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A CHAIN COALGEBRA MODEL FOR THE JAMES MAP

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Abstract

Let EK be the simplicial suspension of a pointed simplicial set K. We construct a chain model of the James map, $\alpha_K\colon CK \to \Omega CEK$. We compute the cobar diagonal on ΩCEK, not assuming that EK is 1-reduced, and show that α_K is comultiplicative. As a result, the natural isomorphism of chain algebras $TCK \cong \Omega CK$ preserves diagonals.

In an appendix, we show that the Milgram map, $\Omega(A \otimes B) \to \Omega A \otimes \Omega B$, where A and B are coaugmented coalgebras, forms part of a strong deformation retract of chain complexes. Therefore, it is a chain equivalence even when A and B are not 1-connected.

1. Introduction

Let L be a 1-reduced simplicial set. Let $G(-)$ and $C(-)$ be the Kan loop group and normalized chain functors respectively. The explicit, natural twisting cochain $t_L\colon CL \to C(GL)$ of Szczarba [**17**] determines a natural morphism of chain algebras $\theta_L\colon \Omega CL \to C(GL)$ that induces an isomorphism in homology, since CL is 1-connected. Here ΩCL is the cobar construction [**1**].

The coassociative, counital diagonal ψ on ΩCL of Baues [**2**] makes ΩCL a Hopf algebra. The Alexander-Whitney diagonal on CL is comultiplicative up to strong homotopy or DCSH [**8**], and hence is the linear part in a morphism of chain algebras, $\tilde{\Omega}\Delta\colon \Omega(CK) \to \Omega(CL \otimes CL)$. By [**11**], $\psi = q \circ \tilde{\Omega}\Delta$, where $q\colon \Omega(CL \otimes CL) \to \Omega CL \otimes \Omega CL$ is the Milgram equivalence [**15**].[1] Furthermore, the Szczarba equivalence is a DCSH morphism.

In the present paper, we consider the special case $L = EK$, where K is a pointed simplicial set and $E(-)$ is the simplicial suspension. Note that we allow the case when K is not reduced, and hence L need not be 1-reduced. Since the Alexander-Whitney diagonal on CEK is trivial, there is a natural isomorphism of chain algebras,

$$\Omega CEK \cong T(\tilde{C}K), \tag{1}$$

where $\tilde{C}K = CK/C(*)$. Our main result states:

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[1] In hindsight, we realize that we had resurrected an idea of Drachman [**3, 4**].

Theorem (Theorem 4.9). The isomorphism (1) preserves diagonals.

To prove the theorem, we construct a chain model of the James map (see below), $\alpha_K\colon CK \to \Omega CEK$. We compute the diagonal ψ_{EK} on ΩCEK, even when EK is not 1-reduced, and show that α_K is comultiplicative. An immediate consequence is that ψ_{EK} is coassociative.

Let JX be the James construction [**12**] on the pointed topological space X. The unit $\eta_X\colon X \to JX$ determines an isomorphism $T\tilde{H}_*(X) \xrightarrow{\cong} H_*(JX)$ of associative algebras when the coefficients are such that $\tilde{H}_*(X)$ is torsion-free. This result has the following implications at the chain level. Let $CS(-)$ be the normalized singular chains functor over $\mathbb{Z}$ (the composite of normalized chains C and total singular complex S). Since JX is a topological monoid, $CS(JX)$ is a chain Hopf algebra. The chain coalgebra morphism $CS(\eta_X)\colon CS(X) \to CS(JX)$, extends to a chain Hopf algebra morphism, $T(CS(X)) \to CS(JX)$ inducing an isomorphism in homology.

If one replaces X by a pointed simplicial set K, our main theorem identifies $T(CK) \cong \Omega CEK$ *as a Hopf algebra.* We calculate the Szczarba equivalence $\Omega CEK \to C(GEK)$ when K is reduced, and show that it factors through the natural equivalence $T(CK) \to C(G^+EK)$ where G^+EK is a simplicial model of $J|K|$. We are interested in G^+EK since in the applications, it is important to be able to treat ΩCEK when K is not necessarily reduced.

We remark that Theorem 4.9 plays an essential role in both [**9**] and [**10**]. In [**9**] Theorem 4.9 is applied in the construction of a simple algebraic model of the free loop space on a suspension, which is the basic building block of a model for calculation of the mod 2 topological cyclic homology of a suspension. Similarly, thanks to Theorem 4.9, the model developed in [**10**] for calculating the homology algebra of double loop spaces has a particularly tractable form when applied to double suspensions.

In the appendix, we show that the Milgram map $q : \Omega(A \otimes C) \to \Omega A \otimes \Omega C$ fits into a strong deformation retract of chain complexes [**8**], and therefore is a chain homotopy equivalence whenever A and C are coaugmented. This result further reinforces the geometric validity of our calculation when K is not reduced.

2. Notation and background

2.1. Simplicial structures

For any $m \leqslant n \in \mathbb{N}$, let $[m,n] = \{j \in \mathbb{N} | m \leqslant j \leqslant n\}$. Let $\mathbf{\Delta}$ denote the category with objects $Ob(\mathbf{\Delta}) = \{[0,n] | n \geqslant 0\}$ and morphisms

$$\mathbf{\Delta}([0,m],[0,n]) = \{[0,m] \xrightarrow{f} [0,n] \mid f \text{ is an order-preserving set map}\}.$$

In [**13**, p.177], it is shown that the classical *coface* and *codegeneracy* maps, namely,

$$d_i^n \ : \ [0,n-1] \to [0,n] \qquad x \mapsto \begin{cases} x & \text{if } x < i \\ x+1 & \text{if } x \geqslant i \end{cases} \qquad \text{and} \qquad \sigma_i^n \ : \ [0,n+1] \to [0,n] \qquad x \mapsto \begin{cases} x & \text{if } x \leqslant i \\ x-1 & \text{if } x > i \end{cases}$$

for $0 \leqslant i \leqslant n$, generate $\mathbf{\Delta}([0,m],[0,n])$. Eilenberg and MacLane in [**5**] associate to a

morphism $f \in \mathbf{\Delta}([0,m],[0,n])$ its *derived* map $f' \in \mathbf{\Delta}([0,m+1],[0,n+1])$ defined by

$$f'(0) = 0 \quad \text{and} \quad f'(i+1) = f(i)+1, \quad i = 0,\dots,m.$$

Clearly, $(d_i^n)' = d_{i+1}^{n+1}$ and $(\sigma_i^n)' = \sigma_{i+1}^{n+1}$.

A *simplicial set* is a contravariant functor $K\colon \mathbf{\Delta} \to Set$. Let $K_n := K([0,n])$, $s_i^n := K(\sigma_i^n)$, and $\partial_i^n := K(d_i^n)$. The maps s_i^n and ∂_i^n are called respectively *degeneracy* and *face* maps and an element $x \in K_n$ is called an *n-simplex*. The dimension superscript will be omitted when the context is clear. We will use extensively the simplicial identities:

$$\begin{aligned}
\partial_i\partial_j &= \partial_{j-1}\partial_i \quad \text{if} \quad i<j,\\
s_i s_j &= s_{j+1}s_i \quad \text{if} \quad i \leqslant j,\\
\partial_i s_j &= s_{j-1}\partial_i \quad \text{if} \quad i<j,\\
\partial_j s_j &= \text{identity} \quad = \quad \partial_{j+1}s_j, \text{ and}\\
\partial_i s_j &= s_j\partial_{i-1} \quad \text{if} \quad i>j+1.
\end{aligned}$$

The derived maps associated to the face and degeneracy maps are

$$(\partial_i^n)' = K((d_i^n)') = \partial_{i+1}^{n+1} \quad \text{and} \quad (s_j^n)' = K((\sigma_j^n)') = s_{j+1}^{n+1}.$$

Naturality implies that the derived map associated to a composition of iterated faces and degeneracies is the composition of the iterated derived face and degeneracy maps, i.e., $(s_{i_1}\dots s_{i_k}\partial_{j_1}\dots\partial_{j_l})' = s_{i_1+1}\dots s_{i_k+1}\partial_{j_1+1}\dots\partial_{j_l+1}$. Given two simplicial sets K and L, we extend the notion of derived map on $K \times L$ componentwise; i.e.,

$$\begin{aligned}
&\big((s_{i_1}\dots s_{i_k}\partial_{j_1}\dots\partial_{j_l}) \times (s_{\alpha_1}\dots s_{\alpha_n}\partial_{\beta_1}\dots\partial_{\beta_m})\big)'\\
&\qquad = (s_{i_1}\dots s_{i_k}\partial_{j_1}\dots\partial_{j_l})' \times (s_{\alpha_1}\dots s_{\alpha_n}\partial_{\beta_1}\dots\partial_{\beta_m})'.
\end{aligned}$$

Let Δ^n denote the *standard geometric n-simplex*. The *singular complex* on a topological space X is the simplicial set $S(X)$ where $S_n(X) = Top(\Delta^n, X)$. Its left adjoint, the *geometric realization* functor, is denoted $|\cdot|$.

More generally a simplicial object in a category $\mathcal{C}$ is a contravariant functor from $\mathbf{\Delta}$ to $\mathcal{C}$. In particular we will be concerned with simplicial monoids and simplicial (abelian) groups.

