

t-MODEL STRUCTURES

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Abstract

For every stable model category \mathcal{M} with a certain extra structure, we produce an associated model structure on the pro-category $\text{pro-}\mathcal{M}$ and a spectral sequence, analogous to the Atiyah-Hirzebruch spectral sequence, with reasonably good convergence properties for computing in the homotopy category of $\text{pro-}\mathcal{M}$. Our motivating example is the category of pro-spectra.

The extra structure referred to above is a t-model structure. This is a rigidification of the usual notion of a t-structure on a triangulated category. A t-model structure is a proper simplicial stable model category \mathcal{M} with a t-structure on its homotopy category together with an additional factorization axiom.

1. Introduction

Recent efforts to understand the homotopy theory of pro-objects have resulted in several different model structures on pro-categories, such as the strict model structure [7] [19], the π_* -model structure on pro-spaces [17], and the π_* -model structure on pro-spectra [20]. The arguments required for establishing these model structures are similar, yet the published proofs are distinct and have an ad hoc flavor. In the accompanying paper [11] we develop a general framework of filtered model categories for giving model structures on pro-categories.

In this paper we explore a particular class of filtered model structures on *stable* model categories. These filtered model structures arise from t-structures on the homotopy category of the stable model category (recall that such a homotopy category is a triangulated category).

More precisely, we work with a *t-model structure*. This is a proper simplicial stable model category with a t-structure on its homotopy category together with a certain kind of lift of the t-structure to the model category itself. This “rigidification” of the t-structure is expressed in terms of an additional factorization axiom for the model category.

If \mathcal{M} is a t-model category, then we produce a model structure on the category $\text{pro-}\mathcal{M}$ (see Theorem 6.3). We show that the associated homotopy category \mathcal{P} of

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pro- \mathcal{M} has a t-structure (see Proposition 9.4). One important property of this t-structure is that an object lies in $\mathcal{P}_{\geq n}$ for all n if and only if it is contractible (see Proposition 9.4 again). This property has at least two important consequences. First, it allows us to construct an Atiyah-Hirzebruch type spectral sequence with reasonable convergence properties (see Theorem 10.3). Second, it allows us to prove results for detecting weak equivalences analogous to the Whitehead theorem (see Theorem 9.13).

Although our main interest in t-model structures is to produce model structures on pro-categories, the notion of a t-model structure is likely to be useful in other contexts. Motivic homotopy theory [23] is a combination of ideas from homotopy theory and from motivic algebraic geometry. Since model categories are important in homotopy theory and since t-structures are important in the study of motives, the interaction between these two notions is probably useful in motivic homotopy theory. See [22] for a possible example.

One specific example of a model structure on a pro-category obtained from a t-model structure is the π_* -model structure for pro-spectra. It is obtained from any reasonable model category of spectra, where the t-structure on the homotopy category of spectra is given by Postnikov sections. The original motivation for this paper was an extension of this model structure to a category of pro- G -spectra when G is a profinite group. That extension is treated in a separate paper [10].

Another example is the H_* -model structure on the category of pro-chain complexes of modules over a unital ring, in which weak equivalences are detected by pro-homology groups. This model structure for pro-chain complexes is obtained from the projective model structure on the category of chain complexes equipped with the standard t-structure on its derived category.

1.1. π_* -model structure on pro-spectra

We provide a summary of our results for the π_* -model structure on pro-spectra. There are similar results for the H_* -model structure on pro-chain complexes.

We remind the reader that the theorems that appear later are much more general. The pro-categorical terminology is established in Section 5.

Definition 1.1. A map f of pro-spectra is a π_* -weak equivalence if:

1. f is an essentially levelwise m -equivalence for some m , and
2. $\pi_n f$ is a pro-isomorphism of pro-groups for all integers n .

We acknowledge that the first condition above appears unnatural at first glance. However, we suspect that it is not possible to construct a model structure on pro-spectra if this condition is not included. In fact, the definition that appears later is different (see Definition 6.2). Here we have given a more concrete equivalent reformulation (see Theorem 9.13).

Theorem 1.2 (Theorem 6.3). *There is a model structure on the category of pro-spectra in which the weak equivalences are the π_* -weak equivalences.*

The cofibrations in this model structure are, up to isomorphism, the levelwise cofibrations. The fibrations are more complicated to describe. Section 6 contains a reasonably concrete characterization of the fibrations.

One of the key observations about this model structure is that the map $X \rightarrow \{P_n X\}$ from a spectrum to its Postnikov tower is a π_* -weak equivalence. In fact, Postnikov towers are the key ingredient in constructing fibrant replacements.

One of the main uses of the previous theorem is the construction of an Atiyah-Hirzebruch spectral sequence for pro-spectra.

Theorem 1.3 (Theorem 10.3). *Let X and Y be pro-spectra. There is a spectral sequence with*

$$E_2^{p,q} = H^p(X; Y^q).$$

The spectral sequence converges conditionally to $[X, Y]^{p+q}$ if:

1. *X is uniformly bounded below (i.e., there exists an integer N such that $\pi_n X_s = 0$ for all $n \leq N$ and all s), or*
2. *X is essentially levelwise bounded below (i.e., for each s , there exists an integer N such that $\pi_n X_s = 0$ for $n \leq N$) and Y is a constant pro-spectrum.*

In the above theorem, the notation $[X, Y]^{p+q}$ refers to weak homotopy classes of maps of degree $p+q$ from X to Y in the homotopy category of pro-spectra. The E_2 -term $H^p(X; Y^q)$ is singular cohomology of the pro-spectrum X with coefficients in the pro-abelian group $Y^q = \pi_{-q} Y$. Recall that the p -th cohomology group $H^p(X; A)$ of a pro-spectrum X with coefficients in an abelian group A is defined to be $\text{colim}_s H^p(X_s; A)$. The definition in the general case is obtained from Definition 2.13 and Propositions 8.4 and 9.11.

1.2. Summary

We summarize the contents of the paper by section.

We give a short review of the basic properties of t-structures in Section 2, assuming no prior knowledge of t-structures. In Section 3, starting with a t-structure on the homotopy category of a stable model category \mathcal{M} , we produce a filtration on the class of morphisms in \mathcal{M} . We reformulate some basic properties of t-structures in this language. In Section 4 we introduce t-model categories.

The rest of the paper is concerned with pro-categories. The basic theory of pro-categories is briefly reviewed in Section 5. We also review the strict model structure and discuss its mapping spaces. In Section 6 we show that a t-model category gives rise to a filtered model category, and we use this to give a model structure on its pro-category. For reasons that will be apparent later, a model structure on a pro-category obtained in this way is called an \mathcal{H}_* -model structure. We describe the cofibrations and fibrations of \mathcal{H}_* -model structures in some detail and discuss Quillen equivalences between pro-categories with \mathcal{H}_* -model structures. We then introduce functorial towers of truncation functors in Section 7. They are used to form fibrant replacements and also to construct the Atiyah-Hirzebruch spectral sequence. We next describe the weak homotopy type of mapping spaces in \mathcal{H}_* -model structures in Section 8. This is used in Section 9 to give a t-structure on the homotopy category of an \mathcal{H}_* -model structure. Under reasonable assumptions, we identify the heart of this t-structure. In Section 10 we construct an Atiyah-Hirzebruch spectral sequence for triangulated categories with a t-structure. The

spectral sequence has reasonably good convergence properties when applied to the t-structure on the homotopy category of an \mathcal{H}_* -model structure.

The last two sections of the paper are devoted to multiplicative structures on pro-categories. In Section 11 we give some basic results concerning the interaction of tensor structures and pro-categories. In Section 12 we discuss tensor model categories and show that we get a partially defined tensor product for some objects in the homotopy category. At the very end, we consider multiplicative structures on the Atiyah-Hirzebruch spectral sequence constructed in Section 10.

1.3. Conventions

We assume that the reader is familiar with model categories. The reference [15] is particularly relevant because of its emphasis on stable model categories, and [14] is also suitable.

We also assume that the reader has a certain practical familiarity with pro-categories. Although a brief review is given in Section 5, see [2], [7], or [17] for additional background.

We use homological grading when working with triangulated categories and t-structures. This disagrees with the more common cohomological grading (see [3] for example), but it is more convenient from the perspective of stable homotopy theory. To emphasize the notational distinction, we use lower subscripts instead of upper subscripts.

Throughout the paper, \mathcal{M} is a proper simplicial stable model category. We always assume that \mathcal{M} has functorial factorizations, even though the model structure on $\text{pro-}\mathcal{M}$ does *not necessarily* have functorial factorizations. We let \mathcal{D} stand for the homotopy category of \mathcal{M} because the notation $\text{Ho}(\mathcal{M})$ is too cumbersome for our purposes. Finally, \mathcal{P} stands for the homotopy category of $\text{pro-}\mathcal{M}$.

The simplicial assumption on \mathcal{M} is probably not necessary for any of our main results, but it is a very convenient hypothesis. Most of the results go through with a weakening of this assumption; see [9, Sec. 5] for more details. On the other hand, the properness assumption on \mathcal{M} is essential for the existence of model structures on $\text{pro-}\mathcal{M}$; see [19] for an explanation.

If \mathcal{C} is any category containing objects X and Y , then $\mathcal{C}(X, Y)$ denotes the set of morphisms from X to Y . Occasionally we will use the notation $[X, Y]$ for the set of morphisms in a homotopy category; in this case, the context makes the precise meaning clear.

2. t-structures

In this section we give a short review of the theory of t-structures on triangulated categories. We only discuss the aspects that are relevant to this work. The original source for this material is [3], but we refer to [12] whenever possible. We assume the reader has a working knowledge of triangulated categories.

Let \mathcal{D} be a triangulated category, and let Σ denote the shift functor such that distinguished triangles are of the form $A \rightarrow B \rightarrow C \rightarrow \Sigma A$.

Definition 2.1. A **t-structure** on \mathcal{D} consists of two strictly full subcategories $\mathcal{D}_{\geq 0}$ and $\mathcal{D}_{\leq 0}$ such that

1. $\mathcal{D}_{\geq 0}$ is closed under Σ , and $\mathcal{D}_{\leq 0}$ is closed under Σ^{-1} .
2. For every object X in \mathcal{D} , there is a distinguished triangle

$$X' \rightarrow X \rightarrow X'' \rightarrow \Sigma X'$$

such that $X' \in \mathcal{D}_{\geq 0}$ and $X'' \in \Sigma^{-1}\mathcal{D}_{\leq 0}$.

3. $\mathcal{D}(X, Y) = 0$ whenever $X \in \mathcal{D}_{\geq 0}$ and $Y \in \Sigma^{-1}\mathcal{D}_{\leq 0}$.

The reader who is already familiar with t-structures should keep in mind that we are using homological grading, not cohomological grading.

Recall that a subcategory is strictly full if it is full and if it is closed under isomorphisms.

In any t-structure, there are two \mathbb{Z} -graded families of strictly full subcategories defined by $\mathcal{D}_{\geq n} = \Sigma^n \mathcal{D}_{\geq 0}$ and $\mathcal{D}_{\leq n} = \Sigma^n \mathcal{D}_{\leq 0}$ for any integer n . By part (1) of Definition 2.1, the categories $\mathcal{D}_{\geq n}$ are a decreasing filtration in the sense that $\mathcal{D}_{\geq n+1}$ is contained in $\mathcal{D}_{\geq n}$, and the categories $\mathcal{D}_{\leq n}$ are an increasing filtration in the sense that $\mathcal{D}_{\leq n}$ is contained in $\mathcal{D}_{\leq n+1}$. Note that parts (2) and (3) of Definition 2.1 can now be rewritten in terms of $\mathcal{D}_{\leq -1}$ instead of $\Sigma^{-1}\mathcal{D}_{\leq 0}$.

Example 2.2. Let \mathcal{D} be the derived category of chain complexes of modules over a ring. The **standard t-structure** on \mathcal{D} [12, IV.4.3] is given by

$$\mathcal{D}_{\geq n} = \{X \mid H_i(X) = 0 \text{ for } i < n\}$$

$$\mathcal{D}_{\leq n} = \{X \mid H_i(X) = 0 \text{ for } i > n\}.$$

The shift functor is defined by $(\Sigma X)_n = X_{n-1}$, and the differential $(\Sigma X)_{n+1} \rightarrow (\Sigma X)_n$ is equal to the negative of the differential $X_n \rightarrow X_{n-1}$.

Example 2.3. Let \mathcal{D} be the homotopy category of spectra. The **Postnikov t-structure** on \mathcal{D} is given by

$$\mathcal{D}_{\geq n} = \{X \mid \pi_i(X) = 0 \text{ for } i < n\}$$

$$\mathcal{D}_{\leq n} = \{X \mid \pi_i(X) = 0 \text{ for } i > n\}.$$

The proof of this fact is classical stable homotopy theory. The main point is to show that $[X, Y] = 0$ when X is (-1) -connected (i.e., $\pi_i X = 0$ for $i < 0$) and Y is 0 -coconnected (i.e., $\pi_i Y = 0$ for $i \geq 0$). See [21, Prop. 3.6], for example.

The two strictly full subcategories $\mathcal{D}_{\geq n}$ and $\mathcal{D}_{\leq n-1}$ of a t-structure determine each other as follows.

Lemma 2.4. *An object X is in $\mathcal{D}_{\geq n}$ if and only if $\mathcal{D}(X, Y) = 0$ for all Y in $\mathcal{D}_{\leq n-1}$, and an object Y is in $\mathcal{D}_{\leq n-1}$ if and only if $\mathcal{D}(X, Y) = 0$ for all X in $\mathcal{D}_{\geq n}$.*

Proof. We prove the first claim; the proof for the second claim is similar. One direction follows immediately from part (3) of Definition 2.1.

For the other direction, suppose that $\mathcal{D}(X, Y) = 0$ for all Y in $\mathcal{D}_{\leq n-1}$. Part (2) of Definition 2.1 says that we can find a distinguished triangle $X' \rightarrow X \rightarrow X'' \rightarrow \Sigma X'$ such that X' belongs to $\mathcal{D}_{\geq n}$ and X'' belongs to $\mathcal{D}_{\leq n-1}$. Now apply $\mathcal{D}(-, X'')$ to obtain a long exact sequence. Since X' and $\Sigma X'$ are both in $\mathcal{D}_{\geq n}$, it follows that $\mathcal{D}(X, X'')$ and $\mathcal{D}(X'', X'')$ are isomorphic. But our assumption implies that the first group is zero, so the second group is also zero. Thus X'' is isomorphic to 0, and $X' \rightarrow X$ is an isomorphism. \square

The next corollary follows immediately from Lemma 2.4.

Corollary 2.5.

1. The zero object 0 belongs to both $\mathcal{D}_{\geq n}$ and $\mathcal{D}_{\leq n}$ for every n .
2. An object that is both in $\mathcal{D}_{\geq n}$ and in $\mathcal{D}_{\leq n-1}$ is isomorphic to 0.
3. The subcategories $\mathcal{D}_{\geq n}$ and $\mathcal{D}_{\leq n}$ of \mathcal{D} are closed under retract.

Corollary 2.6. Let $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$ be a distinguished triangle.

1. If X and Z belong to $\mathcal{D}_{\geq n}$, then so does Y .
2. If X and Z belong to $\mathcal{D}_{\leq n}$, then so does Y .
3. If X belongs to $\mathcal{D}_{\geq n-1}$ and Y belongs to $\mathcal{D}_{\geq n}$, then Z belongs to $\mathcal{D}_{\geq n}$.
4. If X belongs to $\mathcal{D}_{\leq n-1}$ and Y belongs to $\mathcal{D}_{\leq n}$, then Z belongs to $\mathcal{D}_{\leq n}$.
5. If Y belongs to $\mathcal{D}_{\geq n}$ and Z belongs to $\mathcal{D}_{\geq n+1}$, then X belongs to $\mathcal{D}_{\geq n}$.
6. If Y belongs to $\mathcal{D}_{\leq n}$ and Z belongs to $\mathcal{D}_{\leq n+1}$, then X belongs to $\mathcal{D}_{\leq n}$.

Proof. The first two claims follow immediately from Lemma 2.4. The other claims follow from the first two; we illustrate with the fifth statement.

We have an exact triangle

$$\Sigma^{-1}Z \rightarrow X \rightarrow Y \rightarrow Z$$

in which both $\Sigma^{-1}Z$ and Y belong to $\mathcal{D}_{\geq n}$. The result now follows from the first claim. \square

We now give a key lemma about t-structures.

