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DYNAMIC BIFURCATION AND STABILITY IN THE RAYLEIGH-BÉNARD CONVECTION*

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Abstract. We study in this article the bifurcation and stability of the solutions of the Boussinesq equations, and the onset of the Rayleigh-Bénard convection. A nonlinear theory for this problem is established in this article using a new notion of bifurcation called attractor bifurcation and its corresponding theorem developed recently by the authors in [6]. This theory includes the following three aspects. First, the problem bifurcates from the trivial solution an attractor $\mathcal{A}_R$ when the Rayleigh number R crosses the first critical Rayleigh number R_c for all physically sound boundary conditions, regardless of the multiplicity of the eigenvalue R_c for the linear problem. Second, the bifurcated attractor $\mathcal{A}_R$ is asymptotically stable. Third, when the spatial dimension is two, the bifurcated solutions are also structurally stable and are classified as well. In addition, the technical method developed provides a recipe, which can be used for many other problems related to bifurcation and pattern formation.

1. Introduction

Convection is the well known phenomena of fluid motion induced by buoyancy when a fluid is heated from below. It is of course familiar as the driving force in atmospheric and oceanic phenomena, and in the kitchen! The Rayleigh-Bénard convection problem was originated in the famous experiments conducted by H. Bénard in 1900. Bénard investigated a fluid, with a free surface, heated from below in a dish, and noticed a rather regular cellular pattern of hexagonal convection cells. In 1916, Lord Rayleigh [12] developed a theory to interpret the phenomena of Bénard experiments. He chose the Boussinesq equations with some boundary conditions to model Bénard's experiments, and linearized these equations using normal modes. He then showed that the convection would occur only when the non-dimensional parameter, called the Rayleigh number,

$$R = \frac{g\alpha\beta}{\kappa\nu} h^4 \tag{1.1}$$

exceeds a certain critical value, where g is the acceleration due to gravity, α the coefficient of thermal expansion of the fluid, $\beta = |dT/dz| = (\bar{T}_0 - \bar{T}_1)/h$ the vertical temperature gradient with $\bar{T}_0$ the temperature on the lower surface and $\bar{T}_1$ on the upper surface, h the depth of the layer of the fluid, κ the thermal diffusivity and ν the kinematic viscosity.

Since Rayleigh's pioneering work, there have been intensive studies of this problem; see among others Chandrasekhar [1] and Drazin and Reid [2] for linear theories, and Kirchgässner [5], Rabinowitz [11], and Yudovich [13, 14], and the references therein for nonlinear theories. Most, if not all, known results on bifurcation and stability analysis of the Rayleigh-Benard problem are restricted to the bifurcation and stability analysis when the Rayleigh number crosses a simple eigenvalue in certain subspaces of the entire phase space obtained by imposing certain symmetry.

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It is clear that a complete nonlinear bifurcation and stability theory for this problem should at least include:

1) bifurcation theorem when the Rayleigh number crosses the first critical number for all physically sound boundary conditions,
2) asymptotic stability of bifurcated solutions, and
3) the structure/patterns and their stability and transitions in the physical space.

The main difficulties for such a complete theory are two-fold. The first is due to the high nonlinearity of the problem as in other fluid problems, and the second is due to the lack of a theory to handle bifurcation and stability when the eigenvalue of the linear problem has even multiplicity.

The main objective of this article is to try to establish such a nonlinear theory for the Rayleigh-Bénard convection using a new notion of bifurcation, called attractor bifurcation, and the corresponding theory developed recently by the authors in [6]. Part of the results proved in this article is announced in [9]. We now address each aspects of our results in this article following the three aspects of a complete theory for the problem just mentioned along with the main idea and methods used.

First, we show that as the Rayleigh number R crosses the first critical value R_c, the Boussinesq equations bifurcate from the trivial solution an attractor $\mathcal{A}_R$, with dimension between m and $m+1$. Here the first critical Rayleigh number R_c is defined to be the first eigenvalue of the linear eigenvalue problem, and $m+1$ is the multiplicity of this eigenvalue R_c. In comparison with known results, the bifurcation theorem obtained in this article is for all cases with the multiplicity $m+1$ of the critical eigenvalue R_c for the Bénard problem under any set of physically sound boundary conditions. As the trivial solution becomes unstable as the Rayleigh number crosses the critical value R_c, $\mathcal{A}_R$ does not contain this trivial solution.

Second, as an attractor, the bifurcated attractor $\mathcal{A}_R$ has asymptotic stability in the sense that it attracts all solutions with initial data in the phase space outside of the stable manifold, with co-dimension $m+1$, of the trivial solution.

As Kirchgässner indicated in [5], an ideal stability theorem would include all physically meaningful perturbations and establish the local stability of a selected class of stationary solutions, and today we are still far from this goal. On the other hand, fluid flows are normally time dependent. Therefore bifurcation analysis for steady state problems provides in general only partial answers to the problem, and is not enough for solving the stability problem. Hence it appears that the right notion of asymptotic stability after the first bifurcation should be best described by the attractor near, but excluding, the trivial state. It is one of our main motivations for introducing attractor bifurcation, and it is hoped that the stability of the bifurcated attractor obtained in this article contributes to an ideal stability theorem.

Third, another important aspect of a complete nonlinear theory for the Rayleigh-Bénard convection is to classify the structure/pattern of the solutions after the bifurcation. A natural tool to attack this problem is the structural stability of the solutions in the physical space. Since 1997, the authors have made an extensive study toward this goal, and established a systematic theory on structural stability and bifurcation of 2-D divergence-free vector fields; see a survey article by the authors in [8]. Using in particular the structural stability theorem proved in [7], we show in this article that in the two dimensional case, for any initial data outside of the stable manifold of the trivial solution, the solution of the Boussinesq equations will have the roll structure as t is sufficiently large.

Technically speaking, the above results for the Rayleigh-Bénard convection are achieved using a new notion of dynamic bifurcation, called attractor bifurcation, introduced recently by the authors in [6]. The main theorem associated with attractor bifurcation states that as the control parameter crosses a certain critical value when there are $m+1$ ($m \geq 0$) eigenvalues crossing the imaginary axis, the system bifurcates from a trivial steady state solution to an attractor with dimension between m and $m+1$, provided the critical state is asymptotically stable. This new bifurcation concept generalizes the aforementioned known bifurcation concepts. There are a few important features of attractor bifurcation. First, the bifurcation attractor does not include the trivial steady state, and is stable; hence it is physically important. Second, the attractor contains a collection of solutions of the evolution equation, including possibly steady states, periodic orbits, as well as homoclinic and heteoclinic orbits. Third, it provides a unified point of view on dynamic bifurcation and can be applied to many problems in physics and mechanics. Fourth, from the application point of view, the Krasnoselskii-Rabinowitz theorem requires the number of eigenvalues $m+1$ crossing the imaginary axis to be an odd integer, and the Hopf bifurcation is for the case where $m+1=2$. However, the new attractor bifurcation theorem obtained in this article can be applied to cases for all $m \geq 0$. In addition, the bifurcated attractor, as mentioned earlier, is stable, which is another subtle issue for other known bifurcation theorems.

Of course, the price to pay here is the verification of the asymptotic stability of the critical state, in addition to the analysis needed for the eigenvalues problems in the linearized problem. Theorem 3.4 provides a method to obtain asymptotic stability of the critical state for problems with symmetric linearized equations. Thanks to this theorem, the asymptotic stability of the trivial solution to the Rayleigh-Bénard problem is easily established. We remark here that this theorem will be useful in many problems of mathematical physics with symmetric linearized equations.

This article is organized as follows. First in Section 2, we recall the Boussinesq equations, their mathematical setting, and some known existence and uniqueness results of the solutions. Section 3 summaries the main attractor bifurcation theory from [6], and a theorem, Theorem 3.4, for the asymptotic stability of the critical state for problems for an evolution system with symmetric linearized equations. Section 4 states and proves the main attractor bifurcation results from the Rayleigh-Bénard convection. Examples and topological structure of the bifurcated solutions are addressed in Section 5. Corresponding results for the two-dimensional problem are given in Section 6, and the concept and main results on structural stability of 2-D divergence-free vector fields are recalled in the Appendix in Section 7.

2. Boussinesq equations and their mathematical setting

2.1. Boussinesq equations. The Bénard experiment can be modeled by the Boussinesq equations; see among others Rayleigh [12], Drazin and Reid [2] and Chandrasekhar [1]. They read

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu\Delta u + \rho_0^{-1}\nabla p = -gk[1-\alpha(T-\bar{T}_0)], \tag{2.1}$$

$$\frac{\partial T}{\partial t} + (u \cdot \nabla)T - \kappa\Delta T = 0, \tag{2.2}$$

$$\text{div } u = 0, \tag{2.3}$$

where ν, κ, α, g are the constants defined as in (1.1), $u=(u_1,u_2,u_3)$ the velocity field, p the pressure function, T the temperature function, $\bar{T}_0$ a constant representing the lower surface temperature at $x_3 = 0$, and $k = (0, 0, 1)$ the unit vector in x_3-direction; see Figure 2.1.

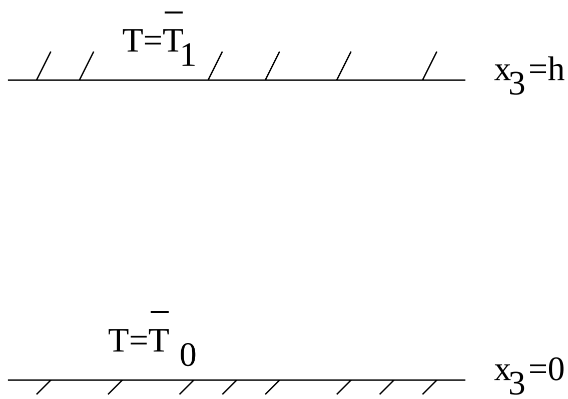


FIG. 2.1. *Flow between two plates heated from below:* $\bar{T}_0 > \bar{T}_1$.