Let $\mathcal{F}_{ab}\colon Set \to Ab$ denote the free abelian group functor. Given a simplicial set K, there is an associated simplicial abelian group, namely $K_{ab} := \mathcal{F}_{ab} \circ K$. We extend linearly the notion of derived map. For all $n > 0$, let $DK_n = \cup_{i=0}^{n-1} s_i(K_{n-1})$, the set of degenerate n-simplices of K. The *normalized chain complex* on K, denoted $C(K)$, is given by

$$C_n(K) = \mathcal{F}_{ab}(K_n)/\mathcal{F}_{ab}(DK_n),$$

where its differential is induced by $\partial = \sum_{i=0}^{n}(-1)^i\partial_i$. The simplicial identities imply that $\partial^2 = 0$ and $\partial(\mathcal{F}_{ab}(DK_n)) \subset \mathcal{F}_{ab}(DK_{n-1})$. As noted in [**5**] the notion of a derived map does not descend to normalized chains. Hence all computations will be carried out with unnormalized chains, and we pass afterwards to the quotient.

Let $*$ be the unique simplicial set generated by a single nondegenerate 0-simplex. For the remainder of this paper we work within the category of *pointed* simplicial

sets. Objects are simplicial maps $\kappa : * \to K$, which boils down to choosing a 0-simplex as basepoint in K. Morphisms are commuting triangles, i.e., basepoint-preserving simplicial maps. The basepoint of K is denoted $k_0 := \kappa(*)$, and $k_n := (s_0)^n k_0$. An n *reduced* simplicial set K is a simplicial set such that $K_i = \{k_i\}$ for $0 \leqslant i \leqslant n$. A 0-reduced simplicial set will simply be called a *reduced* simplicial set. The unique simplicial map $c\colon K \to *$ induces a chain map $C(K) \xrightarrow{\epsilon} C(*) \cong \mathbb{Z}$. The *reduced chain complex* on K will be denoted by $\widetilde{C}(K) = \ker(\epsilon)$. Thus $\widetilde{C}_{>0}(K) = C_{>0}(K)$, while $\kappa : * \to K$ induces a natural splitting of ϵ, from which we obtain a natural basis for $\widetilde{C}_0$, namely the set $\{x - k_0 \mid x \in K_0 \backslash \{k_0\}\}$. Thus $C_0(K) = \mathbb{Z}\{k_0\} \oplus \widetilde{C}_0(K)$.

We now recall three classical constructions on a pointed simplicial set K that are central to the discussion at hand.

(a) Simplicial suspension EK ([**14**, p.124]): Let $E_0(K) = b_0$, while $E_n(K)$ is the set of pairs (i, x), where $i \geqslant 1$ is an integer and $x \in K_{n-i}$, under the identification $(i, k_n) = s_0^{n+i} b_0 = b_{n+i}$. The face and degeneracy operators are generated by

1. $\partial_0(1, x) = b_n$, for all $x \in K_n$,
2. $\partial_1(1, x) = b_0$, for all $x \in K_0$,
3. $\partial_{i+1}(1, x) = (1, \partial_i x)$, for all $x \in K_n$, $n > 0$,
4. $s_0(i, x) = (i + 1, x)$, and
5. $s_{i+1}(1, x) = (1, s_i x)$,

with all other face and degeneracy maps defined by the requirement that EK be a simplicial set.

The reduced suspension of the geometric realization of K is canonically homeomorphic to the geometric realization of EK ([**14**, p.125]). Note that $C_{n+1}(EK)$ is generated by elements $(1, x)$, for nondegenerate $x \in K_n$.

(b) Simplicial loop group $G(K)$ ([**14**, p.118]): Let K be a reduced simplicial set. Define $G_n(K)$ to be the free group generated by the elements of K_{n+1} under the identification $s_0 x = e_n$ for $x \in K_n$, where e_n is the identity element of $G_n(K)$. The face and degeneracy operators are given by

1. $\tau(\partial_0 x)\partial_0 \tau(x) = \tau(\partial_1 x)$,
2. $\partial_i \tau(x) = \tau(\partial_{i+1} x)$, if $i > 0$, and
3. $s_i \tau(x) = \tau(s_{i+1} x)$, if $i \geqslant 0$,

where $\tau(x)$ denotes the class of $x \in K_{n+1}$ in $G_n(K)$. These maps, ∂_i and s_i, extend uniquely to homomorphisms $G_n(K) \to G_{n-1}(K)$ and $G_n(K) \to G_{n+1}(K)$ respectively.

In the appendix of [**16**], Smith shows that the (based) loops on the geometric realization of K is weakly equivalent to the geometric realization of GK.

(c) Simplicial James construction $G^+E(K)$: Notice that since $G(K)$ is a simplicial group, from condition (b.1) we can solve for $\partial_0 \tau(x)$. If we ask that $G_n(K)$ be the free monoid instead of the free group, then condition (b.1) is not enough to determine $\partial_0 \tau(x)$. But if $K = EL$, then condition (b.1) characterizes $\partial_0 \tau(x)$. Indeed,

for $x \in L_{n>0}$, we have either

$$\tau(\partial_0(1,x)) = \tau(b_n) = e_{n-1} \quad \text{or} \quad \partial_0\tau(i+1,x) = \partial_0\tau(s_0(i,x)) = e_{n+i-1}.$$

Hence we have a well-defined functor, $G^+E(-)$, from (arbitrary) pointed simplicial sets to simplicial monoids. An easy calculation shows that for all K, the natural map

$$\begin{aligned} \eta_K \colon K &\to G^+E(K) \\ x &\mapsto \tau(1,x) \end{aligned}$$

is simplicial. In fact, using the universal properties of the James construction, JX, on a topological space X ([**12**]) and the adjunction between $S(-)$ and $|\cdot|$, one can show that η_K is a model of the topological James map $X \to JX$. In particular, $|G^+E(K)| \cong J|K|$. Moreover, when K is reduced, the inclusion $G^+E(K) \hookrightarrow GE(K)$ is a homotopy equivalence. In [**14**, p.126], the last result is said to be valid for countable reduced simplicial sets. Using Proposition 2.4 on page 9 of [**7**], one can extend it to arbitrary reduced simplicial sets.

Remark 2.1. The $G^+E(-)$ construction is isomorphic to Milnor's F^+ construction as remarked in [**16**].

2.2. Differential structures

We recall now a number of basic definitions and constructions related to graded modules and graded (co)algebras over a principal ideal domain R. A graded R-module $V = \oplus_{i\in\mathbb{Z}} V_i$ is *connected* if $V_{<0} = 0$ and $V_0 \cong R$. It is *simply connected* if, in addition, $V_1 = 0$. We write V_+ for $V_{>0}$. Let V be a nonnegatively graded, free R-module. The free associative algebra generated by V is denoted TV; i.e.,

$$TV \cong R \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

where the product $\mu \colon TV \otimes TV \to TV$ is given by word concatenation. We denote the submodule of words of length n by $T^nV = V^{\otimes n}$.

The *suspension* endofunctor s on the category of graded modules is defined on objects $V = \oplus_{i\in\mathbb{Z}} V_i$ by $(sV)_i \cong V_{i-1}$. Given a homogeneous element $v \in V$, we write sv for the corresponding element of sV. The suspension s admits an obvious inverse, which we denote s^{-1}. Observe that $C(EK) \cong sC(K)$ as chain complexes. A map of chain complexes inducing an isomorphism in homology will be called a *quasi-isomorphism*.

Let $f, g \colon (A,d) \to (B,d)$ be two maps of chain algebras. An (f,g)-*derivation* is a linear map $\varphi \colon A \to B$ of degree $+1$ such that $\varphi\mu = \mu(\varphi \otimes g + f \otimes \varphi)$, where μ denotes the multiplication on A and B. A *derivation homotopy* from f to g is an (f,g)-derivation φ that satisfies $d\varphi + \varphi d = f - g$.

Let (C, d, Δ) be a coaugmented chain coalgebra. Let $\overline{C} = \ker \varepsilon$ where $\varepsilon \colon C \to R$ is the counit. The *reduced coproduct* is defined by

$$\overline{\Delta}(c) := \Delta(c) - (c \otimes 1 + 1 \otimes c),$$

for $c \in \overline{C}$.

Definition 2.2 ([**1**])**.** The cobar construction on C, denoted $\Omega(C)$, is the chain algebra $(Ts^{-1}(\overline{C}), d_\Omega)$, where $d_\Omega = -s^{-1}ds + \mu(s^{-1} \otimes s^{-1})\overline{\Delta}s$ on generators.

A word in $T^n(s^{-1}(\overline{C}))$ will be denoted by $[x_1|\ldots|x_n] := s^{-1}x_1 \otimes \ldots \otimes s^{-1}x_n$, while the unit will be denoted by $[\,]$. There is a natural chain algebra morphism

$$q\colon \Omega(C \otimes C') \to \Omega(C) \otimes \Omega(C') \tag{2}$$

specified by $q([x \otimes 1]) = [x] \otimes [\,]$, $q([1 \otimes y]) = [\,] \otimes [y]$, and $q([x \otimes y]) = 0$ for all $x \in \overline{C}$ and $y \in \overline{C}'$. Milgram shows, in [**15**, Theorem 7.4], that if C and C' are 1-connected, then q is a natural quasi-isomorphism of chain algebras. In Appendix A we extend this result to arbitrary coaugmented chain coalgebras.