Lemma 2.7. Let $X' \rightarrow X \rightarrow X'' \rightarrow \Sigma X'$ and $Y' \rightarrow Y \rightarrow Y'' \rightarrow \Sigma Y'$ be two distinguished triangles with X' and Y' in $\mathcal{D}_{\geq n}$ and X'' and Y'' in $\mathcal{D}_{\leq n-1}$. Let $f: X \rightarrow Y$ be a map in \mathcal{D} . Then there are unique maps $f': X' \rightarrow Y'$ and $f'': X'' \rightarrow Y''$ such that there is a commutative diagram

$$\begin{array}{ccccccc} X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & \Sigma X' \\ f' \downarrow & & f \downarrow & & f'' \downarrow & & \downarrow \Sigma f' \\ Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & \Sigma Y' \end{array}$$

Proof. By part (3) of Definition 2.1, we have that $\mathcal{D}(X', Y'') = 0$. Hence there exists a map $f': X' \rightarrow Y'$ making the left square commute, and then also a map $f'': X'' \rightarrow Y''$ such that we get a map of distinguished triangles.

Now we have to show that f' and f'' are unique. Let j be the map $Y' \rightarrow Y$. There is an exact sequence

$$\mathcal{D}(X', \Sigma^{-1}Y'') \longrightarrow \mathcal{D}(X', Y') \xrightarrow{j_*} \mathcal{D}(X', Y).$$

The left group is trivial by part (3) of Definition 2.1 since $\Sigma^{-1}Y''$ belongs to $\mathcal{D}_{\leq n-1}$. Therefore, j_* is injective, so f' is unique. A similar argument involving the map $X \rightarrow X''$ shows that f'' is also unique. \square

The importance of Lemma 2.7 is expressed in the following proposition.

Proposition 2.8. *There are **truncation functors** $\tau_{\geq n}$ and $\tau_{\leq n}$ from \mathcal{D} into $\mathcal{D}_{\geq n}$ and $\mathcal{D}_{\leq n}$ respectively together with natural transformations $\epsilon_n: \tau_{\geq n} \rightarrow 1$, $\eta_n: 1 \rightarrow \tau_{\leq n}$, and $\tau_{\leq n-1} \rightarrow \Sigma\tau_{\geq n}$ such that*

$$\tau_{\geq n}X \rightarrow X \rightarrow \tau_{\leq n-1}X \rightarrow \Sigma\tau_{\geq n}X$$

is a distinguished triangle for all X . Up to canonical isomorphism, these properties determine the truncation functors uniquely.

Notation 2.9. We usually write $\mathbf{X}_{\geq n}$ and $\mathbf{X}_{\leq n}$ for $\tau_{\geq n}X$ and $\tau_{\leq n}X$ respectively.

The functors $\tau_{\geq n}$ and $\tau_{\leq n}$ enjoy many useful properties. Most of the claims in the next two paragraphs are proved in [12, Sec. IV.4]; the rest follow easily. In any case, they are easily verifiable directly for Examples 2.2 and 2.3.

The functors $\tau_{\geq n}$ and $\tau_{\leq m}$ commute (up to natural isomorphism) for all n and m . If $m \geq n$, then $\tau_{\geq n}\tau_{\geq m}$ and $\tau_{\geq m}\tau_{\geq n}$ are both naturally isomorphic to $\tau_{\geq m}$, while $\tau_{\leq n}\tau_{\leq m}$ and $\tau_{\leq m}\tau_{\leq n}$ are both naturally isomorphic to $\tau_{\leq n}$. Also, $\Sigma^n\tau_{\geq 0}$ is naturally isomorphic to $\tau_{\geq n}\Sigma^n$, and $\Sigma^n\tau_{\leq 0}$ is naturally isomorphic to $\tau_{\leq n}\Sigma^n$. Both $\tau_{\leq n-1}\tau_{\geq n}$ and $\tau_{\geq n}\tau_{\leq n-1}$ are isomorphic to the zero functor.

2.1. Hearts and cohomology

Definition 2.10. The **heart** $\mathcal{H}(\mathcal{D})$ of a t-structure \mathcal{D} is the full subcategory $\mathcal{D}_{\geq 0} \cap \mathcal{D}_{\leq 0}$ of \mathcal{D} .

For any t-structure, the heart $\mathcal{H}(\mathcal{D})$ is an abelian category [12, Sec. IV.4].

Definition 2.11. The **n -th homology functor** \mathcal{H}_n associated to a t-structure is defined to be the functor $\tau_{\leq 0}\tau_{\geq 0}\Sigma^{-n}$.

The homology functor \mathcal{H}_n is a covariant functor $\mathcal{D} \rightarrow \mathcal{H}(\mathcal{D})$. The following lemma is proved in [12, Thm. IV.4.11a].

Lemma 2.12. *Let $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$ be a distinguished triangle. There is a long exact sequence*

$$\cdots \rightarrow \mathcal{H}_k X \rightarrow \mathcal{H}_k Y \rightarrow \mathcal{H}_k Z \rightarrow \mathcal{H}_{k-1} X \rightarrow \cdots$$

in the abelian category $\mathcal{H}(\mathcal{D})$.

Definition 2.13. Let E be an object in the heart $\mathcal{H}(\mathcal{D})$. The **n -th cohomology functor** $H^n(-; E)$ with **E -coefficients** is $\mathcal{D}(-, \Sigma^n E)$.

The functor $H^n(-; E)$ is a contravariant functor from \mathcal{D} to the category of abelian groups. The following lemma follows immediately from formal properties of triangulated categories.

Lemma 2.14. *Let $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$ be a distinguished triangle, and let E belong to $\mathcal{H}(\mathcal{D})$. There is a long exact sequence*

$$\cdots \rightarrow H^k(X; E) \rightarrow H^k(Y; E) \rightarrow H^k(Z; E) \rightarrow H^{k+1}(X; E) \rightarrow \cdots$$

in the abelian category $\mathcal{H}(\mathcal{D})$.

Example 2.15. Let \mathcal{D} be the triangulated category of chain complexes with the standard t-structure (see Example 2.2). The heart of \mathcal{D} is isomorphic to the category of abelian groups. The n -th homology $\mathcal{H}_n X$ of a chain complex X is the usual n -th homology

$$\ker(X_n \rightarrow X_{n-1})/\text{im}(X_{n+1} \rightarrow X_n)$$

of X . For any abelian group E , the n -th cohomology with E -coefficients of a chain complex X is the n -th hyperext group $\text{Ext}^n(X, E)$.

Example 2.16. Let \mathcal{D} be the category of spectra with the Postnikov t-structure. The heart of this t-structure is the full subcategory of Eilenberg-MacLane spectra (in degree 0). This category is equivalent to the category of abelian groups. The n -th homology functor \mathcal{H}_n is the usual n th stable homotopy group functor π_n . When E is an abelian group, n -th cohomology with E -coefficients is n -th singular cohomology with coefficients in E .

The following lemma says that the layers in the towers obtained from the two sequences of truncation functors are easily described in terms of the homology functors.

Lemma 2.17. *For every integer n and every X in \mathcal{D} , there are distinguished triangles*

$$X_{\geq n+1} \rightarrow X_{\geq n} \rightarrow \Sigma^n \mathcal{H}_n X \rightarrow \Sigma X_{\geq n+1}$$

and

$$\Sigma^n \mathcal{H}_n X \rightarrow X_{\leq n} \rightarrow X_{\leq n-1} \rightarrow \Sigma^{n+1} \mathcal{H}_n X.$$

Proof. We construct the first distinguished triangle. The construction of the second one is similar.

Start with the object $X_{\geq n}$ of \mathcal{D} . There is a distinguished triangle

$$\tau_{\geq n+1} \tau_{\geq n} X \rightarrow \tau_{\geq n} X \rightarrow \tau_{\leq n} \tau_{\geq n} X \rightarrow \Sigma \tau_{\geq n+1} \tau_{\geq n} X.$$

This is equal to the distinguished triangle

$$\tau_{\geq n+1} X \rightarrow \tau_{\geq n} X \rightarrow \Sigma^n \mathcal{H}_n X \rightarrow \Sigma \tau_{\geq n+1} X. \quad \square$$

The following lemma tells us when homology and cohomology theories detect trivial objects. It is also proved in [12, IV.4.11(b)].

Lemma 2.18. *Let \mathcal{D} be a triangulated category equipped with a t -structure such that $\cap_n \mathcal{D}_{\geq n}$ consists only of the objects isomorphic to 0. Assume that X is in $\mathcal{D}_{\geq m}$ for some m . Then the following are equivalent:*

1. X is isomorphic to 0.
2. $\mathcal{H}_n(X) = 0$ for all n .
3. $H^n(X; E) = 0$ for all n and all E in $\mathcal{H}(\mathcal{D})$.

Proof. Condition (1) implies conditions (2) and (3), so we need to show that either condition (2) or (3) implies condition (1).

Assuming either condition (2) or (3), we prove by induction on n that $X_{\geq n} \rightarrow X$ is an isomorphism for all n . Our assumption on $\cap_n \mathcal{D}_{\geq n}$ then implies condition (1). Since X is in $\mathcal{D}_{\geq m}$, the natural map $X_{\geq n} \rightarrow X$ is an isomorphism whenever $n \leq m$; this is the base case of the induction.

Now suppose for induction that the map $X_{\geq n} \rightarrow X$ is an isomorphism. Condition (2) and the first part of Lemma 2.17 gives that the composition $X_{\geq n+1} \rightarrow X_{\geq n} \rightarrow X$ is an isomorphism.

On the other hand, condition (3) and our inductive assumption implies that $\mathcal{D}(X_{\geq n}, \Sigma^n \mathcal{H}_n X)$ is zero. Now apply the functor $\mathcal{D}(-, \Sigma^n \mathcal{H}_n X)$ to the first triangle in Lemma 2.17 and use part 3 of Definition 2.1 to conclude that

$$\mathcal{D}(\Sigma^n \mathcal{H}_n(X), \Sigma^n \mathcal{H}_n(X))$$

is zero. It follows that $\mathcal{H}_n(X)$ is isomorphic to 0. As in the previous paragraph, this implies that $X_{\geq n+1} \rightarrow X$ is an isomorphism. \square

3. n -equivalences and co- n -equivalences

From now on, we no longer consider just triangulated categories but rather proper simplicial stable model categories \mathcal{M} (with functorial factorization). We write \mathcal{D} for the homotopy category of \mathcal{M} , which is automatically a triangulated category because \mathcal{M} is stable [15, 7.1]. We briefly review the main properties of stable model categories that we need. See [15, Ch. 7] for more details.

Recall that a stable model category \mathcal{M} is pointed. In addition to unreduced tensors $X \otimes K$ and cotensors $\text{Map}(K, X)$, a pointed simplicial model category also has **reduced tensors** $X \wedge K$ and **reduced cotensors** $\text{Map}_*(K, X)$ for any pointed simplicial set K and any object X of \mathcal{M} .

The **suspension** of any object X of \mathcal{M} is defined to be $X \wedge S^1$ [15, 6.1.1]. Note that this construction is homotopically correct only if X is cofibrant. In general, one must first take a cofibrant replacement for X . In a simplicial stable model category, suspension is a left Quillen functor that induces an automorphism on \mathcal{D} . Its associated right Quillen functor is reduced cotensor with S^1 , which is also known as **loops** [15, 6.1.1].

The homotopy category of a stable model category is a triangulated category whose shift functor Σ is the left derived functor of suspension. We use the symbol Ω for the right derived functor of loops. Note that Ω induces the inverse shift Σ^{-1} on \mathcal{D} . Homotopy cofiber sequences and homotopy fiber sequences are the same, and they induce the distinguished triangles in the homotopy category.

Since \mathcal{M} has functorial factorizations, there are functorial constructions of homotopy fibers and homotopy cofibers in \mathcal{M} . We write $\mathbf{hocofib} f$ and $\mathbf{hofib} f$ for the functorial homotopy cofiber and homotopy fiber of a map f . Using the properness assumption, there are natural maps $\mathbf{hofib} f \rightarrow X$ and $Y \rightarrow \mathbf{hocofib} f$ for any map $f: X \rightarrow Y$. In \mathcal{D} , $\mathbf{hocofib} f$ is isomorphic to $\Sigma \mathbf{hofib} f$.

These constructions induce functors on the homotopy category \mathcal{D} of \mathcal{M} . The functoriality of these constructions is one of the chief advantages of working with stable model categories rather than just triangulated categories, where homotopy cofibers and homotopy fibers are only defined up to non-canonical isomorphism.

We now lift the full subcategories given by a t-structure on \mathcal{D} to full subcategories on \mathcal{M} .

Definition 3.1. Let \mathcal{M} be a proper simplicial stable model category whose homotopy category \mathcal{D} is equipped with a t-structure. Let $\mathcal{M}_{\geq n}$ be the full subcategory of \mathcal{M} consisting of those objects whose weak homotopy types belong to $\mathcal{D}_{\geq n}$, and let $\mathcal{M}_{\leq n}$ be the full subcategory of \mathcal{M} consisting of those objects in \mathcal{M} whose weak homotopy types belong to $\mathcal{D}_{\leq n}$.

Many properties of $\mathcal{D}_{\geq n}$ and $\mathcal{D}_{\leq n}$ from Section 2 carry over to the classes $\mathcal{M}_{\geq n}$ and $\mathcal{M}_{\leq n}$. For example, $\mathcal{M}_{\geq n}$ is closed under Σ , and $\mathcal{M}_{\leq n}$ is closed under Ω . Also, an object X belongs to $\mathcal{M}_{\geq n}$ if and only if $\Omega^n X$ belongs to $\mathcal{M}_{\geq 0}$ (and similarly for $\mathcal{M}_{\leq n}$). Finally, if X belongs to $\mathcal{M}_{\geq n}$ and Y belongs to $\mathcal{M}_{\leq n-1}$, then $\mathcal{D}(X, Y)$ is zero.

To explore the interaction between a stable model structure and a t-structure on its homotopy category, we find it more convenient to work with subclasses of morphisms associated to the t-structure rather than the subclasses of objects given by the t-structure.

Definition 3.2. Let \mathcal{M} be a proper stable model category whose homotopy category \mathcal{D} is equipped with a t-structure. The class of **n -equivalences** in \mathcal{M} is

$$\mathbf{W}_n = \{f \mid \mathbf{hofib} f \in \mathcal{M}_{\geq n}\} = \{f \mid \mathbf{hocofib} f \in \mathcal{M}_{\geq n+1}\}.$$

The class of **co- n -equivalences** in \mathcal{M} is

$$\mathbf{coW}_n = \{f \mid \mathbf{hocofib} f \in \mathcal{M}_{\leq n}\} = \{f \mid \mathbf{hofib} f \in \mathcal{M}_{\leq n-1}\}.$$

Example 3.3. Consider the standard t-structure on the derived category of chain complexes from Example 2.2. A map is an n -equivalence if and only if it induces a homology isomorphism in degrees strictly less than n and induces a surjection in degree n . Similarly, a map is a co- n -equivalence if and only if it induces a homology isomorphism in degrees strictly greater than n and induces an injection in degree n .

Example 3.4. Let \mathcal{M} be a model category of spectra with the Postnikov t-structure on its homotopy category (see Example 2.3). A map is an n -equivalence if and only if it induces isomorphisms in homotopy groups below dimension n and induces a surjection in dimension n . Similarly, a map is a co- n -equivalence if and only if it induces isomorphisms in homotopy groups above dimension n and induces an injection in dimension n .

We now translate many of the results from Section 2 into properties of n -equivalences and co- n -equivalences.

Lemma 3.5.

1. The class W_n is closed under Σ , and the class coW_n is closed under Ω .
2. The class W_n is contained in W_{n-1} , and the class coW_{n-1} is contained in coW_n .
3. The classes W_n and coW_n contain all weak equivalences.
4. If a map is both an n -equivalence and a co- n -equivalence, then it is a weak equivalence.
5. The classes W_n and coW_n are closed under retract.

Proof. The first two claims follow from the properties of $\mathcal{M}_{\geq n}$ and $\mathcal{M}_{\leq n}$ stated after Definition 3.1 and the fact that the functors Σ and Ω commute with the functors hofib and hocofib up to weak equivalence.

The next two claims follow immediately from parts (1) and (2) of Corollary 2.5, together with the fact that a map is a weak equivalence if and only if its homotopy fiber is contractible.

For the fifth claim, let g be a retract of a map f . Then $\text{hofib } g$ is a retract of $\text{hofib } f$ because homotopy fibers are functorial. Now we just need to use part (3) of Corollary 2.5. \square

Lemma 3.6. *Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be two maps.*

1. If f and g both belong to W_n , then so does gf .
2. If f and g both belong to coW_n , then so does gf .
3. If f belongs to W_{n-1} and gf belongs to W_n , then g belongs to W_n .
4. If f belongs to coW_{n-1} and gf belongs to coW_n , then g belongs to coW_n .
5. If g belongs to W_{n+1} and gf belongs to W_n , then f belongs to W_n .
6. If g belongs to coW_{n+1} and gf belongs to coW_n , then f belongs to coW_n .

