

# A NEW CONSTRUCTION OF COMPACT 8-MANIFOLDS WITH HOLONOMY $\text{Spin}(7)$

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## 1. Introduction

In Berger's classification [1] of holonomy groups of Riemannian manifolds there are two special cases, the exceptional holonomy groups  $G_2$  in 7 dimensions and  $\text{Spin}(7)$  in 8 dimensions. Bryant [2] and Bryant and Salamon [3] showed that such metrics exist locally, and wrote down explicit, complete metrics with holonomy  $G_2$  and  $\text{Spin}(7)$  on noncompact manifolds.

The first examples of metrics with holonomy  $G_2$  and  $\text{Spin}(7)$  on *compact* 7- and 8-manifolds were constructed by the author in [10], [11], [12]. The survey paper [13] provides a good introduction to these constructions. Here is a brief description of the method used in [10] to construct compact 8-manifolds with holonomy  $\text{Spin}(7)$ , divided into four steps.

- (a) We start with a flat  $\text{Spin}(7)$ -structure  $(\Omega_0, g_0)$  on the 8-torus  $T^8$ , and a finite group  $\Gamma$  of isometries of  $T^8$  preserving  $(\Omega_0, g_0)$ . Then  $T^8/\Gamma$  is an *orbifold*, a singular manifold with only quotient singularities.
- (b) For certain  $\Gamma$  one can resolve the singularities of  $T^8/\Gamma$  in a natural way, using complex geometry. This gives a nonsingular, compact 8-manifold  $M$ , and a projection  $\pi : M \rightarrow T^8/\Gamma$ .

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- (c) We write down a 1-parameter family of Spin(7)-structures  $(\Omega_t, g_t)$  on  $M$  for  $t \in (0, \epsilon)$ , such that  $(\Omega_t, g_t)$  has small torsion when  $t$  is small, and converges to the singular Spin(7)-structure  $\pi^*(\Omega_0, g_0)$  as  $t \rightarrow 0$ .
- (d) Using analysis we prove that for small  $t$ , the Spin(7)-structure  $(\Omega_t, g_t)$  can be deformed to a nearby Spin(7)-structure  $(\tilde{\Omega}_t, \tilde{g}_t)$  on  $M$ , with zero torsion. Then  $\tilde{g}_t$  has holonomy Spin(7).

In this paper we will describe a new method for constructing compact 8-manifolds with holonomy Spin(7), in which one starts not with a torus  $T^8$  but with a *Calabi–Yau 4-orbifold*  $Y$  with isolated singular points  $p_1, \dots, p_k$ . We use algebraic geometry to find a number of suitable complex orbifolds  $Y$ , which in the simplest cases are hypersurfaces in *weighted projective spaces*  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$ .

Then, instead of a finite group  $\Gamma$ , we suppose we have an antiholomorphic, isometric involution  $\sigma : Y \rightarrow Y$ , whose only fixed points are  $p_1, \dots, p_k$ . This involution does not preserve the SU(4)-structure on  $Y$ , but it does preserve the induced Spin(7)-structure. We think of  $\sigma$  as breaking the structure group of  $Y$  from SU(4) down to Spin(7). Define  $Z = Y/\langle\sigma\rangle$ . Then  $Z$  is an orbifold with isolated singular points  $p_1, \dots, p_k$ , and the Calabi–Yau structure on  $Y$  induces a torsion-free Spin(7)-structure on  $Z$ .

If the singularities of  $Z$  are of a suitable kind, we can resolve them to get a compact 8-manifold  $M$  with holonomy Spin(7), as in steps (b)–(d) above. To perform the resolution we need to find *Asymptotically Locally Euclidean Spin(7)-manifolds* corresponding to the singularities of  $Z$ , which are a special class of noncompact Spin(7)-manifolds asymptotic to quotient singularities  $\mathbb{R}^8/G$ .

Our construction then yields new examples of compact 8-manifolds  $M$  with holonomy Spin(7). We calculate the Betti numbers  $b^k(M)$  in each case. They turn out to be rather different to the Betti numbers arising from the previous construction in [10]. In particular, in this new construction the middle Betti number  $b^4$  tends to be rather large, as big as 11 662 in one example, whereas the manifolds of [10] all satisfied  $b^4 \leq 162$ .

Sections 2 and 3 introduce the holonomy group Spin(7) and Calabi–Yau orbifolds, and §4 defines the idea of ALE Spin(7)-manifold, and gives a number of examples. Section 5 then proves our main result, that given a Calabi–Yau 4-orbifold  $Y$  and an antiholomorphic involution

$\sigma : Y \rightarrow Y$  satisfying certain conditions, we can construct a compact 8-manifold  $M$  with holonomy  $\text{Spin}(7)$ .

We explain in §6 how to use the construction in practice, and ways of computing the Betti numbers of the resulting 8-manifolds  $M$ . Sections 7–10 apply the construction to generate new examples of compact 8-manifolds with holonomy  $\text{Spin}(7)$ , and we finish in §11 with a discussion of our results.

The material in this paper will be discussed in the author's book [14], which pays much attention to the exceptional holonomy groups, and also gives a more sophisticated version of the original construction [10] of compact 8-manifolds with holonomy  $\text{Spin}(7)$ .