Let (C, d, Δ) be a chain coalgebra, and let (A, d, μ) be a chain algebra. A *twisting cochain* from C to A is a degree -1 map $t\colon C \to A$ of graded modules such that

$$dt + td = \mu(t \otimes t)\Delta.$$

If C is connected, then any twisting cochain $t\colon C \to A$ induces a chain algebra map $\theta\colon \Omega(C) \to A$ by setting $\theta([c]) = t(c)$. It is equally clear that any chain algebra map $\theta\colon \Omega(C) \to A$ gives rise to a twisting cochain via the composition

$$C_+ \xrightarrow{s^{-1}} s^{-1}C_+ \hookrightarrow Ts^{-1}C_+ \xrightarrow{\theta} A.$$

In Section 4 of this paper, we work in the category **DCSH** [**8**]. Its objects are augmented, connected coassociative chain coalgebras. A DCSH-morphism from C to C' is a map of chain algebras $\Omega(C) \to \Omega(C')$. In a slight abuse of terminology, we say that a chain map between chain coalgebras $f\colon C \to C'$ is a DCSH-*map* if there is a morphism in $\mathbf{DCSH}(C, C')$ of which f is the linear part. In other words, for $c \in \overline{C}$, there is a map of chain algebras $g\colon \Omega(C) \to \Omega(C')$ such that

$$g([c]) = [f(c)] + \text{higher-order terms}.$$

In a further abuse of notation, we sometimes write $\widetilde{\Omega}f\colon \Omega C \to \Omega C'$ to indicate one choice of chain algebra map of which f is the linear part.

2.3. Homological perturbation theory

We now recall those elements of homological perturbation theory that we need for this article.

Definition 2.3. Suppose that $\nabla\colon (X, \partial) \to (Y, d)$ and $f\colon (Y, d) \to (X, \partial)$ are morphisms of chain complexes. If $f\nabla = 1_X$ and there exists a chain homotopy $\varphi\colon (Y, d) \to (Y, d)$ such that

1. $d\varphi + \varphi d = \nabla f - 1_Y$,
2. $\varphi\nabla = 0$,
3. $f\varphi = 0$, and
4. $\varphi^2 = 0$,

then $(X, d) \underset{f}{\overset{\nabla}{\rightleftarrows}} (Y, d) \circlearrowleft \varphi$ is a *strong deformation retract (SDR) of chain complexes.*

The following notion was introduced by Gugenheim and Munkholm [**8**].

Definition 2.4. An SDR $(X,d) \underset{f}{\overset{\nabla}{\rightleftarrows}} (Y,d) \circlearrowleft \varphi$ is called *Eilenberg-Zilber (E-Z) data* if (Y,d,Δ_Y) and (X,d,Δ_X) are chain coalgebras and ∇ is a morphism of coalgebras.

Observe that in this case

$$(d \otimes 1_X + 1_X \otimes d)\big((f \otimes f)\Delta_Y \varphi\big) + \big((f \otimes f)\Delta_Y \varphi\big)d = \Delta_X f - (f \otimes f)\Delta_Y;$$

i.e., f is a map of coalgebras up to chain homotopy. In fact, as the following theorem of Gugenheim and Munkholm shows, f is usually a DCSH map.

Theorem 2.5 ([**8**, Theorem 4.1])**.** *Let $(X,d) \underset{f}{\overset{\nabla}{\rightleftarrows}} (Y,d) \circlearrowleft \varphi$ be E-Z data such that X is simply connected and Y is connected. Let $F_0 = 0$ and F_1 be the composite*

$$Y \xrightarrow{f} X \to X_+ \xrightarrow{s^{-1}} s^{-1}X_+,$$

and construct inductively $F_k \colon Y \to T^k(s^{-1}X_+)$ by the formula

$$F_k = -\sum_{i+j=k} (F_i \otimes F_j)\Delta_Y \varphi.$$

Then $F = \prod_{k \geqslant 1} F_k = \bigoplus_{k \geqslant 1} F_k$ is a twisting cochain. Similarly, let Φ_0 be the natural augmentation on Y and Φ_1 be the composite

$$Y \xrightarrow{\varphi} \underbrace{Y \to Y_+ \xrightarrow{s^{-1}} s^{-1}Y_+}_{\rho_Y},$$

and construct inductively $\Phi_k \colon Y \to T^k(s^{-1}Y_+)$ by the formula

$$\Phi_k = \left(\Phi_{k-1} \otimes \rho_Y + \sum_{i+j=k} \Big(\Omega(\nabla)F_j \otimes \Phi_i\Big)\right)\Delta_Y \varphi.$$

Then $\Phi = \prod_{k \geqslant 0} \Phi_k = \bigoplus_{k \geqslant 0} \Phi_k$ is a twisting (homotopy) cochain. Moreover,

$$\Omega(X,d) \underset{\widetilde{\Omega} f}{\overset{\Omega\nabla}{\rightleftarrows}} \Omega(Y,d) \circlearrowleft \widetilde{\Omega}\varphi$$

is an SDR, where $\Omega\nabla$ is the algebra morphism determined by the coalgebra morphism ∇, $\widetilde{\Omega} f$ is the algebra morphism determined by the twisting cochain F, and $\widetilde{\Omega}\varphi$ is the derivation homotopy determined by the twisting cochain Φ.

Remark 2.6. Since X is simply connected, the key, as Gugenheim and Munkholm noted, is that on elements of degree n, $F_k = 0$ when $k \geqslant n$. Thus F is a well-defined map into $\bigoplus_{k \geqslant 1} T^k(s^{-1}X_+)$ and not merely into $\prod_{k \geqslant 1} T^k(s^{-1}X_+)$. That is the only place where the hypothesis X is used. Thus that condition can be removed, and hence Theorem 2.5 still applies if for some other reason (geometric, algebraic, ...) F turns out to be locally nilpotent.

2.4. Relating simplicial and differential structures

Let K and L be simplicial sets. The natural *Alexander-Whitney map* is the morphism of chain complexes $f_{K,L}\colon C(K\times L)\to C(K)\otimes C(L)$ given on nondegenerate $x\times y\in(K\times L)_n$ by

$$f_{K,L}(x\times y)=\sum_{i=0}^{n}\tilde{\partial}^{n-i}x\otimes\partial_0^i y,$$

where $\tilde{\partial}^{n-i}:=\partial_{i+1}\dots\partial_n$. The simplicial diagonal $K\xrightarrow{\Lambda}K\times K$ together with the Alexander-Whitney map endow the normalized chains on K with a natural coproduct $\Delta_K=f_{K,K}\circ C(\Lambda)$ ([**5, 6, 14**]). If $x\in K_n$ is nondegenerate, then

$$\Delta_K(x)=\sum_{i=0}^{n}\tilde{\partial}^{n-i}x\otimes\partial_0^i x.$$

It is well known that the Alexander-Whitney map is a chain equivalence [**6**]. A natural chain homotopy inverse is the *Eilenberg-Zilber map* $\nabla_{K,L}\colon C(K)\otimes C(L)\to C(K\times L)$ defined by

$$\nabla_{K,L}(x\otimes y)=\sum_{(\mu,\nu)}(-1)^{\epsilon(\mu)}s_{\nu_q}\dots s_{\nu_1}x\ \times\ s_{\mu_p}\dots s_{\mu_1}y$$

for nondegenerate $x\in K_p$ and $y\in L_q$, where the sum is taken over all (p,q)-shuffles (μ,ν) and $\epsilon(\mu)=\sum_{i=1}^{p}[\mu_i-(i-1)]$ is the signature of the corresponding permutation. Recall that a (p,q)-shuffle is a permutation π of $\{0,\dots,p+q-1\}$ such that $\pi(i)<\pi(j)$ if $0\leqslant i<j\leqslant p-1$ or $p\leqslant i<j\leqslant p+q-1$. We use Eilenberg and MacLane's convention where we let $\mu_i=\pi(i-1)$, $1\leqslant i\leqslant p$, and $\nu_j=\pi(j+p-1)$, $1\leqslant j\leqslant q$. Clearly π is determined by μ and ν, and we let $\pi=(\mu,\nu)$. Notice that, in contrast to the Alexander-Whitney map, $\nabla_{K,L}$ is a coalgebra map.

There is a natural chain homotopy, $\varphi_{K,L}$ such that

$$C(K)\otimes C(L)\underset{f_{K,L}}{\overset{\nabla_{K,L}}{\rightleftharpoons}}C(K\times L)\circlearrowleft\varphi_{K,L}\tag{3}$$

constitutes E-Z data. If K and L are 1-reduced, then Theorem 2.5 implies that $f_{K,L}$ is naturally a DCSH map. In [**6**] Eilenberg and MacLane give a recursive formula for $\varphi_{K,L}$ that we reproduce here. Set $\varphi(C_0(K\times L))=0$. For $q>0$ we let

$$\varphi(x\times y)=-\varphi'(x\times y)\ +\ (\nabla_{K,L}\circ f_{K,L})'s_0(x\times y),$$

for $x\times y\in(K\times L)_q$ nondegenerate.

2.5. The cobar diagonal

We now describe the coproduct structure on $\Omega(C(K))$ for a 1-reduced simplicial set K. It was defined in [**11**] and shown there to be identical to the Baues coproduct. Recall that the Alexander-Whitney map $f_{K,L}$ is naturally a DCSH map. We are thus led to define

$$\psi_K=q\circ\tilde{\Omega}f_{K,K}\circ\Omega C(\Lambda)$$

where q is the Milgram equivalence (see (2) p.214). The cobar diagonal, ψ_K, is strictly coassociative and cocommutative up to derivation homotopy. Furthermore,

the Szczarba quasi-isomorphism of chain algebras $\theta_K\colon \Omega C(K) \to C(GK)$ is a DCSH map, in a way compatible with the algebra structures.