To make the equations non-dimensional, let

$$\begin{aligned} x &= hx', \\ t &= h^2 t'/\kappa, \\ u &= \kappa u'/h, \\ T &= \beta h(T'/\sqrt{R}) + \bar{T}_0 - \beta h x_3', \\ p &= \rho_0 \kappa^2 p'/h^2 + p_0 - g\rho_0(hx_3' + \alpha\beta h^2 (x_3')^2/2), \\ P_r &= \nu/\kappa. \end{aligned}$$

Here the Rayleigh number R is defined by (2.1), and $P_r = \nu/\kappa$ is the Prandtl number.

Omitting the primes, the equations (2.2)-(2.4) can be rewritten as follows

$$\frac{1}{P_r}\left[\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p\right] - \Delta u - \sqrt{R}Tk = 0, \tag{2.4}$$

$$\frac{\partial T}{\partial t} + (u \cdot \nabla)T - \sqrt{R}u_3 - \Delta T = 0, \tag{2.5}$$

$$\text{div } u = 0. \tag{2.6}$$

The non-dimensional domain is $\Omega = D \times (0, 1) \subset \mathbb{R}^3$, where $D \subset \mathbb{R}^2$ is an open set. The coordinate system is given by $x = (x_1, x_2, x_3) \in \mathbb{R}^3$.

The Boussinesq equations (2.4)—(2.6) are basic equations to study the Rayleigh-Bénard problem in this article. They are supplemented with the following initial value conditions

$$(u, T) = (u_0, T_0) \qquad \text{at } t = 0. \tag{2.7}$$

Boundary conditions are needed at the top and bottom and at the lateral boundary $\partial D \times (0, 1)$. At the top and bottom boundary ($x_3 = 0, 1$), either the so-called rigid or free boundary conditions are given

$$T=0, \quad u=0 \quad \text{(rigid boundary)}, \tag{2.8}$$

$$T=0, \quad u_3=0, \quad \frac{\partial(u_1,u_2)}{\partial x_3}=0 \quad \text{(free boundary)}. \tag{2.9}$$

Different combinations of top and bottom boundary conditions are normally used in different physical setting such as *rigid-rigid*, *rigid-free*, *free-rigid*, and *free-free*.

On the lateral boundary $\partial D\times[0,1]$, one of the following boundary conditions are usually used:

1. Periodic condition:

$$(u,T)(x_1+k_1L_1,x_2+k_2L_2,x_3)=(u,T)(x_1,x_2,x_3), \tag{2.10}$$

   for any $k_1,k_2\in\mathbb{Z}$.

2. Dirichlet boundary condition:

$$u=0, \quad T=0 \quad (\text{or } \frac{\partial T}{\partial n}=0); \tag{2.11}$$

3. Free boundary condition:

$$T=0, \quad u_n=0, \quad \frac{\partial u_\tau}{\partial n}=0, \tag{2.12}$$

   where n and τ are the unit normal and tangent vectors on $\partial D\times[0,1]$ respectively, and $u_n=u\cdot n$, $u_\tau=u\cdot\tau$.

For simplicity, we proceed in this article with the following set of boundary conditions, and all results hold true as well for other combinations of boundary conditions.

$$\begin{cases} T=0, \quad u=0 & \text{at } x_3=0,1,\\ (u,T)(x_1+k_1L_1,x_2+k_2L_2,x_3,t)=(u,T)(x,t), \end{cases} \tag{2.13}$$

for any $k_1,k_2\in\mathbb{Z}$.

2.2. Functional setting and properties of solutions. We recall here the functional setting of equations (2.4)-(2.6) with initial and boundary conditions (2.7) and (2.13) and refer the interested readers to Foias, Manley and Temam [3] for details. To this end, let

$$\begin{aligned} H=\{(u,T)\in L^2(\Omega)^3\times L^2(\Omega) \quad | \quad & \mathrm{div} u=0, u_3|_{x_3=0,1}=0, \\ & u_i \text{ is periodic in the } x_i \text{ direction } (i=1,2)\}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} V=\{(u,T)\in H^1_0(\Omega)^4 \quad | \quad & \mathrm{div} u=0, \\ & u_i \text{ is periodic in the } x_i \text{ direction } (i=1,2)\}, \end{aligned} \tag{2.15}$$

where $H^1_0(\Omega)$ is the space of functions in $H^1(\Omega)$, which vanish at $x_3=0,1$ and are periodic in the x_i-directions $(i=1,2)$. Here $H^1(\Omega)$ is the usual Sobolev space.

Then the results concerning the existence of a solution for (2.4)-(2.6) with initial and boundary conditions (2.7) and (2.13) are classical. For every $(\phi_0,T_0)\in H$, (2.4)-(2.6) with (2.7) and (2.13) possesses a weak solution

$$(u,T)\in L^\infty([0,\tau];H)\cap L^2(0,\tau;V) \quad \forall\tau>0. \tag{2.16}$$

If $(u_0, T_0) \in V$, (2.4)-(2.6) with (2.7) and (2.13) possesses a unique solution on some interval $[0, \tau_1]$,

$$(u, T) \in C([0, \tau_1]; V) \cap L^2(0, \tau_1; H^2(\Omega)^4 \cap V), \tag{2.17}$$

where $\tau_1 = \tau_1(M)$ depends on a bound of the V norm of (ϕ_0, T_0):

$$||(u_0, T_0)|| \leq M.$$

In addition, for any $||(\phi_0, T_0)|| \leq \delta$ small, (2.4)-(2.6) with (2.7) and (2.13) possesses a unique global (in time) solution

$$(u, T) \in C([0, \tau]; V) \cap L^2(0, \tau; H^2(\Omega)^4 \cap V), \quad \forall \tau > 0. \tag{2.18}$$

Thanks to these existence results, we can define a semi-group

$$S(t) : (u_0, T_0) \to (u(t), T(t)),$$

which enjoys the semi-group properties.

3. Dynamic bifurcation of nonlinear evolution equations

In this section, we shall recall some results of dynamic bifurcation of abstract nonlinear evolution equations developed by the authors in [6], which is crucial in the study of the Bénard problem in this paper. In fact, we shall provide in this section a recipe for proving dynamic bifurcations for problems with symmetric linear operators.

3.1. Attractor bifurcation. Let H and H_1 be two Hilbert spaces, and $H_1 \hookrightarrow H$ be a dense and compact inclusion. We consider the following nonlinear evolution equations

$$\frac{du}{dt} = L_\lambda u + G(u, \lambda), \tag{3.1}$$

$$u(0) = u_0, \tag{3.2}$$

where $u : [0, \infty) \to H$ is the unknown function, $\lambda \in \mathbb{R}$ is the system parameter, and $L_\lambda : H_1 \to H$ are parameterized linear completely continuous fields continuously depending on $\lambda \in \mathbb{R}^1$, which satisfy

$$\begin{cases} L_\lambda = -A + B_\lambda & \text{is a sectorial operator,} \\ A : H_1 \to H & \text{a linear homeomorphism,} \\ B_\lambda : H_1 \to H & \text{the parameterized linear compact operators.} \end{cases} \tag{3.3}$$

It is easy to see [4, 10] that L_λ generates an analytic semi-group $\{e^{-tL_\lambda}\}_{t \geq 0}$. Then we can define fractional power operators L_λ^α for any $0 \leq \alpha \leq 1$ with domain $H_\alpha = D(L_\lambda^\alpha)$ such that $H_{\alpha_1} \subset H_{\alpha_2}$ if $\alpha_1 > \alpha_2$, and $H_0 = H$.

Furthermore, we assume that the nonlinear terms $G(\cdot, \lambda) : H_\alpha \to H$ for some $1 > \alpha \geq 0$ are a family of parameterized C^r bounded operators ($r \geq 1$) continuously depending on the parameter $\lambda \in \mathbb{R}^1$, such that

$$G(u, \lambda) = o(\|u\|_{H_\alpha}), \quad \forall\ \lambda \in \mathbb{R}^1. \tag{3.4}$$

In the applications, we are interested in the sectorial operator $L_\lambda = -A + B_\lambda$ such that there exist a real eigenvalue sequence $\{\rho_k\} \subset \mathbb{R}^1$ and and an eigenvector sequence $\{e_k\} \subset H_1$ of A:

$$\begin{cases} Ae_k = \rho_k e_k, \\ 0 < \rho_1 \leq \rho_2 \leq \cdots, \\ \rho_k \to \infty \ (k \to \infty) \end{cases} \tag{3.5}$$

such that $\{e_k\}$ is an orthogonal basis of H.

For the compact operator $B_\lambda : H_1 \to H$, we also assume that there is a constant $0 < \theta < 1$ such that

$$B_\lambda : H_\theta \longrightarrow H \text{ bounded}, \ \forall \ \lambda \in \mathbb{R}^1. \tag{3.6}$$

Let $\{S_\lambda(t)\}_{t \geq 0}$ be an operator semi-group generated by the equation (3.1) which enjoys the properties

(i) For any $t \geq 0$, $S_\lambda(t) : H \to H$ is a linear continuous operator,
(ii) $S_\lambda(0) = I : H \to H$ is the identity on H, and
(iii) For any $t, s \geq 0$, $S_\lambda(t + s) = S_\lambda(t) \cdot S_\lambda(s)$

Then the solution of (3.1) and (3.2) can be expressed as

$$u(t) = S_\lambda(t) u_0, \qquad t \geq 0.$$

Definition 3.1. *A set $\Sigma \subset H$ is called an invariant set of (3.1) if $S(t)\Sigma = \Sigma$ for any $t \geq 0$. An invariant set $\Sigma \subset H$ of (3.1) is said to be an attractor if Σ is compact, and there exists a neighborhood $U \subset H$ of Σ such that for any $\varphi \in U$ we have*

$$\lim_{t \to \infty} \text{dist}_H(u(t, \varphi), \Sigma) = 0. \tag{3.7}$$

The largest open set U satisfying (3.7) is called the basin of attraction of Σ.

Definition 3.2.

