borel subgroup* B of the connected algebraic group G is a connected solvable closed subgroup of G that is maximal relative to these properties. The main results about them are:

(a) G/B is a projective variety,
(b) any two Borel subgroups of G are conjugate,
(c) the Borel groups cover G.

Although similar results in the context of Lie groups were in the air, Borel's results in arbitrary characteristic were surprisingly elegant and the proofs were surprisingly simple. A crucial ingredient for the proof is the "orbit lemma": if the algebraic group G acts on an algebraic variety X (in the sense of algebraic geometry), there exists a point of X whose G-orbit is closed.

Borel once told me that at first he was doubtful about the lemma, because in complex analytic geometry—then more familiar—such a result is completely false. But a conversation with Weil removed the doubts.[4]

[3] An algebraic group is unipotent if all its elements are unipotent.
[4] This must be the conversation alluded to in [Œ, vol. IV, p. 660].

Here is a sketch of how the main results (a) and (b) follow from the orbit lemma. Assume that $G \subset GL_n(k)$. There is a natural projective variety X on which G acts, namely the flag variety of $V = k^n$. A *flag* in V is a sequence of subspaces $(V_1, V_2, \ldots, V_{n-1}, V_n)$ of V with $\dim V_i = i$ for all i and $V_i \subset V_{i+1}$ for $1 \leq i \leq n-1$. The set F_n of all flags has a structure of projective variety (this is a fact from classical algebraic geometry) on which the algebraic group $GL_n(k)$ acts. Hence G acts on F_n. The orbit lemma implies that there is a closed subgroup C of G such that G/C is a projective variety. Then C is a subgroup of $GL_n(k)$ that is triangular relative to some basis of k^n, hence is solvable. If G itself is solvable, such quotients are affine varieties, hence must be finite. If moreover G is connected, it follows that G fixes a flag, i.e., can be put in triangular form. This is the Lie-Kolchin theorem.

By a similar argument Borel proves a fixed point theorem, which now carries his name: a connected solvable algebraic group G acting on a projective (or complete) variety X fixes a point of X.

From the fixed point theorem it is not hard to see that the identity component B of C (the component containing the identity element) is a Borel subgroup. The conjugacy of Borel subgroups follows by another application of the fixed point theorem, proving (a) and (b).

Now let S be a subtorus of G (a closed subgroup that is a torus), and let B be a Borel subgroup. Applying the fixed point theorem to S and G/B, one sees that S is conjugate to a subgroup of B. From the fact that maximal tori in B are conjugate, it follows that two maximal tori of G are conjugate. (This is an analog of the conjugacy of Cartan subgroups of a compact Lie group.)

The proof of (c) proceeds as follows. Let B be a Borel subgroup of G. To establish (c), one has to show that the sets gBg^{-1}, for $g \in G$, cover G. To prove this, first a geometric analysis is made of the union of the conjugates of a given closed subgroup of G. In the case of a Borel group B this leads to the following construction (Borel formulates things a bit differently).

Let $\widetilde{G}$ be the quotient of $G \times B$ by the B-action $b.(g, b') = (gb^{-1}, bb'b^{-1})$, and let $\pi : \widetilde{G} \to G$ be the morphism induced by the map $(g, b) \mapsto gbg^{-1}$. The image of π is the union of all Borel subgroups. So property (c) says that π is surjective. As π is a proper morphism, surjectivity will follow if the image of π is dense. This is proved by showing that the conjugates of a subgroup of B (namely the connected centralizer of a maximal torus) fill up a dense subset.

The map π appears for the first time, implicitly, in Borel's work. Further study of π and of its fibers has led to interesting insights, discussed in [Slo]. For example, let G_u be the set of unipotent elements of G. This is an irreducible closed subvariety of G. Then (say over $\mathbb{C}$) the restriction of π to $(\pi)^{-1}(G_u)$ is a resolution of singularities of G_u. It has been much studied.

Chevalley's Work. What is not proved in [Œ 39] and what Borel did not know at the time of writing is the normalizer theorem: a Borel subgroup B of G coincides with its normalizer; i.e., if $g \in G$ is such that $gBg^{-1} = B$, then $g \in B$.