Proof. We have a distinguished triangle

$$\text{hofib } f \rightarrow \text{hofib } gf \rightarrow \text{hofib } g \rightarrow \Sigma \text{hofib } f$$

in the homotopy category \mathcal{D} of \mathcal{M} . Corollary 2.6 gives the desired results. \square

Lemma 3.7. *The classes W_n and coW_n are both closed under base changes along fibrations and cobase changes along cofibrations.*

Proof. In a proper model structure, the homotopy fiber of a map is weakly equivalent to the homotopy fiber of its base change along a fibration. Similarly, the homotopy cofiber of a map is weakly equivalent to the homotopy cofiber of its cobase change along a cofibration. \square

The next result gives a general setting in which homology and cohomology functors detect weak equivalences in \mathcal{M} .

Theorem 3.8. *Assume that $\cap_n W_n$ is equal to the class of weak equivalences. Let $f: X \rightarrow Y$ be an m -equivalence for some m . The following are equivalent:*

1. f is a weak equivalence.
2. $\mathcal{H}_n(f)$ is an isomorphism in the heart $\mathcal{H}(\mathcal{D})$ for all n .
3. $H^n(Y; E) \rightarrow H^n(X; E)$ is an isomorphism for all n and all E in $\mathcal{H}(\mathcal{D})$.

Applied to the Postnikov t-structure on spectra from Example 2.3, this is the usual Whitehead theorem for detecting weak equivalences with stable homotopy groups or with ordinary cohomology.

Proof. Note first that $\cap_n W_n$ is equal to the class of weak equivalences if and only if $\cap_n \mathcal{D}_{\geq n}$ consists only of contractible objects. This follows immediately from the definitions and the fact that a map is a weak equivalence if and only if its homotopy fiber is contractible.

Now Lemma 2.18 gives the desired result, using the long exact sequences of Lemmas 2.12 and 2.14. \square

Remark 3.9. In Theorem 3.8, an additional assumption on the t-structure allows one to avoid the assumption that f is an m -equivalence. Namely, if both $\cap_n W_n$ and $\cap_n \text{co}W_n$ are equal to the class of weak equivalences, then a map f is a weak equivalence in \mathcal{M} if and only if $\mathcal{H}_n(f)$ is an isomorphism for all n ; the proof of [12, IV.4.11] can be easily adapted to show this. Both the standard t-structure on chain complexes from Example 2.2 and the Postnikov t-structure on spectra from Example 2.3 satisfy this condition.

Our assumptions in Theorem 3.8 are dictated by the t-structures we consider on homotopy categories of pro-categories. In that case we have that $\cap_n W_n$ is equal to the class of weak equivalences, while $\cap_n \text{co}W_n$ is never equal to the class of weak equivalences except in trivial cases. See Lemma 9.5 for more details.

Remark 3.10. Consider the situation of a t-structure on a triangulated category \mathcal{D} that is not associated to a stable model category. Even though homotopy fibers and homotopy cofibers are not well-defined in \mathcal{D} , one can still define classes of n -equivalences and $\text{co-}n$ -equivalences in \mathcal{D} as in Definition 3.2. The point is that homotopy fibers and homotopy cofibers are well-defined up to non-canonical isomorphism, and that is good enough for the purposes of Definition 3.2. All of the lemmas of this section remain true except for Lemma 3.7, which does not make sense without a model structure.

4. t-model structures

We continue to work in a proper simplicial stable model category \mathcal{M} whose homotopy category \mathcal{D} has a t-structure. We need to assume that the t-structure on \mathcal{D} can be rigidified in a certain sense.

Definition 4.1. A **t-model structure** is a proper simplicial stable model category \mathcal{M} equipped with a t-structure on its triangulated homotopy category \mathcal{D} together with functorial factorizations of maps in \mathcal{M} into n -equivalences followed by $\text{co-}n$ -equivalences.

There is a t-model structure on the category of chain complexes that induces the standard t-structure of Example 2.2 (except, possibly, for the simplicial structure). This example is dealt with in greater detail in [9, 5,6]. More importantly for our applications, all reasonable model categories of spectra have t-model structures that induce the Postnikov t-structure on the stable homotopy category. To factor a map into an n -equivalence followed by a co- n -equivalence, apply the small object argument to the set of maps consisting of all generating acyclic cofibrations and also generating cofibrations whose cofibers are spheres of dimension greater than n .

Lemma 4.2. *The truncation functors $\tau_{\geq n}: \mathcal{D} \rightarrow \mathcal{D}_{\geq n}$ and $\tau_{\leq n}: \mathcal{D} \rightarrow \mathcal{D}_{\leq n}$ can be lifted to functors $\tau_{\geq n}: \mathcal{M} \rightarrow \mathcal{M}_{\geq n}$ and $\tau_{\leq n}: \mathcal{M} \rightarrow \mathcal{M}_{\leq n}$. Similarly, the natural transformations $\epsilon_n: \tau_{\geq n} \rightarrow 1$ and $\eta_n: 1 \rightarrow \tau_{\leq n}$ can be lifted to natural transformations on \mathcal{M} such that ϵ_n is a natural co- $(n-1)$ -equivalence, η_n is a natural $(n+1)$ -equivalence, and $\tau_{\geq n}X \xrightarrow{\epsilon_n} X \xrightarrow{\eta_{n-1}} \tau_{\leq n-1}X$ is a natural homotopy fiber sequence in \mathcal{M} .*

Proof. Given any object X of \mathcal{M} , factor the map $* \rightarrow X$ functorially into an $(n-1)$ -equivalence $* \rightarrow X'$ followed by a co- $(n-1)$ -equivalence $X' \rightarrow X$. Define $\tau_{\geq n}X$ to be X' , and $\epsilon_n(X)$ to be the natural map $X' \rightarrow X$. Define $\tau_{\leq n}X$ to be the homotopy cofiber of $\epsilon_{n+1}(X)$ and $\eta_n(X)$ to be the map $X \rightarrow \tau_{\leq n}X$. We have that X' belongs to $\mathcal{M}_{\geq n}$ since $* \rightarrow X'$ is an $(n-1)$ -equivalence, and $\tau_{\leq n}X$ belongs to $\mathcal{M}_{\leq n}$ since the co- n -equivalences are defined to have homotopy cofibers in $\mathcal{M}_{\leq n}$. \square

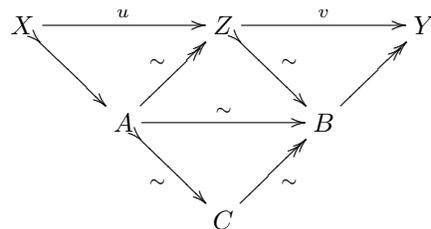
Definition 4.3. Let \mathcal{M} be a t-model structure. A map in \mathcal{M} is an **n -cofibration** if it is both a cofibration and an n -equivalence. A map in \mathcal{M} is a **co- n -fibration** if it is both a fibration and a co- n -equivalence.

Lemma 4.4. *The classes of n -cofibrations and co- n -fibrations are closed under composition and retract.*

Proof. This follows immediately from the fact that the classes of cofibrations, fibrations, n -equivalences, and co- n -equivalences are all closed under composition and retract by part (5) of Lemma 3.5 and parts (1) and (2) of Lemma 3.6. \square

Lemma 4.5. *There is a functorial factorization of maps in \mathcal{M} into n -cofibrations followed by co- n -fibrations.*

Proof. We construct a factorization explicitly. Let $f: X \rightarrow Y$ be a map in \mathcal{M} . We have a natural diagram



obtained as follows. First, factor f into an n -equivalence $u: X \rightarrow Z$ followed by a co- n -equivalence $v: Z \rightarrow Y$. Next, factor u into a cofibration $X \rightarrow A$ followed by

an acyclic fibration $A \rightarrow Z$, and factor v into an acyclic cofibration $Z \rightarrow B$ followed by a fibration $B \rightarrow Y$. Now the composition $A \rightarrow Z \rightarrow B$ is a weak equivalence, so it can be factored into an acyclic cofibration $A \rightarrow C$ followed by an acyclic fibration $C \rightarrow B$.

The composition $X \rightarrow A \rightarrow C$ is a cofibration because it is a composition of two cofibrations, and it is an n -equivalence by Lemma 3.6.

Similarly, the composition $C \rightarrow B \rightarrow Y$ is a co- n -fibration. □

We next prove that the classes of n -cofibrations and co- n -fibrations determine each other via lifting properties.

Lemma 4.6. *A map is an n -cofibration if and only if it has the left lifting property with respect to all co- n -fibrations. A map is a co- n -fibration if and only if it has the right lifting property with respect to all n -cofibrations.*

Proof. Let i be an n -cofibration and p a co- n -fibration. We use the abstract obstruction theory of [6] to show that there exists a lift $B \rightarrow X$ in the diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & & \downarrow p \\ B & \longrightarrow & Y. \end{array}$$

A lift exists in the diagram if the obstruction group $\mathcal{D}(\text{hofib } i, \text{hofib } p)$ vanishes [6, 8.4]. By definition, $\text{hofib } i$ belongs to $\mathcal{M}_{\geq n}$, and $\text{hofib } p$ belongs to $\mathcal{M}_{\leq n-1}$. Hence the obstruction group vanishes because there are only trivial maps in \mathcal{D} from objects in $\mathcal{M}_{\geq n}$ to objects in $\mathcal{M}_{\leq n-1}$.

Now suppose that a map i has the left lifting property with respect to all co- n -fibrations. Lemma 4.5 allows us to apply the retract argument and conclude that i is a retract of an n -cofibration. But n -cofibrations are preserved by retract by Lemma 4.4, so i is an n -cofibration.

A similar argument shows that if p has the right lifting property with respect to all n -cofibrations, then p is a co- n -fibration. □

Corollary 4.7. *The class of n -cofibrations is closed under arbitrary cobase change. The class of co- n -fibrations is closed under arbitrary base change.*

Proof. This follows immediately from Lemma 4.6 together with the facts that cobase changes preserve left lifting properties and base changes preserve right lifting properties. □

Lemma 4.8. *Every acyclic fibration is a co- n -fibration. Every acyclic cofibration is an n -cofibration. If $n \geq m$, then every n -cofibration is an m -cofibration, and every co- m -fibration is a co- n -fibration.*

Proof. This follows from part (2) and (3) of Lemma 3.5. □

Lemma 4.9. *Let f be a cofibration. Then f is an n -cofibration if and only if $f \wedge S^1$ is an $(n + 1)$ -cofibration.*

Proof. First, $-\wedge S^1$ preserves cofibrations because the model structure on \mathcal{M} is simplicial. Let C be the cofiber of f ; this is also the homotopy cofiber of f because f is a cofibration. Note that C is cofibrant. Now $C \wedge S^1$ is the cofiber (and also the homotopy cofiber) of $f \wedge S^1$ because the model structure on \mathcal{M} is simplicial. Since C is cofibrant, $C \wedge S^1$ is homotopically correct and is a model for ΣC in \mathcal{D} . Now C belongs to $\mathcal{M}_{\geq n+1}$ if and only if $C \wedge S^1$ belongs to $\mathcal{M}_{\geq n+2}$. \square

We next show that n -cofibrations interact appropriately with the simplicial structure. This will be needed to show that our later constructions behave well simplicially.

Proposition 4.10. *Suppose that $f: A \rightarrow B$ is an n -cofibration and $i: K \rightarrow L$ is a cofibration of simplicial sets. Then the map*

$$g: A \otimes L \amalg_{A \otimes K} B \otimes K \rightarrow B \otimes L$$

is also an n -cofibration.

Proof. The map i is a transfinite composition of cobase changes of maps of the form $\partial\Delta[j] \rightarrow \Delta[j]$. Therefore, the map g is a transfinite composition of cobase changes of maps of the form

$$A \otimes \Delta[j] \amalg_{A \otimes \partial\Delta[j]} B \otimes \partial\Delta[j] \rightarrow B \otimes \Delta[j].$$

Since n -cofibrations are characterized by a left lifting property (see Lemma 4.6), n -cofibrations are preserved by cobase changes and transfinite compositions. Therefore, we may assume that i is the map $\partial\Delta[j] \rightarrow \Delta[j]$.

Since \mathcal{M} is a simplicial model category and f is a cofibration, g is also a cofibration. We need only show that g is an n -equivalence.

Let C be the cofiber of the n -cofibration f , so C belongs to $\mathcal{M}_{\geq n+1}$. Then the cofiber of g is $C \wedge S^j$, where the simplicial set S^j is the sphere $\Delta[j]/\partial\Delta[j]$ based at the image of $\partial\Delta[j]$. We need to show that $C \wedge S^j$ also belongs to $\mathcal{M}_{\geq n+1}$. But $C \wedge S^j$ is a model for $\Sigma^j C$ in \mathcal{D} because C is cofibrant, so $C \wedge S^j$ belongs to $\mathcal{M}_{\geq n+1}$ because $\mathcal{M}_{\geq n+1}$ is closed under Σ . \square

Note that the reduced version of Proposition 4.10 also holds. Namely, if $f: A \rightarrow B$ is an n -cofibration and $i: K \rightarrow L$ is a cofibration of pointed simplicial sets, then the map

$$A \wedge L \amalg_{A \wedge K} B \wedge K \rightarrow B \wedge L$$

is also an n -cofibration. The proof is identical.

Corollary 4.11. *Let $A \rightarrow B$ be an n -cofibration and let $X \rightarrow Y$ be a co- n -fibration. The map*

$$f: \text{Map}(B, X) \rightarrow \text{Map}(A, X) \times_{\text{Map}(A, Y)} \text{Map}(B, Y)$$

is an acyclic fibration of simplicial sets.

Proof. This follows from the lifting property characterization of acyclic fibrations, adjointness, and Proposition 4.10. \square

4.1. Producing t-model categories

We give some elementary results for constructing t-model structures.

Lemma 4.12. *Assume that \mathcal{D} is the homotopy category of a proper simplicial stable model category \mathcal{M} . Let $\mathcal{D}_{\geq 0}$ be a strictly full subcategory of \mathcal{D} that is closed under Σ . Define $\mathcal{D}_{\geq n}$ to be $\Sigma^n \mathcal{D}_{\geq 0}$. Let W_n be defined as in Definition 3.2, and set $F_n = \text{inj } C \cap W_n$. Let $\mathcal{D}_{\leq n-1}$ be the full subcategory of \mathcal{D} whose objects are isomorphic to $\text{hofib}(g)$ for all g in F_n . If there is a functorial factorization of any map in \mathcal{M} as a map in $C \cap W_n$ followed by a map in F_n , then $\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0}$ is a t-structure on \mathcal{D} , and hence we get a t-model structure on \mathcal{M} .*

Proof. We verify that $\mathcal{D}_{\geq 0}, \mathcal{D}_{\leq 0}$ satisfy the three axioms of a t-structure on \mathcal{D} given in Definition 2.1. Axiom 1 holds since $\mathcal{D}_{\geq 0}$ is closed under Σ . The factorization applied to $* \rightarrow X$ (or $X \rightarrow *$) gives a natural triangle, fulfilling Axiom 2 for a t-structure.

Now assume that $X \in \mathcal{D}_{\geq 0}$ and that $Y \in \mathcal{D}_{\leq -1}$. We can assume that X is cofibrant. Factor $X \rightarrow *$ into a cofibration $g: X \rightarrow Z$ followed by an acyclic fibration $Z \rightarrow *$. We have that g is in $C \cap W_0$ since g is a cofibration with homotopy cofiber in $\mathcal{D}_{\geq 1}$. By our assumption Y is weakly equivalent to the pullback Y' of a fibration $p: E \rightarrow B$ with fibrant target having the right lifting property with respect to g . For any map $f: X \rightarrow Y'$ there are commutative squares

$$\begin{array}{ccccc}
 X & \longrightarrow & Y' & \longrightarrow & E \\
 \downarrow g & & \downarrow & & \downarrow p \\
 Z & \xrightarrow{\sim} & * & \longrightarrow & B.
 \end{array}$$

We get that the left square lifts by our assumptions. Hence any map $f: X \rightarrow Y'$ factors through a contractible object. Since X is cofibrant and Y' is fibrant we get that $\mathcal{D}(X, Y') = 0$. Hence we conclude that $\mathcal{D}(X, Y) = 0$ whenever $X \in \mathcal{D}_{\geq 0}$ and $Y \in \mathcal{D}_{\leq -1}$. □

Proposition 4.13. *Let \mathcal{M} be a proper simplicial stable cofibrantly generated model category with homotopy category \mathcal{D} . Let I be a set of generating cofibrations and let J be a set of generating acyclic cofibrations. Let K_n be subsets of J for $n \in \mathbb{Z}$. Let $C(n)$ be the class of retracts of relative $I \cup K_n$ -cell complexes. Let W_n be the corresponding class of n -equivalences defined as the class of maps that is the composite of a map in $C(n)$ followed by an acyclic fibration.*

If W_n is equal to $\Sigma^n W_0$ for all n , then the structure defined above is a t-model structure and $C(n) = C \cap W_n$.