## 2. Background on the holonomy group $\text{Spin}(7)$

We now collect together some facts we will need about the holonomy group  $\text{Spin}(7)$ , taken from the books by Salamon [18, Ch. 12] and the author [14, Ch. 10]. First we define  $\text{Spin}(7)$  as a subgroup of  $\text{GL}(8, \mathbb{R})$ .

**Definition 2.1.** Let  $\mathbb{R}^8$  have coordinates  $(x_1, \dots, x_8)$ . Write  $\mathbf{dx}_{ijkl}$  for the 4-form  $dx_i \wedge dx_j \wedge dx_k \wedge dx_l$  on  $\mathbb{R}^8$ . Define a 4-form  $\Omega_0$  on  $\mathbb{R}^8$  by

$$(1) \quad \begin{aligned} \Omega_0 = & \mathbf{dx}_{1234} + \mathbf{dx}_{1256} + \mathbf{dx}_{1278} + \mathbf{dx}_{1357} - \mathbf{dx}_{1368} \\ & - \mathbf{dx}_{1458} - \mathbf{dx}_{1467} - \mathbf{dx}_{2358} - \mathbf{dx}_{2367} - \mathbf{dx}_{2457} \\ & + \mathbf{dx}_{2468} + \mathbf{dx}_{3456} + \mathbf{dx}_{3478} + \mathbf{dx}_{5678}. \end{aligned}$$

The subgroup of  $\text{GL}(8, \mathbb{R})$  preserving  $\Omega_0$  is  $\text{Spin}(7)$ . It is a compact, connected, simply-connected, semisimple, 21-dimensional Lie group, which is isomorphic as a Lie group to the double cover of  $\text{SO}(7)$ . This group also preserves the orientation on  $\mathbb{R}^8$  and the Euclidean metric  $g_0 = dx_1^2 + \dots + dx_8^2$  on  $\mathbb{R}^8$ .

Let  $M$  be an 8-manifold. For each  $p \in M$ , define  $\mathcal{A}_p M$  to be the subset of 4-forms  $\Omega \in \Lambda^4 T_p^* M$  for which there exists an isomorphism between  $T_p M$  and  $\mathbb{R}^8$  identifying  $\Omega$  and the 4-form  $\Omega_0$  of (1). Let  $\mathcal{A}M$  be the bundle with fibre  $\mathcal{A}_p M$  at each  $p \in M$ . Then  $\mathcal{A}M$  is a subbundle of  $\Lambda^4 T^* M$  with fibre  $\text{GL}(8, \mathbb{R})/\text{Spin}(7)$ . It is not a vector subbundle, and has codimension 27 in  $\Lambda^4 T^* M$ . We say that a 4-form  $\Omega$  on  $M$  is *admissible* if  $\Omega|_p \in \mathcal{A}_p M$  for each  $p \in M$ .

Now the conventional definition of a  $\text{Spin}(7)$ -structure on an 8-manifold  $M$  (which we will not use) is a principal subbundle  $Q$  of the frame bundle  $\mathcal{F}$  with structure group  $\text{Spin}(7)$ . There is a 1-1 corre-

spondence between Spin(7)-structures  $Q$  in this sense, and admissible 4-forms  $\Omega \in C^\infty(\mathcal{AM})$  on  $M$ . Each Spin(7)-structure  $Q$  induces a 4-form  $\Omega$ , a metric  $g$  and an orientation on  $M$ , corresponding to  $\Omega_0$ ,  $g_0$  and the orientation on  $\mathbb{R}^8$ .

**Definition 2.2.** Let  $M$  be an 8-manifold,  $\Omega$  an admissible 4-form on  $M$ , and  $g$  the associated metric. We shall abuse notation by referring to the pair  $(\Omega, g)$  as a Spin(7)-*structure* on  $M$ . Let  $\nabla$  be the Levi-Civita connection of  $g$ . We call  $\nabla\Omega$  the *torsion* of  $(\Omega, g)$ , and we say that  $(\Omega, g)$  is *torsion-free* if  $\nabla\Omega = 0$ . A triple  $(M, \Omega, g)$  is called a Spin(7)-*manifold* if  $M$  is an 8-manifold, and  $(\Omega, g)$  a torsion-free Spin(7)-structure on  $M$ .

Let  $(\Omega, g)$  be a Spin(7)-structure on an 8-manifold  $M$ . Then  $(\Omega, g)$  is torsion-free if and only if  $d\Omega = 0$ . If  $(\Omega, g)$  is torsion-free then  $g$  is Ricci-flat, and  $M$  is spin and has a constant positive spinor. If  $M$  is compact and  $\text{Hol}(g) = \text{Spin}(7)$  then the positive Dirac operator

$$D_+ : C^\infty(S_+) \rightarrow C^\infty(S_-)$$

has kernel  $\mathbb{R}$  and cokernel 0. Thus  $D_+$  has index 1.