3. A chain model for the simplicial James map

Let K be an arbitrary pointed simplical set. Recall that EK is reduced, so the chain coalgebra $C(EK)$ is connected and $\Omega C(EK)$ is defined. Define a map

$$\alpha\colon C(K) \to \Omega C(EK) \tag{4}$$

by $\alpha(k_0) = [\,]$, $\alpha(y - k_0) = [(1, y)]$ if $y \in K_0 \backslash \{k_0\}$, and $\alpha(x) = [(1, x)]$ if $x \in K_{\geqslant 1}$.

We proceed to show that α is a chain map. As in the geometric case, the coproduct on $C(EK)$ is as simple as possible.

Lemma 3.1. *The diagonal on $C(EK)$ is primitive, i.e., $\overline{\Delta}_{EK} = 0$.*

Proof. The face map ∂_0 applied to any element $(1, x) \in E_{n+1}(K)$ is the basepoint b_n, and b_n is degenerate unless $n = 0$. Therefore

$$\begin{aligned}\Delta(1, x) &= \sum_{j=0}^{n+1} \partial_{j+1} \cdots \partial_{n+1}(1, x) \otimes \partial_0^j(1, x)\\ &= b_0 \otimes (1, x) + (1, x) \otimes b_0\end{aligned}$$

as desired. □

Corollary 3.2. *The differential in $\Omega C(EK)$ is linear, and*

$$d_\Omega[(1, x)] = \sum_{j=0}^{n} (-1)^j [(1, \partial_j x)],$$

for $x \in K_n$.

Proof. The quadratic part of the differential vanishes by Lemma 3.1. A straightforward computation gives the result. □

Theorem 3.3. *The map α is a chain map.*

Proof. We only check for $x \in K_1$ since it is obvious in other degrees. Recall that $(1, k_0) = b_1$ is degenerate in EK. We have

$$\begin{aligned}\alpha(\partial x) &= \alpha(\partial_0 x - \partial_1 x)\\ &= \alpha(\partial_0 x - k_0) - (\partial_1 x - k_0)\\ &= \alpha(\partial_0 x - k_0) - \alpha(\partial_1 x - k_0)\\ &= \begin{cases} [(1, \partial_0 x)] - [(1, \partial_1 x)] & \text{if } \partial_0 x \neq k_0 \text{ and } \partial_1 x \neq k_0\\ [(1, \partial_0 x)] & \text{if } \partial_0 x \neq k_0 \text{ and } \partial_1 x = k_0\\ 0 & \text{if } \partial_0 x = k_0 \text{ and } \partial_1 x = k_0\end{cases}\\ &= d_\Omega \alpha(x).\end{aligned}$$

□

The chain map α is in fact a chain model of the simplicial James map $K \to G^+EK$. More precisely, the diagram below commutes

$$\begin{tikzcd} & \Omega C(EK) \arrow[d, "\gamma"] \\ C(K) \arrow[ur, "\alpha"] \arrow[r, "C(\eta_K)"'] & C(G^+EK), \end{tikzcd} \tag{5}$$

where the chain algebra map γ is induced by the twisting cochain $(1,x) \mapsto \tau(1,x)$. Hence diagram (5) induces a commuting chain algebra triangle

$$\begin{tikzcd} & \Omega C(EK) \arrow[d, "\gamma"] \\ T\widetilde{C}(K) \arrow[ur, "\widehat{\alpha}"] \arrow[r, "\widehat{C}(\eta_K)"'] & C(G^+EK), \end{tikzcd} \tag{6}$$

where $\widehat{\alpha}$ and $\widehat{C}(\eta_K)$ are the chain algebra morphisms induced by α and $C(\eta_K)$ respectively. Since $C(\eta_K)$ is a coalgebra morphism, $\widehat{C}(\eta_K)$ is a Hopf algebra morphism. A homological argument shows that $\widehat{C}(\eta_K)$ is a quasi-isomorphism while $\widehat{\alpha}$ is obviously an isomorphism. Hence γ is a quasi-isomorphism.

4. Bott-Samelson, Szczarba and the cobar diagonal

In this section, we extend the definition of the cobar diagonal to the suspension of an arbitrary pointed simplicial set K. Indeed, we show that we have a factorisation

$$\begin{tikzcd} C(EK) \arrow[r, "C(\Lambda)"] \arrow[dr, dotted] & C(EK \times EK) \arrow[r, "\prod_{k\geqslant 1} F_k"] & \prod_{n\geqslant 0} T^n s^{-1}(C(EK)\otimes C(EK))_+ \\ & \Omega(C(EK)\otimes C(EK)) \arrow[r, "q"] \arrow[ur, hook] & \Omega(C(EK))\otimes\Omega(C(EK)), \end{tikzcd}$$

where $\{F_k\}_{k\geqslant 1}$ is the family of higher homotopies associated to the Alexander-Whitney map $f_{EK,EK}$ as given by Theorem 2.5. Moreover, we show that the composite of the dotted arrow together with q, denoted ξ_K, is a twisting cochain. We define the (extended) *cobar diagonal* to be the induced chain algebra map

$$\begin{aligned} \psi_K \colon \Omega(C(EK)) &\longrightarrow \Omega(C(EK))\otimes\Omega(C(EK)) \\ [(1,x)] &\longmapsto \xi_K\big((1,x)\big). \end{aligned}$$

Note that when K is reduced, then $\psi_K = q \circ \tilde{\Omega} f_{K,K} \circ \Omega C(\Lambda)$ as on page 216.

Finally, we prove Theorem 4.9, a chain-level Bott-Samelson theorem with respect to the (extended) cobar diagonal. We then show that the Szczarba equivalence is a strict morphism of chain Hopf algebras for suspensions. In the process we establish a few helpful combinatorial identities involving face maps in simplicial suspensions.

Remark 4.1. From Lemma 3.1 we deduce that Δ_{EK} is a coalgebra map. Thus one can directly obtain a diagonal on $\Omega C(EK)$ as $q \circ \Omega(\Delta_{EK})$. If $\psi_K = q \circ \Omega(\Delta_{EK})$ it would imply, however, that the higher homotopies were trivial. As we show below, it is not true in general: the homotopy F_2 is usually nonzero. Endowed with the "wrong" coproduct, $\Omega(C(EK))$ is indeed a chain Hopf algebra, but it is not weakly equivalent to $C(G^+EK)$.

4.1. The (extended) cobar diagonal

To show that F_2 is nonzero in general, we need first to study the Eilenberg-MacLane homotopy $\varphi\colon C(EK \times EL) \to C(EK \times EL)$ ([**6**]). Since we will eventually be interested only in the image of $C(\Lambda)$, i.e., simplices of the form $(1,x) \times (1,x)$ for nondegenerate $(n-1)$-simplices x, we will assume that $n \geqslant 1$ and concentrate on simplices of the form $(1,x) \times (1,y) \in (EK \times EL)_n$ for nondegenerate $x \in K_{n-1}$ and $y \in L_{n-1}$. Recall that since EK and EL are reduced, $C(EK \times EL)$ is a connected coalgebra and so the reduced diagonal is defined.

Proposition 4.2. *On simplices of the form $(1,x) \times (1,y)$, we have*

$$\bar{\Delta}_{EK\times EL} \circ \varphi = \bar{\Delta}_{EK\times EL} \circ (\nabla_{EK,EL} \circ f_{EK,EL})' \circ s_0.$$

Proof. Recall that $\overline{\Delta} = \Big(\sum_{j=1}^{n} \tilde{\partial}^{n+1-j} \otimes \partial_0^j\Big) \circ C(\Lambda_{EK\times EL})$ on elements of degree $n+1$, and that $\varphi_n = -\varphi'_{n-1} + (\nabla f)' s_0$. The map φ'_{n-1} is the sum of simplicial operators whose component face and degeneracy maps all have index at least 1. Thus, as noted by Eilenberg and MacLane, $\partial_0 \varphi'_{n-1} = \varphi_{n-1}\partial_0$. On one hand if $(1,x) \times (1,y) \in C_1(EK \times EL)$, then $\varphi_0\partial_0((1,x) \times (1,y)) = \varphi_0(b_0 \times b_0) = 0$ by construction. On the other hand if $(1,x) \times (1,y) \in C_{k>1}(EK \times EL)$, then $\partial_0((1,x) \times (1,y)) = b_{k-1} \times b_{k-1}$ which is degenerate; whence $\bar{\Delta}\varphi'_{n-1}$ vanishes. □

Thus we only need to consider the differential operator $(\nabla f)' s_0$. In dimension $n > 1$, $(\nabla f)' s_0$ is the sum of terms of the form

$$\begin{array}{cl} s_0^{n+1}\partial_1 \cdots \partial_n \times s_0 & \text{when } j = 0 \\ s_{\nu_{n-j}+1} \cdots s_{\nu_1+1} s_0 \tilde{\partial}^{n-j} \times s_{\mu_j+1} \cdots s_{\mu_1+1} \partial_1^{j-1} & \text{when } 0 < j < n \\ s_0 \times s_n \cdots s_1 \partial_1^{n-1} & \text{when } j = n, \end{array}$$

where (μ,ν) is a $(j, n-j)$-shuffle, and j is the running index in the definition of the Alexander-Whitney map ($0 \leqslant j \leqslant n$). In dimension $n = 1$, we have

$$(\nabla f)' s_0 = s_0^2 \partial_1 \times s_0 + s_0 \times s_1.$$

We notice that when $n \geqslant 1$ the term corresponding to $j = 0$ is degenerate and so vanishes. Thus from now on we will consider $j \geqslant 1$.