Proof. We have functorial factorization of any map as a map in $C(n)$ followed by a map in $\text{inj-}C(n)$ [14, 10.5, 11.1.2]. The result follows from Lemma 4.12 by letting $\mathcal{D}_{\geq 0}$ be the full subcategory of \mathcal{D} consisting of objects isomorphic to the homotopy fibers of maps in W_0 . □

5. Review of pro-categories

We give a brief review of pro-categories. This section contains mostly standard material on pro-categories [1] [2] [7].

Definition 5.1. For any category \mathcal{C} , the objects of the category **pro- \mathcal{C}** are all cofiltering diagrams in \mathcal{C} , and

$$\text{pro-}\mathcal{C}(X, Y) = \lim_t \text{colim}_s \mathcal{C}(X_s, Y_t).$$

Composition is defined in the natural way.

A **constant** pro-object is one indexed by the category with one object and one (identity) map. Let $\mathbf{c}: \mathcal{C} \rightarrow \text{pro-}\mathcal{C}$ be the functor taking an object X to the constant pro-object with value X . Note that this functor makes \mathcal{C} a full subcategory of **pro- \mathcal{C}** . The limit functor **lim**: **pro- \mathcal{C}** $\rightarrow \mathcal{C}$ is the right adjoint of \mathbf{c} .

A **level map** $X \rightarrow Y$ is a pro-map that is given by a natural transformation (so X and Y must have the same indexing category); this is a very special kind of pro-map. Up to pro-isomorphism, every map is a level map [2, App. 3.2].

Let M be a collection of maps in \mathcal{C} . A level map g in **pro- \mathcal{C}** is a **levelwise M -map** if each g_s belongs to M . A pro-map is an **essentially levelwise M -map** if it is isomorphic to a levelwise M -map.

We say that a level map is directed (resp. cofinite directed) if its indexing category is a directed set (resp. cofinite directed set). Recall that a directed indexing set S is cofinite if for all $s \in S$, the set $\{t \in S \mid t < s\}$ is finite.

Definition 5.2. A map in **pro- \mathcal{C}** is a **special M -map** if it is isomorphic to a cofinite directed level map $f = \{f_s\}_{s \in S}$ with the property that for each $s \in S$, the map

$$M_s f: X_s \rightarrow \lim_{t < s} X_t \times \lim_{t < s} Y_t \rightarrow Y_s$$

belongs to M .

5.1. Strict model structures

If \mathcal{M} is a proper model category, then **pro- \mathcal{M}** has a **strict model structure** [7] [19]. The **strict cofibrations** are the essentially levelwise cofibrations, the **strict weak equivalences** are the essentially levelwise weak equivalences, and the **strict fibrations** are retracts of special fibrations (see Definition 5.2).

The functors \mathbf{c} and **lim** are a Quillen adjoint pair between \mathcal{M} and **pro- \mathcal{M}** . The right derived functor of **lim** is **holim** [7, Rem. 4.2.11]. To see why this is true, recall that $R\text{lim} X$ is defined to be $\lim \hat{X}$, where \hat{X} is a strict fibrant replacement for X . Now $\hat{X} \rightarrow *$ is a special fibration if and only if \hat{X} is a “Reedy fibrant” diagram [14, Ch. 15]. This shows that $\lim \hat{X}$ is one of the usual models for **holim** X [5, XI].

Proposition 5.3. *Let X be a cofibrant object of **pro- \mathcal{M}** . Let Y be any levelwise fibrant object of **pro- \mathcal{M}** with strict fibrant replacement \hat{Y} . Then the homotopically correct mapping space $\text{Map}(X, \hat{Y})$ in the strict model structure is weakly equivalent to $\text{holim}_t \text{colim}_s \text{Map}(X_s, Y_t)$.*

Proof. We may reindex Y so that it is cofinite directed and still levelwise fibrant [7, Thm. 2.1.6]. Since $\text{Map}(X, \hat{Y})$ is homotopically correct, it does not matter which strict fibrant replacement \hat{Y} we consider. Therefore we may choose one with particularly good properties. Use the method of [19, Lem. 4.7] to factor the map $Y \rightarrow *$ into a strict acyclic cofibration $Y \rightarrow \hat{Y}$ followed by a special fibration $\hat{Y} \rightarrow *$. This particular construction gives that $Y \rightarrow \hat{Y}$ is a levelwise weak equivalence and that \hat{Y} is levelwise fibrant.

Define a new pro-space Z by setting $Z_t = \text{colim}_s \text{Map}(X_s, \hat{Y}_t)$. The map $\hat{Y}_t \rightarrow \lim_{u < t} \hat{Y}_u$ is a fibration because \hat{Y} is strict fibrant. Since finite limits and directed colimits of simplicial sets commute, we get that the map $Z_t \rightarrow \lim_{u < t} Z_u$ is a fibration, and Z is a strict fibrant pro-space. Therefore, the simplicial set $\text{Map}(X, \hat{Y}) = \lim_t Z_t$ is weakly equivalent to the simplicial set $\text{holim}_t Z_t$ because homotopy limit is the derived functor of limit.

The map $\text{colim}_s \text{Map}(X_s, Y_t) \rightarrow \text{colim}_s \text{Map}(X_s, \hat{Y}_t)$ is a weak equivalence because $Y_t \rightarrow \hat{Y}_t$ is a weak equivalence between fibrant objects. Homotopy limits preserve levelwise weak equivalences, so the map

$$\text{holim}_t \text{colim}_s \text{Map}(X_s, Y_t) \rightarrow \text{holim}_t \text{colim}_s \text{Map}(X_s, \hat{Y}_t)$$

is a weak equivalence. □

6. Model structures on pro-categories

The goal of this section is to construct a certain model structure on $\text{pro-}\mathcal{M}$ when \mathcal{M} is a t -model structure. First, we make a connection between t -model structures and **filtered model structures**. A filtered model structure is a highly technical generalization of a model structure that is useful for producing interesting model structures on pro-categories [11].

We denote by F_n the class of co- n -fibrations in the t -model structure \mathcal{M} , and we write $C_n = C$ for the class of cofibrations in \mathcal{M} . Note that C_n does not really depend on n and that it is *not* the class of n -cofibrations. Recall from Definition 3.2 that W_n is the class of n -equivalences.

Proposition 6.1. *Let \mathcal{M} be a t -model structure. Then (W_n, C_n, F_n) is a proper simplicial filtered model structure on \mathcal{M} , where the indexing set is \mathbb{Z} with its usual ordering.*

Proof. We showed in part (2) of Lemma 3.5 that W_n is contained in W_m whenever $n \leq m$. The second half of part (2) of Lemma 3.5 implies that F_m is contained in F_n whenever $n \geq m$.

The class $\text{inj-}C$ of maps that have the right lifting property with respect to C is equal to the class of acyclic fibrations, while the class $\text{proj-}F_n$ of maps that have the left lifting property with respect to F_n is equal to the class of n -cofibrations by Lemma 4.6. These observations are central to verification of the axioms for a filtered model structure.

The axiom numbers below refer to [11, Sec. 4]. Axiom 4.2 follows from Lemma 3.6. Axiom 4.3 follows from part (5) of Lemma 3.5, Lemma 4.4, and Corollary 4.7.

The first half of Axiom 4.4 follows from our identification of $\text{inj-}C$ and by part (3) of Lemma 3.5, while the second half is immediate from the description of $\text{proj-}F_n$ as $C \cap W_n$. The first half of Axiom 4.5 is provided by factorizations into cofibrations followed by acyclic fibrations, while the second half is Lemma 4.5. For Axiom 4.6, we can factor an n -equivalence into a cofibration followed by an acyclic fibration; then the cofibration is necessarily an n -cofibration. Axioms 4.9 and 4.10 are established in Lemma 3.7. The non-trivial part of Axiom 4.12 is Proposition 4.10. \square

Definition 6.2. A map in $\text{pro-}\mathcal{M}$ is an \mathcal{H}_* -weak equivalence if it is an essentially levelwise W_n -equivalence for all n . Let F_∞ be the union $\cup_n F_n$. A map in $\text{pro-}\mathcal{M}$ is an \mathcal{H}_* -fibration if it is a retract of a special F_∞ -map.

A justification for the terminology is given by Theorem 9.13, where we show that the \mathcal{H}_* -weak equivalences can be detected by the homology functors \mathcal{H}_n . The notation F_∞ reflects the fact that F_{n+1} contains F_n for all integers n .

The cofibrations in $\text{pro-}\mathcal{M}$ are the essentially levelwise cofibrations. They are the same as the strict cofibrations (see Section 5.1), so we do not need a new name for them.

Theorem 6.3. *Let \mathcal{M} be a t -model structure. The essentially levelwise cofibrations, \mathcal{H}_* -weak equivalences, and \mathcal{H}_* -fibrations are a proper simplicial model structure on $\text{pro-}\mathcal{M}$.*

This model structure on $\text{pro-}\mathcal{M}$ is called the \mathcal{H}_* -model structure.

Proof. This follows immediately from Proposition 6.1 and [11, Thms. 5.15, 5.16]. \square

Theorem 6.3 applied to the Postnikov t -model structure on a category of spectra gives the model structure on the category of pro-spectra described in the introduction.

Remark 6.4. A t -structure is **constant** if $\mathcal{D}_{\geq 0} = \mathcal{D}_{\geq 1}$. Constant t -structures correspond to triangulated localization functors. A localization functor is a functor L together with a natural transformation $\eta: 1 \rightarrow L$ such that $L\eta(X) = \eta(LX)$ and these maps are isomorphisms for all $X \in \mathcal{D}$. The functor $\tau_{\leq 0}$ together with the natural transformation $\eta_0: 1 \rightarrow \tau_{\leq 0}$ is always a localization functor. It is triangulated exactly when the t -structure is constant. A constant t -model structure on a model category \mathcal{M} is a functorial left Bousfield localization of \mathcal{M} with respect to the class of maps W_0 [14, 3.3.1].

The \mathcal{H}_* -model structure associated to a constant t -model structure on a category \mathcal{M} is the strict model structure on $\text{pro-}\mathcal{M}$ obtained from the localized model structure on \mathcal{M} .

In order for the \mathcal{H}_* -model structure to be useful, one needs a better understanding of cofibrant objects and fibrant objects. Moreover, an understanding of the \mathcal{H}_* -acyclic cofibrations and \mathcal{H}_* -fibrations is also useful. We study these issues next.

Cofibrant objects are easy to describe. They are just essentially levelwise cofibrant objects.

The following proposition gives useful criteria for detecting \mathcal{H}_* -acyclic cofibrations.

Proposition 6.5. *Let f be a map in $\text{pro-}\mathcal{M}$. The following are equivalent:*

1. f is an \mathcal{H}_* -acyclic cofibration.
2. f is an essentially levelwise n -cofibration for every n .
3. f has the left lifting property with respect to all constant pro-maps $cX \rightarrow cY$ in which $X \rightarrow Y$ is a co- m -fibration for some m .

Proof. This follows from [11, Prop. 4.11, 4.12] and the lifting property characterization of n -cofibrations given in Lemma 4.6. \square

The \mathcal{H}_* -fibrations are more difficult to describe. The \mathcal{H}_* -fibrations are strict fibrations since F_∞ is contained in the class of all fibrations. We take the strict fibrations as our starting point and characterize the \mathcal{H}_* -fibrations among them.

Lemma 6.6. *Let $p: X \rightarrow Y$ be a special fibration indexed on a cofinite directed set S . Then p is a special F_∞ -map if and only if for each $s \in S$ the map p_s is a co- n -fibration for some n .*

Proof. We need to show that each $M_s p: X_s \rightarrow \lim_{t < s} X_t \times \lim_{t < s} X_t Y_s$ is in F_∞ if and only if each p_s is in F_∞ .

By induction, it suffices to prove that if both $M_t p$ and p_t are in F_∞ for all $t < s$, then $M_s p$ is in F_∞ if and only if p_s is in F_∞ . We have a pullback diagram

$$\begin{array}{ccc} \lim_{t < s} X_t \times \lim_{t < s} X_t Y_s & \longrightarrow & Y_s \\ \downarrow & & \downarrow \\ \lim_{t < s} X_t & \longrightarrow & \lim_{t < s} Y_t \end{array}$$

in which the lower horizontal map is in F_∞ since it is a finite composition of base changes of the maps $M_t p$ for $t < s$ [11, Lem. 2.3]. Hence its base change $\lim_{t < s} X_t \times \lim_{t < s} X_t Y_s \rightarrow Y_s$ is also in F_∞ by Corollary 4.7.

Now p_s is the composition $X_s \xrightarrow{M_s p} \lim_{t < s} X_t \times \lim_{t < s} X_t Y_s \rightarrow Y_s$. If $M_s p$ belongs to F_∞ , then p_s is the composition of two maps in F_∞ and is therefore in F_∞ . On the other hand, if p_s belongs to F_∞ , then Lemma 3.6 implies that $M_s p$ is a co- n -equivalence for some n . Since $M_s p$ was assumed to be a fibration, this means that $M_s p$ is a co- n -fibration and hence in F_∞ . \square

We now give a characterization of \mathcal{H}_* -fibrations. In order to do this, let $\text{co}W_\infty$ be the union $\cup_n \text{co}W_n$. The notation reminds us that $\text{co}W_{n+1}$ contains $\text{co}W_n$ for all integers n .

Proposition 6.7. *A map in $\text{pro-}\mathcal{M}$ is an \mathcal{H}_* -fibration if and only if it is a strict fibration and an essentially levelwise $\text{co}W_\infty$ -map.*

Proof. If $f: X \rightarrow Y$ is a \mathcal{H}_* -fibration, then f is a strict fibration since F_∞ is contained in the class of all fibrations. Since f is a retract of a special F_∞ -map by definition, it is a retract of an essentially levelwise F_∞ -map because special F_∞ -maps are essentially levelwise F_∞ -maps [11, Lem. 5.14]. Hence f is itself an essentially levelwise F_∞ -map since retracts preserve essentially levelwise F_∞ -maps [18, Cor. 5.6]. Finally, just observe that F_∞ is contained in $\text{co}W_\infty$.

For the other direction, we may assume that $f: X \rightarrow Y$ is a levelwise $\text{co}W_\infty$ -map indexed on a cofinite directed set. Factor f into a levelwise acyclic cofibration $i: X \rightarrow Z$ followed by a special fibration $p: Z \rightarrow Y$ using the method of [19, Lem. 4.7]. We have that f is a retract of p .

By Lemma 6.6, we just need to show that each map p_s is a co- n -fibration for some n . Each p_s is a fibration because special fibrations are levelwise fibrations [11, Lem. 5.14]. The map p_s is a co- n -equivalence for some n by Lemma 3.6 (3) since $p_s i_s$ is f_s and i_s is a weak equivalence. \square

Finally, we are ready to identify the \mathcal{H}_* -fibrant objects.

Definition 6.8. An object X of \mathcal{M} is **bounded above** if it belongs to $\mathcal{M}_{\leq n}$ for some n , and **bounded below** if it belongs to $\mathcal{M}_{\geq n}$ for some n .

Proposition 6.9. *An object of $\text{pro-}\mathcal{M}$ is \mathcal{H}_* -fibrant if and only if it is strict fibrant and essentially levelwise bounded above.*

Proof. This is immediate from Proposition 6.7, once we note that an object X of \mathcal{M} is bounded above if and only if the map $X \rightarrow *$ belongs to $\text{co}W_\infty$. \square

The following corollary simplifies the construction of \mathcal{H}_* -fibrant replacements.

Corollary 6.10. *If Y is an essentially levelwise bounded above pro-object, then there is a strict fibrant replacement \hat{Y} for Y such that \hat{Y} is also a \mathcal{H}_* -fibrant replacement for Y .*

Proof. We may assume that Y is levelwise bounded above and indexed on a cofinite directed set. Factor the map $Y \rightarrow *$ into a levelwise acyclic cofibration $i: Y \rightarrow \hat{Y}$ followed by a special fibration $p: \hat{Y} \rightarrow *$ using the method of [19, Lem. 4.7]. Since i is a levelwise weak equivalence, it follows that p is a levelwise $\text{co}W_\infty$ -map and thus a levelwise F_∞ -map because it is a levelwise fibration [11, Lem. 5.14]. Hence p is a special F_∞ -map by Proposition 6.6, so \hat{Y} is \mathcal{H}_* -fibrant. \square

6.1. Quillen adjunctions

In this section we give some conditions that guarantee that a Quillen adjunction between two t-model structures \mathcal{M} and \mathcal{M}' gives a Quillen adjunction between the \mathcal{H}_* -model structures on $\text{pro-}\mathcal{M}$ and $\text{pro-}\mathcal{M}'$. We use this to show that the \mathcal{H}_* -model structure on $\text{pro-}\mathcal{M}$ is stable.