Chevalley proved this a little later and then developed a structure theory of semisimple groups.[5] He gave a complete classification of simple algebraic groups over any algebraically closed field k. It is the "same" as the Cartan-Killing classification

[5] The radical (respectively, unipotent radical) of the algebraic group G is the maximal connected normal solvable closed subgroup of G. G is semisimple if its radical is trivial. Replacing "solvable" by "unipotent", one has the the definition of the unipotent radical of G and of G being reductive.

of simple Lie algebras over $\mathbb{C}$.

Borel tells in [2, p. 158] that he gave Chevalley a copy of his paper in the summer of 1955. The next summer Chevalley told him that after reading the paper he had proved the normalizer theorem, after which "the rest followed by analytic continuation."

Chevalley also introduced the combinatorial ingredients from Lie theory, such as root system and Weyl group. His work was published in the Paris Seminar Notes [Che]; they were for several years the standard text about the theory of algebraic groups.

The Notes also contain (without naming it) the first published version of the "Borel-Weil theorem" in the context of algebraic groups [Che, exp. 15]. The theorem asserts that in characteristic 0 the irreducible representations of a semisimple algebraic group G can be realized in spaces of sections of suitable line bundles on G/B, where B is a Borel subgroup of G.

Borel and Weil in 1954 dealt with the representations of compact Lie groups, which was seemingly a somewhat different context. In the meantime it has become clear that the representation theory of compact Lie groups is equivalent to the representation theory of reductive algebraic groups over $\mathbb{C}$.

Borel's own notes on the Borel-Weil Theorem remained unpublished until their appearance in [Œ] as [Œ 30]. The insight that representations of a Lie group or algebraic group G may be constructed using sections (or, more generally, cohomology groups) of line bundles on suitable varieties with a G-action has turned out to be quite fruitful.

Reminiscences. I entered the area of algebraic groups by a back entrance, so to speak. In the 1950s I had been interested in questions about quadratic forms and special algebraic groups, working more or less on my own. It was clear to me that I was getting involved with algebraic groups, but I did not know well the general theory. I was not yet versed in the technicalities of Lie theory (what A. Weil calls "digging roots and lifting weights").

It was a great surprise when I received in the fall of 1960 a letter from Borel, inviting me to the Institute for Advanced Study. Of course, I accepted the invitation gratefully. I spent the academic year 1961–62 at the Institute.

In that year Borel had a weekly seminar on algebraic groups. As I recall it, the seminar was more in the nature of a course, as he was the only speaker. The seminar was oriented towards rationality questions, i.e., questions involving a ground field F. This was at the time largely new territory. Borel expounded his own as-yet-unpublished work. A little later he joined forces with J. Tits, who worked in the same direction. I shall return to their joint work below.

The seminar was very stimulating, and I learned a great deal. Moreover, Borel gave very good advice.

Soon during my stay at the Institute I got to know Robert Steinberg, who also attended the seminar. We discussed a problem of "fusion" of conjugacy classes over finite fields, which he was interested in. This is as follows. Let G be a connected algebraic group over a finite field F, and let k be an algebraic closure of F. Suppose $a, b \in G(F), g \in G$ are such that $gag^{-1} = b$. Is there $g \in G(F)$ with this property? A theorem of Lang of 1956 showed that this is the case if the centralizer of a in G is connected. So a general question arose: what can one say about connectedness of centralizers in a connected algebraic group G, assumed to be semisimple?