1. *We say that the equation (3.1) bifurcates from $(u, \lambda) = (0, \lambda_0)$ an invariant set Ω_λ, if there exists a sequence of invariant sets $\{\Omega_{\lambda_n}\}$ of (3.1), $0 \notin \Omega_{\lambda_n}$ such that*

$$\lim_{n \to \infty} \lambda_n = \lambda_0,$$
$$\lim_{n \to \infty} \max_{x \in \Omega_{\lambda_n}} |x| = 0.$$

2. *If the invariant sets Ω_λ are attractors of (3.1), then the bifurcation is called attractor bifurcation.*
3. *If Ω_λ are attractors and are homotopy equivalent to an m–dimensional sphere S^m, then the bifurcation is called S^m–attractor bifurcation.*

A complex number $\beta = \alpha_1 + i\alpha_2 \in \mathbb{C}$ is called an eigenvalue of L_λ if there are $x, y \in H_1$ such that

$$L_\lambda x = \alpha_1 x - \alpha_2 y,$$
$$L_\lambda y = \alpha_2 x + \alpha_1 y.$$

Now let the eigenvalues (counting the multiplicity) of L_λ be given by

$$\beta_1(\lambda), \beta_2(\lambda), \cdots, \beta_k(\lambda) \in \mathbb{C},$$

where $\mathbb{C}$ is the complex space. Suppose that

$$Re\beta_i(\lambda) = \begin{cases} < 0, & \lambda < \lambda_0 \\ = 0, & \lambda = \lambda_0 \\ > 0, & \lambda > \lambda_0 \end{cases} \qquad (1 \le i \le m+1) \tag{3.8}$$

$$Re\beta_j(\lambda_0) < 0, \qquad \forall\, m+2 \le j. \tag{3.9}$$

Let the eigenspace of L_λ at λ_0 be

$$E_0 = \cup_{1 \le i \le m+1} \left\{ u \in H_1 \mid (L_{\lambda_0} - \beta_i(\lambda_0))^k u = 0,\, k = 1, 2, \cdots \right\}.$$

It is known that $\dim E_0 = m + 1$.

The following dynamic bifurcation theorems for the (3.1) were proved in [6].

Theorem 3.3 (Attractor Bifurcation, [6]). *Assume that the conditions (3.3), (3.4), (3.8) and (3.9) hold true, and $u = 0$ is a locally asymptotically stable equilibrium point of (3.1) at $\lambda = \lambda_0$. Then the following assertions hold true.*

1. *(3.1) bifurcates from $(u, \lambda) = (0, \lambda_0)$ an attractor $\mathcal{A}_\lambda$ for $\lambda > \lambda_0$, with $m \le \dim \mathcal{A}_\lambda \le m+1$, which is connected as $m > 0$;*
2. *the attractor $\mathcal{A}_\lambda$ is a limit of a sequence of $(m+1)$–dimensional annulus M_k with $M_{k+1} \subset M_k$; especially if $\mathcal{A}_\lambda$ is a finite simplicial complex, then $\mathcal{A}_\lambda$ has the homotopy type of S^m;*
3. *For any $u_\lambda \in \mathcal{A}_\lambda$, u_λ can be expressed as*
$$u_\lambda = v_\lambda + o(\|v_\lambda\|_{H_1}), \quad v_\lambda \in E_0;$$
4. *If $G : H_1 \to H$ is compact, and the equilibrium points of (3.1) in $\mathcal{A}_\lambda$ are finite, then we have the index formula*
$$\sum_{u_i \in \mathcal{A}_\lambda} \text{ind}\,[-(L_\lambda + G), u_i] = \begin{cases} 2 & \text{if } m = \text{odd}, \\ 0 & \text{if } m = \text{even}. \end{cases}$$
5. *If $u = 0$ is globally stable for (3.1) at $\lambda = \lambda_0$, then for any bounded open set $U \subset H$ with $0 \in U$ there is an $\varepsilon > 0$ such that as $\lambda_0 < \lambda < \lambda_0 + \varepsilon$, the attractor $\mathcal{A}_\lambda$ bifurcated from $(0, \lambda_0)$ attracts U/Γ in H, where Γ is the stable manifold of $u = 0$ with co-dimension $m + 1$. In particular, if (3.1) has global attractor for all λ near λ_0, then the ε here can be chosen independently of U.*

3.2. Asymptotical stability at critical states. To apply the above dynamic bifurcation theorems, it is crucial to verify the asymptotic stability of the critical states. We establish in this subsection a theorem to verify the needed asymptotic stability for equations with symmetric linear parts.

Let the linear operator L_λ in (3.1) be symmetric, i.e.

$$\langle L_\lambda u, v\rangle_H = \langle u, L_\lambda v\rangle_H, \qquad \forall\, u, v \in H_1.$$

Then all eigenvalues of L_λ are real numbers. Let the eigenvalues $\{\beta_k\}$ of L_λ at $\lambda = \lambda_0$ satisfy

$$\begin{cases} \beta_i = 0, & 1 \le i \le m+1 \quad (m \ge 0), \\ \beta_j < 0, & m+2 \le j < \infty. \end{cases} \tag{3.10}$$

Set

$$
\begin{aligned}
&E_0 = \{u \in H_1 \mid L_{\lambda_0} u = 0\}\,,\\
&E_1 = E_0^{\perp} = \{u \in H_1 \mid \langle u, v\rangle_H = 0 \quad \forall\, v \in E_0\}\,,\\
&P_1 \;:\; H \longrightarrow E_1 \text{ the projection.}
\end{aligned}
$$

By (3.10), $\dim E_0 = m + 1$.

THEOREM 3.4. *Let L_λ in (3.3) be symmetric with spectrum given by (3.10) hold true, and $G_{\lambda_0} : H_1 \to H$ satisfies the following orthogonal condition:*

$$
\langle G_{\lambda_0} u, u\rangle_H = 0, \qquad \forall\, u \in H_1. \tag{3.11}
$$

Then exactly one and only one of the following two assertions holds true:

1. *There exists a sequence of invariant sets $\{\Gamma_n\} \subset E_0$ of (3.1) at $\lambda = \lambda_0$ such that*

$$
0 \notin \Gamma_n, \qquad \lim_{n\to\infty} \operatorname{dist}(\Gamma_n, 0) = 0;
$$

2. *the trivial steady state solution $u = 0$ for (3.1) at $\lambda = \lambda_0$ is locally asymptotically stable under the H–norm.*

Furthermore, if (3.1) has no invariant sets in E_0 except the trivial one $\{0\}$, then $u = 0$ is globally asymptotically stable.

Proof. We proceed in the following four steps.

STEP 1. It is easy to see that Assertions (1) and (2) in Theorem 3.4 can not be true at the same time.

Hereafter in this proof, we always work on the case where $\lambda = \lambda_0$. In this case, direct energy estimates imply that that the solutions u of (3.1) satisfy that

$$
\frac{d}{dt}\|u\|_H^2 = 2 < L_{\lambda_0} u, u >= \sum_{n=m+2}^{\infty} \beta_i |u_i|^2 \le 0, \tag{3.12}
$$

$$
\|u\|_H^2 \le \|u(0)\|_H^2 - 2\,|\beta_{m+2}| \int_0^t \|v\|_H^2 d\tau, \tag{3.13}
$$

where

$$
\begin{aligned}
&u = w + v \in H = E_0 \oplus E_0^{\perp},\\
&v = \sum_{i=m+2}^{\infty} u_i \in E_0^{\perp},\\
&w = \sum_{i=1}^{m+1} u_i \in E_1 = E_0.
\end{aligned}
$$

It is easy to see that for any $\varphi \in H_1$ the solution $u(t, \varphi)$ of (5.1) is non-increasing, i.e.

$$
\|u(t_2, \varphi)\| \le \|u(t_1, \varphi)\|, \quad \forall\, t_1 < t_2 \text{ and } \varphi \in H_1. \tag{3.14}
$$

Hence $\lim_{t\to\infty} \|u(t, \varphi)\|$ exists.

STEP 2. For any $\varphi \in H_1$, we have

$$\lim_{t\to\infty} \|u(t,\varphi)\| = \lim_{t\to\infty} \|v(t,\varphi) + w(t,\varphi)\| = \delta \le ||\varphi||.$$

Then the ω-limit set, which is an invariant set, satisfies that

$$\omega(\varphi) \subset S_\delta = \{u \in H \mid \|u\| = \delta\}.$$

Since $\omega(\varphi)$ is an invariant set, for an $\psi \in \omega(\varphi)$ we have

$$u(t,\psi) \subset \omega(\varphi) \subset S_\delta \quad \forall\ t \ge 0.$$

Hence if $\psi = \bar{v} + \bar{w} \in E_0^\perp \oplus E_0$ with $\bar{v} \ne 0$, then by (3.12), for any $t > 0$,

$$\|u(t,\psi)\| < \|\psi\| = \delta,$$

a contradication. Namely, for any $\varphi \in H_1$

$$\omega(\varphi) \subset E_0. \tag{3.15}$$

STEP 3. If Assertion (2) is false, then there exists $u_n \in H_1$ with $u_n \to 0$ as $n \to \infty$ such that $0 \notin \omega(u_n) \subset E_0$, and

$$\lim_{n\to\infty} \mathrm{dist}(\omega(u_n), 0) = 0.$$

Namely, Assertion (1) holds true.

STEP 4. If Assertion (1) is not true, there exist a neighborhood $U \subset H$ of 0 such that for any $\phi \in U$,

$$\lim_{t\to\infty} \|u(t,\varphi)\| = 0.$$

Namely, Assertion (2) holds true. The rest part of the proof is trivial, and the proof is complete. ∎

4. Attractor bifurcation of the Bénard problem

4.1. Main theorems. The linearized equations of (2.4)-(2.6) are given by

$$\begin{cases} -\Delta u + \nabla p - \sqrt{R}Tk = 0, \\ -\Delta T - \sqrt{R}u_3 = 0, \\ \mathrm{div}\, u = 0, \end{cases} \tag{4.1}$$

where R is the Rayleigh number. These equations are supplemented with the same boundary conditions (2.13) as the nonlinear Boussinesq system. This eigenvalue problem for the Rayleigh number R is symmetric. Hence, we know that all eigenvalues R_k with multiplicities m_k of (4.1) with (2.13) are real numbers, and

$$0 < R_1 < \cdots < R_k < R_{k+1} < \cdots. \tag{4.2}$$

The first eigenvalue R_1, also denoted by $R_c = R_1$, is called the critical Rayleigh number. Let the multiplicity of R_c be $m_1 = m+1$ ($m \ge 0$), and the first eigenvectors $\Psi_1 = (e_1(x), T_1), \cdots, \Psi_{m+1} = (e_{m+1}, T_{m+1})$ of (4.1) be orthonormal:

$$\langle \Psi_i, \Psi_j \rangle_H = \int_\Omega [e_i \cdot e_j + T_i T_j] dx = \delta_{ij}.$$

For simplicity, let E_0 be the first eigenspace of (4.1) with with (2.13)

$$E_0 = \left\{ \sum_{k=1}^{m+1} \alpha_k \Psi_k \mid \alpha_k \in \mathbb{R},\ 1 \leq k \leq m+1 \right\}. \tag{4.3}$$

The main results in this section are the following theorems.