Recall that if $F: \mathcal{C} \rightarrow \mathcal{C}'$ is a functor, then F induces another functor $\text{pro-}\mathcal{C} \rightarrow \text{pro-}\mathcal{C}'$ defined by applying F levelwise. We will abuse notation and write F also for this functor. If G is the right adjoint of F , then the induced functor $G: \text{pro-}\mathcal{C}' \rightarrow \text{pro-}\mathcal{C}$ is the right adjoint of $F: \text{pro-}\mathcal{C} \rightarrow \text{pro-}\mathcal{C}'$.

Proposition 6.11. *Let \mathcal{M} and \mathcal{M}' be two t-model categories and let $L: \mathcal{M} \rightarrow \mathcal{M}'$ be a left adjoint of $R: \mathcal{M}' \rightarrow \mathcal{M}$. The following are equivalent:*

1. *The induced functors $L: \text{pro-}\mathcal{M} \rightarrow \text{pro-}\mathcal{M}'$ and $R: \text{pro-}\mathcal{M}' \rightarrow \text{pro-}\mathcal{M}$ are a Quillen adjoint pair with respect to the \mathcal{H}_* -model structures on $\text{pro-}\mathcal{M}$ and $\text{pro-}\mathcal{M}'$.*

2. $L: \mathcal{M} \rightarrow \mathcal{M}'$ preserves cofibrations, and for every n' , there is an n such that L takes n -cofibrations in \mathcal{M} to n' -cofibrations in \mathcal{M}' .
3. $R: \mathcal{M}' \rightarrow \mathcal{M}$ preserves acyclic fibrations and also preserves the class of maps that are co- n -fibrations for some n , i.e., $R(F'_\infty)$ is contained in F_∞ .

Proof. This follows from [11, Lem. 3.7], [11, Thm. 6.1], and [11, Prop. 6.2]. Note that \mathcal{M} is a pointed model category, and the classes of n -cofibrations are closed under retracts and arbitrary small coproducts by Lemma 4.6. \square

Proposition 6.12. *Let \mathcal{M} and \mathcal{M}' be two t -model categories, and let $L: \mathcal{M} \rightarrow \mathcal{M}'$ be a left adjoint of $R: \mathcal{M}' \rightarrow \mathcal{M}$ such that the induced functors $L: \text{pro-}\mathcal{M} \rightarrow \text{pro-}\mathcal{M}'$ and $R: \text{pro-}\mathcal{M}' \rightarrow \text{pro-}\mathcal{M}$ are a Quillen adjoint pair with respect to the \mathcal{H}_* -model structures on $\text{pro-}\mathcal{M}$ and $\text{pro-}\mathcal{M}'$. Assume also that:*

1. *For every n' , there is an n such that if $X \rightarrow RY$ is in W_n with X cofibrant in \mathcal{M} and Y fibrant in \mathcal{M}' , then the adjoint map $LX \rightarrow Y$ is in $W'_{n'}$.*
2. *For every n , there is an n' such that if $LX \rightarrow Y$ is in W'_n with X cofibrant in \mathcal{M} and Y fibrant in \mathcal{M}' , then the adjoint map $X \rightarrow RY$ is in W_n .*

Then L and R induce a Quillen equivalence between the \mathcal{H}_ -model structures on $\text{pro-}\mathcal{M}$ and $\text{pro-}\mathcal{M}'$.*

Proof. This follows from [11, Thm. 6.3]. \square

For any pro-object X , the suspension $X \wedge S^1$ is defined to be the levelwise suspension of X , and the loops $\text{Map}_*(S^1, X)$ are defined to be the levelwise loops of X .

The next theorem says that the \mathcal{H}_* -model structure on $\text{pro-}\mathcal{M}$ is a *stable* model structure. In particular, the \mathcal{H}_* -homotopy category $\text{Ho}(\text{pro-}\mathcal{M})$ is a triangulated category.

Theorem 6.13. *The functors $-\wedge S^1$ and $\text{Map}_*(S^1, -)$ are a Quillen equivalence from the \mathcal{H}_* -model structure on $\text{pro-}\mathcal{M}$ to itself.*

Proof. On \mathcal{M} , the functor $-\wedge S^1$ preserves cofibrations because \mathcal{M} is a simplicial model category. By Lemma 4.9, $-\wedge S^1$ takes n -cofibrations in \mathcal{M} to $(n+1)$ -cofibrations. Hence $-\wedge S^1$ is a left Quillen adjoint by Proposition 6.11. The goal of the rest of the proof is to show that the conditions of Proposition 6.12 are satisfied.

Let $g: X \rightarrow \text{Map}_*(S^1, Y)$ be any map in \mathcal{M} with X cofibrant and Y fibrant. Factor g into a cofibration $i: X \rightarrow Z$ followed by an acyclic fibration $p: Z \rightarrow \text{Map}_*(S^1, Y)$. The adjoint map $f: X \wedge S^1 \rightarrow Y$ factors as $X \wedge S^1 \rightarrow Z \wedge S^1 \rightarrow Y$, where the first map is $i \wedge S^1$ and the second is adjoint to p . Note that the second map is a weak equivalence because \mathcal{M} is a stable model category.

Now g is an n -equivalence if and only if i is an n -cofibration. Lemma 4.9 implies that this occurs if and only if $i \wedge S^1$ is an $(n+1)$ -cofibration, and this happens if and only if f is an $(n+1)$ -equivalence. Thus g is an n -equivalence if and only if f is an $(n+1)$ -equivalence, and both conditions of Proposition 6.12 are satisfied. \square

7. Functorial towers of truncation functors

Recall that in Lemma 4.2, we showed that there are rigid truncation functors $\tau_{\leq n}: \mathcal{M} \rightarrow \mathcal{M}_{\leq n}$ and $\tau_{\geq n}: \mathcal{M} \rightarrow \mathcal{M}_{\geq n}$. Although these functors are well-defined on the model category \mathcal{M} (not just on the homotopy category \mathcal{D}), they have a major defect as defined previously. Namely, there are no natural transformations $\tau_{\leq n+1} \rightarrow \tau_{\leq n}$ and $\tau_{\geq n+1} \rightarrow \tau_{\geq n}$. In this section, we will modify the definition of the truncation functors so that these natural transformations do exist.

Let X be an object of \mathcal{M} . We will define objects $T_{\leq n}X$ for $n \geq 0$ inductively.

When $n = 0$, factor $X \rightarrow *$ into a 1-cofibration $X \rightarrow T_{\leq 0}X$ followed by a co-1-fibration $T_{\leq 0}X \rightarrow *$. The object $T_{\leq 0}X$ belongs to $\mathcal{M}_{\leq 0}$ by definition of co-1-equivalences.

To define $T_{\leq 1}X$, factor the map $X \rightarrow T_{\leq 0}X$ into a 2-cofibration $X \rightarrow T_{\leq 1}X$ followed by a co-2-fibration $T_{\leq 1}X \rightarrow T_{\leq 0}X$. Note that the map $T_{\leq 1}X \rightarrow *$ is a composition of a co-2-fibration with a co-1-fibration, so it is a co-2-equivalence by Lemma 3.6. Therefore $T_{\leq 1}X$ belongs to $\mathcal{M}_{\leq 1}$ as desired.

Inductively, to construct $T_{\leq n}X$ for $n > 0$, assume that $T_{\leq n-1}X$ and a natural map $X \rightarrow T_{\leq n-1}X$ have already been constructed. Factor this map into an $(n + 1)$ -cofibration $X \rightarrow T_{\leq n}X$ followed by a co- $(n + 1)$ -fibration $T_{\leq n}X \rightarrow T_{\leq n-1}X$.

The construction is summarized by the tower

$$\cdots \rightarrow T_{\leq 2}X \rightarrow T_{\leq 1}X \rightarrow T_{\leq 0}X.$$

Note that we have not defined $T_{\leq n}X$ for $n < 0$. As far as we know, it is not possible to define $T_{\leq n}$ for all n so that the desired natural transformations between these functors exist.

Note also that $T_{\leq n}$ takes values in fibrant objects, and the natural map $T_{\leq n}X \rightarrow T_{\leq n-1}X$ is a co- $(n + 1)$ -fibration.

To define $T_{\geq n}X$ for $n \geq 1$, first recall that maps in \mathcal{M} have functorial homotopy fibers. Then $T_{\geq n}X$ is defined to be the homotopy fiber of the map $X \rightarrow T_{\leq n-1}X$. As before, we end up with a tower

$$\cdots \rightarrow T_{\geq 3}X \rightarrow T_{\geq 2}X \rightarrow T_{\geq 1}X,$$

and each $T_{\geq n}X$ belongs to $\mathcal{M}_{\geq n}$.

The following lemma shows that $T_{\leq n}X$ and $T_{\geq n}X$ have the desired homotopy types.

Lemma 7.1. *The objects $T_{\leq n}X$ and $\tau_{\leq n}X$ of \mathcal{M} are weakly equivalent. Similarly, the objects $T_{\geq n}X$ and $\tau_{\geq n}X$ are weakly equivalent.*

Proof. There is a homotopy fiber sequence $T_{\geq n}X \rightarrow X \rightarrow T_{\leq n-1}X$ with the property that $T_{\leq n-1}X$ belongs to $\mathcal{M}_{\leq n-1}$ and $T_{\geq n}X$ belongs to $\mathcal{M}_{\geq n}$. On \mathcal{D} , $\tau_{\leq n-1}$ and $\tau_{\geq n}$ are the unique functors with this property (see Lemma 2.7). Therefore $T_{\leq n-1}$ induces $\tau_{\leq n-1}$ on \mathcal{D} , which means that $T_{\leq n-1}X$ and $\tau_{\leq n-1}X$ are weakly equivalent for all X . Similarly, $T_{\geq n}$ induces $\tau_{\geq n}$ on \mathcal{D} , so $T_{\geq n}X$ and $\tau_{\geq n}X$ are weakly equivalent. □

Lemma 7.2. *Let Y be a pro-object indexed by a cofiltered category I . Consider the pro-object Z indexed on $I \times \mathbb{N}$ such that $Z_{s,n} = T_{\leq n} Y_s$. The natural map $Y \rightarrow Z$ is an \mathcal{H}_* -weak equivalence.*

Proof. For each s and n , the map $Y_s \rightarrow Z_{s,n}$ has homotopy fiber $T_{\geq n+1} Y_s$, so this map is an $(n + 1)$ -equivalence. This shows that the map $Y \rightarrow Z$ is an essentially levelwise k -equivalence for all k . \square

The next result shows how the functors $T_{\leq n}$ are of tremendous value in constructing \mathcal{H}_* -fibrant replacements.

Proposition 7.3. *Let Y be a pro-object indexed by a cofiltered category I . Consider the pro-object Z indexed on $I \times \mathbb{N}$ such that $Z_{s,n} = T_{\leq n} Y_s$. A strict fibrant replacement for Z is an \mathcal{H}_* -fibrant replacement for Y .*

Proof. In order to construct an \mathcal{H}_* -fibrant replacement for Y , Lemma 7.2 says that we may construct an \mathcal{H}_* -fibrant replacement for Z instead. Finally, Corollary 6.10 says that a strict fibrant replacement for Z is the desired \mathcal{H}_* -fibrant replacement. \square

8. Homotopy classes of maps of pro-spectra

We continue to work in a t-model structure \mathcal{M} . Recall that \mathcal{D} is the homotopy category of \mathcal{M} . Let \mathcal{P} be the \mathcal{H}_* -homotopy category of pro- \mathcal{M} .

The mapping space $\text{Map}(X, Y)$ is related to homotopy classes in the following way [15, 6.1.2]. For every cofibrant X and \mathcal{H}_* -fibrant Y , $\mathcal{P}(X, Y)$ is isomorphic to $\pi_0 \text{Map}(X, Y)$.

Lemma 8.1. *When pro- \mathcal{M} is equipped with the \mathcal{H}_* -model structure, the constant pro-object functor $c: \mathcal{M} \rightarrow \text{pro-}\mathcal{M}$ and the limit functor $\text{lim}: \text{pro-}\mathcal{M} \rightarrow \mathcal{M}$ are a Quillen adjoint pair.*

Proof. Note that c preserves cofibrations and acyclic cofibrations. \square

Proposition 8.2. *The right derived functor $R\text{lim}$ of $\text{lim}: \text{pro-}\mathcal{M} \rightarrow \mathcal{M}$ is given by $R\text{lim} Y = \text{holim}_{s,n} T_{\leq n} Y_s$.*

Proof. We may assume that Y is indexed by a cofinite directed set I . Let Z be the pro-object indexed by $I \times \mathbb{N}$ such that $Z_{s,n}$ equals $T_{\leq n} Y_s$. Recall from Lemma 7.2 that the natural map $Y \rightarrow Z$ is an \mathcal{H}_* -weak equivalence.

Let \hat{Z} be a strict fibrant replacement for Z . Corollary 6.10 says that \hat{Z} is an \mathcal{H}_* -fibrant replacement for Y , so $R\text{lim} Y$ is equal to $\text{lim} \hat{Z}$. As observed in Section 5.1, $\text{lim} \hat{Z}$ is the same as $\text{holim} \hat{Z}$. \square

Corollary 8.3. *There is a natural isomorphism*

$$\mathcal{P}(cX, Y) \cong \mathcal{D}(X, \text{holim}_{t,n} T_{\leq n} Y_t)$$

for all X in \mathcal{M} and all Y in pro- \mathcal{M} . There is a natural isomorphism

$$\mathcal{P}(cX, cY) \cong \mathcal{D}(X, \text{holim}_{n \rightarrow \infty} T_{\leq n} Y)$$

for all X and Y in \mathcal{M} .

Consequently, if $Y \rightarrow \text{holim}_{n \rightarrow \infty} T_{\leq n} Y$ is a weak equivalence for all Y in \mathcal{M} , then the homotopy category of \mathcal{M} embeds into the \mathcal{H}_* -homotopy category on $\text{pro-}\mathcal{M}$.

Proposition 8.4. *Let X and Y be objects in $\text{pro-}\mathcal{M}$. Let \hat{X} be a cofibrant replacement of X . The homotopically correct mapping space of maps from X to Y in the \mathcal{H}_* -model structure is weakly equivalent to $\text{holim}_{t,n} \text{colim}_s \text{Map}(\hat{X}_s, T_{\leq n} Y_t)$.*

Proof. This follows by Propositions 5.3 and 7.3. □

Let X and Y be objects in $\text{pro-}\mathcal{M}$. In general the group $\mathcal{P}(X, Y)$ is quite different from $\text{pro-}\mathcal{D}(X, Y)$. There is not even a canonical map from one to the other. The next lemma says that under some strong conditions, the homsets in \mathcal{P} and $\text{pro-}\mathcal{D}$ agree. We choose to use conceptual proofs rather than the higher derived limit spectral sequence relating $\mathcal{P}(X, Y)$ to higher derived limits of the inverse system $\{\text{colim}_s \mathcal{D}(X_s, T_{\leq n} Y_t)\}_{t,n}$ of abelian groups.

Lemma 8.5. *Let X and Y be two pro-objects such that X_s is in $\mathcal{M}_{\geq n}$ for all s and Y_t is in $\mathcal{M}_{\leq n}$ for all t . Then $\mathcal{P}(X, Y)$ is isomorphic to $\text{lim}_t \text{colim}_s \mathcal{D}(X_s, Y_t)$.*

Proof. We may assume that X is cofibrant. By taking a levelwise fibrant replacement, we may assume that each Y_s is fibrant. Corollary 6.10 and Proposition 5.3 imply that the homotopically correct mapping space of maps from X to Y is $\text{holim}_t \text{colim}_s \text{Map}(X_s, Y_t)$, and we want to compute π_0 of this space.

For $k \geq 1$, the only map $\Sigma^k X_s \rightarrow Y_t$ in \mathcal{D} is the trivial map because $\Sigma^k X_s$ belongs to $\mathcal{D}_{\geq n+k}$ while Y_t belongs to $\mathcal{D}_{\leq n}$. Therefore, $\pi_0 \text{Map}(\Sigma^k X_s, Y_t) = \pi_k \text{Map}(X_s, Y_t)$ is trivial. We have just shown that $\text{Map}(X_s, Y_t)$ is a homotopy-discrete space.