Borel, when consulted about this question, put us on the right track by pointing out that in his paper [Œ 53] on torsion in compact Lie groups, which had just appeared, connectedness of centralizers was proved for a compact semisimple Lie group under the assumption that the group is simply connected. In the proof the affine Weyl group makes its appearance, with a reference to E. Stiefel, who introduced Borel to these matters. In a footnote Borel remarks that he also proved that the centralizer of a semisimple element of a complex semisimple simply connected Lie group is connected. But he did not publish the proof. An analysis of the connectedness of centralizers of semisimple elements in semisimple groups over any algebraically closed field was given by Steinberg [St2, §§8-9] in 1968. In particular, he proved connectedness in a simply connected group.[6]

Borel's paper [Œ 53] on torsion is of interest also for other reasons. It introduces for the first time the "bad" primes for the simple types of Lie groups (for example 2, 3, 5 for type E_8). Subsequently these showed up in the theory of algebraic groups in several other places, for example in Galois cohomology and in the study of unipotent elements.

The paper is also unique among Borel's papers in having a literary quotation in the introduction,[7] an ironic comment on the technicalities of the paper.

Galois Cohomology. The last part of Borel's seminar of 1961–62 was devoted to Galois cohomology of linear algebraic groups, at the time a new topic. Let G be an algebraic group over the field F, and let E/F be a finite Galois extension with group Γ. A 1-cocycle of Γ with values in $G(E)$ is a $G(E)$-valued function z on Γ satisfying $z(st) = z(s).s(z(t))$. Two such cocycles z, z' are equivalent if there is $g \in G(E)$ with $z'(s) = g^{-1}z(s)(s.(g))$ for $s, t \in \Gamma$. The set of equivalence classes is denoted by $H^1(E/F, G)$. Taking a limit over all finite extensions E/F contained in a given separable closure of F, one obtains the Galois cohomology set $H^1(F, G)$. It has a distinguished element 1.

Several problems about algebraic groups involving a ground field have a convenient formulation in terms of Galois cohomology. An example is the problem of describing F-isomorphism classes of algebraic groups over a field . The theorem of Lang, which was alluded to before, when formulated in Galois cohomology terms, states that $H^1(F, G) = 1$ if F is finite and G is connected.

At the time of the seminar, Borel and Serre were preparing the paper [Œ 64] on Galois cohomology, which appeared in 1964. Borel explained part of the material of the paper, such as the formalism of exact sequences and the proof of finiteness of $H^1(F, G)$ when F is a p-adic field.

He also introduced us to Serre's Conjectures I and II. Conjecture I states that $H^1(F, G) = 1$ if F has cohomological dimension 1 and G is connected. During the seminar I proved Conjecture I for a particular class of fields. For perfect fields it was proved by Steinberg in 1963 in [St1, §10] as a consequence of work on conjugacy classes. The general case was proved in [Œ 80]. Conjecture II about fields of dimension 2 is still open as far as I know, although proved in many particular cases.

Borel did not further pursue Galois cohomology. The notes of Serre's lectures on this topic at the Collège de France in 1962–63 have been very influential. They have gone through several editions and were translated into English [Ser]. Galois

[6] An isogeny $G \to G'$ of semisimple groups is a surjective homomorphism of algebraic groups with finite kernel. G is simply connected (respectively, adjoint) if any isogeny $G' \to G$ (respectively, $G \to G'$) is an isomorphism of algebraic groups.

[7] Namely, a quotation from G. B. Shaw, *Arms and the Man*: "You have a low shopkeeping mind. You think of things that would never come into a gentleman's head." "That's the Swiss national character, dear lady."

cohomology is still an interesting and active subject. I owe my early introduction to it to Borel.

Grothendieck's Results. I mentioned already that in his seminar that Borel expounded his results on rationality questions. But he had to assume perfectness of the ground field F. This is an undesirable restriction, as it excludes interesting fields such as global fields of nonzero characteristic. In subsequent work with Tits the restriction could be removed, thanks to work of Grothendieck (in 1964 in his Seminar on Group Schemes [SGA3, exp. XII, XIII, XIV]). Grothendieck's new results were:

(a) an algebraic group G defined over F contains a maximal torus defined over G;
(b) if moreover G is connected reductive and F is infinite, then $G(F)$ is Zariski-dense in G.

"Groupes Réductifs". I now come to the work of Borel and Tits on algebraic groups over an arbitrary field F, published in 1965 in their paper "Groupes réductifs" [Œ 66]. I think that Borel was motivated by questions of an arithmetical nature, such as the construction of boundary pieces of the compactification of a quotient by an arithmetic group (cf. [Œ 59]). Tits's interest had its origin in geometry.