THEOREM 4.1. *For the Bénard problem (2.4-2.6) with (2.13), the following assertions hold true.*

1. *When the Rayleigh number is less than or equal to the critical Rayleigh number: $R \leq R_c$, the steady state $(u, T) = 0$ is a globally asymptotically stable equilibrium point of the equations.*
2. *The equations bifurcate from $((u, T), R) = (0, R_c)$ an attractor $\mathcal{A}_R$ for $R > R_c$, with $m \leq \dim \mathcal{A}_R \leq m + 1$, which is connected when $m > 0$.*
3. *For any $(u, T) \in \mathcal{A}_R$, the velocity field u can be expressed as*

$$u = \sum_{k=1}^{m+1} \alpha_k e_k + o\left(\sum_{k=1}^{m+1} \alpha_k e_k\right) \tag{4.4}$$

where e_k are the velocity fields of the first eigenvectors in E_0.

4. *The attractor $\mathcal{A}_R$ has the homotopy type of an m-dimensional sphere S^m provided $\mathcal{A}_R$ is a finite simplicial complex.*
5. *There are an open neighborhood $U \subset H$ of $(u, T) = 0$ and an $\varepsilon > 0$ such that as $R_c < R < R_c + \varepsilon$, the attractor $\mathcal{A}_R$ attracts U/Γ in H, where Γ is the stable manifold of $(u, T) = 0$ with co-dimension $m + 1$.*

THEOREM 4.2. *If the first eigenvalue of L_{λ_0} is simple, i.e. $\dim E_0 = 1$, then the bifurcated attractor $\mathcal{A}_R$ of the Bénard problem (2.4-2.6) with (2.13) consists of exactly two points, $\bar{\phi}_1, \bar{\phi}_2 \in H_1 = V \cap H^2(\Omega)^4$ given by*

$$\bar{\phi}_1 = \alpha\Psi_1 + o(|\alpha|), \quad \bar{\phi}_2 = -\alpha\Psi_1 + o(|\alpha|),$$

for some $\alpha \neq 0$, where Ψ_1 is the first eigenvector generating E_0 in (4.3). Moreover, for any bounded open set $U \in H$ with $0 \in U$, there is an $\varepsilon > 0$, as $R_c < R < R_c + \varepsilon$, U can be decomposed into two open sets U_1 and U_2 such that

1. *$\bar{U} = \bar{U}_1 + \bar{U}_2$, $U_1 \cap U_2 = \emptyset$ and $0 \in \partial U_1 \cap \partial U_2$,*
2. *$\bar{\phi}_i \subset U_i$ $(i = 1, 2)$, and*
3. *for any $\phi_0 \in U_i$ $(i = 1, 2)$, $\lim_{t\to\infty} S_\lambda(t)\phi_0 = \bar{\phi}_i$, where $S_\lambda(t)\phi_0$ is the solution of the Bénard problem (2.4-2.6) with (2.13) with initial data $\phi_0 = (u_0, T_0)$.*

A few remarks are now in order.

REMARK 4.3. As we shall see in next section, (4.4) in Theorem 4.1 is crucial for studying the topological structure of the Rayleigh-Bénard convection.

REMARK 4.4. Theorem 4.2 corresponds to the classical pitchfork bifurcation. The main advantage of this theorem is that we know the stability of these bifurcated steady states.

REMARK 4.5. Both theorems hold true for Boussinesq equations (2.4-2.6) with different combinations of boundary conditions as described in Section 2.

4.2. Proof of Theorem 4.1. We shall use the abstract results in Section 3 to prove Theorem 4.1, and proceed in the following steps.

Step 1. First of all, without loss of generality, we assume the Prandtl number

$$P_r = 1; \tag{4.5}$$

otherwise, we only have to consider the following form of (2.4)-(2.6), and the proof is the same.

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p - P_r \Delta u - \sqrt{R}\sqrt{P_r}\theta k = 0, \\ \dfrac{\partial \theta}{\partial t} + (u \cdot \nabla)\theta - \sqrt{R}\sqrt{P_r} u_3 - \Delta\theta = 0, \\ \text{div } u = 0, \end{cases} \tag{4.6}$$

where $\theta = \sqrt{P_r}T$.

Now let H be the function space defined by (2.14) and let H_1 be the intersection of H with H^2 Sobolev space, i.e.

$$H_1 = H \cap (H^2(\Omega))^4.$$

Then let $G : H_1 \to H$, and $L_\lambda = -A + B_\lambda : H_1 \to H$ be defined by

$$\begin{cases} G(\phi) = (-P[(u \cdot \nabla)u], -(u \cdot \nabla)T), \\ A\phi = (-P(\Delta u), -\Delta T), \\ B_\lambda \phi = \lambda(P(Tk), u_3), \end{cases} \tag{4.7}$$

for any $\phi \in H$. Here $\lambda = \sqrt{R}$, and $P : L^2(\Omega)^3 \to H$ the Leray projection. Then it is easy to see that these operators enjoy the following properties:

1. the linear operators A, B_λ and L_λ are all symmetric operators,
2. the nonlinear operator G is orthogonal, i.e.

$$\langle G(\phi), \phi \rangle_H = 0. \tag{4.8}$$

3. the conditions (3.3)—(3.6) hold true for these operators defined in (4.7).

Then the Boussinesq equations (2.4) can be rewritten in the following operator form

$$\frac{d\phi}{dt} = L_\lambda \phi + G(\phi), \qquad \phi = (u, T). \tag{4.9}$$

Step 2. Now, we need to check the conditions (3.8) and (3.9). Consider the engenvalue problem

$$L_\lambda \phi = \beta(\lambda)\phi, \qquad \phi = (u, T) \in H_1. \tag{4.10}$$

This eigenvalue problem is equivalent to

$$\begin{cases} -\Delta u + \nabla p - \lambda T k + \beta(\lambda) u = 0, \\ -\Delta T - \lambda u_3 + \beta(\lambda) T = 0, \\ \text{div}\, u = 0. \end{cases} \tag{4.11}$$

It is known that the eigenvalues β_k $(k = 1, 2, \cdots)$ of (4.11) are real numbers satisfying

$$\begin{cases} \beta_1(\lambda) \geq \beta_2(\lambda) \geq \cdots \geq \beta_k(\lambda) \geq \cdots, \\ \lim_{k\to\infty} \beta_k(\lambda) = -\infty, \end{cases} \tag{4.12}$$

and the first eigenvalue $\beta_1(\lambda)$ of (4.11) and the first eigenvalue $\lambda_1 = \sqrt{R_c}$ of (4.1) have the relation:

$$\beta_1(\lambda) \begin{cases} < 0 & \text{as } 0 \leq \lambda < \lambda_1, \\ = 0 & \text{as } \lambda = \lambda_1. \end{cases} \tag{4.13}$$

Step 3. To prove (3.8) and (3.9), by (4.12) and (4.13), it suffices to prove that

$$\beta_1(\lambda) > 0 \quad \text{as } \lambda > \lambda_1. \tag{4.14}$$

We know that the first eigenvalue $\beta_1(\lambda)$ of (4.11) has the minimal property

$$-\beta_1(\lambda) = \min_{(u,T)\in H_1} \frac{\int_\Omega \left[|\nabla u|^2 + |\nabla T|^2 - 2\lambda T u_3\right] dx}{\int_\Omega [T^2 + u^2] dx}. \tag{4.15}$$

It is clear that the first eigenvectors $(e, \varphi) \in H_1$ satisfy

$$\int_\Omega \left[|\nabla e|^2 + |\nabla \varphi|^2 - 2\lambda e_3 \varphi\right] dx = \begin{cases} 0, & \lambda = \lambda_1 \\ < 0, & \lambda > \lambda_1. \end{cases} \tag{4.16}$$

From (4.15) and (4.16) we infer (4.14). Thus the conditions (3.8) and (3.9) are achieved.

Step 4. Finally, in order to use Theorems 3.3 to prove Theorem 4.1, we need to show that $(u, T) = 0$ is a globally asymptotically stable equilibrium point of (2.4)-(2.6) at the critical Rayleigh number $\lambda_1 = \sqrt{R_c}$. By Theorem 3.4, it suffices to prove that the equations (2.4)-(2.6) have no invariant sets except the steady state $(u, T) = 0$ in the first eigenspace E_0 of (4.1).

We know that the Boussinesq equations (2.4)-(2.6) have a bounded absorbing set in H; hence, all invariant sets have the same bound in H as the absorbing set. Assume (2.4)-(2.6) have an invariant $B \subset E_0$ with $B \neq \{0\}$ at $\lambda_1 = \sqrt{R_c}$. Then restricted in B, which contains eigenfuctions of the linear part corresponding to the eigenvalue 0, the Boussinesq equations (2.4)-(2.6) can be rewritten as

$$\begin{cases} \dfrac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p = 0, \\ \dfrac{\partial T}{\partial t} + (u \cdot \nabla) T = 0, \end{cases} \tag{4.17}$$

It is easy to see that for the solutions $(u, T) \in B$ of (4.17), $(\widetilde{u}, \widetilde{T}) = \alpha(u(\alpha t), T(\alpha t)) \in \alpha B \subset E_0$ are also solutions of (4.17). Namely, for any real number $\alpha \in R$, the set $\alpha B \subset E_0$ is an invariant set of (4.18). Thus, we infer that (2.4)-(2.6) have an unbounded invariant set, which is a contradiction to the existence of absorbing set. Hence the invariant set B can only consist of $(u, T) = 0$. The proof is complete. $\square$

4.3. Proof of Theorem 4.2. By Theorem 4.1, it suffices to prove that the stationary equations of (4.1) will bifurcate exactly two singular points in H_1 as $R > R_c$. We use the Lyapunov–Schmidt method to prove this assertion.