But the index of  $D_+$  is the  $\hat{A}$ -genus  $\hat{A}(M)$ , and is given by

$$(2) \quad 24\hat{A}(M) = -1 + b^1(M) - b^2(M) + b^3(M) + b_+^4(M) - 2b_-^4(M),$$

where  $b^k = b^k(M)$  are the Betti numbers of  $M$ . Thus a compact 8-manifold  $M$  with holonomy Spin(7) must satisfy  $b^3 + b_+^4 = b^2 + b_-^4 + 25$ . As in [10, Th. C], one can use this to show:

**Theorem 2.3.** *Let  $(M, \Omega, g)$  be a compact Spin(7)-manifold. Then  $\text{Hol}(g) = \text{Spin}(7)$  if and only if  $M$  is simply-connected, and  $b^3 + b_+^4 = b^2 + b_-^4 + 25$ .*

The following result [10, Th. D] describes the moduli space of holonomy Spin(7) metrics.

**Theorem 2.4.** *Let  $M$  be a compact 8-manifold admitting metrics with holonomy Spin(7). Then the moduli space of metrics with holonomy Spin(7) on  $M$ , up to diffeomorphisms isotopic to the identity, is a smooth manifold of dimension  $1 + b_-^4(M)$ .*

Our next proposition follows from the ideas of [14, §10.6].

**Proposition 2.5.** *Let  $M$  be an 8-manifold. Then there exists a tubular open neighbourhood  $\mathcal{TM}$  of  $\mathcal{AM}$  in  $\Lambda^4 T^*M$  which is a fibration*

over  $M$ , a smooth map of fibre bundles  $\Theta : \mathcal{T}M \rightarrow \mathcal{A}M$ , and positive constants  $\rho, C$ , such that

- (i) If  $(\Omega, g)$  is a Spin(7)-structure and  $\xi$  a 4-form on  $M$  with  $|\xi - \Omega|_g \leq \rho$ , then  $\xi \in C^\infty(\mathcal{T}M)$ .
- (ii) Suppose  $(\Omega, g)$  is a Spin(7)-structure on  $M$ , and  $\xi$  a 4-form on  $M$  with  $|\xi - \Omega|_g \leq \rho$ . Write  $\Omega' = \Theta(\xi)$ , and let  $(\Omega', g')$  be the associated Spin(7)-structure. Then  $|\xi - \Omega'|_{g'} \leq |\xi - \Omega|_g$ . If  $(\Omega, g)$  is also torsion-free, then  $|\nabla'(\xi - \Omega')|_{g'} \leq C|\nabla(\xi - \Omega)|_g$ .

Here  $\nabla, \nabla'$  are the Levi-Civita connections of  $g$  and  $g'$ , and  $|\cdot|_g, |\cdot|_{g'}$  the norms defined using  $g$  and  $g'$ .

This is an entirely local result, involving calculations at a point, and  $\rho, C$  are independent of  $M$ . The inequality  $|\xi - \Omega'|_{g'} \leq |\xi - \Omega|_g$  in part (ii) should be understood as saying that  $\Omega' = \Theta(\xi)$  is the Spin(7)-form closest to  $\xi$ . That is,  $\mathcal{T}M$  is a small open neighbourhood of  $\mathcal{A}M$  in  $\Lambda^4 T^*M$ , and  $\Theta$  is the projection from  $\mathcal{T}M$  to the nearest point in  $\mathcal{A}M$ . But as we have not fixed a metric on  $M$ , we do not have a way to measure distance in  $\Lambda^4 T^*M$ , and so we use the metrics  $g, g'$  associated to the Spin(7)-forms  $\Omega, \Omega'$  to do this.

Our final result is proved in [10, Th. A & Th. B], and also in [14, Ch. 13].

**Theorem 2.6.** *Let  $\lambda, \mu, \nu$  be positive constants. Then there exist positive constants  $\kappa, K$  such that whenever  $0 < t \leq \kappa$ , the following is true.*

*Let  $M$  be a compact 8-manifold, and  $(\Omega, g)$  a Spin(7)-structure on  $M$ . Suppose that  $\phi$  is a smooth 4-form on  $M$  with  $d\Omega + d\phi = 0$ , and*

- (i)  $\|\phi\|_{L^2} \leq \lambda t^{9/2}$  and  $\|d\phi\|_{L^{10}} \leq \lambda t$ ,
- (ii) the injectivity radius  $\delta(g)$  satisfies  $\delta(g) \geq \mu t$ , and
- (iii) the Riemann curvature  $R(g)$  satisfies  $\|R(g)\|_{C^0} \leq \nu t^{-2}$ .

*Then there exists a smooth, torsion-free Spin(7)-structure  $(\tilde{\Omega}, \tilde{g})$  on  $M$  with  $\|\tilde{\Omega} - \Omega\|_{C^0} \leq K t^{1/2}$ .*

Here is how to interpret this result. As  $\nabla\Omega = 0$  if and only if  $d\Omega = 0$  and  $d\phi + d\Omega = 0$ , the torsion  $\nabla\Omega$  is determined by  $d\phi$ . Thus we can think of  $\phi$  as a *first integral of the torsion* of  $(\Omega, g)$ . So  $\|\phi\|_{L^2}$  and  $\|d\phi\|_{L^{10}}$  are both measures of the torsion of  $(\Omega, g)$ . As  $t$  is small, part (i) of the theorem says that  $(\Omega, g)$  has *small torsion* in a certain sense.