Filtered colimits preserve homotopy-discrete spaces; moreover, they commute with π_0 . Similarly, homotopy limits preserve homotopy-discrete spaces and respect π_0 in the sense that the set of components of a homotopy limit is the ordinary limit of the sets of components of each space. This implies that $\pi_0 \text{holim}_t \text{colim}_s \text{Map}(X_s, Y_t)$ is equal to $\text{lim}_t \text{colim}_s \pi_0 \text{Map}(X_s, Y_t)$, which is the desired result. □

Corollary 8.6. *Let X and Y be two pro-objects such that X_s is in $\mathcal{M}_{\geq n}$ for all s and Y_t is in $\mathcal{M}_{\leq n-1}$ for all t . Then $\mathcal{P}(X, Y)$ is trivial.*

Proof. This follows from Lemma 8.5 and part (3) of Definition 2.1. □

Corollary 8.7. *If Y is a bounded above object in \mathcal{M} and X is any object in $\text{pro-}\mathcal{M}$, then $\mathcal{P}(X, cY)$ is isomorphic to $\text{colim}_s \mathcal{D}(X_s, Y)$.*

Proof. The proof is nearly the same as the proof of Lemma 8.5. We need to compute $\pi_0 \text{colim}_s \text{Map}(X_s, Y)$. We just need to observe that π_0 commutes with filtered colimits. □

Since $\text{pro-}\mathcal{M}$ is a model category, one may consider homotopy limits internal to $\text{pro-}\mathcal{M}$. In other words, given a diagram of pro-objects, one can form the homotopy limit of this diagram and obtain another pro-object. We will need the following basic result about homotopy limits of countable towers later when we discuss convergence of spectral sequences.

Lemma 8.8. *Let \mathcal{M} be a simplicial model category, and let \mathcal{D} be its homotopy category. Let X belong to \mathcal{M} , and let*

$$\dots \rightarrow Y^2 \rightarrow Y^1 \rightarrow Y^0$$

be a countable tower in \mathcal{M} . There is a natural short exact sequence

$$\lim_k^1 \mathcal{D}(\Sigma X, Y^k) \rightarrow \mathcal{D}(X, \text{holim}_k Y^k) \rightarrow \lim_k \mathcal{D}(X, Y^k).$$

Proof. We may assume that X is cofibrant, that each Y^k is \mathcal{H}_* -fibrant, and that each map $Y^k \rightarrow Y^{k-1}$ is an \mathcal{H}_* -fibration. We have that $\text{Map}(X, \lim_k Y_k)$ is isomorphic to $\lim_k \text{Map}(X, Y_k)$ since $\text{Map}(X, -)$ is right adjoint to tensoring with X . Since $- \otimes X$ sends acyclic cofibrations of simplicial sets to acyclic cofibrations in \mathcal{M} , we get that the tower $\text{Map}(X, Y_k)$ is a tower of fibrations between fibrant simplicial sets.

Hence $\text{Map}(X, \lim_k Y_k)$ is equivalent to $\text{holim}_k \text{Map}(X, Y^k)$. The claim now follows by the \lim^1 short exact sequence for simplicial sets [5, IX.3.1]. \square

9. t-model structure for pro-categories

We now define a t-structure on the \mathcal{H}_* -homotopy category \mathcal{P} of $\text{pro-}\mathcal{M}$.

Definition 9.1. Let $(\text{pro-}\mathcal{M})_{\leq 0}$ be the full subcategory of $\text{pro-}\mathcal{M}$ on all objects that are \mathcal{H}_* -weakly equivalent to a pro-object X such that each X_s belongs to $\mathcal{M}_{\leq 0}$. Let $(\text{pro-}\mathcal{M})_{\geq 0}$ be the full subcategory of $\text{pro-}\mathcal{M}$ on all objects that are \mathcal{H}_* -weakly equivalent to a pro-object X such that each X_s belongs to $\mathcal{M}_{\geq 0}$.

We define $(\text{pro-}\mathcal{M})_{\leq n}$ and $(\text{pro-}\mathcal{M})_{\geq n}$ to be the subcategories $\Sigma^n(\text{pro-}\mathcal{M})_{\leq 0}$ and $\Sigma^n(\text{pro-}\mathcal{M})_{\geq 0}$ respectively. Recall that here Σ refers to the levelwise suspension functor on pro-objects.

Lemma 9.2. *The subcategory $(\text{pro-}\mathcal{M})_{\leq n}$ is the full subcategory of $\text{pro-}\mathcal{M}$ on all objects that are \mathcal{H}_* -weakly equivalent to a pro-object X such that each X_s belongs to $\mathcal{M}_{\leq n}$. Similarly, the subcategory $(\text{pro-}\mathcal{M})_{\geq n}$ is the full subcategory of $\text{pro-}\mathcal{M}$ on all objects that are \mathcal{H}_* -weakly equivalent to a pro-object X such that each X_s belongs to $\mathcal{M}_{\geq n}$.*

Proof. We prove the first claim. The proof of the second claim is similar.

First suppose that X is a pro-object such that each X_s belongs to $\mathcal{M}_{\leq n}$. Since Σ^{-n} takes $\mathcal{M}_{\leq n}$ to $\mathcal{M}_{\leq 0}$, it follows that $\Sigma^{-n}X$ belongs to $\mathcal{M}_{\leq 0}$ levelwise. Therefore $\Sigma^{-n}X$ belongs to $(\text{pro-}\mathcal{M})_{\leq 0}$, and X belongs to $\Sigma^n(\text{pro-}\mathcal{M})_{\leq 0}$.

Now suppose that Y belongs to $\Sigma^n(\text{pro-}\mathcal{M})_{\leq 0}$. It follows that $\Sigma^{-n}Y$ belongs to $(\text{pro-}\mathcal{M})_{\leq 0}$, so it is \mathcal{H}_* -weakly equivalent to a pro-object X such that each X_s belongs to $\mathcal{M}_{\leq 0}$. Note that $\Sigma^n X$ belongs to $\mathcal{M}_{\leq n}$ levelwise. But $\Sigma^n X$ is \mathcal{H}_* -weakly equivalent to Y . This is the desired result. \square

Definition 9.3. Let $\mathcal{P}_{\leq n}$ be the full subcategory of \mathcal{P} on all objects whose \mathcal{H}_* -weak homotopy types belong to $(\text{pro-}\mathcal{M})_{\leq n}$. Let $\mathcal{P}_{\geq n}$ be the full subcategory of \mathcal{P} on all objects whose \mathcal{H}_* -weak homotopy types belong to $(\text{pro-}\mathcal{M})_{\geq n}$.

Proposition 9.4. *The classes $\mathcal{P}_{\geq 0}$ and $\mathcal{P}_{\leq 0}$ are a t-structure on the \mathcal{H}_* -homotopy category \mathcal{P} of pro- \mathcal{M} . Moreover, $\cap_n \mathcal{P}_{\geq n}$ consists only of contractible objects.*

Proof. We verify the axioms in Definition 2.1.

For part (1), suppose that X belongs to $\mathcal{P}_{\geq 0}$. We may assume that each X_s belongs to $\mathcal{M}_{\geq 0}$. Now each ΣX_s belongs to $\mathcal{M}_{\geq 0}$, so ΣX belongs to $\mathcal{M}_{\geq 0}$ levelwise. Thus ΣX lies in $\mathcal{P}_{\geq 0}$. To show that $\mathcal{P}_{\leq 0}$ is closed under Σ^{-1} , use the dual argument.

Lemma 9.2 implies that $\Sigma^{-1}\mathcal{P}_{\leq 0}$ is the full subcategory of \mathcal{P} on all objects that are \mathcal{H}_* -weakly equivalent to an object X such that each X_s belongs to $\mathcal{M}_{\leq -1}$. This will be needed in parts (2) and (3) below.

For part (2), let X be any object in pro- \mathcal{M} . Apply the truncation functors $\tau_{\geq 0}$ and $\tau_{\leq -1}$ to obtain a levelwise homotopy cofiber sequence $\tau_{\geq 0}X \rightarrow X \rightarrow \tau_{\leq -1}X$. Finally, observe that levelwise homotopy cofiber sequences are homotopy cofiber sequences in the \mathcal{H}_* -model structure because levelwise cofibrations are cofibrations. Note that we need Lemma 9.2 to conclude that $\tau_{\leq -1}X$ belongs to $\mathcal{P}_{\leq -1}$.

Part (3) is Corollary 8.6, again using Lemma 9.2 to identify $\mathcal{P}_{\leq -1}$.

For the last claim, suppose that X belongs to $\cap_n \mathcal{P}_{\geq n}$. Fix a value of n . Then we may assume that each X_s belongs to $\mathcal{M}_{\geq n}$, so the map $X \rightarrow *$ is a levelwise n -equivalence. Thus $X \rightarrow *$ is an \mathcal{H}_* -weak equivalence, so X is contractible. \square

The subcategory $\cap_n \mathcal{P}_{\geq n}$ contains only contractible objects even if $\cap_n \mathcal{D}_{\geq n}$ contains non-contractible objects. On the other hand, $\cap_n \mathcal{P}_{\leq n}$ contains only contractible objects if and only if $\mathcal{D} = \mathcal{D}_{\geq 0}$.

Lemma 9.5. *If all the objects of $\cap_n \mathcal{P}_{\leq n}$ are contractible, then \mathcal{D} is equal to $\mathcal{D}_{\geq 0}$.*

Proof. Assume that there are noncontractible elements $X_m \in \mathcal{M}_{\leq m}$ in each degree m . Define a pro-object $\{Y_n\}$ by letting $Y_n = \coprod_{m \leq n} X_m$ and letting the map $Y_{n-1} \rightarrow Y_n$ be the canonical map (\mathcal{M} has a zero object). Now $\{Y_n\}$ is in $\cap_n(\text{pro-}\mathcal{M})_{\leq n}$, but $\{Y_n\}$ is noncontractible in pro- \mathcal{M} : If there is a weak equivalence between $\{Y_n\}$ and $*$ in \mathcal{P} , then for every n there are integers n' and m such that in the homotopy category \mathcal{D} of \mathcal{M} the map $(Y_{n'})_{\leq m} \rightarrow (Y_n)_{\leq m}$ is the zero map. This gives a contradiction since $(Y_{n'})_{\leq m}$ is not contractible in \mathcal{D} for any m . \square

We can now identify the heart $\mathcal{H}(\mathcal{P})$ of the t-structure from Proposition 9.4.

Lemma 9.6. *The category $\mathcal{P}_{\leq 0} \cap \mathcal{P}_{\geq 0}$ is the \mathcal{H}_* -homotopy category of the subcategory pro- $(\mathcal{M}_{\leq 0} \cap \mathcal{M}_{\geq 0})$ of pro- \mathcal{M} .*

Proof. Let X be an object of $\mathcal{P}_{\leq 0} \cap \mathcal{P}_{\geq 0}$. We need to show that X is \mathcal{H}_* -weakly equivalent to an object Y of pro- $(\mathcal{M}_{\leq 0} \cap \mathcal{M}_{\geq 0})$.

We may assume that X belongs to $\mathcal{M}_{\geq 0}$ levelwise. If we apply the functors $\tau_{\geq 1}$ and $\tau_{\leq 0}$ to X levelwise, we obtain a levelwise homotopy cofiber sequence $\tau_{\geq 1}X \rightarrow X \rightarrow \tau_{\leq 0}X$. This is a homotopy cofiber sequence in pro- \mathcal{M} because cofibrations are defined to be levelwise cofibrations. Therefore

$$\tau_{\geq 1}X \rightarrow X \rightarrow \tau_{\leq 0}X \rightarrow \Sigma\tau_{\geq 1}X$$

is a distinguished triangle in \mathcal{P} . The object $\tau_{\geq 1}X$ belongs to both $\mathcal{P}_{\geq 1}$ and to $\mathcal{P}_{\leq 0}$, so it is contractible. This means that the map $X \rightarrow \tau_{\leq 0}X$ is an \mathcal{H}_* -weak equivalence.

Finally, $\tau_{\leq 0}X$ belongs to $\mathcal{M}_{\leq 0}$ levelwise by definition of $\tau_{\leq 0}$, and it also belongs to $\mathcal{M}_{\geq 0}$ levelwise because X belongs to $\mathcal{M}_{\geq 0}$ levelwise. Thus, $\tau_{\leq 0}X$ is the desired pro-object Y . \square

As in Definition 3.2, we define an **n -equivalence** (resp. **co- n -equivalence**) in $\text{pro-}\mathcal{M}$ to be a map whose homotopy fiber belongs to $(\text{pro-}\mathcal{M})_{\geq n}$ (resp. homotopy cofiber belongs to $(\text{pro-}\mathcal{M})_{\leq n}$). By definition, the n -equivalences in $\text{pro-}\mathcal{M}$ are different from the levelwise n -equivalences. A similar warning applies to co- n -equivalences. However, we will show below in Lemma 9.9 that actually they coincide.

Unfortunately, we cannot conclude that $\text{pro-}\mathcal{M}$ has a t-model structure. Although we can factor any map in $\text{pro-}\mathcal{M}$ into an n -equivalence followed by a co- n -equivalence, it does not seem to be possible to make this factorization functorial. Absence of functorial factorizations is a general problem with pro-categories. However, we will prove a slightly weaker result below in Proposition 9.8.

Lemma 9.7. *If f is an essentially levelwise n -equivalence in $\text{pro-}\mathcal{M}$, then f is an n -equivalence in $\text{pro-}\mathcal{M}$. If f is an essentially levelwise co- n -equivalence in $\text{pro-}\mathcal{M}$, then f is a co- n -equivalence in $\text{pro-}\mathcal{M}$.*

Proof. Let f be a levelwise n -equivalence. Homotopy cofibers of pro-maps can be computed levelwise because cofibrations are defined levelwise. It follows that $\text{hocofib } f$ belongs to $\mathcal{M}_{\geq n+1}$ levelwise. Lemma 9.2 says that $\text{hocofib } f$ belongs to $(\text{pro-}\mathcal{M})_{\geq n+1}$. By definition, f is an n -equivalence.

A similar argument proves the second claim. \square

Proposition 9.8. *The \mathcal{H}_* -model structure on $\text{pro-}\mathcal{M}$ and the t -structure on \mathcal{P} of Definition 9.1 are a non-functorial t -model structure on $\text{pro-}\mathcal{M}$ in the sense that all the axioms of a t -model structure are satisfied except that the factorizations in the model structure and the factorizations into n -equivalences followed by co- n -equivalences might not necessarily be functorial.*

Proof. We showed in Theorems 6.3 and 6.13 that the \mathcal{H}_* -model structure is simplicial, proper, and stable. We showed in Proposition 9.4 that Definition 9.1 is a t -structure on \mathcal{P} .

It remains only to produce (non-functorial) factorizations into n -equivalences followed by co- n -equivalences. Let $f: X \rightarrow Y$ be any map in $\text{pro-}\mathcal{M}$, which we may assume is a levelwise map. Using that the t -model structure on \mathcal{M} has functorial factorizations, we may factor f into a levelwise n -equivalence $g: X \rightarrow Z$ followed by a levelwise co- n -equivalence $h: Z \rightarrow Y$. Finally, Lemma 9.7 implies that g is an n -equivalence in $\text{pro-}\mathcal{M}$, and h is a co- n -equivalence in $\text{pro-}\mathcal{M}$. \square

Lemma 9.9. *A map in $\text{pro-}\mathcal{M}$ is an n -equivalence if and only if it is an essentially levelwise n -equivalence. A map in $\text{pro-}\mathcal{M}$ is a co- n -equivalence if and only if it is an essentially levelwise co- n -equivalence.*

Proof. We prove the first claim. The proof of the second claim is dual.

One direction was already proved in Lemma 9.7. For the other direction, suppose that $f: X \rightarrow Y$ is an n -equivalence in $\text{pro-}\mathcal{M}$. We may assume that f is a directed

cofinite level map. Factor f into a levelwise cofibration $i: X \rightarrow Z$ followed by a special acyclic fibration $p: Z \rightarrow Y$. The map p is an \mathcal{H}_* -weak equivalence, so i is also an n -equivalence in $\text{pro-}\mathcal{M}$. Moreover, the map p is a levelwise weak equivalence. The class of essentially levelwise n -equivalences is closed under composition (the proof of [19, Lem. 3.5] applies), so it suffices to show that i is an essentially levelwise n -equivalence.