Since the operator $L_\lambda : H_1 \to H$ defined by (4.7) is a symmetric completely continuous field, H_1 can be decomposed into

$$\begin{aligned} &H_1 = E_1^\lambda \oplus E_2^\lambda, \\ &E_1^\lambda = \{\alpha\Psi_1(\lambda) \mid \alpha \in \mathbb{R},\ \Psi_1(\lambda) \text{ the first eigenvector of } L_\lambda + G\}, \\ &E_2^\lambda = \{\phi \in H_1 \mid \langle \phi, \Psi_1 \rangle_H = 0\}. \end{aligned}$$

Furthermore, E_1^λ and E_2^λ are invariant subspaces of $L_\lambda + G$.

Let $P_1 : H_1 \to E_1^\lambda$ be the canonical projection, and

$$\phi = x\Psi_1 + y, \quad x \in \mathbb{R}, \quad y \in E_2^\lambda.$$

Then the equations $L_\lambda \phi + G(\phi) = 0$ can be decomposed into

$$\beta(\lambda)x + \langle G(\phi), \Psi_1(\lambda)\rangle_H = 0, \tag{4.18}$$

$$L_\lambda y + P_1 G(u) = 0. \tag{4.19}$$

By the assumption, the eigenvalues $\beta_j(\lambda)$ of $L_\lambda \phi = \beta(\lambda)\phi$ satisfy that $\beta_j(\lambda_1) \neq 0$ for $j \geq 2$, and $\lambda_1 = \sqrt{R_c}$. Hence the restriction

$$L_\lambda \mid_{E_2^\lambda} : E_2^\lambda \longrightarrow E_2^\lambda$$

is invertible. By the implicit function theorem, from (4.19) it follows that y is a function of x:

$$y = y(x, \lambda), \tag{4.20}$$

which satisfies (4.19). Since $G(u) = G(x\Psi_1 + y)$ is an analytic function of u, the function (4.20) is also analytic. Hence, the function

$$f(x,\lambda) = \langle G(x\Psi_1 + y(x,\lambda)), \Psi_1\rangle_H \tag{4.21}$$

is analytic. Thus, the equation (4.18) has the expansion

$$\beta(\lambda)x + f(x,\lambda) = \beta(\lambda)x + \alpha(\lambda)x^k + o(|x|^k) = 0, \tag{4.22}$$

for some $\alpha(\lambda) \in \mathbb{R}$ such that $\alpha(\lambda_1) \neq 0$ and $k > 1$, where $\lambda_1 = R_c$ is the critical Rayleigh number. By assumption

$$\beta(\lambda) \begin{cases} < 0 & \text{as } \lambda < \lambda_1, \\ = 0 & \text{as } \lambda = \lambda_1, \\ > 0 & \text{as } \lambda > \lambda_1. \end{cases}$$

In addition, by Theorem 4.1, as $\lambda \leq \lambda_1$ (i.e. $R \leq R_c$) and $\lambda_1 - \lambda$ is small, the equations (4.18) and (4.19) have no non-zero solutions, which implies that $\alpha(\lambda_1) < 0$ and $k =$ odd.

Thus, we derive that the equation (4.22) has exactly two solutions

$$x_\pm = \pm\left(\frac{\beta(\lambda)}{|\alpha|}\right)^{1/k} + o\left(\left(\frac{\beta(\lambda)}{|\alpha|}\right)^{1/k}\right),$$

for $\lambda > \lambda_1$ with $\lambda - \lambda_1$ sufficiently small. Namely, we have proved that as $\lambda > \lambda_1$, or $R > R_c$, with $\lambda - \lambda_1$ sufficiently small, the stationary equations of (2.4)-(2.6) bifurcate from $(\phi, \lambda) = (0, \lambda_1)$ exactly two solutions

$$\phi_\lambda = x_\pm \Psi_1 + o(|x_\pm|).$$

Thus, this theorem is proved. □

5. Remarks on topological structure of solutions of the Rayleigh-Bénard problem

As we mentioned before, the structure of the eigenvectors of the linearized problem (4.1) plays an important role for studying the onset of the Rayleigh-Bénard convection. The dimension $m+1$ of the eigenspace E_0 determines the dimension of the bifurcated attractor $\mathcal{A}_R$ as well. Hence in this section we examine in detail the first eigenspace for different geometry of the spatial domain and for different boundary conditions.

5.1. Solutions of the eigenvalue problem. Hereafter, we always consider the Bénard problem on the rectangular region: $\Omega = (0, L_1) \times (0, L_2) \times (0, 1)$, and the boundary condition taken as the free boundary condition

$$u \cdot n = 0, \qquad \frac{\partial u \cdot \tau}{\partial n} = 0 \text{ on } \partial\Omega, \tag{5.1}$$

$$T = 0 \quad \text{at} \quad x_3 = 0, 1, \tag{5.2}$$

$$\frac{\partial T}{\partial n} = 0 \quad \text{at} \quad x_1 = 0, L_1 \text{ or } x_2 = 0, L_2. \tag{5.3}$$

For the eigenvalue equations (4.1) with the boundary condition (5.1)—(5.3), we take the separation of variables as follows

$$\begin{cases} (u_1, u_2) = \dfrac{1}{a^2}\left(\dfrac{\partial f(x_1,x_2)}{\partial x_1}, \dfrac{\partial f(x_1,x_2)}{\partial x_2}\right)\dfrac{dH(x_3)}{dx_3}, \\ u_3 = f(x_1,x_2)H(x_3), \\ T = f(x_1,x_2)\alpha(x_3), \end{cases} \tag{5.4}$$

where $a^2 > 0$ is an arbitrary constant.

It follows from (4.1) with (5.1)—(5.3) that the functions f, H, α satisfy

$$\begin{cases} -\Delta_1 f = a^2 f, \\ \dfrac{\partial f}{\partial x_1} = 0 & \text{at } x_1 = 0, L_1, \\ \dfrac{\partial f}{\partial x_2} = 0 & \text{at } x_2 = 0, L_2; \end{cases} \tag{5.5}$$

and

$$\begin{cases} \left(\dfrac{d^2}{dz^2} - a^2\right)^2 H = a^2\lambda\alpha, \\ \left(\dfrac{d^2}{dz^2} - a^2\right)\alpha = -\lambda H, \end{cases} \tag{5.6}$$

supplemented with the boundary conditions

$$\begin{cases} \varphi(0) = \varphi(1) = 0, \\ H(0) = H(1) = 0, \quad H''(0) = H''(1) = 0. \end{cases} \tag{5.7}$$

It is clear that the solutions of (5.5) are given by

$$\begin{cases} f(x_1, x_2) = \cos(a_1 x_1)\cos(a_2 x_2), \\ a_1^2 + a_2^2 = a^2, \quad (a_1, a_2) = (k_1\pi/L_1, k_2\pi/L_2)\,, \end{cases} \tag{5.8}$$

for any $k_1, k_2 = 0, 1 \cdots$.

Let $a_1^2 + a_2^2 = a^2$. It is easy to see that for each given a^2, the first eigenvalue $\lambda_0(a)$ and the eigenvectors of (5.6) and (5.7) are given by

$$\begin{cases} \lambda_0(a) = \dfrac{(\pi^2 + a^2)^{3/2}}{a}, \\ (H, \alpha) = \left(\sin \pi x_3,\ \dfrac{1}{a}\sqrt{\pi^2 + a^2}\ \sin \pi x_3\right). \end{cases} \tag{5.9}$$

It is easy to see that the first eigenvalue $\lambda_1 = \sqrt{R_c}$ of (4.1) with (5.1)—(5.3) is the minimum of $\lambda_0(a)$:

$$\begin{aligned} R_c &= \min_{a^2 = a_1^2 + a_2^2} \lambda_0^2(a) \qquad (5.10) \\ &= \min_{k_1, k_2 \in \mathbb{Z}} \left[\pi^4 \left(1 + \frac{k_1^2}{L_1^2} + \frac{k_1^2}{L_2^2}\right)^3 \Big/ \left(\frac{k_1^2}{L_1^2} + \frac{k_2^2}{L_2^2}\right)\right]. \end{aligned}$$

Thus the first eigenvectors of (4.1) with (5.1)—(5.3) can be directly derived from (5.4), (5.8) and (5.9):

$$\begin{cases} u_1 = -\dfrac{a_1\pi}{a^2}\sin(a_1x_1)\cos(a_2x_2)\cos(\pi x_3), \\ u_2 = -\dfrac{a_2\pi}{a^2}\cos(a_1x_1)\sin(a_2x_2)\cos(\pi x_3), \\ u_3 = \cos(a_1x_1)\cos(a_2x_2)\sin(\pi x_3), \\ T = \dfrac{1}{a}\sqrt{\pi^2 + a^2}\cos(a_1x_1)\cos(a_2x_2)\sin(\pi x_3), \end{cases} \tag{5.11}$$

where $a^2 = a_1^2 + a_2^2$ satisfies (5.10).

By Theorem 4.1, the topological structure of the bifurcated solutions of the Bénard problem (2.4–2.6) with (5.1)—(5.3) is determined by that of (5.11), and which depends, by (5.10), on the horizontal length scales L_1 and L_2. Namely, the pattern of convection in the Bénard problem depends on the size and form of the containers of fluid. This will be illustrated in the remaining part of this section.

5.2. Roll structure. By (5.10) and (5.11) we know that when the length scales L_1 and L_2 are given, the wave numbers k_1 and k_2 are derived, and the structure of the eigenvectors u of (4.1) are determined.