To simplify notation we set $s_{\mu+1} = s_{\mu_j+1} \cdots s_{\mu_1+1}$, and analogously for s_ν.

Lemma 4.3. *Let k be an integer such that $1 \leqslant k \leqslant n$, and let (μ,ν) be a $(j, n-j)$-shuffle different from the one characterized by $\nu_{n-j} = n-j-1$. Then*

$$\partial_0^k(s_{\nu+1} s_0 \tilde{\partial}^{n-j} \times s_{\mu+1} \partial_1^{j-1}) \tag{7}$$

is degenerate on simplices $(1,x) \times (1,y) \in (EK \times EL)_n$.

Proof. We use the simplicial identities to move one ∂_0 past the second factor $s_{\mu+1}\partial_1^{j-1}$; i.e.,

$$\partial_0 s_{\mu+1}\partial_1^{j-1} = s_\mu \partial_0 \partial_1^{j-1} = s_\mu \partial_0^j.$$

Thus the second factor is the basepoint b_{n-k+1} and is degenerate since $n-k+1 \neq 0$ by assumption. It therefore suffices to show that the first factor in (7) is degenerate. Because of the simplicial identities we have

$$\partial_0^k s_{\nu+1} s_0 \tilde{\partial}^{n-j} = \partial_0^{k-1} s_\nu \partial_0 s_0 \tilde{\partial}^{n-j} = \partial_0^{k-1} s_\nu \tilde{\partial}^{n-j}. \tag{8}$$

Notice that if $k=1$, then (8) is degenerate. Let $k \geqslant 2$, so that $n \geqslant 2$. There exists an integer $0 \leqslant l < n-j$ such that if $l=0$, then $\nu_1 \geqslant 1$, or if $l>0$, then $\nu_l = l-1$ and $\nu_{l+1} > l$, so that (8) becomes

$$\begin{aligned}\partial_0^{k-1} s_\nu \tilde{\partial}^{n-j} &= \partial_0^{k-1} s_{\nu_{n-j}} \cdots s_{\nu_{l+1}} s_0^l \tilde{\partial}^{n-j} \\ &= \begin{cases} \partial_0^{k-l-1} s_{\nu_{n-j}-l} \cdots s_{\nu_{l+1}-l} \tilde{\partial}^{n-j} & \text{if} \quad l < k-1 \\ s_{\nu_{n-j}-k+1} \cdots s_{\nu_{l+1}-k+1} s_0^{l-k+1} \tilde{\partial}^{n-j} & \text{if} \quad l \geqslant k-1. \end{cases}\end{aligned}$$

If $l < k-1$, then at least one ∂_0 can slide through without interference to the right of the expression; i.e.,

$$\partial_0^{k-l-1} s_{\nu_{n-j}-l} \cdots s_{\nu_{l+1}-l} \tilde{\partial}^{n-j} = \partial_0^{k-l-2} s_{\nu_{n-j}-l-1} \cdots s_{\nu_{l+1}-l-1} \tilde{\partial}^{n-j} \partial_0.$$

Applying this operator to $(1,x)$ we obtain the basepoint b_{n+1-k}, which is degenerate since $k \leqslant n$. If $l \geqslant k-1$, then the expression is clearly degenerate since $l < n-j$. □

Using Lemma 4.3, we establish the following equality.

Proposition 4.4. *On simplices of the form* $(1,x)\times(1,y)$*, we have*

$$\bar{\Delta}\varphi\big((1,x)\times(1,y)\big) = \sum_{j=1}^{n} (-1)^{j(n-1)} \left(b_{n-j+1} \times \partial_1^{j-1}(1,y)\right) \otimes \left(\tilde{\partial}^{n-j}(1,x) \times b_j\right).$$

Proof. Notice that the degree of the image under φ of $(1,x)\times(1,y)$ is $n+1$. Thus

$$\overline{\Delta} = \left(\sum_{k=1}^{n} \tilde{\partial}^{n+1-k} \otimes \partial_0^k\right) \circ C(\Lambda_{EK\times EL}), \tag{9}$$

where $\tilde{\partial}^{n+1-k} = \partial_{k+1}\ldots\partial_{n+1}$. By Lemma 4.3 we only need to consider the cases where the $(j,n-j)$-shuffle (μ,ν) in (7) is characterized by $\nu_{n-j} = n-j-1$. Notice that the signature of that shuffle is $\epsilon(\mu) \equiv j(n-1) \bmod 2$. Thus on the right side of the tensor in (9), we have

$$\begin{aligned}&\partial_0^k\big(s_{\nu+1}s_0\tilde{\partial}^{n-j}(1,x) \times s_{\mu+1}\partial_1^{j-1}(1,y)\big) \\ &\qquad = \partial_0^{k-1} s_\nu \tilde{\partial}^{n-j}(1,x) \times \partial_0^{k-1} s_\mu \partial_0^j(1,y) \\ &\qquad = \partial_0^{k-1} s_{n-j-1}\ldots s_0 \tilde{\partial}^{n-j}(1,x) \times b_{n-k+1} \\ &\qquad = \partial_0^{k-1} (s_0)^{n-j} \tilde{\partial}^{n-j}(1,x) \times b_{n-k+1}\end{aligned}$$

which equals $\tilde{\partial}^{n-j}(1,x)\times b_j$, if $k-1 \neq n-j$, or else, it is degenerate. On the left

side of the tensor in (9), assuming that $k-1=n-j$ and $\nu_{n-j}=n-j-1$, we have $\mu_1=n-j,\dots,\mu_j=n-1$, and so $\tilde{\partial}^{n+1-k}s_{\mu+1}=id$. Hence

$$\begin{aligned}\tilde{\partial}^{n+1-k}\big(s_{\nu+1}s_0\tilde{\partial}^{n-j}(1,x)\times s_{\mu+1}\partial_1^{j-1}(1,y)\big)&\\ &= \partial_{n-j+2}\dots\partial_{n+1}\big(s_{\nu+1}s_0\tilde{\partial}^{n-j}(1,x)\times s_{\mu+1}\partial_1^{j-1}(1,y)\big)\\ &= \partial_{n-j+2}\dots\partial_{n+1}\big(s_0^{n-j+1}\tilde{\partial}^{n-j}(1,x)\times s_{\mu+1}\partial_1^{j-1}(1,y)\big)\\ &= s_0^{n-j+1}\partial_1\dots\partial_n(1,x)\times\partial_1^{j-1}(1,y)\\ &= b_{n-j+1}\times\partial_1^{j-1}(1,y).\end{aligned}$$

The desired result is obtained by extending linearly. □

We now tackle $\prod_{k\geqslant 1}F_k$ associated to the E-Z data of equation (3) on page 216. The first thing to notice is that the F_k are of degree -1, and hence vanish on elements of degree 0.

Lemma 4.5. *If $m\geqslant 2$, then $F_m(b_n\times(1,y))=F_m((1,x)\times b_n)=0$.*

Proof. The argument of Proposition 4.2 shows that

$$\bar{\Delta}\varphi(b_n\times(1,y))=\bar{\Delta}(\nabla f)'s_0(b_n\times(1,y)).$$

We can go further and show that in fact $\bar{\Delta}\varphi(b_n\times(1,y))=0$. Indeed, as mentioned just after Proposition 4.2, $(\nabla f)'s_0$ is the sum of terms

$$s_{\nu+1}s_0\tilde{\partial}^{n-j}\times s_{\mu+1}\partial_1^{j-1},$$

where $j\geqslant 1$ and (μ,ν) is a $(j,n-j)$-shuffle. Applying such a term to $b_n\times(1,y)$, we obtain

$$s_{\nu+1}s_0\tilde{\partial}^{n-j}b_n\times s_{\mu+1}\partial_1^{j-1}(1,y)=b_{n+1}\times s_{\mu+1}\partial_1^{j-1}(1,y).$$

Since $j\geqslant 1$, the right factor is degenerate. Since the left factor is the basepoint, the whole expression is degenerate. Therefore $\bar{\Delta}\varphi(b_n\times(1,y))=0$, from which it follows that $F_m(b_n\times(1,y))=0$ for $m\geqslant 2$. For $(1,x)\times b_n$ the argument is symmetric. □

Proposition 4.6. $\prod_{k\geqslant 1}F_k=F_1\oplus F_2$ *on simplices of the form* $(1,x)\times(1,y)$. *Moreover,*

$$\begin{aligned}&q\circ(F_1\oplus F_2)\big((1,x)\times(1,y)\big)\\ &\quad=[(1,x)]\otimes[\,]+\left(\sum_{j=1}^{n}[\tilde{\partial}^{n-j}(1,x)]\otimes[\partial_1^{j-1}(1,y)]\right)+[\,]\otimes[(1,y)].\end{aligned}$$