Parts (ii) and (iii) say that the injectivity radius of  $g$  should not be too small, and its curvature not too large. When a metric becomes singular, in general its injectivity radius goes to zero and its curvature becomes infinite. So we can interpret (ii) and (iii) as saying that  $g$  is not too close to being singular.

Thus, the theorem as a whole says that if the torsion of  $(\Omega, g)$  is small enough, and  $g$  is not too singular, then we can deform  $(\Omega, g)$  to a nearby, torsion-free Spin(7)-structure  $(\tilde{\Omega}, \tilde{g})$  on  $M$ . We can hence use Theorem 2.3 to show that if  $M$  is simply-connected and  $b^3 + b_+^4 = b^2 + b_-^4 + 25$ , then  $\tilde{g}$  has holonomy Spin(7).

We prove Theorem 2.6 using analysis: we write the condition that  $(\tilde{\Omega}, \tilde{g})$  be torsion-free as a nonlinear elliptic p.d.e., which can be approximated by a linear elliptic p.d.e. when  $\tilde{\Omega} - \Omega$  is small. Then we use tools such as Sobolev spaces, the Sobolev Embedding Theorem and elliptic regularity to show that this nonlinear elliptic p.d.e. has a smooth solution.

### 3. Calabi–Yau manifolds and orbifolds

We now give a brief introduction to Calabi–Yau geometry, and the relation between Calabi–Yau 4-folds and Spin(7)-manifolds. Some suitable references are Salamon [18, Ch. 8] and the author [14, Ch. 6].

**Definition 3.1.** A *Calabi–Yau manifold* or *orbifold* is a compact Kähler manifold or orbifold  $(Y, J, g)$  of dimension  $m$ , with  $\text{Hol}(g) = \text{SU}(m)$ .

Now Calabi–Yau manifolds and orbifolds are nearly the same thing as Ricci-flat Kähler manifolds and orbifolds, as we see in the next proposition. It follows from elementary properties of holonomy groups and Kähler geometry.

**Proposition 3.2.** *Any Calabi–Yau orbifold  $(Y, J, g)$  is Ricci-flat. Conversely, let  $(Y, J, g)$  be a compact Ricci-flat Kähler orbifold of dimension  $m$ , with singular set  $S$ . Suppose that  $Y \setminus S$  is simply-connected and  $h^{p,0}(Y) = 0$  for  $0 < p < m$ . Then  $\text{Hol}(g) = \text{SU}(m)$ , so  $Y$  is a Calabi–Yau orbifold.*

But using Yau’s proof of the Calabi conjecture [20], one can show that suitable complex orbifolds admit Ricci-flat Kähler metrics.

**Theorem 3.3.** *Let  $(Y, J)$  be a compact complex orbifold admit-*

ting Kähler metrics, with  $c_1(Y) = 0$ . Then there is a unique Ricci-flat Kähler metric in each Kähler class on  $Y$ .

Now the action of  $SU(m)$  on  $\mathbb{C}^m$  fixes the complex  $m$ -form  $dz_1 \wedge \cdots \wedge dz_m$ . It follows by general principles of Riemannian holonomy that any Riemannian manifold or orbifold with holonomy  $SU(m)$  admits a complex  $m$ -form  $\theta$  corresponding to  $dz_1 \wedge \cdots \wedge dz_m$  which is constant under the Levi-Civita connection  $\nabla$ . So we get:

**Proposition 3.4.** *Let  $(Y, J, g)$  be a Calabi–Yau manifold or orbifold of dimension  $m$ , with Kähler form  $\omega$ . Then there exists a constant  $(m, 0)$ -form  $\theta$  on  $Y$ , such that near every point  $p \in Y$  we can choose complex coordinates  $(z_1, \dots, z_m)$  in which*

$$(1) \quad \begin{aligned} g &= |dz_1|^2 + \cdots + |dz_m|^2, \\ \omega &= \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \cdots + dz_m \wedge d\bar{z}_m), \\ \text{and } \theta &= dz_1 \wedge \cdots \wedge dz_m \end{aligned}$$

at  $p$ . This form  $\theta$  is unique up to multiplication by  $e^{i\phi}$  for some  $\phi \in [0, 2\pi)$ .

We call  $\theta$  the *holomorphic volume form* of  $Y$ . Now we restrict our attention to complex dimension 4. Here is a criterion for a complex 4-orbifold to be Calabi–Yau.