Consider the case where

$$L_1 = L_2 = L, \quad \text{and} \quad 0 < L^2 < \frac{2 - 2^{1/3}}{2^{1/3} - 1} \simeq 3. \tag{5.12}$$

We remark here that $L = hL/h$ is the aspect ratio between the horizontal scale and the vertical scale of the domain. In this case, the wave numbers (k_1, k_2) are given by

$$(k_1, k_2) = (1, 0) \text{ and } \quad (0, 1),$$

and the eigenspace E_0 defined by (4.3) for the linearized Bousinesq equation (4.1) with boundary conditions (5.1-5.3) is two-dimensional and is given by

$$E_0 = \{\alpha_1\Psi_1 + \alpha_2\Psi_2 \quad | \quad \alpha_1, \alpha_2 \in \mathbb{R}\},$$

where

$$\begin{aligned}
&\Psi_i = (e_i, T_i) \quad i = 1, 2,\\
&e_1 = \left(-L\sin\left(\frac{\pi x_1}{L}\right)\cos(\pi x_3), 0, \cos\left(\frac{\pi x_1}{L}\right)\sin(\pi x_3)\right),\\
&e_2 = \left(0, -L\sin\left(\frac{\pi x_2}{L}\right)\cos(\pi x_3), \cos\left(\frac{\pi x_2}{L}\right)\sin(\pi x_3)\right),\\
&T_1 = \sqrt{L^2+1}\cos\left(\frac{\pi x_1}{L}\right)\sin(\pi x_3),\\
&T_2 = \sqrt{L^2+1}\cos\left(\frac{\pi x_2}{L}\right)\sin(\pi x_3).
\end{aligned}$$

When $\alpha_1, \alpha_2 \neq 0$, the structure of $\phi = \alpha_1\Psi^1 + \alpha_2\Psi^2 \in E_0$ is given schematically by Figure 5.1(a)-(d).

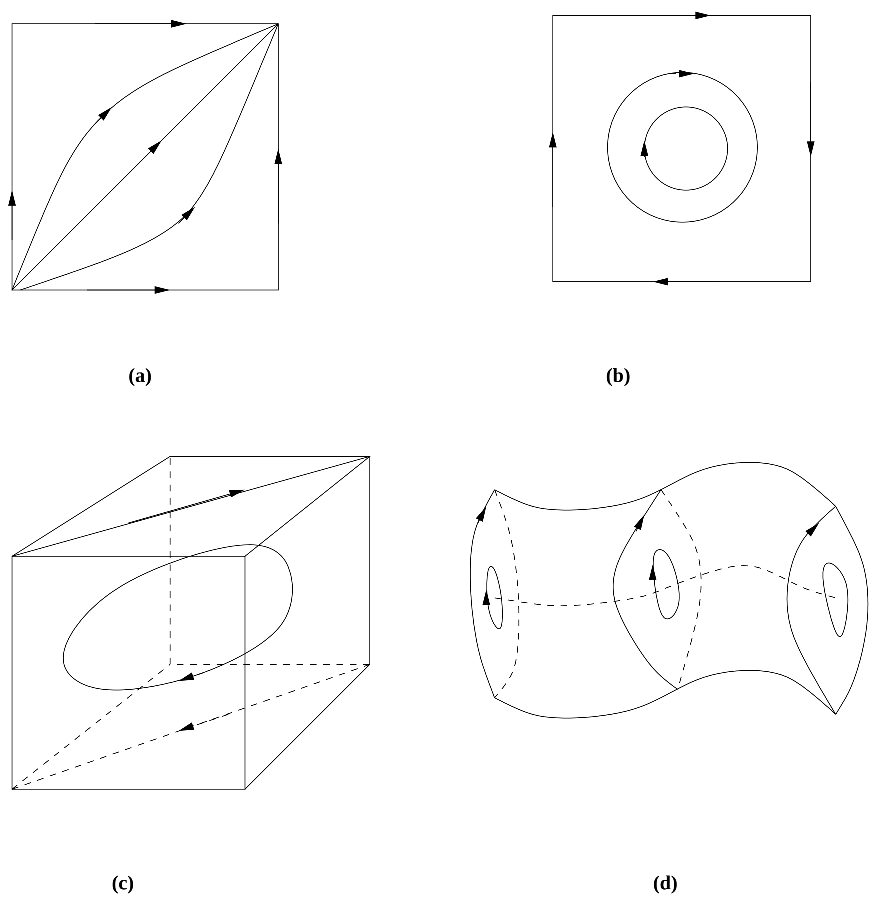


FIG. 5.1. *Roll structure: (a) Flow structure on $z = 1$, (b) flow structure on $x = 1$ or $y = 0$, (c) an elevation of the flow, and (d) flow structure in the interior of the cube.*

The roll structure of $\phi = \alpha_1\Psi^1 + \alpha_2\Psi^2 \in E_0$ has a certain stability, although it is not the structural stability, i.e. under a perturbation the roll trait remains invariant; we shall report on this new stability elsewhere.

Furthermore, the critical Rayleigh number is

$$R_c = \frac{\pi^4(1+L^2)^3}{L^4}. \tag{5.13}$$

By Theorem 4.1, we have the following results.

1. When the Rayleigh number $R \leq R_c$, the trivial solution $\phi = 0$ is globally asymptotically stable in H;
2. When the Rayleigh number $R_c < R < R_c + \varepsilon$ for some $\varepsilon > 0$, or when the temperature gradient satisfies

$$\frac{\kappa\nu}{g\alpha}\ \frac{\pi^4(1+L^2)^3}{(Lh)^4} < \beta = \frac{T_0 - T_1}{h} < \frac{\kappa\nu}{g\alpha}\ \frac{\pi^4(1+L^2)^3}{(Lh)^4} + \varepsilon_1, \tag{5.14}$$

   the Bénard problem bifurcates from the trivial state $\phi = 0$ an attractor $\mathcal{A}_R$ with $1 \leq \dim\mathcal{A}_R \leq 2$.
3. All solutions in $\mathcal{A}_R$ are small perturbations of the eigenvectors in E_0, having the roll structure.
4. As an attractor, $\mathcal{A}_R$ attracts $H-\Gamma$, where $\Gamma \subset H$ is a co-dimension 2 manifold. Hence, $\mathcal{A}_R$ is stable in the Lyapunov sense. Consequently, for any initial value $\varphi_0 \in H - \Gamma$, the solution $S_R(t)\varphi_0$ of the Boussinesq equations with (5.1)—(5.3) converges to $\mathcal{A}_R$, which approximates the roll structure.

REMARK 5.1. Since the eigenvector eigenspace E_0 has dimension two, the bifurcated attractor $\mathcal{A}_R$ has the homotopy type of cycle S^1. In fact, it is possible that the bifurcated attractor is S^1. Since the spaces $E_1 = \{(u,\theta) \in H_1 \mid u_1 = 0\}$ and $E_2 = \{(u,\theta) \in H_1 \mid u_2 = 0\}$ is invariant for the equation (4.1), the bifurcated attractor Σ contains at least four singular points. If $\Sigma = S^1$, then Σ has exactly four singular points, and two of which are the minimal attractors; see [6] for details.

REMARK 5.2. As dim $E_0 = 2$, both the Krasnselskii-Rabinowitz theory and the Hopf bifurcation theorem, which requires complex eigenvalues, cannot be applied to this case for the Rayleigh-Bénard convection.

REMARK 5.3. By (5.13), the critical Rayleigh number R_c depends on the aspect ratio; see also Remark 5.4 below.

5.3. Coupled roll structure. Consider the case

$$L_1 = L_2 = L, \quad \text{and} \quad \frac{2-2^{1/3}}{2^{1/3}-1} < L^2 < 2 \times \frac{2-2^{1/3}}{2^{1/3}-1}\ . \tag{5.15}$$

In this case, the wave numbers are $(k_1, k_2) = (1, 1)$, and the eigenvalue is simple:

$$\begin{cases} E_0 = \text{Span}\,\{\Psi_1\}, \\ \Psi_1 = (e_1, T_1), \\ e_1 = (u_1, u_2, u_3), \\ u_1 = -\dfrac{L}{2} \sin \dfrac{\pi x_1}{L} \cos \dfrac{\pi x_2}{L} \cos \pi x_3, \\ u_2 = -\dfrac{L}{2} \cos \dfrac{\pi x_1}{L} \sin \dfrac{\pi x_2}{L} \cos \pi x_3, \\ u_3 = \cos \dfrac{\pi x_1}{L} \cos \dfrac{\pi x_2}{L} \sin \pi x_3, \\ T_1 = \sqrt{\dfrac{L^2}{2} + 1} \cos \dfrac{\pi x_1}{L} \cos \dfrac{\pi x_2}{L} \sin \pi x_3. \end{cases} \tag{5.16}$$

The topological structure of (5.16) is as shown in Figure 5.2(a)-(c)

From the topological viewpoint, the structure of (5.16) consists of two rolls with the reverse orientation. The axes of both rolls are $\{(L/2, x_2, 1/2) \mid 0 \le x_2 \le L/2\} \cup \{(x_1, L/2, 1/2) \mid 0 \le x_1 \le L/2\}$ and $\{(x_1, \frac{L}{2}, \frac{1}{2}) \mid \frac{1}{2} \le x_1 \le L\} \cup \{(\frac{L}{2}, x_2, \frac{1}{2}) \mid \frac{L}{2} \le x_2 \le L\}$, respectively.