Proof. A direct calculation shows that

$$qF_1((1,x)\times(1,y)) = [(1,x)]\otimes[\,] + [\,]\otimes[(1,y)].$$

Using Proposition 4.4 and the fact that q is a graded algebra morphism, we see that

$$\begin{aligned}
&q\circ F_2((1,x)\times(1,y))\\
&\quad= -\,q\circ F_1\otimes F_1\left(\sum_{j=1}^{n}(-1)^{j(n-1)}\left(b_{n-j+1}\times\partial_1^{j-1}(1,y)\right)\otimes\left(\tilde\partial^{n-j}(1,x)\times b_j\right)\right)\\
&\quad= q\sum_{j=1}^{n}(-1)^{(n-j)+j(n-1)}\left([b_0\otimes\partial_1^{j-1}(1,y)\,|\,\tilde\partial^{n-j}(1,x)\otimes b_0]\right)\\
&\quad= \sum_{j=1}^{n}(-1)^{j(n-j)+j(n-1)}\,[\tilde\partial^{n-j}(1,x)]\otimes[\partial_1^{j-1}(1,y)],
\end{aligned}$$

where $j(n-j)+j(n-1)\equiv j(j+1)\equiv 0 \bmod 2$. It remains to show that $F_m((1,x)\times(1,y)) = 0$ if $m\geqslant 3$. But, by Proposition 4.4,

$$\begin{aligned}
F_m((1,x)\times(1,y)) &= -\sum_{i+j=m}(F_i\otimes F_j)\bar\Delta\varphi((1,x)\times(1,y))\\
&= \sum_{i+j=m}\sum_{k=1}^{n}(-1)^{k(n-1)+(n-k)}\\
&\quad\times F_i(b_{n-k+1}\times\partial_1^{k-1}(1,y))\otimes F_j(\tilde\partial^{n-k}(1,x)\times b_k).
\end{aligned}$$

If $m\geqslant 3$, then either $i\geqslant 2$ or $j\geqslant 2$; hence Lemma 4.5 implies that the above expression vanishes. □

Corollary 4.7. *The map $\xi_K = q\circ\big(F_2\oplus F_1\big)\circ C(\Lambda)$ is a twisting cochain.*

Proof. We need to show that $d\xi_K+\xi_K d = \mu(\xi_K\otimes\xi_K)\Delta$. By Lemma 3.1 and the fact that $\xi_K = 0$ on dimension 0 elements, we have that $\mu(\xi_K\otimes\xi_K)\Delta = 0$. Thus we need to show that ξ_K is a chain map of degree -1. But, by Proposition 4.6, the same proof which shows that the Alexander-Whitney map $f_{K,L}$ is a chain map gives the desired result. □

Hence the (extended) cobar diagonal for an arbitrary pointed simplicial set K is given by

$$\psi_{EK}\big([(1,x)]\big) = [(1,x)]\otimes[\,] + \left(\sum_{j=1}^{n}[(1,\tilde\partial^{n-j}x)]\otimes[(1,\partial_0^{j-1}x)]\right) + [\,]\otimes[(1,x)],$$

on nondegenerate simplices $(1,x)\in EK_n$.

The next corollary follows immediately from this formula.

Corollary 4.8. *For any pointed simplicial set K the map $\alpha : C(K)\to\Omega C(EK)$ is a coalgebra morphism with respect to the (extended) cobar diagonal.*

Proof. We first check that α is augmentation preserving. Let ϵ_K and ϵ_{EK} denote the augmentations on $C(K)$ and $C(EK)$ respectively. The augmentation on $\Omega(C(EK))$ is $\Omega(\epsilon_{EK})$ (since ϵ_{EK} is a coalgebra map). On one hand, we have $\Omega(\epsilon_{EK})(\alpha(k_0)) = \Omega(\epsilon_{EK})([\,]) = 1 = \epsilon_K(k_0)$. On the other hand, for $y \in K_0 \backslash \{k_0\}$, $\Omega(\epsilon_{EK})(\alpha(y - k_0)) = \Omega(\epsilon_{EK})\big([(1,y)]\big) = 0 = \epsilon_K(y - k_o)$.

Secondly, we check that $\alpha \otimes \alpha \circ \Delta_K = \psi_{EK} \circ \alpha$. Let $x \in C_{n\geqslant 1}(K)$ be a nondegenerate simplex. Then

$$\begin{aligned}
&\alpha \otimes \alpha \circ \Delta_K(x) \\
&\quad = \alpha \otimes \alpha\Big(\tilde{\partial}^n x \otimes x + \left(\sum_{j=1}^{n-1} \tilde{\partial}^{n-j} x \otimes \partial_0^j x\right) + x \otimes \partial_0^n x\Big) \\
&\quad = \alpha \otimes \alpha\Big(k_0 \otimes x + (\tilde{\partial}^n x - k_0) \otimes x \\
&\qquad + \left(\sum_{j=1}^{n-1} \tilde{\partial}^{n-j} x \otimes \partial_0^j x\right) + x \otimes (\partial_0^n x - k_0) + x \otimes k_0\Big) \\
&\quad = [\,] \otimes [(1,x)] + \alpha(\tilde{\partial}^n x - k_0) \otimes [(1,x)] \\
&\qquad + \left(\sum_{j=1}^{n-1} [(1, \tilde{\partial}^{n-j} x)] \otimes [(1, \partial_0^j x)]\right) + [(1,x)] \otimes \alpha(\partial_0^n x - k_0) + [(1,x)] \otimes [\,].
\end{aligned}$$

As in the proof of Theorem 3.3, there are four cases to consider. We will compute the case when $\tilde{\partial}^n x \neq k_0$ and $\partial_0^n x \neq k_0$, leaving the three other cases to the reader (one should recall that $(1, k_0)$ is degenerate). In our case, we have

$$\alpha(\tilde{\partial}^n x - k_0) = [(1, \tilde{\partial}^n x) \quad \text{and} \quad \alpha(\partial_0^n x - k_0) = [(1, \partial_0^n x)].$$

Now by inspection we have $\alpha \otimes \alpha \circ \Delta_K(x) = \psi_{EK}(\alpha(x))$.

Finally, let $y \in K_0$. By construction, we have $\psi_{EK}([\,]) = [\,] \otimes [\,]$. Hence, clearly, $\psi_{EK}(\alpha(k_0)) = (\alpha \otimes \alpha)\Delta_K(k_0)$. Let $y \in K_0 \backslash \{k_0\}$. An easy computation shows that

$$\begin{aligned}
\psi_{EK}(\alpha(y - k_0)) &= \psi_{EK}\big([(1,y)]\big) \\
&= [\,] \otimes [(1,y)] + [(1,y)] \otimes [(1,y)] + [(1,y)] \otimes [\,],
\end{aligned}$$

while

$$\begin{aligned}
&\alpha \otimes \alpha \circ \Delta_K(y - k_0) \\
&\quad = \alpha \otimes \alpha(y \otimes y - k_0 \otimes k_0) \\
&\quad = \alpha \otimes \alpha\Big(\big((y - k_0) + k_0\big) \otimes \big((y - k_0) + k_0\big) - k_0 \otimes k_0\Big) \\
&\quad = \alpha \otimes \alpha\Big((y - k_0) \otimes (y - k_0) + k_0 \otimes (y - k_0) + (y - k_0) \otimes k_0\Big) \\
&\quad = [(1,y)] \otimes [(1,y)] + [\,] \otimes [(1,y)] + [(1,y)] \otimes [\,].
\end{aligned}$$

Hence we have the result. □

Corollary 4.8 directly implies the chain-level Bott-Samelson theorem.

Theorem 4.9. *Let K be any pointed simplicial set. There is a natural isomorphism of chain Hopf algebras*

$$\widehat{\alpha}\colon (T\widetilde{C}(K), \hat{d}, \widehat{\Delta}_K) \to (\Omega C(EK), \psi_{EK})$$

where $\hat{d}$ and $\widehat{\Delta}_K$ denote, respectively, the derivation determined by the differential d of $C(K)$ and the algebra morphism determined by Δ_K.

Therefore two of the three morphisms in (6) are Hopf algebra morphisms.

4.2. The Szczarba twisting cochain

In this section we will assume that K is a reduced pointed simplicial set.

Since $(\Omega C(EK), \psi_{EK})$ has a particularly simple form, it is reasonable to expect that the Szczarba map also behaves better for suspensions. We show below that our expectations are indeed fulfilled: θ_{EK} is a strict map of chain Hopf algebras.

In [**17**], Szczarba defines a natural twisting cochain $t_L\colon C(L) \longrightarrow C(GL)$, inducing a quasi-isomorphism of chain algebras

$$\theta_L\colon \Omega C(L) \longrightarrow C(GL)$$

for any 1-reduced simplicial set L. Explicitly, when $L = EK$, t_{EK} is given by

$$t_{EK}(1,x) = \sum_{i=1}^{(n-1)!} (-1)^{\varepsilon(i,n)} D^n_{0,i}\tau((1,x))^{-1}$$

since $\partial_0(1,x) = b_{n-1}$ for $x \in K_{n-1}$. The $D^n_{0,i}$ are simplicial operators defined inductively as follows.

$$D^1_{0,1} = \mathrm{id}$$

$$D^{n+1}_{0,i+k(n-1)!} = \begin{cases} (D^n_{0,i})' s_0 \partial_k & k > 0 \\ (D^n_{0,i})' & k = 0. \end{cases}$$

The operator $D^n_{0,1}$ is therefore the identity for $n \geqslant 1$. The signature function ϵ is given by

$$\begin{aligned} \epsilon(1,1) &= 0, \\ \epsilon(i + k(n-1)!, n+1) &= \epsilon(i,n) + k + 1 \pmod 2 \\ &\qquad \text{whenever } 1 \leqslant i \leqslant (n-1)!,\ 0 \leqslant k \leqslant n-1; \\ \epsilon(i,n) &= 0 \quad \text{otherwise.} \end{aligned}$$

Lemma 4.10. *$D^n_{0,i}$ begins with a degeneracy for all $n \geqslant 1$, $2 \leqslant i \leqslant (n-1)!$.*

Proof. We proceed by induction on n. If $n = 1$, then the statement is true vacuously. Suppose inductively that $D^n_{0,i}$ begins with a degeneracy if $2 \leqslant i \leqslant (n-1)!$. For $2 \leqslant i \leqslant n!$, consider the operator $D^{n+1}_{0,i}$. Write $i = k(n-1)! + \ell$, where $1 \leqslant \ell \leqslant (n-1)!$.