Thanks to Grothendieck's results, Borel and Tits could deal with an arbitrary base field F. The paper of Borel and Tits contains a wealth of interesting material. I mention only a few of its results.

Let G be a connected algebraic group defined over $F \subset k$, as before. A subgroup P of G containing a Borel subgroup is called a *parabolic subgroup.* Then $P \subset G$ is parabolic if and only if G/P is a complete variety, as follows via an application of Borel's fixed point theorem. The parabolic subgroups containing a given Borel subgroup can be described by combinatorial data extracted from the root system of G.

Now let G be reductive. In the theory over F the role of Borel subgroups over k is taken over by the minimal parabolic subgroups defined over F. It is shown that two of these are conjugate by an element of $G(F)$. If there are no proper parabolic subgroups over F, the group G is called *anisotropic.* For example, such is the case with the orthogonal group defined by an anisotropic quadratic form over F, whence the name.

The role of maximal tori over k is taken over by subtori of G that are defined over F and F-split, i.e., F-isomorphic to a group of diagonal matrices, and maximal for these properties. Let S be such a torus. Then G is anisotropic if and only if S lies in the center of G. This implies that the centralizer M of S, which is a connected group over F, is anisotropic. Out of M and a half-space in the character group of S, which is a free abelian group of rank $\dim S$, one can construct a minimal parabolic subgroup over F. The split torus S defines a "small" Weyl group, an ingredient of a "Tits system" on $G(F)$.

It is also shown that two maximal split F-tori are $G(F)$-conjugate. In fact, this is true for any F-group G, not necessarily reductive. The proof is sketched in the later paper [Œ 110] by Borel and Tits.

The Boulder Conference. Borel and G. D. Mostow were the organizers of a Summer Institute of the American Mathematical Society "Algebraic Groups and Discontinuous Subgroups", held in Boulder (Colorado) in the summer of 1965. Looking today at the proceedings of the Institute [1], one is struck by the taste and foresight shown in the choice of the subjects. Several of them have had impressive developments in the intervening years. Such are: Hasse principles, Tamagawa numbers, Eisenstein series, Shimura varieties, Bruhat-Tits theory (of algebraic groups over local fields, the theory having been conceived in Boulder), rigidity of arithmetic groups.

I took part in the Boulder Institute. Shortly before, I had tried to understand Grothendieck's new results, mentioned above. An interesting feature was his use of the Lie algebra $\mathfrak{g}$ of the algebraic group G over F. So far, there had been a feeling that it was not of much use.

I noticed that there is an additive Jordan decomposition for elements of $\mathfrak{g}$ and that in characteristic 0 the fixed point group in G (acting on $\mathfrak{g}$ by the adjoint representation) of a well-chosen semisimple element of $\mathfrak{g}(F)$ contains a maximal torus of G that is defined over F (if F is infinite). I wondered if an argument of this kind could work for arbitrary F to prove Grothendieck's result (a). (The argument cannot be used in G, since for a nonperfect field F one does not know a priori that nontrivial semisimple elements of $G(F)$ exist; for $\mathfrak{g}$ there is no problem.) A difficulty was that the fixed point group might be all of G.

In Boulder I discussed the matter with Borel. Very soon he saw how the difficulty could be removed: by exploiting a result of Serre (used by him in the context of abelian varieties), passing to a quotient not by a subgroup but by a subalgebra of $\mathfrak{g}$. This led to the short paper [Œ 76] in [1]. A further outgrowth was [Œ 80], in which Grothendieck's result (b) is also dealt with.

Further Work with Tits. Subsequently Borel continued his collaboration with Tits. In [Œ 92] it is shown how to associate canonically to a unipotent subgroup U of a reductive group G a parabolic subgroup whose unipotent radical contains U (in [Œ 92] fields of definition are also taken into account). A consequence is the following nice result, which was known already in characteristic 0: a maximal proper closed subgroup of G is either parabolic or reductive.