The critical Rayleigh number is

$$R_c = \pi^4 (L^2 + 2)^3 / L^4. \tag{5.17}$$

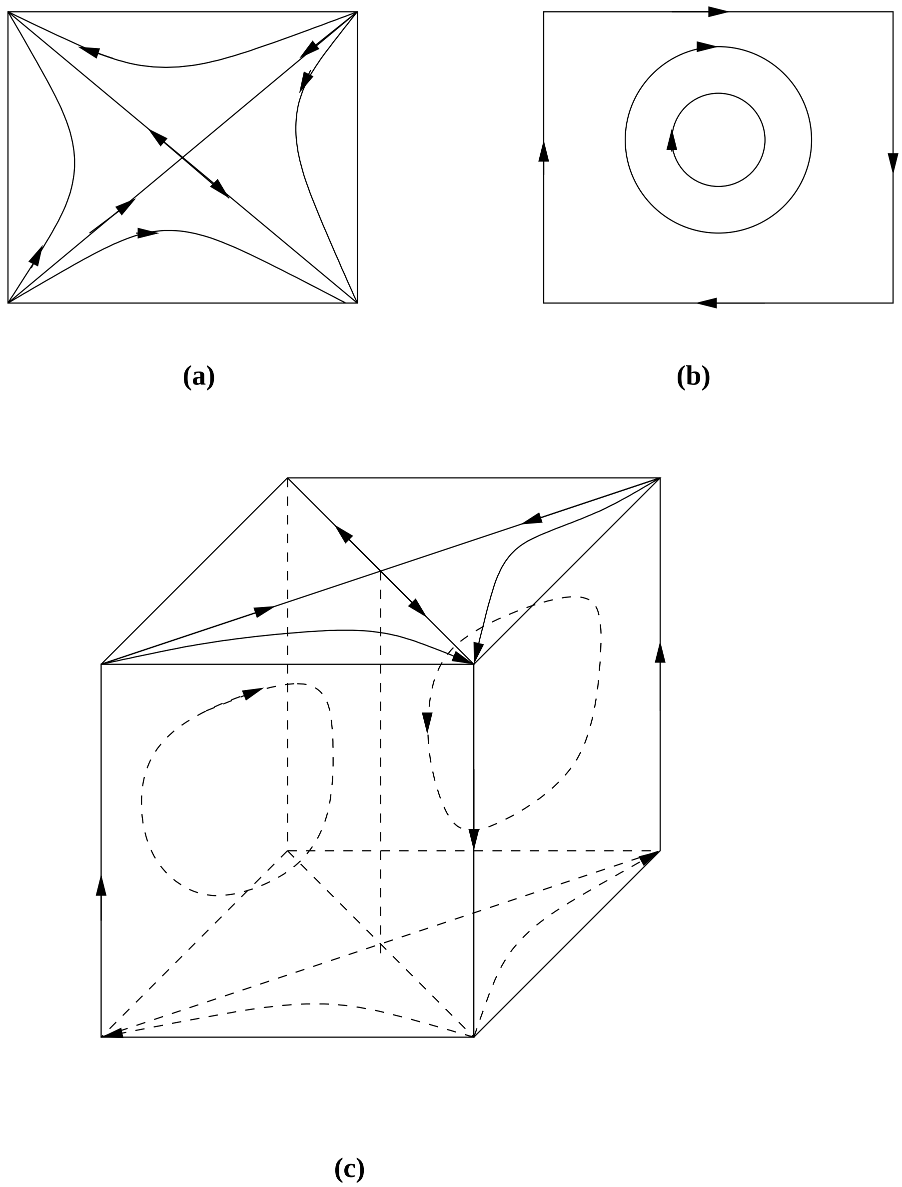
By Theorem 4.2, the following assertions hold true.

1. When the Rayleigh number $R \le R_c$, the trivial solution $\phi = 0$ is globally asymptotically stable in H;
2. When the Rayleigh number $R_c < R < R_c + \varepsilon$ for some $\varepsilon > 0$, the Bénard problem bifurcates from the trivial state $\phi = 0$ two attracted regions U_1 and U_2, such that the solution $S_R(t)\phi_0$ has the coupled roll structure as $t > 0$ sufficiently large, with orientation depending on the initial value ϕ_0 taken in U_1 or U_2, respectively.

Remark 5.4. Both cases of (5.12) and (5.15) are consistent with physical experiments. As we boil water in a container, when the rate of the diameter and the height is smaller than $\sqrt{3}$ (the condition (5.12)), then the convection of heating water takes the roll pattern, and if the rate is between $\sqrt{3}$ and $\sqrt{6}$ (condition (5.15)), then the convection takes the coupled roll pattern.

5.4. Honeycomb structure. As in the Bénard experiments, if the horizontal length scales L_1 and L_1 are sufficiently large, then it is reasonable to consider the periodic boundary condition in the (x_1, x_2)-plane as follows:

$$\begin{cases} (u, T)(x_1 + k_1 L_1, x_2 + k_2 L_2, x_3) = (u, T)(x), \\ T = 0, \quad u_3 = 0, \quad \dfrac{\partial(u_1, u_2)}{\partial x_3} = 0, \quad \text{at } x_3 = 0, 1. \end{cases} \tag{5.18}$$



(a) (b) (c)

FIG. 5.2. *Coupled roll structure: (a) Flow structure on $x_3 = 1$, (b) Flow structure on $x_2 = 0$, and (c) An elevation of flows.*

In this case, the critical Rayleigh number R_c takes the minimum of $\lambda_0^2(a)$ defined by (5.9):

$$R_c = \min_a \lambda_0^2(a) = 657.5, \tag{5.19}$$

where $a_c = \frac{\pi}{\sqrt{2}}$ is the critical wave number, representing the size of the cells in the Bénard convection. Hence, the number $r = 2^{3/2}\pi h/a_c$ can be regarded as the radius of the cells. Thus the first eigenspace E_0 of (4.1) is generated by eigenvectors of the following type:

$$
\begin{cases}
\Psi = (e, T), \\
e = (u_1, u_2, u_3), \\
u_1 = \dfrac{\pi}{a_c^2} \dfrac{\partial f}{\partial x_1} \cos \pi x_3, \\
u_2 = \dfrac{\pi}{a_c^2} \dfrac{\partial f}{\partial x_2} \cos \pi x_3, \\
u_3 = f(x_1, x_2) \sin \pi x_3, \\
T = \dfrac{\sqrt{\pi^2 + a_c^2}}{a_c} f(x_1, x_2) \sin \pi x_3,
\end{cases} \tag{5.20}
$$

where $f(x_1, x_2)$ is any one of the following functions

$$
\begin{aligned}
&\cos(\frac{2\pi k_1 x_1}{L_1}) \cos(\frac{2\pi k_2 x_2}{L_2}), \qquad &\cos(\frac{2\pi k_1 x_1}{L_1}) \sin(\frac{2\pi k_2 x_2}{L_2}), \\
&\sin(\frac{2\pi k_1 x_1}{L_1}) \cos(\frac{2\pi k_2 x_2}{L_2}), \qquad &\sin(\frac{2\pi k_1 x_1}{L_1}) \sin(\frac{2\pi k_2 x_2}{L_2}),
\end{aligned}
$$

with the periods L_1 and L_2 satisfying

$$
\frac{4\pi k_1^2 x_1^2}{L_1^2} + \frac{4\pi k_2^2 x_2^2}{L_2^2} = a_c^2 = \frac{\pi^2}{2}, \qquad k_1, k_2 \in \mathbb{Z};
$$

namely,

$$
\frac{k_1^2}{L_1^2} + \frac{k_2^2}{L_2^2} = \frac{1}{8}, \qquad k_1, k_2 \in \mathbb{Z}. \tag{5.21}
$$

It is clear that the dimension of the first eigenspace E_0 is determined by the given periods L_1 and L_2 satisfying (5.21), and

$$
\dim E_0 = \text{ even } \geq 4.
$$

Various solutions having the honeycomb structure are found in E_0. For convenience, we list two examples as follows.

Square cells. The solution of eigenvalue equation (4.1) given by

$$
\begin{cases}
\Psi_1 = (e_1, T_1), \\
e_1 = (u_1, u_2, u_3), \\
u_1 = -\dfrac{4}{L_1} \sin \dfrac{2\pi x_1}{L_1} \cos \dfrac{2\pi x_2}{L_2} \cos \pi x_3, \\
u_2 = -\dfrac{4}{L_2} \cos \dfrac{2\pi x_1}{L_1} \sin \dfrac{2\pi x_2}{L_2} \cos \pi x_3, \\
u_3 = \cos \dfrac{2\pi x_1}{L_1} \cos \dfrac{2\pi x_2}{L_2} \sin \pi x_3, \\
T = \sqrt{3} \cos \dfrac{2\pi x_1}{L_1} \cos \dfrac{2\pi x_2}{L_2} \sin \pi x_3, \\
\dfrac{1}{L_1^2} + \dfrac{1}{L_2^2} = \dfrac{1}{8},
\end{cases} \tag{5.22}
$$

is a rectangular cells with sides of lengths L_1 and L_2.

HEXAGONAL CELLS. A solution in E_0 having the hexagonal pattern was found by Christopherson in 1940, and is given by

$$\begin{cases} \Psi = (e, T), \\ e = (u_1, u_2, u_3), \\ u_1 = -\dfrac{2}{\sqrt{6}} \sin \dfrac{3\pi x_1}{2\sqrt{6}} \cos \dfrac{\pi x_2}{2\sqrt{2}} \cos \pi x_3, \\ u_2 = -\dfrac{2}{3\sqrt{2}} \left(\cos \dfrac{3\pi x_1}{2\sqrt{6}} + 2\cos \dfrac{\pi x_2}{2\sqrt{2}} \right) \sin \dfrac{\pi x_2}{2\sqrt{2}} \cos \pi x_3, \\ u_3 = \dfrac{1}{3} \left(2\cos \dfrac{3\pi x_1}{2\sqrt{6}} \cos \dfrac{\pi x_2}{2\sqrt{2}} + \cos \dfrac{\pi x_2}{\sqrt{2}} \right) \sin \pi x_3, \\ T = \dfrac{1}{\sqrt{3}} \left(2\cos \dfrac{3\pi x_1}{2\sqrt{6}} \cos \dfrac{\pi x_2}{2\sqrt{2}} + \cos \dfrac{\pi x_2}{\sqrt{2}} \right) \sin \pi x_3. \end{cases} \tag{5.23}$$

This solution is the case where the periods are taken as $L_2 = \sqrt{3}L_1$ and $L_1 = 4\sqrt{6}/3$, and the wave numbers are $(k_1, k_2) = (1, 1)$ and $(0, 1)$.