If $k \geqslant 1$, then

$$D^n_{0,i} = (D^n_{0,\ell})' s_0 \partial_k = s_0 D^n_{0,\ell} \partial_k$$

since $D^n_{0,\ell}$ contains no ∂_0 ([**17**] Lemmas 1.2 and 3.1).

If $k = 0$, then $2 \leqslant i \leqslant (n-1)!$, so

$$D^{n+1}_{0,i} = (D^n_{0,i}),'$$

which starts with a degeneracy by the inductive hypothesis. □

An easy induction shows that $\epsilon(1,n) = (-1)^{n+1}$. The formula for Szczarba's twisting cochain thus becomes

$$t_{EK}(1,x) = (-1)^{n+1}\big(\tau((1,x))\big)^{-1}.$$

Now a straightforward calculation proves the following theorem.

Theorem 4.11. *The Szczarba equivalence*

$$\theta_{EK} \colon (\Omega C(EK), \psi_{EK}) \xrightarrow{\simeq} C(GEK)$$

is comultiplicative, and therefore a Hopf algebra quasi-isomorphism.

Remark 4.12. We now have two natural chain Hopf algebra quasi-isomorphisms $\Omega C(EK) \to C(GEK)$ when K is reduced. The first is θ_{EK}. The second is the composite

$$\Omega(CEK) \xrightarrow{\gamma} C(G^+EK) \to C(GEK)$$

where the second arrow is induced by the obvious inclusion. Note that these two morphisms are not in general homotopic because the identity and the inversion maps are not homotopic as simplicial maps in GEK.

Appendix A. The Milgram equivalence as a natural SDR

Let A and B be coaugmented chain coalgebras. Recall that the Milgram map $q \colon \Omega(A \otimes B) \to \Omega A \otimes \Omega B$ on page 214 is a quasi-isomorphism if A and B are 1-connected. The purpose of this appendix is to prove the following generalization.

Theorem A.1. *Let A and B be coaugmented chain coalgebras. There exists a strong deformation retract of chain complexes,*

$$\Omega A \otimes \Omega B \underset{q}{\overset{\sigma}{\rightleftarrows}} \Omega(A \otimes B) \circlearrowleft h.$$

In particular, q is a chain homotopy equivalence.

A.1. Notation and definitions

Throughout, we work over an arbitrary commutative ground ring R.

Let $(A, \Delta, \varepsilon, \eta)$ and $(B, \Delta, \varepsilon, \eta)$ be coaugmented chain coalgebras. We adapt Sweedler's notation for the diagonal by writing

$$\Delta a = a \otimes 1 + a_1 \otimes a_2 + 1 \otimes a.$$

Note that we only use the notation for the reduced diagonal, and we suppress the summation sign. In $\Omega(A \otimes B)$, we simplify further by omitting the tensor product symbol and the unit. Thus $[ab] = s^{-1}(a \otimes b)$, $[a] = s^{-1}(a \otimes 1)$, and $[b] = s^{-1}(1 \otimes b)$, so $d[a] = -[da] + (-1)^{\deg a_1}[a_1|a_2]$.

Let $i_A = A \otimes \eta_B \colon A \cong A \otimes R \to A \otimes B$ and $i_B = \eta_A \otimes B \colon B \cong R \otimes B \to A \otimes B$. Let σ be the composite

$$\Omega A \otimes \Omega B \xrightarrow{\Omega i_A \otimes \Omega i_B} \big(\Omega(A \otimes B)\big)^{\otimes 2} \xrightarrow{\text{mult.}} \Omega(A \otimes B).$$

A calculation shows that $q\sigma$ is the identity. We show that q and σ fit into a natural strong deformation retract. As a result, the Milgram map is a homotopy equivalence for any pair of coaugmented coalgebras, not just those that are 1-connected.

A.2. Definition of the homotopy

In this section we construct a natural map of degree $+1$, $h \colon \Omega(A \otimes B) \to \Omega(A \otimes B)$. In the next section we show that $h \colon \sigma q \simeq 1$.

Let $w \in \Omega(A \otimes B)$. Write $w = [\omega|\beta]$, where ω is a word that does not end in an element of B and β is possibly empty. Define $\sharp w$ to be the total number of letters in ω from B.

Define $h[\,] = 0$. If w is not the empty word but ends in ab, then set $h(w) = 0$.

Consider $w = [b|a_1|\cdots|a_m]$. To define $h(w)$, we take the iterated diagonal of b and distribute the factors among the a_i's. We make an equivalent definition that is easier to work with inductively. Suppose $\bar{\Delta} b = b_1 \otimes b_2$ (again, summation is understood). Define $h[b|a] = -(-1)^{(\deg a+1)\deg b}[ab]$. Suppose inductively that we have defined $h[b|\alpha]$ for any $b \in B$ and any $\alpha = [a_1|\cdots|a_{m-1}]$. Define

$$\begin{aligned} h[b|\alpha|a] = &-(-1)^{\deg b(\deg \alpha + \deg a + 1)}[\alpha|ab] + (h[b|\alpha])[a] \\ &- (-1)^{\deg b_2(\deg \alpha + \deg a + 1)}(h[b_1|\alpha])[ab_2]. \end{aligned}$$

Example A.2. A calculation of $h[b|a|a'|a'']$, signs suppressed:

$$\begin{aligned} h[b|a|a'|a''] &= [a|a'|a''b] + (h[b|a|a'])[a''] + (h[b_1|a|a'])[a''b_2] \\ &= [a|a'|a''b] + [a|a'b|a''] + [ab|a'|a''] \\ &\quad + [ab_1|a'b_2|a''] + [a|a'b_1|a''b_2] + [ab_1|a'|a''b_2] \\ &\quad + [ab_{11}|a'b_{12}|a''b_2]. \end{aligned}$$

Suppose that we have constructed h for all words u such that $\sharp u < n$. Let w be a word with $\sharp w = n$. We may suppose that w does not end in ab (else we set $h(w) = 0$). Thus we may write $w = [\zeta|\alpha|\beta]$, where one of α or β may be the empty word, but not both, and ζ is either empty or $\zeta = [\omega|x]$ with $x = b$ or $x = ab$. If $\alpha = [\,]$ then $x = ab$. If $\zeta = [\,]$ or if $x = ab$ then set $h(w) = 0$. Otherwise, set

$$h[\omega|b|\alpha|\beta] = (-1)^{\deg \omega}[\omega](h[b|\alpha])[\beta] + (-1)^{\deg \alpha(\deg b+1)}h([\omega|\alpha|b|\beta]).$$

Since $\sharp[\omega|\alpha|b|\beta] = \sharp w - 1$, $h[\omega|\alpha|b|\beta]$ has been defined.

A.3. Induction

In this section we show that $h \colon \sigma q \simeq 1$. To simplify notation we work modulo 2.

Lemma A.3. $(dh + hd)[b|a_1|\cdots|a_n] = [a_1|\cdots|a_n|b] + [b|a_1|\cdots|a_n]$.

Proof. We proceed by induction on n. When $n = 1$, we have

$$\begin{aligned}(dh + hd)[b|a] &= d[ab] + h[db|a] + h[b|da] + h[b_1|b_2|a] + h[b|a_1|a_2] \\ &= [(da)b] + [a(db)] + [a|b] + [b|a] + [ab_1|b_2] + [b_1|ab_2] \\ &\quad + [a_1b|a_2] + [a_1|a_2b] + [a_1b_1|a_2b_2] + [a(db)] + [(da)b] \\ &\quad + [b_1|ab_2] + [ab_1|b_2] + [a_1|a_2b] + [a_1b|a_2] + [a_1b_1|a_2b_2] \\ &= [a|b] + [b|a].\end{aligned}$$