**Proposition 3.5.** *Let  $(Y, J)$  be a compact complex 4-orbifold with  $c_1(Y) = 0$ , admitting Kähler metrics. Suppose  $Y \setminus S$  is simply-connected, where  $S$  is the singular set of  $Y$ , and  $h^{2,0}(Y) = 0$ . Then each Kähler class on  $Y$  contains a unique metric  $g$  such that  $(Y, J, g)$  is a Calabi–Yau 4-orbifold.*

*Proof.* As  $\pi_1(Y \setminus S) = 0$  we have  $b^1(Y) = 0$ , so that  $h^{1,0}(Y) = 0$ . Since  $\pi_1(Y \setminus S) = 0$  and  $c_1(Y) = 0$  the canonical bundle  $K_Y$  of  $Y$  is trivial, and this implies that  $h^{p,0}(Y) = h^{4-p,0}(Y)$ . Thus  $h^{3,0}(Y) = 0$ . But we are given that  $h^{2,0}(Y) = 0$ . Hence  $h^{p,0}(Y) = 0$  for  $0 < p < 4$ , and the proposition follows from Proposition 3.2 and Theorem 3.3.    q.e.d.

A Calabi–Yau 4-fold  $Y$  has holonomy  $SU(4)$ , and so carries a natural torsion-free  $SU(4)$ -structure. Since  $SU(4) \subset \text{Spin}(7) \subset \text{SO}(8)$ , this  $SU(4)$ -structure induces a  $\text{Spin}(7)$ -structure on  $Y$ , which is also torsion-free.

**Proposition 3.6.** *Suppose  $(Y, J, g)$  is a Calabi–Yau 4-orbifold, with Kähler form  $\omega$  and holomorphic volume form  $\theta$ . Define a 4-form  $\Omega$  on*

$Y$  by  $\Omega = \frac{1}{2}\omega \wedge \omega + \text{Re}(\theta)$ . Then  $(\Omega, g)$  is a torsion-free Spin(7)-structure on  $Y$ .

*Proof.* Let  $p$  be a point in  $Y$ . Then by Proposition 3.4 we can choose complex coordinates  $(z_1, \dots, z_4)$  near  $p$  such that  $g, \omega$  and  $\theta$  are given by (1) at  $p$ , with  $m = 4$ . Define real coordinates  $(x_1, \dots, x_8)$  on  $Y$  near  $p$  such that  $(z_1, \dots, z_4) = (x_1 + ix_2, x_3 + ix_4, x_5 + ix_6, x_7 + ix_8)$ . Then from (1) we see that  $g, \omega$  and  $\text{Re}(\theta)$  are given at  $p$  by

$$g = dx_1^2 + \dots + dx_8^2, \quad \omega = d\mathbf{x}_{12} + d\mathbf{x}_{34} + d\mathbf{x}_{56} + d\mathbf{x}_{78}$$

and

$$\begin{aligned} \text{Re}(\theta) = & d\mathbf{x}_{1357} - d\mathbf{x}_{1368} - d\mathbf{x}_{1458} - d\mathbf{x}_{1467} \\ & - d\mathbf{x}_{2358} - d\mathbf{x}_{2367} - d\mathbf{x}_{2457} + d\mathbf{x}_{2468}, \end{aligned}$$

where  $d\mathbf{x}_{ij\dots l} = dx_i \wedge dx_j \wedge \dots \wedge dx_l$ .

It follows from this equation that  $\Omega = \frac{1}{2}\omega \wedge \omega + \text{Re}(\theta)$  coincides with the 4-form  $\Omega_0$  defined in (1). As this holds for all  $p \in Y$ , we see that  $(\Omega, g)$  is a Spin(7)-structure on  $Y$ , in the sense of Definition 2.2. Now  $\nabla\omega = \nabla\theta = 0$ , where  $\nabla$  is the Levi-Civita connection of  $g$ , and so  $\nabla\Omega = 0$ . But  $\nabla\Omega$  is the torsion of  $(\Omega, g)$ , so that  $(\Omega, g)$  is torsion-free, as we want. q.e.d.

Thus Calabi-Yau 4-folds are also Spin(7)-manifolds.

#### 4. ALE Spin(7)-manifolds

*ALE manifolds*, or *Asymptotically Locally Euclidean manifolds*, are a class of noncompact Riemannian manifolds with one end modelled asymptotically on a quotient singularity  $\mathbb{R}^n/G$ .

**Definition 4.1.** Let  $G$  be a finite subgroup of  $\text{SO}(n)$  which acts freely on  $\mathbb{R}^n \setminus \{0\}$ . Let  $X$  be a noncompact  $n$ -manifold and  $\pi : X \rightarrow \mathbb{R}^n/G$  a continuous, surjective map, such that  $\pi^{-1}(0)$  is a compact subset of  $X$ , and  $\pi : X \setminus \pi^{-1}(0) \rightarrow (\mathbb{R}^n/G) \setminus \{0\}$  is a diffeomorphism. Then we call  $(X, \pi)$  a *real resolution* of  $\mathbb{R}^n/G$ .

A metric  $g$  on  $X$  is called *Asymptotically Locally Euclidean*, or *ALE*, if

$$\nabla^l(\pi_*(g) - g_0) = O(r^{-n-l}) \quad \text{on } \{x \in \mathbb{R}^n/G : r(x) > R\}, \text{ for all } l \geq 0.$$

Here  $g_0$  is the Euclidean metric on  $\mathbb{R}^n/G$ ,  $r$  is the radius function on  $\mathbb{R}^8/G$ , and  $R > 0$  is a constant. We say that  $(X, g)$  is *asymptotic to  $\mathbb{R}^n/G$* .

