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# Kähler–Einstein metrics of negative Ricci curvature on general quasi–projective manifolds

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In this paper, we give sufficient and necessary conditions for the existence of a Kähler–Einstein metric on a quasi-projective manifold of finite volume, bounded Riemannian sectional curvature and Poincaré growth near the boundary divisor. These conditions are obtained by solving a degenerate Monge–Ampère equation and deriving the asymptotics of the solution.

## 1. Introduction

Let $M$ be a quasi-projective manifold which can be compactified by adding a divisor $D$ with *simple normal crossings*, which means that $D = \sum_{i=1}^{p} D_i$, where the irreducible components $D_i$ are smooth and intersect transversely. Under certain positivity conditions of the adjoint bundle over the compactification, the existence of a complete Kähler–Einstein metric on the quasi-projective manifold was first addressed by Yau (see, for example, [29, p. 166]), right after his resolution of the Calabi conjecture [28]. This program has been followed by many authors, for example, [1, 5, 8, 22, 24], and [31].

In fact, notice that the second part of [28] is essentially devoted to construct Kähler–Einstein metrics on algebraic manifolds of general type. In general such a metric has singularities. In a sense, Yau's motivation for this program may be viewed as to understand these singularities in many important cases from the differential-geometric point of view. This appears in his later papers joint with Cheng [8], and with Tian [22]. In the last paper, they proved the following result.

Let

$$K_{\overline{M}} + \sum_{i=1}^{q} [D_i] + \sum_{j=q+1}^{p} \frac{m_j - 1}{m_j} [D_j] \tag{1.1}$$

be big, nef, and ample modulo $\sum_{i=1}^{q} D_i$, where $K_{\overline{M}}$ denotes the canonical bundle over $\overline{M}$, $[D_i]$ is the line bundle induced from $D_i$, $m_j \in \mathbb{N}$ and $m_j \geq 2$. Then there exists a unique (almost) complete Kähler–Einstein metric with negative Ricci curvature on $M$. (See Section 7 here for definition of almost completeness. The completeness is later settled by Yau [31, p. 474].) As a consequence, the logarithmic version of Miyaoka–Yau inequality is established on $M$ and a numerical characterization of ball quotients is given (see [22, p. 626]).

Note that the condition of (1.1) is equivalent to the following (see [22, p. 612]): There exist $\mu_i \in \mathbb{Q}$, $\mu \in (0,1]$, for $i = 1, \ldots, q$, and $m_j \in \mathbb{N}$, $m_j \geq 2$ such that

$$K_{\overline{M}} + \sum_{i=1}^{q} \mu_i [D_i] + \sum_{j=q+1}^{p} \frac{m_j - 1}{m_j} [D_j] > 0 \tag{1.2}$$

on $\overline{M}$ and

$$K_{\overline{M}} + \sum_{i=1}^{q} [D_i] + \sum_{j=q+1}^{p} \frac{m_j - 1}{m_j} [D_j] \quad \text{is nef.} \tag{1.3}$$

The positivity condition (1.2) assures that, on $M$, there exists a natural complete Kähler metric, which has Poincaré growth near the divisor $\sum_{i=1}^{q} D_i$. This background metric can be deformed to a complete Kähler–Einstein metric by solving a degenerate Monge–Ampère equation on $M$. (For the condition corresponding to a nondegenerate Monge–Ampère equation, the existence has also been studied from the viewpoint of Ricci flow, see Chau [4], which is parallel to Cao's work [3] on the compact case.)

In [22], a modified continuity method is introduced to solve the equation. The nef condition (1.3) is imposed so that the deformation of metrics would not be out of control until $t = 1$. But then it is nontrivial to show the completeness of the resulting metric. The key observation in [31] is that the Kähler–Einstein metric can dominate a Poincaré-type metric which has negative holomorphic curvature at least in the normal direction. This is proved by the argument in Yau's Schwarz lemma [30] together with the property of almost completeness.

In this paper, we would like to consider a general positivity assumption, under which $M$ admits a Kähler–Einstein metric of a negative Ricci curvature. Under the assumption, it turns out that the information coming from the boundary divisor is crucial for the Kähler–Einstein geometry over the whole quasi-projective manifold. Indeed, let us relax the positivity

(1.2) to the following condition: There exist real numbers $\alpha_i \in [-1, +\infty)$, $i = 1, \dots, p$, so that

$$K_{\overline{M}} - \sum_{i=1}^{p} \alpha_i [D_i] > 0 \tag{1.4}$$

on $\overline{M}$. This assumption is pretty general for the existence of a Kähler–Einstein metric with negative Ricci curvature on $M$. In fact, if we allow some coefficient $\alpha_k < -1$, then it is possible that there is no Kähler–Einstein metric of negative Ricci curvature on $M$. For instance, let $\overline{M} = \mathbb{C}P^n$ and $D$ be a smooth hypersurface of degree $n+1$. Then (1.4) holds for any $\alpha < -1$. But by [23] there exists a complete Kähler–Ricci flat metric on $M$. Thus, $M$ does not admit any Kähler–Einstein metric of negative Ricci curvature, in view of Yau's Schwarz lemma.

Under an assumption like (1.4), it is standard to construct a complete Kähler metric $\omega$ on $M$ with Poincaré growth near $D$. Furthermore, $\omega$ has bounded sectional curvature, and indeed, has bounded geometry in the sense of Cheng–Yau (see, for example, [26, p. 800–802]). Let $\mathcal{R}(M)$ be the Cheng–Yau's Hölder ring (see Section 2 for definition). Our first result is on the existence of a negative Kähler–Einstein metric on $M$, which is parallel to part of Yau's work on the compact degenerate case [28, p. 364–389].

**Theorem 1.1.** *Under* (1.4)*, there exists a Kähler–Einstein metric $\omega_{\mathrm{KE}}$ on $M$ of finite volume and negative Ricci curvature, satisfying that*

$$C^{-1} \prod_{i=1}^{p} |s_i|^{2\Lambda} \omega < \omega_{\mathrm{KE}} < C \prod_{i=1}^{p} |s_i|^{-2\Lambda} \omega,$$

*where $s_i$ is the holomorphic defining section of $D_i$, $|\cdot|$ is a metric on $[D_i]$, $i = 1, \dots, p$, and $C$ and $\Lambda$ are positive constants.*

Here, the finite volume property follows directly from Yau's Schwarz lemma ([30, p. 202]); see also Lemma 3.4 of this paper. The proof of existence amounts to solving a degenerate Monge–Ampère equation, similar to the previous work. However, without any extra assumption like (1.3), the continuity method will break down. So instead we use an approximation method modified from the second part of Yau's celebrated paper [28, pp. 364–389]. It is unclear whether such a Kähler–Einstein metric is complete or not in the general setting, which is the main drawback of the theorem.

On the other hand, for a Kähler–Einstein metric of negative Ricci curvature, if complete, then it must be unique, and depends only on the complex