Suppose true for $n - 1$. Let $\alpha = [a_1|\cdots|a_{n-1}]$. Then

$$\begin{aligned}&(dh + hd)[b|\alpha|a] \\ &= d\big([\alpha|ab] + (h[b|\alpha])[a] + (h[b_1|\alpha])[ab_2]\big) \\ &\quad + h\big([db|\alpha|a] + [b_1|b_2|\alpha|a] + [b](d[\alpha])[a] + [b|\alpha](d[a])\big) \\ &= (d[\alpha])[ab] + [\alpha|d(ab)] + [\alpha|a|b] + [\alpha|b|a] \\ &\quad + [\alpha|a_1b|a_2] + [\alpha|a_1|a_2b] + [\alpha|ab_1|b_2] + [\alpha|b_1|ab_2] \\ &\quad + [\alpha|a_1b_1|a_2b_2] + (dh[b|\alpha])[a] + (h[b|\alpha])d[a] + (dh[b_1|\alpha])[ab_2] \\ &\quad + (h[b_1|\alpha])\big([d(ab_2)] + [a|b_2] + [b_2|a] + [a_1b_2|a_2] \\ &\quad + [a_1|a_2b_2] + [ab_{21}|b_{22}] + [b_{21}|ab_{22}] + [a_1b_{21}|a_2b_{22}]\big) \\ &\quad + [\alpha|a(db)] + (h[db]\alpha)[a] + (h[db_1|\alpha])[ab_2] + (h[b_1|\alpha])[a(db_2)] \\ &\quad + [b_1](h[b_2|\alpha])[a] + [b_1|\alpha|ab_2] + [b_1](h[b_{21}|\alpha])[ab_{22}] + [\alpha|ab_1|b_2] \\ &\quad + (h[b_1|\alpha])[a|b_2] + (h[b_{11}|\alpha])[ab_{12}|b_2] + (d[\alpha])[ab] + (h[b]d[\alpha])[a] \\ &\quad + (h[b_1]d[\alpha])[ab_2] + [\alpha|(da)b] + (h[b|\alpha])[da] + (h[b_1|\alpha])[(da)b_2] \\ &\quad + [\alpha|a_1|a_2b] + [\alpha|a_1b|a_2] + (h[b|\alpha])[a_1|a_2] + (h[b_1|\alpha])[a_1b_2|a_2] \\ &\quad + [\alpha|a_1b_1|a_2b_2] + (h[b_1|\alpha])[a_1|a_2b_2] + (h[b_{11}|\alpha])[a_1b_{12}|a_2b_2] \\ &= [\alpha|a|b] + [b|\alpha|a] \\ &\quad + \big((dh[b|\alpha])[a] + [\alpha|b|a] + (h[b_1|\alpha])[b_2|a] + (h[db|\alpha])[a] \\ &\quad + [b_1](h[b_2|\alpha])[a] + (h[b]d[\alpha])[a] + [b|\alpha|a]\big) \\ &\quad + \big((dh[b_1|\alpha])[ab_2] + [\alpha|b_1|ab_2] + (h[b_1|\alpha])[b_{21}|ab_{22}] + (h[db_1|\alpha])[ab_2] \\ &\quad + [b_1](h[b_{21}|\alpha])[ab_{22}] + (h[b_1]d[\alpha])[ab_2] + [b_1|\alpha|ab_2]\big) \\ &= (\sigma q + 1)[b|\alpha|a]\end{aligned}$$

by the inductive hypothesis and using the fact that $b_{11} \otimes b_{12} \otimes b_2 = b_1 \otimes b_{21} \otimes b_{22}$ by coassociativity. □

Lemma A.4. $h([\omega](d[b|\alpha])[\beta]) = [\omega](hd[b|\alpha])[\beta] + h([\omega](d[\alpha|b])[\beta])$.

Proof.

$$\begin{aligned}
h\big([\omega](d[b|alpha])[\beta]\big) &= h\big([\omega|db|\alpha|\beta] + [\omega|b_1|b_2|\alpha|\beta] + [\omega|b](d[\alpha])[\beta]\big) \\
&= [\omega](h[db|\alpha])[\beta] + h[\omega|\alpha|db|\beta] \\
&\quad [\omega|b_1](h[b_2|\alpha])[\beta] + [\omega](h[b_1|\alpha])[b_2|\beta] \\
&\quad + h[\omega|\alpha|b_1|b_2|\beta] + [\omega](h[b]d[\alpha])[\beta] \\
&\quad + h([\omega](d[\alpha])[b|\beta]) \\
&= [\omega](hd[b|\alpha])[\beta] + h([\omega](d[\alpha|b])[\beta]). \qquad \square
\end{aligned}$$

Lemma A.5. $h[b|a|\alpha] = [ab|\alpha] + [a]h[b|\alpha] + [ab_1]h[b_2|\alpha]$.

Proof. Induct on length of α. First,

$$\begin{aligned}
h[b|a|a'] &= [a|a'b] + [ab|a'] + [ab_1|a'b_2] \\
&= [ab|a'] + [a]h[b|a'] + [ab_1]h[b_2|a'].
\end{aligned}$$

Consider $\alpha = [\alpha'|a']$, and suppose the lemma holds for $[b|a|\alpha']$. Then

$$\begin{aligned}
h[b|a|\alpha] &= h[b|a|\alpha'|a'] \\
&= [a|\alpha'|a'b] + (h[b|a|\alpha'])[a'] + (h[b_1|a|\alpha'])[a'] \\
&= [a|\alpha'|a'b] + [ab|\alpha'|a'] + [a](h[b|\alpha'])[a'] \\
&\quad + [ab_1](h[b_2|\alpha'])[a'] + [ab_1|\alpha'|a'b_2] \\
&\quad + [a](h[b_1|\alpha'])[a'b_2] + [ab_{11}](h[b_{12}|\alpha'])[a'b_2] \\
&= [ab|\alpha] + [a]\big([\alpha'|a'b] + (h[b|\alpha'])[a'] + (h[b_1|\alpha'])[a'b_2]\big) \\
&\quad + [ab_1]\big((h[b_2|\alpha'])[a'] + [\alpha'|a'b_2] + (h[b_{21}|\alpha'])[a'b_2]\big) \\
&= [ab|\alpha] + [a]h[b|\alpha] + [ab_1]h[b_2|\alpha].
\end{aligned}$$

In the above calculation we used the fact that

$$b_1 \otimes b_{21} \otimes b_{22} = b_{11} \otimes b_{12} \otimes b_2$$

by coassociativity. $\square$

Theorem A.6. $dh + hd = \sigma q + 1$.

Proof. Let $w \in \Omega(A \otimes B)$. Write $w = [\zeta|\alpha|\beta]$, where ζ, α, and β are possibly empty words, α is a word from A, β is a word from B, and ζ is a word that does not end in a letter from A. If $\zeta = [\,]$, then $(dh + hd)[\alpha|\beta] = 0 = (\sigma q + 1)[\alpha|\beta]$. If $\zeta \neq [\,]$, then

write $\zeta = [\omega|x]$, with $x = ab$ or $x = b$. If $x = ab$, then

$$\begin{aligned}
(dh + hd)[\omega|ab|\alpha|\beta] &= h\big((d[\omega])[ab|\alpha|\beta] + [\omega]d[ab|\alpha|\beta]\big) \\
&= h\big([\omega|d(ab)|\alpha|\beta] + [\omega|a|b|\alpha|\beta] \\
&\quad + [\omega|b|a|\alpha|\beta] + [\omega|a_1b|a_2|\alpha|\beta] \\
&\quad + [\omega|a_1|a_2b|\alpha|\beta] + [\omega|ab_1|b_2|\alpha|\beta] \\
&\quad + [\omega|b_1|ab_2|\alpha|\beta] + [\omega|a_1b_1|a_2b_2|\alpha|\beta]\big) \\
&= [\omega|a](h[b|\alpha])[\beta] + h[\omega|a|\alpha|b|\beta] \\
&\quad + [\omega|ab|\alpha|\beta] + [\omega|a](h[b|\alpha])[\beta] \\
&\quad + [\omega|ab_1](h[b_2|\alpha])[\beta] + h([\omega|a|\alpha|b|\beta]) \\
&\quad + [\omega|ab_1](h[b_2|\alpha])[\beta] + h[\omega|ab_1|\alpha|b_2|\beta] \\
&= [\omega|ab|\alpha|\beta].
\end{aligned}$$

where we used Lemma A.5.

Suppose that $x = b$. We induct on $\sharp w$. The case $\sharp w = 1$, ω and β empty was treated in Lemma A.3 and the general case of $\sharp w = 1$ is an easy calculation using Lemma A.4.

Suppose that $(dh + hd)(u) = \sigma q(u) + u$ for all words u such that $\sharp u < n$. Let w be a word with $\sharp w = n$. Write $w = [\omega|b|\alpha|\beta]$. Note that the inductive hypothesis applies to $[b|\alpha]$ and to $[\omega|\alpha|b|\beta]$. Thus

$$\begin{aligned}
(dh + hd)(w) &= d\big([w](h[b|\alpha])[\beta]\big) + dh[\omega|\alpha|b|\beta] \\
&\quad + h\big((d[\omega])[b|\alpha|\beta] + [\omega](d[b|\alpha])[\beta] + [\omega|b|\alpha]d[\beta]\big) \\
&= (d[\omega])(h[b|\alpha])[\beta] + [\omega](dh[b|\alpha])[\beta] + [\omega](h[b|\alpha])d[\beta] \\
&\quad + dh[\omega|\alpha|b|\beta] + (d[\omega])(h[b|\alpha])[\beta] + h\big((d[\omega])[\alpha|b|\beta]\big) \\
&\quad + [\omega](hd[b|\alpha])[\beta] + h\big([\omega](d[\alpha|b])[\beta]\big) + [\omega](h[b|\alpha])d[\beta] \\
&\quad + h\big([\omega|\alpha|b]d[\beta]\big) \\
&= [\omega]\big([\alpha|b] + [b|\alpha]\big)[\beta] + \sigma q[\omega|\alpha|b|\beta] + [\omega|\alpha|b|\beta] \\
&= (\sigma q + 1)(w)
\end{aligned}$$

where we have used Lemma A.4 and the fact that $\sigma q[\omega|\alpha|b|\beta] = \sigma q(w)$. □

A.4. Elementary properties

Proposition A.7. *The natural homotopy h satisfies*

1. $qh = 0$,
2. $h\sigma = 0$, *and*
3. $h^2 = 0$.

Proof. Properties (1) and (2) follow from the definitions. Property (3) is proved by an induction on $\sharp w$ and the length of the last block of letters from A. □

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