One reason ALE manifolds are interesting is that if you have an ALE manifold  $(X, g_X)$  asymptotic to  $\mathbb{R}^n/G$ , and a compact Riemannian orbifold  $(Y, g_Y)$  with isolated singularities modelled on  $\mathbb{R}^n/G$ , then you can glue  $X$  and  $Y$  together to get a nonsingular, compact Riemannian manifold  $(M, g_M)$ . We think of this as resolving the singularities of  $Y$  using  $X$ .

This technique is particularly valuable when  $X$  and  $Y$  both have special holonomy, so that  $\text{Hol}(g_X)$  and  $\text{Hol}(g_Y)$  both lie in some holonomy group  $H \subset \text{SO}(n)$ , as then we can hope to construct a metric  $g_M$  on  $M$  with  $\text{Hol}(g_M) \subseteq H$ . So ALE manifolds  $(X, g_X)$  with  $\text{Hol}(g_X) \subseteq H$  are ingredients in a construction for compact manifolds with holonomy  $H$ .

In fact the only interesting candidates for the holonomy group  $H$  are  $U(m)$  and  $\text{SU}(m)$  for  $m \geq 2$ , and  $\text{Spin}(7)$ . Kronheimer [16], [17] constructed and classified all ALE 4-manifolds with holonomy  $\text{SU}(2)$ . Calabi [4, p. 285] found an explicit family of ALE manifolds with holonomy  $\text{SU}(m)$  asymptotic to  $\mathbb{C}^m/\mathbb{Z}_m$ , and more generally the author [15], [14, Ch. 8] gave existence theorems for ALE manifolds with holonomy  $\text{SU}(m)$ . No examples of ALE 8-manifolds with holonomy  $\text{Spin}(7)$  are known, at the time of writing.

However, we can construct compact 8-manifolds with holonomy  $\text{Spin}(7)$  using only ALE 8-manifolds whose holonomy is a proper subgroup of  $\text{Spin}(7)$  such as  $\text{SU}(4)$  or  $\mathbb{Z}_2 \times \text{SU}(4)$ , and many examples of these can be found using the results of [15]. To discuss these, it is useful to define the idea of *ALE Spin(7)-manifold*, as in [14, Ch. 13].

**Definition 4.2.** Let  $G$  be a finite subgroup of  $\text{Spin}(7)$  which acts freely on  $\mathbb{R}^8 \setminus \{0\}$ , let  $(X, \pi)$  be a real resolution of  $\mathbb{R}^8/G$ , and  $(\Omega, g)$  a torsion-free  $\text{Spin}(7)$ -structure on  $X$ . We call  $(X, \Omega, g)$  an *ALE Spin(7)-manifold* if

$$\nabla^l(\pi_*(\Omega) - \Omega_0) = O(r^{-8-l}) \quad \text{on } \{x \in \mathbb{R}^8/G : r(x) > R\}, \text{ for all } l \geq 0.$$

Here  $\Omega_0$  is the  $\text{Spin}(7)$  4-form on  $\mathbb{R}^8/G$  given in (1),  $r$  the radius function on  $\mathbb{R}^8/G$ , and  $R > 0$  a constant.

In the rest of the section we give some examples of ALE  $\text{Spin}(7)$ -manifolds.

#### 4.1 An example of an ALE Spin(7)-manifold

We define a finite group  $G \subset \text{Spin}(7)$ , such that  $\mathbb{R}^8/G$  has an isolated singularity at 0, and construct two topologically distinct ALE Spin(7)-manifolds  $(X_1, \Omega_1, g_1)$  and  $(X_2, \Omega_2, g_2)$  asymptotic to  $\mathbb{R}^8/G$ . These will be used in §5 as part of a construction of compact 8-manifolds with holonomy Spin(7).

Let  $\mathbb{R}^8$  have coordinates  $(x_1, \dots, x_8)$  and Spin(7)-structure  $(\Omega_0, g_0)$ , as in Definition 2.1. Use the complex coordinates

$$(z_1, z_2, z_3, z_4) = (x_1 + ix_2, x_3 + ix_4, x_5 + ix_6, x_7 + ix_8)$$

to identify  $\mathbb{R}^8$  with  $\mathbb{C}^4$ . Then  $g_0 = |dz_1|^2 + \dots + |dz_4|^2$ , and  $\Omega_0 = \frac{1}{2}\omega_0 \wedge \omega_0 + \text{Re}(\theta_0)$ , where  $\omega_0$  is the Kähler form of  $g_0$  and  $\theta_0 = dz_1 \wedge \dots \wedge dz_4$  the complex volume form on  $\mathbb{C}^4$ .

Define  $\alpha, \beta : \mathbb{C}^4 \rightarrow \mathbb{C}^4$  by

$$(4) \quad \begin{aligned} \alpha &: (z_1, \dots, z_4) \mapsto (iz_1, iz_2, iz_3, iz_4), \\ \beta &: (z_1, \dots, z_4) \mapsto (\bar{z}_2, -\bar{z}_1, \bar{z}_4, -\bar{z}_3). \end{aligned}$$

Then  $\alpha \in \text{SU}(4) \subset \text{Spin}(7)$  and  $\beta \in \text{Spin}(7)$ , and  $\alpha, \beta$  satisfy  $\alpha^4 = \beta^4 = 1$ ,  $\alpha^2 = \beta^2$  and  $\alpha\beta = \beta\alpha^3$ . Let  $G = \langle \alpha, \beta \rangle$ . Then  $G$  is a finite nonabelian subgroup of Spin(7) of order 8 which acts freely on  $\mathbb{R}^8 \setminus \{0\}$ .