We know that i is an n -cofibration in $\text{pro-}\mathcal{M}$, so it has the left lifting property with respect to all co- n -fibrations. Factor the map i into a levelwise n -cofibration $j: X \rightarrow W$ followed by a special co- n -fibration $q: W \rightarrow Z$. The map q is a co- n -fibration in $\text{pro-}\mathcal{M}$, so i has the left lifting property with respect to q , by Lemma 4.6. Thus the retract argument shows that i is a retract of j . But essentially levelwise n -cofibrations are closed under retract [18, Cor. 5.6], so i is an essentially levelwise n -cofibration and thus an essentially levelwise n -equivalence. \square

Lemma 9.10. *The strict homotopy category of $\text{pro-}(\mathcal{M}_{\leq 0} \cap \mathcal{M}_{\geq 0})$ is the same as the \mathcal{H}_* -homotopy category of $\text{pro-}(\mathcal{M}_{\leq 0} \cap \mathcal{M}_{\geq 0})$.*

Proof. In order to compute strict weak homotopy classes from X to Y , one needs to take a strict cofibrant replacement \tilde{X} of X and a strict fibrant replacement \tilde{Y} of Y . But \tilde{X} is also an \mathcal{H}_* -cofibrant replacement for X , and Corollary 6.10 says that \tilde{Y} is an \mathcal{H}_* -fibrant replacement for Y . \square

We would like a description of the heart of the t-structure on \mathcal{P} . We have not been able to identify the heart in complete generality. However, in the primary applications to chain complexes or to spectra, we can identify it using Proposition 9.11.

Proposition 9.11. *Suppose that there is a “rigidification” functor $K: \mathcal{H}(\mathcal{D}) \rightarrow \mathcal{M}_{\leq 0} \cap \mathcal{M}_{\geq 0}$ such that the composition $\mathcal{H}(\mathcal{D}) \rightarrow \mathcal{M}_{\leq 0} \cap \mathcal{M}_{\geq 0} \rightarrow \mathcal{H}(\mathcal{D})$ is the identity. Then the heart $\mathcal{H}(\mathcal{P})$ of the \mathcal{H}_* -homotopy category on $\text{pro-}\mathcal{M}$ is equivalent to the category $\text{pro-}\mathcal{H}(\mathcal{D})$.*

For spectra, the functor K takes an abelian group A to a functorial model for the Eilenberg-MacLane spectrum HA . For chain complexes, K takes an R -module A to the chain complex with value A concentrated in degree 0.

Proof. Since K extends to a levelwise functor $\text{pro-}\mathcal{H}(\mathcal{D}) \rightarrow \text{pro-}(\mathcal{M}_{\leq 0} \cap \mathcal{M}_{\geq 0})$, this gives us a functor $F: \text{pro-}\mathcal{H}(\mathcal{D}) \rightarrow \mathcal{H}(\mathcal{P})$ after composition with the usual quotient functor (because the quotient functor takes $\text{pro-}(\mathcal{M}_{\leq 0} \cap \mathcal{M}_{\geq 0})$ into both $\mathcal{P}_{\leq 0}$ and $\mathcal{P}_{\geq 0}$).

On the other hand, the quotient functor $\mathcal{M}_{\leq 0} \cap \mathcal{M}_{\geq 0} \rightarrow \mathcal{H}(\mathcal{D})$ extends to a levelwise functor $\text{pro-}(\mathcal{M}_{\leq 0} \cap \mathcal{M}_{\geq 0}) \rightarrow \text{pro-}\mathcal{H}(\mathcal{D})$. This functor takes levelwise weak equivalences to (levelwise) isomorphisms, so the functor factors through the strict homotopy category of $\text{pro-}(\mathcal{M}_{\leq 0} \cap \mathcal{M}_{\geq 0})$. Lemma 9.10 implies that the functor also factors through the \mathcal{H}_* -homotopy category of $\text{pro-}(\mathcal{M}_{\leq 0} \cap \mathcal{M}_{\geq 0})$, which is the same as $\mathcal{H}(\mathcal{P})$ by Lemma 9.6. Thus we obtain a functor $G: \mathcal{H}(\mathcal{P}) \rightarrow \text{pro-}\mathcal{H}(\mathcal{D})$.

It remains to show that F and G are inverse equivalences. The composition FG is the identity because of the original assumption on K . On the other hand, for every pro-object X , $GF X$ is levelwise weakly equivalent to X . Thus GF is isomorphic to the identity. \square

Without a rigidification functor, the most we can say is stated in the following lemma.

Lemma 9.12. *The functor $G: \mathcal{H}(\mathcal{P}) \rightarrow \text{pro-}\mathcal{H}(\mathcal{D})n$, from the proof of Proposition 9.11, is fully faithful.*

Proof. See Lemma 8.5. □

As promised in Section 6, we now recharacterize \mathcal{H}_* -weak equivalences in terms of the pro-homology functors \mathcal{H}_n .

Theorem 9.13. *A map $f: X \rightarrow Y$ in $\text{pro-}\mathcal{M}$ is an \mathcal{H}_* -weak equivalence if and only if it is an essentially levelwise m -equivalence for some m and $\mathcal{H}_n(f)$ is an isomorphism in $\text{pro-}\mathcal{H}(\mathcal{D})$ for all n .*

Proof. This is an application of Theorem 3.8 to the \mathcal{H}_* -model structure. The hypothesis of that theorem is proved at the end of Proposition 9.4. We also need Lemma 9.9 to identify the m -equivalences in $\text{pro-}\mathcal{M}$. Finally, we need Lemma 9.12 to recognize that a map g is an isomorphism in $\mathcal{H}(\mathcal{P})$ if and only if $G(g)$ is an isomorphism in $\text{pro-}\mathcal{H}(\mathcal{D})$ (where G is the functor of Lemma 9.12). □

10. The Atiyah-Hirzebruch spectral sequence

In this section we construct a spectral sequence for computing in the homotopy category of a t-model structure. We will also specialize this construction to the case of \mathcal{H}_* -model structures on pro-categories. Applied to the homotopy category of spectra with the Postnikov t-structure, we recover the Atiyah-Hirzebruch spectral sequence for spectra.

Recall from Definition 2.11 that $\tau_{\leq q}\tau_{\geq q}Y$ is isomorphic to $\Sigma^q\mathcal{H}_q(Y)$, where $\mathcal{H}_q(Y)$ is the q -th homology of Y with values in the heart of the t-structure. Also recall from Definition 2.13 that $H^{-p}(X; E)$ is the $-p$ -th cohomology of X with E -coefficients, where E belongs to the heart $\mathcal{H}(\mathcal{D})$.

For brevity we write $[X, Y]_n$ instead of $\mathcal{D}(X, \Sigma^{-n}Y)$.

Theorem 10.1. *For any X and Y in a t-model category \mathcal{M} , there is a spectral sequence with*

$$E_{p,q}^2 = H^{-p}(X; \mathcal{H}_q(Y)).$$

The spectral sequence conditionally converges to $[X, Y]_{p+q}$ if $\text{holim}_{q \rightarrow \infty} T_{\geq q}Y$ is contractible and if X is bounded below.

The construction of the spectral sequence is standard (for example, see [4, Sec. 12] or [13, App. B]). Conditional convergence of spectral sequences is defined in [4, Defn. 5.10].

To set up the spectral sequence, we only use the t-structure on the homotopy category, but we use homotopy theory to state the convergence criterion. We are

using the functors $T_{\geq q}$ rather than the functors $\tau_{\geq q}$ so that we obtain an actual tower

$$\cdots \rightarrow T_{\geq q+1}Y \rightarrow T_{\geq q}Y \rightarrow T_{\geq q-1}Y \rightarrow \cdots$$

whose homotopy limit we can consider. Recall that there are not necessarily maps $\tau_{\geq q+1}Y \rightarrow \tau_{\geq q}Y$.

Proof. Consider the filtration

$$\cdots \rightarrow T_{\geq q+1}Y \rightarrow T_{\geq q}Y \rightarrow T_{\geq q-1}Y \rightarrow \cdots$$

of Y . We have a distinguished triangle

$$T_{\geq q+1}Y \rightarrow T_{\geq q}Y \rightarrow \Sigma^q \mathcal{H}_q(Y) \rightarrow \Sigma T_{\geq q+1}Y$$

in \mathcal{D} by Lemma 2.17. If we apply the functor $[X, -]_{p+q}$ and set $D_{p,q}^2 = [X, T_{\geq q}Y]_{p+q}$ and $E_{p,q}^2 = [X, \Sigma^q \mathcal{H}_q(Y)]_{p+q}$, we get an exact couple

$$\begin{array}{ccc} D^2 & \xrightarrow{(1,-1)} & D^2 \\ & \swarrow (-2,1) & \searrow (0,0) \\ & E^2 & \end{array}$$

with the bidegrees of the maps indicated. This gives a spectral sequence where d^r has bidegree $(-r, r-1)$. This follows from the definition of the differentials given after [4, 0.6].

Now we consider conditional convergence. Recall from [4, 5.10] that we need to show that the limit $\lim_{p \rightarrow -\infty} D_{p,n-p}^2$ and the derived limit $\lim_{p \rightarrow -\infty}^1 D_{p,n-p}^2$ are both zero, while the map $\operatorname{colim}_{p \rightarrow \infty} D_{p,n-p}^2 \rightarrow [X, Y]^n$ is an isomorphism.

For the limit and the derived limit, Lemma 8.8 gives us a short exact sequence

$$\lim_{p \rightarrow -\infty}^1 [X, T_{\geq n-p}Y]_{n+1} \rightarrow [X, \operatorname{holim}_{p \rightarrow -\infty} T_{\geq n-p}Y]_n \rightarrow \lim_{p \rightarrow -\infty} [X, T_{\geq n-p}Y]_n.$$

The middle group is zero by our assumption, so the first and last groups are also zero.

For the colimit, we claim that the map $\operatorname{colim}_{p \rightarrow \infty} D_{p,n-p}^2 \rightarrow [X, Y]_n$ is an isomorphism for all n if and only if $\operatorname{colim}_{q \rightarrow \infty} [X, T_{\leq n-q}Y]_n$ is zero for all n . This follows from the distinguished triangle $T_{\geq n-p}Y \rightarrow Y \rightarrow T_{\leq n-p-1}Y \rightarrow \Sigma T_{\geq n-p}Y$ and the fact that directed colimits of abelian groups respect exact sequences. Under our assumption, X is weakly equivalent to $\tau_{\geq m}X$ for some m , so $[X, T_{\leq n-q}Y]_n$ is zero whenever $q > -m$. Thus $\operatorname{colim}_{q \rightarrow \infty} [X, T_{\leq n-q}Y]_n$ is zero. \square

We now specialize to the homotopy category \mathcal{P} of the \mathcal{H}_* -model structure on $\operatorname{pro}\mathcal{M}$. We first give a lemma which shows that one of the conditions in Theorem 10.1 is always satisfied.

Lemma 10.2. *For any Y in $\operatorname{pro}\mathcal{M}$, $\operatorname{holim}_{q \rightarrow \infty} T_{\geq q}Y$ is contractible in the \mathcal{H}_* -homotopy category \mathcal{P} .*

Proof. Each map $T_{\geq q+1}Y \rightarrow T_{\geq q}Y$ is a fibration, so the homotopy limit is the same as the ordinary limit $\lim_{q \rightarrow \infty} T_{\geq q}Y$. If I is the indexing category for Y , then one

model for this limit is the pro-object Z indexed by $I \times \mathbb{N}$ such that $Z_{s,q} = T_{\geq q} Y_s$ [18, 4.1].

The map $Z_{s,q} \rightarrow *$ is an n -equivalence whenever $q \geq n$. This shows that $Z \rightarrow *$ is an essentially levelwise n -equivalence for all n , so it is an \mathcal{H}_* -weak equivalence. \square

Theorem 10.3. *Let \mathcal{M} be a t -model category. Let X and Y be objects in $\text{pro-}\mathcal{M}$. There is a spectral sequence with*

$$E_{p,q}^2 = H^{-p}(X; \mathcal{H}_q(Y))$$

The spectral sequence converges conditionally to $\mathcal{P}(X, \Sigma^{-p-q}Y)$ if:

1. X is uniformly essentially levelwise bounded below (i.e., each X_s belongs to $\mathcal{M}_{\geq n}$ for some fixed n), or
2. if Y is a constant pro-object and X is essentially levelwise bounded above (i.e., each X_s belongs to $\mathcal{M}_{\geq n}$ for some n depending on s).

Recall that the object $\mathcal{H}_q(Y)$ by definition belongs to the heart $\mathcal{H}(\mathcal{P})$ of the \mathcal{H}_* -homotopy category. However, when the conditions of Proposition 9.11 are satisfied, we can also view $\mathcal{H}_q(Y)$ as the object of $\text{pro-}\mathcal{H}(\mathcal{D})$ obtained by applying \mathcal{H}_q to Y levelwise. When Y is a constant pro-object (i.e., belongs to \mathcal{M} , not $\text{pro-}\mathcal{M}$), then $\mathcal{H}_q(Y)$ belongs to $\mathcal{H}(\mathcal{D})$.

Proof. The spectral sequence and conditional convergence under the first hypothesis follow from Theorem 10.1 and Lemma 10.2. Observe that an object X in $\text{pro-}\mathcal{M}$ is bounded below (in the sense that it belongs to $\mathcal{P}_{\geq n}$ for some n if and only if X is uniformly essentially levelwise bounded above); this follows from Lemma 9.2.

As in the last paragraph of the proof of Theorem 10.1, it remains to show that under the second hypothesis, $\text{colim}_{q \rightarrow \infty} \mathcal{P}(X, \Sigma^{-n} T_{\leq n-q} Y)$ vanishes for all n . Because Y is a constant pro-object, Corollary 8.7 implies that the colimit is isomorphic to

$$\text{colim}_{q \rightarrow \infty} \text{colim}_s \mathcal{D}(X_s, \Sigma^{-n} T_{\leq n-q} Y).$$

Now exchange the colimits. By hypothesis, X_s belongs to $\mathcal{M}_{\geq m}$ for some m . Then $\mathcal{D}(X_s, \Sigma^{-n} T_{\leq n-q} Y) = 0$ for $q > -m$, so $\text{colim}_{q \rightarrow \infty} \mathcal{D}(X_s, \Sigma^{-n} T_{\leq n-q} Y)$ vanishes for each s . \square

11. Tensor structures on pro-categories

In this section we give some basic results about tensor structures on pro-categories.

Let \mathcal{C} be a tensor category. There is a **levelwise tensor structure** on $\text{pro-}\mathcal{C}$ given by letting $\{X_a\} \otimes \{Y_b\}$ be the pro-object $\{X_a \otimes Y_b\}$. The unit object of $\text{pro-}\mathcal{C}$ is the constant pro-object with value the unit object in \mathcal{C} . We only consider tensor structures on $\text{pro-}\mathcal{C}$ that are levelwise tensor structures inherited from a tensor structure on \mathcal{C} .

If \mathcal{C} is a cocomplete category, then $\text{pro-}\mathcal{C}$ is a cocomplete category [17, 11.1]. We recall the description of arbitrary direct sums and of coequalizers in $\text{pro-}\mathcal{C}$.

Let A be an indexing set and let $X^\alpha \in \text{pro-}\mathcal{C}$ for $\alpha \in A$ be a set of pro-objects in \mathcal{C} . Let I_α be the cofiltered indexing category of the pro-object X^α . The coproduct $\coprod_{\alpha \in A} X_\alpha$ in $\text{pro-}\mathcal{M}$ is the pro-object

$$\{\coprod_{\alpha \in A} X_{i_\alpha}^\alpha\}$$

indexed on the cofiltered category $\prod_\alpha I_\alpha$.

Up to isomorphism we can assume that a coequalizer diagram is given by levelwise maps $\{X_a\} \rightrightarrows \{Y_a\}$. The coequalizer is the pro-object $\{\text{coeq}(X_a \rightrightarrows Y_a)\}$ obtained by forming the coequalizer levelwise in \mathcal{C} .

We now consider how direct sums and tensor products interact. Let Y be a pro-object indexed on J . We have that $\coprod_{\alpha \in A} (X^\alpha \otimes Y)$ is the pro-object

$$\{\coprod_\alpha (X_{i_\alpha}^\alpha \otimes Y_{j_\alpha})\}$$

indexed on $\prod_\alpha (I_\alpha \times J)$. We also have that $(\prod_\alpha X^\alpha) \otimes Y$ is the pro-object

$$\{(\prod_\alpha X_{i_\alpha}^\alpha) \otimes Y_j\}$$

indexed on $(\prod_\alpha I_\alpha) \times J$. There is a canonical map from $\coprod_\alpha (X^\alpha \otimes Y)$ to $(\prod_\alpha X^\alpha) \otimes Y$.

Lemma 11.1. *Let \mathcal{C} be a cocomplete tensor category. If the tensor product in \mathcal{C} commutes with finite direct sums (coequalizers), then the tensor product in $\text{pro-}\mathcal{C}$ also commutes with finite direct sums (coequalizers).*

Proof. This follows by cofinality arguments. □

The tensor product on $\text{pro-}\mathcal{C}$ might not commute with arbitrary direct sums even if \mathcal{C} is a closed tensor category. In particular, the tensor structure on $\text{pro-}\mathcal{C}$ is typically not closed.