The starting point of the long joint paper [Œ 97] is the problem of determining the automorphisms of the abstract group $G(F)$, where G is a connected semisimple group over F. More generally, the homomorphisms are studied of $G(F)$ into a similar group $G'(F')$. The problem had been around since the 1920s and had been solved in many particular cases, under restrictions on G or F ([2, pp. 134–140]).

G is assumed to be semisimple. The important standing hypothesis is: G is isotropic over F, so contains proper parabolic subgroups over F. Let G^+ be the subgroup of $G(F)$ generated by the groups $U(F)$, where U runs through the unipotent radicals of the parabolic subgroups over F of G (G^+ is a "large" subgroup of $G(F)$ but does not always coincide with it). If $\phi : F \to F'$ is a homomorphism of fields, one can transport G via ϕ to a group ${}^{\phi}G$ over F' and there is a canonical homomorphism $\phi^0 : G(F) \to {}^{\phi}G(F')$.

I shall not try to fully describe the results of [Œ 97]. Here is a typical example. Assume that G is simple (and isotropic over F). Let G' be a simple (nontrivial) algebraic group over F', and let $\alpha : G(F) \to G'(F')$ be a homomorphism such that $\alpha(G^+)$ is Zariski-dense in G'. Then there exist a homomorphism $\phi : F \to F'$ and an isomorphism of algebraic groups $\beta : {}^{\phi}G \to G'$ over F' such that $\alpha(g) = \beta(\phi^0(g))$ for $g \in G^+$.

Another topic of the paper is the analysis of an irreducible representation (in the algebraic sense) $\rho : G(F) \to PGL_n(k')$, k' being algebraically closed. It is shown that ρ can be built up from irreducible projective representations of the algebraic group G.

The paper exploits the properties of parabolic subgroups established in [Œ 66]. There are many technicalities, in particular in characteristic 2. The restriction to isotropic groups made in [Œ 97] is very essential. For the case of anisotropic groups, there is, as far as I know, as yet no general theory.

The short note [Œ 110] of Borel and Tits announces a number of results about a connected algebraic group G over a nonperfect field F that is "reductive relative to F", i.e., that has no connected normal unipotent closed subgroup defined over F (the assumptions in [Œ 110] are a bit more general). Analogs of the results of [Œ 97] are announced, such as a theory of "pseudo-parabolic" subgroups and the existence of a Tits system on $G(F)$. Subsequently, Tits has extended the work to obtain classification results. He has lectured about these matters at the Collège de France, but full proofs of his results and of those of [Œ 110] have not appeared. (I have treated some of the results of Borel and Tits in [Sp, Ch. 15].)

Epimorphic Groups. Borel's last contributions to the general theory of linear algebraic groups are in joint work with Bien and Kollár in the 1990s [Œ 147, 148, 158]. Let G be an algebraic group over k, and let H be a closed subgroup. In the cited papers the situation is studied where the variety G/H is quasi-complete; i.e., its regular functions are constant. The papers [Œ 147, 148] study the groups H with this property (called "epimorphic"). It is conjectured that if H is epimorphic and if E is a finite-dimensional rational H-module, the induced G-module is also finite dimensional. In [Œ 158], which studies questions of rational connectedness of homogeneous spaces, it is shown that the conjectured property holds if G/H contains sufficiently many one-dimensional images of $\mathbb{P}^1$. This last paper is methodologically interesting: it applies recent work on rational curves on algebraic varieties.

Borel was an excellent expositor. This does not mean that his research papers, often about difficult topics, are easy to read. But his presentation of the topics is excellent, striving for maximal clarity in the exposition. The same was true for his talks on his own work or on the work of others.

I knew Borel for more than forty years. During some periods we had quite frequent contacts, and not only about mathematics. In later years the contacts were less frequent, but they never ceased. The last time he wrote me was in May of 2003. This was about mathematics, but he also wrote that he was reasonably well. Hence it was a shock to hear in August about his demise.

I remember him with admiration and gratitude.

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