Now  $\mathbb{C}^4/\langle \alpha \rangle$  is a complex singularity, as  $\alpha \in \text{SU}(4)$ . Let  $(Y_1, \pi_1)$  be the blow-up of  $\mathbb{C}^4/\langle \alpha \rangle$  at 0. Then  $Y_1$  is the unique crepant resolution of  $\mathbb{C}^4/\langle \alpha \rangle$ . The action of  $\beta$  on  $\mathbb{C}^4/\langle \alpha \rangle$  lifts to a free antiholomorphic map  $\beta : Y_1 \rightarrow Y_1$  with  $\beta^2 = 1$ . Define  $X_1 = Y_1/\langle \beta \rangle$ . Then  $X_1$  is a nonsingular 8-manifold, and the projection  $\pi_1 : Y_1 \rightarrow \mathbb{C}^4/\langle \alpha \rangle$  pushes down to  $\pi_1 : X_1 \rightarrow \mathbb{R}^8/G$ .

By [15, Th. 3.3, Th. 3.4] there exist ALE Kähler metrics  $g_1$  on  $Y_1$  with holonomy SU(4), which were in fact written down explicitly by Calabi [4, p. 285]. Each such  $g_1$  is invariant under the action of  $\beta$  on  $Y_1$ . Let  $\omega_1$  be the Kähler form of  $g_1$ , and  $\theta_1 = \pi_1^*(\theta_0)$  the holomorphic volume form on  $Y_1$ . Then Proposition 3.6 defines a torsion-free Spin(7)-structure  $(\Omega_1, g_1)$  on  $Y_1$  with  $\Omega_1 = \frac{1}{2}\omega_1 \wedge \omega_1 + \text{Re}(\theta_1)$ .

As  $\beta^*(\omega_1) = -\omega_1$  and  $\beta^*(\theta_1) = \theta_1$ , we see that  $\beta$  preserves  $(\Omega_1, g_1)$ . Thus  $(\Omega_1, g_1)$  pushes down to a torsion-free Spin(7)-structure  $(\Omega_1, g_1)$  on  $X_1$ . Then  $(X_1, \Omega_1, g_1)$  is an ALE Spin(7)-manifold asymptotic to  $\mathbb{R}^8/G$ . The Betti numbers of  $X_1$  are  $b^1 = b^2 = b^3 = 0$  and  $b^4 = 1$ , and  $\pi_1(X_1) = \mathbb{Z}_2$ .

## 4.2 A second ALE Spin(7)-manifold asymptotic to $\mathbb{R}^8/G$

Define new complex coordinates  $(w_1, \dots, w_4)$  on  $\mathbb{R}^8$  by

$$(w_1, w_2, w_3, w_4) = (-x_1 + ix_3, x_2 + ix_4, -x_5 + ix_7, x_6 + ix_8).$$

Then  $g_0 = |dw_1|^2 + \dots + |dw_4|^2$  and  $\Omega_0 = \frac{1}{2}\omega'_0 \wedge \omega'_0 + \text{Re}(\theta'_0)$ , where  $\omega'_0$  is the Kähler form of  $g_0$  with respect to the complex structure induced by the  $w_j$ , and  $\theta'_0 = dw_1 \wedge \dots \wedge dw_4$  is the complex volume form on  $\mathbb{C}^4$ .

As the action of  $\text{SU}(4)$  on  $\mathbb{R}^8 = \mathbb{C}^4$  induced by the  $w_j$  preserves  $g_0, \omega'_0$  and  $\theta'_0$ , it preserves  $(\Omega_0, g_0)$ . Thus the action of  $\text{SU}(4)$  on  $\mathbb{R}^8$  compatible with the coordinates  $w_j$  is a subgroup of  $\text{Spin}(7)$ . Note that this is a *different*  $\text{SU}(4)$  subgroup of  $\text{Spin}(7)$  to that considered above, induced by the  $z_j$ . In the coordinates  $w_j$ , we find that  $\alpha, \beta$  act by

$$(5) \quad \begin{aligned} \alpha &: (w_1, \dots, w_4) \mapsto (\bar{w}_2, -\bar{w}_1, \bar{w}_4, -\bar{w}_3), \\ \beta &: (w_1, \dots, w_4) \mapsto (iw_1, iw_2, iw_3, iw_4). \end{aligned}$$

Observe that (4) and (5) are the same, except that the rôles of  $\alpha, \beta$  are reversed. Therefore we can use the ideas above again.