Example 11.2. Let \mathcal{C} be a tensor category with arbitrary direct sums. Assume that the tensor product on \mathcal{C} commutes with arbitrary direct sums. Then the tensor product with a constant pro-object in $\text{pro-}\mathcal{C}$ respects arbitrary direct sums. In general, however, tensor product with a pro-object does not commute with arbitrary direct sums. Let X be a pro-object indexed on natural numbers. The canonical map $(\prod_0^\infty c(I)) \otimes X \rightarrow \prod_0^\infty (c(I) \otimes X)$ is the map

$$\{\prod_{i=0}^\infty X_{n_i}\}_{\{n_i\} \in \mathbb{N}^\mathbb{N}} \rightarrow \{\prod_{i=0}^\infty X_n\}_{n \in \mathbb{N}}.$$

Assume that this map is an isomorphism; then there is an integer n and integers $n_i \geq 0$ such that $\prod_{i=0}^\infty X_{n_i+n+i} \rightarrow \prod_{i=0}^\infty X_i$ factors through $\prod_{i=0}^\infty X_n$. Hence the map is typically not a pro-isomorphism.

In particular, the tensor product on the category of pro-abelian groups does not respect arbitrary sums.

In a closed tensor category \mathcal{C} , the tensor product with any object in \mathcal{C} respects epic maps. The same is true for a tensor product on $\text{pro-}\mathcal{C}$, even though the tensor product is not closed.

Lemma 11.3. *Let \mathcal{C} be a closed tensor category. Then the tensor product on $\text{pro-}\mathcal{C}$ respects epic maps.*

Proof. A pro-map $f: X \rightarrow Y$ is epic if and only if for any b in the indexing category B of Y and any two maps $Y_b \rightrightarrows Z$, for some $Z \in \mathcal{C}$, which equalize f composed with the projection to Y_b , there is a map $Y_{b'} \rightarrow Y_b$ which also equalizes the two maps to Z .

Let $f: X \rightarrow Y$ be an epic map. Assume that $f \otimes 1_W$ equalizes the two maps $Y \otimes W \rightrightarrows Z$ where Z is an object in \mathcal{C} . Given two maps $Y_b \otimes W_k \rightrightarrows Z$. Assume that $f \otimes 1_W$ composed with the projection to $Y_b \otimes W_k$ equalize the two maps. Then there is an a and a k' so that $X_a \otimes W_{k'} \rightarrow Y_b \otimes W_k$ equalizes these two maps. We have that $X_a \rightarrow Y_b$ equalizes the two adjoint maps $Y_b \rightrightarrows F(W_{k'}, Z)$, where F denote the inner hom functor. Hence by the assumption that f is epic there is a map $Y_{b'} \rightarrow Y_b$ which also equalizes the two maps. Now use the adjunction one more time to get the conclusion that $Y_{b'} \otimes W_{k'} \rightarrow Y_b \otimes W_k$ equalizes the two maps $Y_b \otimes W_k \rightrightarrows Z$. \square

One might consider monoids in $\text{pro-}\mathcal{C}$. This is a more flexible notion than pro-objects in the category of pro- $(\mathcal{C}$ -monoids). The category of monoids in $\text{pro-}\mathcal{C}$ is the category of algebras for the monad $\mathbb{T}X = \coprod_{n \geq 0} X^{\otimes n}$. The category of commutative monoids in $\text{pro-}\mathcal{C}$ is the category of algebras for the monad $\mathbb{P}X = \coprod_{n \geq 0} X^{\otimes n} / \Sigma_n$.

Lemma 11.4. *Let \mathcal{C} be a complete and cocomplete closed tensor category. Then the category of (commutative) monoids in $\text{pro-}\mathcal{C}$ is complete and cocomplete.*

Proof. We can follow [8, II.7]. The proof of Proposition II.7.2 in [8] only uses that the tensor product commutes with finite colimits and respects epimorphisms. This holds by Lemmas 11.1 and 11.4. Hence the result follows from [8, II.7.4]. \square

12. Tensor model categories

We give conditions that guarantee that a tensor product on a model category \mathcal{M} induces a tensor product on the homotopy category of \mathcal{M} . We also give more specific conditions for a t-model category which guarantee that the induced tensor product respects the triangulated structure and the t-structure on its homotopy category.

The **pushout product axiom** for cofibrations says that if $f: X \rightarrow Y$ and $f': X' \rightarrow Y'$ are cofibrations, then the **pushout product map**

$$(X \otimes Y') \amalg_{(X \otimes X')} (Y \otimes X') \rightarrow (Y \otimes Y')$$

is a cofibration, and if in addition f or g is a weak equivalence, then the pushout product map is also a weak equivalence.

Definition 12.1. A **tensor model category** \mathcal{M} is a model category with a tensor product such that

1. \mathcal{M} satisfies the **pushout product axiom** for cofibrations, and
2. the functors $- \otimes C$ and $C \otimes -$ take weak equivalences to weak equivalences for all cofibrant objects C in \mathcal{M} .

See Hovey [15, 4.2.6] for more details on tensor model categories. Our definition is slightly stronger than his. If \mathcal{M} is a tensor model category, then there is a tensor

product on the homotopy category \mathcal{D} of \mathcal{M} [15, 4.3.2]. The homotopically correct tensor product is given by first making a cofibrant replacement of at least one of the two objects and then forming the tensor product.

Let \mathcal{M} be a pointed simplicial model category and a symmetric tensor category. Let ρ be the functor from simplicial sets to \mathcal{M} obtained by applying the simplicial tensorial structure on \mathcal{M} to the unit object in \mathcal{M} .

Definition 12.2. We say that the tensor structure and the simplicial structure on \mathcal{M} are compatible if there is a natural isomorphism between the simplicial tensorial structure and the functor $Id \otimes \rho$, restricted to finite simplicial complexes.

Lemma 12.3. *Let \mathcal{M} be a simplicial t-model category and a tensor category. If the tensor product and the simplicial structure are compatible, then the simplicial structure and the levelwise tensor structure on $\text{pro-}\mathcal{M}$ with the strict or \mathcal{H}_* -model structures are also compatible.*

Proof. For finite simplicial complexes, the simplicial tensorial structure on $\text{pro-}\mathcal{M}$ is given by applying the simplicial structure on \mathcal{M} levelwise [17, Sec. 16]. Hence for a finite simplicial set K , we have that $X \otimes \rho(K)$ is naturally isomorphic to the simplicial tensor of X with K . \square

We only consider the most naïve compatibility of the tensor structure and the triangulated structure on the homotopy category of a stable model category.

Lemma 12.4. *Let \mathcal{M} be a symmetric tensor model category with a compatible based simplicial structure. Assume the tensor product respects pushouts. Then there is a tensor triangulated structure on the homotopy category \mathcal{D} in the sense of a nonclosed version of [16, A.2].*

We make use of the following property of a tensor triangulated category in Proposition 12.11: There are natural isomorphisms $(\Sigma X) \otimes Y \rightarrow \Sigma(X \otimes Y)$ and $X \otimes \Sigma Y \rightarrow \Sigma(X \otimes Y)$ such that the following holds. If $X \rightarrow Y \rightarrow Z \rightarrow \Sigma X$ is a distinguished triangle, then $X \otimes W \rightarrow Y \otimes W \rightarrow Z \otimes W \rightarrow \Sigma(X \otimes W)$ is again a distinguished triangle, where the last map is $Z \otimes W \rightarrow (\Sigma X) \otimes W \rightarrow \Sigma(X \otimes W)$, and similarly when the triangle is tensored by W from the left.

Proof. The unit and associativity conditions stated in [16, A.2] follows from the corresponding results for the tensor product. The results stated above follows since the tensor product respects homotopy cofibers by our assumption. \square

Next we give a compatibility of the tensor structure with respect to the t-model structure. The conditions are used Proposition 12.11 to get a multiplicative structure on the Atiyah-Hirzebruch spectral sequence.

Definition 12.5. Let \mathcal{D} be a triangulated category. A t-structure $\mathcal{D}_{\geq 0}$ and $\mathcal{D}_{\leq 0}$ and a tensor structure on \mathcal{D} are compatible if $\mathcal{D}_{\geq 0}$ is closed under the tensor product.

Thus for all integers i and j we have that if $X \in \mathcal{D}_{\geq i}$ and $Y \in \mathcal{D}_{\geq j}$, then $X \otimes Y \in \mathcal{D}_{\geq i+j}$.

Remark 12.6. If the t-structure on \mathcal{D} is not constant, then the unit object of any tensor structure compatible with the t-structure on \mathcal{D} must be an object in $\mathcal{D}_{\geq 0}$.

Proposition 12.7. *Let \mathcal{M} be a tensor model category. Then $\text{pro-}\mathcal{M}$ with the strict model structure is also a tensor model category. In particular, there is an induced tensor structure on its homotopy category. If in addition, the simplicial structure is compatible with the tensor product, then the homotopy category is a triangulated tensor category.*

Proof. The pushout product axiom holds, since the pushout product map can be defined levelwise.

Let f be a weak equivalence in $\text{pro-}\mathcal{M}$. We can assume that f is a levelwise weak equivalence $\{f_s: X_s \rightarrow Y_s\}$. We can furthermore assume that the cofibrant object is a levelwise cofibrant pro-object $\{Z_t\}$ indexed on a directed set T . We get that $\{f_s\} \otimes \{Z_t\}$ and $\{Z_t\} \otimes \{f_s\}$ are levelwise weak equivalences. The last statement follows from Lemmas 12.3 and 12.4. \square

We do not get an induced tensor structure on the homotopy category \mathcal{P} of $\text{pro-}\mathcal{M}$ with the \mathcal{H}_* -model structure when \mathcal{M} is a t-model category and a tensor model category. This does not even hold when the t-structure and the tensor structure on \mathcal{D} respect each other. But in this case we do get a tensor product on the full subcategory of \mathcal{P} consisting of objects that are essentially levelwise bounded below.

Definition 12.8. Let \mathcal{M} be a t-model category. Let $\mathcal{M}_{>-\infty}$ be the full subcategory of \mathcal{M} with objects X such that $X \in \mathcal{M}_{\geq n}$ for some n .

The category $\text{pro-}\mathcal{M}_{>-\infty}$ is the strictly full subcategory of $\text{pro-}\mathcal{M}$ consisting of objects that are essentially levelwise bounded below. It is larger than the category $(\text{pro-}\mathcal{M})_{>-\infty}$.

Lemma 12.9. *Let \mathcal{M} be a t-model category. Then the category $\mathcal{M}_{>-\infty}$ inherits a t-model structure from \mathcal{M} .*

The model structure $\mathcal{M}_{>-\infty}$ has only finite colimits and limits. The classes of cofibrations, weak equivalences, and fibrations are all inherited from the full inclusion functor $\mathcal{M}_{>-\infty} \rightarrow \mathcal{M}$.

Proof. If $f: X \rightarrow Y$ is a map in $\mathcal{M}_{>-\infty}$, and $X \xrightarrow{g} Z \rightarrow Y$ is a factorization of f as an n -equivalence followed by a co- n -equivalence in \mathcal{M} , then Z is also in $\mathcal{M}_{>-\infty}$: Assume that $X \in \mathcal{M}_{\geq m}$ for some m . In the homotopy category of \mathcal{M} we have a triangle $\text{hofib}(g) \rightarrow X \xrightarrow{g} Z$. Hence by Corollary 2.6 we have that Z is in $\mathcal{D}_{\geq \min\{m,n\}}$, so $Z \in \mathcal{M}_{>-\infty}$. A similar argument shows that we have functorial factorizations of any map in $\mathcal{M}_{>-\infty}$ as an acyclic cofibration followed by a fibration and as a cofibration followed by an acyclic fibration. The rest of the t-model category axioms are inherited from \mathcal{M} . \square

Proposition 12.10. *Let \mathcal{M} be a t-model category with a tensor model structure such that the tensor product on \mathcal{D} is compatible with the t-structure. Then the model category $\text{pro-}\mathcal{M}_{>-\infty}$ is a tensor model category and the tensor product on*

its homotopy category is compatible with the t -structure. If in addition, the simplicial structure and the tensor structure on \mathcal{M} are compatible, then $\text{Ho}(\text{pro-}\mathcal{M}_{>-\infty})$ is a tensor triangulated category.

Proof. Let f be a weak equivalence in $\text{pro-}\mathcal{M}$. We can assume that f is a levelwise map $\{f_s: X_s \rightarrow Y_s\}$ indexed on a directed set S such that for all n there is an s_n such that f_s is n -connected for all $s \geq s_n$ [11, 3.2]. We can assume that the cofibrant object is a pro-object $\{Z_t\}$ indexed on a directed set T such that $Z_t \in \mathcal{D}_{\geq n_t}$ and each Z_t is cofibrant. We use that tensoring with a cofibrant object has the correct homotopy type. The indexing set $\{s, t \in S \times T \mid \text{conn}(f_s) + \text{conn}(X_t) \geq n\}$ is cofinal in $S \times T$. Hence we have that $\{f_s\} \otimes \{Z_t\}$ is an essentially levelwise n -equivalence for all n .

The first part of the pushout-product axiom follows by considering two levelwise cofibrations. When one of the maps is a levelwise acyclic cofibration, we use the previous paragraph and Lemma 3.7 to show that the pushout-product map is also a weak equivalence. The last statement follows from Lemmas 12.3 and 12.4. \square

12.1. Multiplicativity in the Atiyah-Hirzebruch spectral sequence

We show that if Y is a monoid in a tensor triangulated category with a t -structure that is compatible with the tensor structure, then the Atiyah-Hirzebruch spectral sequence is multiplicative.

Proposition 12.11. *Let \mathcal{D} be a symmetric tensor triangulated category with a compatible t -structure. Let Y be a monoid in \mathcal{D} . Then the spectral sequence in 10.1 is multiplicative.*

Proof. For convenience, let h_n denote $\tau_{\leq n} \tau_{\geq n} \cong \Sigma^n \mathcal{H}_n$. It suffices to prove that we have unique dotted maps

$$\begin{array}{ccc} Y_{\geq i} \otimes Y_{\geq j} & \xrightarrow{f} & Y_{\geq i+j} \\ \downarrow & & \downarrow \\ h_i(Y) \otimes h_j(Y) & \xrightarrow{g} & h_{i+j}(Y) \end{array}$$

where f is compatible with the multiplication on Y . Consider the square

$$\begin{array}{ccc} Y_{\geq i} \otimes Y_{\geq j} & \xrightarrow{f} & Y_{\geq i+j} \\ \downarrow & & \downarrow \\ Y \otimes Y & \longrightarrow & Y \longrightarrow Y_{\leq i+j-1}. \end{array}$$

Since $Y_{\geq i} \otimes Y_{\geq j} \in \mathcal{D}_{\geq i+j}$, we get that the map from $Y_{\geq i} \otimes Y_{\geq j}$ to $Y_{\leq i+j-1}$ vanishes. Hence there is a lift to $Y_{\geq i+j}$. This lift is unique since the difference of two lifts factors through $\Sigma^{-1}Y_{\leq i+j-1} \in \mathcal{D}_{\leq i+j-2}$. Hence there is a unique map f . We now

prove that there is a unique map g between the cohomology. Consider the square

$$\begin{array}{ccccc}
 Y_{\geq i+1} \otimes Y_{\geq j} & \longrightarrow & Y_{\geq i} \otimes Y_{\geq j} & \longrightarrow & Y_{\geq i+j} \\
 & & \downarrow & & \downarrow \\
 & & h_i(Y) \otimes Y_{\geq j} & \xrightarrow{\quad \quad \quad} & h_{i+j}(Y).
 \end{array}$$

Since $Y_{\geq i+1} \otimes Y_{\geq j} \in \mathcal{D}_{\geq i+j+1}$ and $\Sigma Y_{\geq i+1} \otimes Y_{\geq j} \in \mathcal{D}_{\geq i+j+2}$, we get that there is a unique map making the diagram commute. A similar argument with the distinguished triangle $Y_{\geq j+1} \rightarrow Y_{\geq j} \rightarrow h_j(Y) \rightarrow \Sigma Y_{\geq j+1}$ tensored from the right by $h_i(Y)$ gives a unique map $h_i(Y) \otimes h_j(Y) \rightarrow h_{i+j}(Y)$ compatible with $Y_{\geq i} \otimes Y_{\geq j} \rightarrow Y_{\geq i+j}$. \square

The following remark says that it suffices to consider monoids in the homotopy category of $\text{pro-}\mathcal{M}$ with the strict model structure to get a multiplicative structure on the Atiyah-Hirzebruch spectral sequence of Theorem 10.3.

Remark 12.12. We can apply Theorem 10.1 to the homotopy category of $\text{pro-}\mathcal{M}$ with the strict model structure and with the levelwise t-structure. We get a spectral sequence that is isomorphic to the spectral sequence obtained from \mathcal{P} with the levelwise t-structure. This is seen by inspecting the definition of D^2 , E^2 and the differentials using Propositions 5.3 and 8.4.

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