In summary, for any fixed periods L_1 and L_2, the first eigenspace E_0 of (4.1) has dimension determined by

$$\dim E_0 = \begin{cases} 6 & \text{if } L_2 = \sqrt{k^2 - 1}L_1, k = 2, 3, \cdots, \\ 4 & \text{otherwise.} \end{cases}$$

Therefore, by the attractor bifurcation theorem, Theorems 4.1, we have the following results:

1. When the Rayleigh number $R \leq R_c$, the trivial solution $\phi = 0$ is globally asymptotically stable in H;
2. When the Rayleigh number $R_c < R < R_c + \varepsilon$ for some $\varepsilon > 0$, the Bénard problem bifurcates from the trivial state $\phi = 0$ an attractor $\mathcal{A}_R$ with dimension satisfying
$$\begin{aligned} &5 \leq \dim \mathcal{A}_R \leq 6 && \text{if } L_2 = \sqrt{k^2 - 1}L_1, k = 2, 3, \cdots, \\ &3 \leq \dim \mathcal{A}_R \leq 4 && \text{otherwise.} \end{aligned}$$
3. All solutions in $\mathcal{A}_R$ are small perturbations of the eigenvectors in E_0, having the honeycomb structure.
4. As an attractor, $\mathcal{A}_R$ attracts $H - \Gamma$, where $\Gamma \subset H$ is the stable manifold of the trivial solution with co-dimension 6 if $L_2 = \sqrt{k^2 - 1}L_1, k = 2, 3, \cdots$, and with co-dimension 4 otherwise. Hence, $\mathcal{A}_R$ is stable in the Lyapunov sense. Consequently, for any initial value $\varphi_0 \in H - \Gamma$, the solution $S_R(t)\varphi_0$ of the Boussinesq equations with (5.18) converges to $\mathcal{A}_R$, which approximates the honeycomb structure.

6. Two-Dimensional Rayleigh-Bénard convection: asymptotic and structural stabilities of bifurcated solution

The main objective of this section is to study the dynamic bifurcation and the structural stability of the bifurcated solutions of the 2-D Boussinesq equations related to the Rayleigh-Bénard convection. It is easy to see that both Theorems 4.1 and

4.1 hold true for the 2D Boussinesq equations with any combination of boundary conditions as discussed in Section 2. Hence we focus in this section on structural stability in the physical space of the bifurcated solutions, justifying the roll pattern formation in the Rayleigh-Bénard convection.

Technically speaking, we see from (5.10) that as L_2/L_1 is small, the wave number $k_2 = 0$. Hence the 3–D Bénard problem is reduced to the two dimensional one. Furthermore due to the symmetry on the xy–plane of the honeycomb structure of the Bénard convection, from the viewpoint of a cross section, the 3–D Bénard convection can be well understood by the two dimensional version.

For consistency, we always assume that the domain $\Omega = [0, L] \times [0, 1]$ with coordinate system $x = (x_1, x_3)$. The 2-D Boussinesq equations for the 2-D Bénard convection take the same form as the 3-D Boussinesq equations (2.4-2.6):

$$\begin{cases} \dfrac{1}{P_r}\left[\dfrac{\partial u}{\partial t} + (u\cdot\nabla)u + \nabla p\right] - \Delta u - \sqrt{R}Tk = 0, \\ \dfrac{\partial T}{\partial t} + (u\cdot\nabla)T - \sqrt{R}u_3 - \Delta T = 0, \\ \text{div } u = 0, \end{cases} \tag{6.1}$$

where the velocity field being replaced by $u = (u_1, u_3)$, and the operators are the corresponding 2-D operators in the $x = (x_1, x_3)$ coordinate system. For simplicity, we consider here only the free-free boundary conditions as follows:

$$\begin{cases} u\cdot n = 0, \quad \dfrac{\partial u_\tau}{\partial n} = 0, \quad \text{on } \partial\Omega, \\ T = 0 \text{ at } x_3 = 0, 1, \quad \dfrac{\partial T}{\partial x_1} = 0, \text{ at } x_1 = 0, L. \end{cases} \tag{6.2}$$

In this case, the function space H defined by (2.14) is replaced here by

$$H = \{(u, T) \in L^2(\Omega)^3 \quad | \quad \text{div} u = 0, u_3|_{x_3=0,1} = 0, \quad u_1|_{x_1=0,L} = 0\}.$$

By (5.10) and (5.11), for the equation (6.1) with the free boundary condition, the wave number k and the critical Rayleigh number are

$$\begin{aligned} &k \simeq a_c\, L/\pi = \frac{L}{\sqrt{2}}, \\ &R_c = \pi^4(k^2 + L^2)^3/L^4, \end{aligned}$$

and the first eigenspace E_0 is one-dimensional, and is given by

$$\begin{cases} E_0 = \text{Span}\,\{\Psi_1 = (e_1, T_1)\}, \\ e_1 = \left(-\dfrac{L}{k}\sin\dfrac{k\pi x_1}{L}\cos\pi x_3, \cos\dfrac{k\pi x_1}{L}\sin\pi x_3\right), \\ T = \dfrac{1}{k}\sqrt{L^2+k^2}\,\cos\dfrac{k\pi x_1}{L}\sin\pi x_3. \end{cases} \tag{6.3}$$

The topological structure of e_1 in (6.3) consists of k vortices as shown in Figure 6.1(a) and (b)

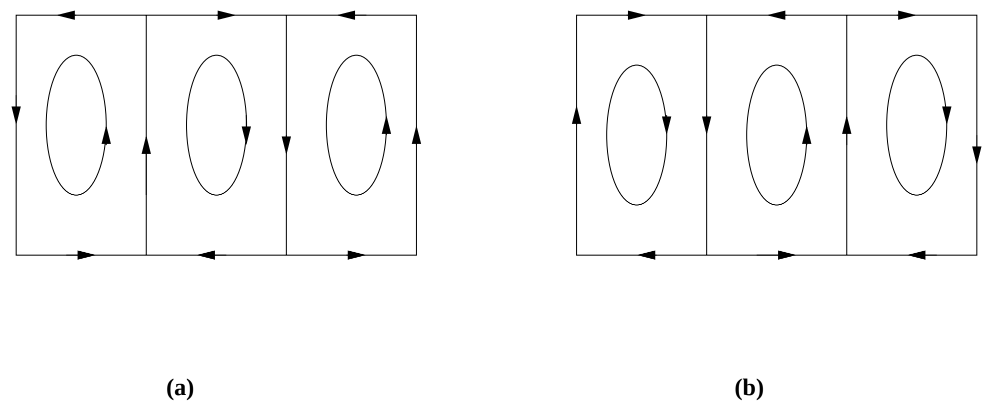


FIG. 6.1. *Rolls with reverse orientations*

By Theorem 7.3, the first eigenvectors (6.3) are structurally stable; therefore, from Theorem 4.2 we immediately obtain the following result.

THEOREM 6.1. *For any bounded open set $U \subset H$ with $0 \in U$, there is an $\varepsilon > 0$, as the Rayleigh number $R_c < R < R_c + \varepsilon$, U can be decomposed into two open sets U_1 and U_2 depending on R such that*

1. $\bar{U} = \bar{U}_1 + \bar{U}_2, \quad U_1 \cap U_2 = \emptyset, \quad 0 \in \partial U_1 \cap \partial U_2$;
2. *for any initial value $\phi_0 \in U_i$ $(i = 1, 2)$ there exists a time $t_0 > 0$ such that the solution $S_R(t)\phi_0$ of (6.1) with (6.2) is topologically equivalent to either the structure as shown in Figure 6.1(a) or that as shown in (b) for all $t > t_0$.*

7. Appendix: Structural Stability for Divergence-Free Vector Fields

Let $C^r(\Omega, \mathbb{R}^2)$ be the space of all C^r $(r \geq 1)$ vector fields on Ω. We consider a subspace of $C^r(\Omega, \mathbb{R}^2)$:

$$B^r(\Omega, \mathbb{R}^2) = \left\{ v \in C^r(\Omega, \mathbb{R}^2) \mid \operatorname{div} v = 0,\, v_n = \frac{\partial v_\tau}{\partial n} = 0 \text{ on } \partial\Omega \right\}.$$

DEFINITION 7.1. *Two vector fields $u, v \in B^r(\Omega, \mathbb{R}^2)$ are called topologically equivalent if there exists a homeomorphism of $\varphi : \Omega \to \Omega$, which takes the orbits of u to orbits of v and preserves their orientation.*

DEFINITION 7.2. *A vector field $v \in B^r(\Omega, \mathbb{R}^2)$ is called structurally stable in $B^r(\Omega, R^2)$ if there exists a neighborhood $U \subset B^r(\Omega, \mathbb{R}^2)$ of v such that for any $u \in U$, u and v are topologically equivalent.*

We recall next some basic facts and definitions on divergence–free vector fields. Let $v \in B^r(\Omega, \mathbb{R}^2)$.

1. A point $p \in \Omega$ is called a singular point of v if $v(p) = 0$; a singular point p of v is called non-degenerate if the Jacobian matrix $Dv(p)$ is invertible; v is called regular if all singular points of v are non-degenerate.
2. An interior non-degenerate singular point of v can be either a center or a saddle, and a non-degenerate boundary singularity must be a saddle.
3. Saddles of v must be connected to saddles. An interior saddle $p \in \Omega$ is called self–connected if p is connected only to itself, i.e., p occurs in a graph whose topological form is that of the number 8.

The following theorem was proved in [7], providing necessary and sufficient conditions for structural stability of a divergence–free vector field.

THEOREM 7.3. *Let $v \in B^r(\Omega, \mathbb{R}^2)$ $(r \geq 1)$. Then v is structurally stable in $B^r(\Omega, \mathbb{R}^2)$ if and only if*

1. *v is regular;*
2. *all interior saddles of v are self-connected; and*
3. *each boundary saddle point is connected to boundary saddle points on the same connected component of the boundary.*

Moreover, the set of all structurally stable vector fields is open and dense in $B^r(\Omega, \mathbb{R}^2)$.

REMARK 7.4. The structural stability theorems for the divergence–free vector fields with the Dirichlet boundary condition and the Hamiltonian vector fields on a torus $\mathbb{T}^2$ have been proved; see [8].

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