Let  $Y_2$  be the crepant resolution of  $\mathbb{C}^4/\langle\beta\rangle$ . The action of  $\alpha$  on  $\mathbb{C}^4/\langle\beta\rangle$  lifts to a free antiholomorphic involution of  $Y_2$ . Let  $X_2 = Y_2/\langle\alpha\rangle$ . Then  $X_2$  is nonsingular, and as above there exists a torsion-free  $\text{Spin}(7)$ -structure  $(\Omega_2, g_2)$  on  $X_2$ , making  $(X_2, \Omega_2, g_2)$  into an ALE  $\text{Spin}(7)$ -manifold asymptotic to  $\mathbb{R}^8/G$ .

Now  $(X_1, \Omega_1, g_1), (X_2, \Omega_2, g_2)$  are clearly isomorphic as  $\text{Spin}(7)$ -manifolds, but they should be regarded as *topologically distinct* ALE manifolds, because the isomorphism between them acts nontrivially on  $\mathbb{R}^8/G$ . Thus, we have found two topologically distinct ALE  $\text{Spin}(7)$ -manifolds  $(X_1, \Omega_1, g_1), (X_2, \Omega_2, g_2)$  asymptotic to the same singularity  $\mathbb{R}^8/G$ .

## 4.3 Other examples of ALE Spin(7)-manifolds

We can use the ideas above to construct other ALE  $\text{Spin}(7)$ -manifolds too. Here we very briefly describe two infinite families of ALE  $\text{Spin}(7)$ -manifolds  $X_1^n, X_2^n$  for  $n = 1, 3, 5, \dots$ . For simplicity they will not be used in the rest of the paper, although they easily could be.

Identify  $\mathbb{R}^8$  and  $\mathbb{C}^4$  as in §4.1. Let  $n \geq 1$  be an odd integer, and define  $\alpha, \beta, \gamma : \mathbb{C}^4 \rightarrow \mathbb{C}^4$  by

$$\begin{aligned} \alpha &: (z_1, \dots, z_4) \mapsto (e^{2\pi i/n} z_1, e^{-2\pi i/n} z_2, e^{2\pi i/n} z_3, e^{-2\pi i/n} z_4), \\ \beta &: (z_1, \dots, z_4) \mapsto (iz_1, iz_2, iz_3, iz_4), \\ \gamma &: (z_1, \dots, z_4) \mapsto (\bar{z}_2, -\bar{z}_1, \bar{z}_4, -\bar{z}_3). \end{aligned}$$

Then  $\alpha, \beta \in \mathrm{SU}(4)$  and  $\gamma \in \mathrm{Spin}(7)$ , and  $G^n = \langle \alpha, \beta, \gamma \rangle$  is a finite nonabelian subgroup of  $\mathrm{Spin}(7)$  of order  $8n$  which acts freely on  $\mathbb{R}^8 \setminus \{0\}$ . Note that  $G^1$  coincides with the group  $G$  of §4.1-§4.2.

We can construct a family of ALE  $\mathrm{Spin}(7)$ -manifolds asymptotic to  $\mathbb{R}^8/G^n$  as follows. The complex singularity  $\mathbb{C}^4/\langle \alpha, \beta \rangle$  has a unique crepant resolution  $Y_1^n$ , which can be described explicitly using toric geometry. The action of  $\gamma$  on  $\mathbb{C}^4/\langle \alpha, \beta \rangle$  lifts to a free antiholomorphic involution  $\gamma : Y_1^n \rightarrow Y_1^n$ , so that  $X_1^n = Y_1^n/\langle \gamma \rangle$  is a nonsingular 8-manifold with a projection  $\pi_1^n : X_1^n \rightarrow \mathbb{R}^8/G^n$ .

By the results of [15], there exist ALE Kähler metrics  $g_1^n$  on  $Y_1^n$  with holonomy  $\mathrm{SU}(4)$ . We can choose  $g_1^n$  to be  $\gamma$ -invariant, and then the induced  $\mathrm{Spin}(7)$ -structure  $(\Omega_1^n, g_1^n)$  on  $Y_1^n$  is also  $\gamma$ -invariant, and pushes down to  $X_1^n$ , making  $(X_1^n, \Omega_1^n, g_1^n)$  into an ALE  $\mathrm{Spin}(7)$ -manifold asymptotic to  $\mathbb{R}^8/G^n$ . Using the idea of §4.2, we can also construct a second ALE  $\mathrm{Spin}(7)$ -manifold  $(X_2^n, \Omega_2^n, g_2^n)$  asymptotic to  $\mathbb{R}^8/G^n$ .

## 5. Proof of the construction

Starting with a Calabi–Yau 4-orbifold  $Y$  with isolated singularities of a certain kind, and an antiholomorphic involution  $\sigma$  on  $Y$ , we will now construct a compact 8-manifold  $M$  by resolving  $Z = Y/\langle \sigma \rangle$ , and prove that there exist torsion-free  $\mathrm{Spin}(7)$ -structures  $(\tilde{\Omega}, \tilde{g})$  on  $M$ , which have holonomy  $\mathrm{Spin}(7)$  if  $M$  is simply-connected.

### 5.1 A class of $\mathrm{Spin}(7)$ -orbifolds $Z$

We set out below the ingredients in our construction, and the assumptions they must satisfy.

**Condition 5.1.** Let  $(Y, J)$  be a compact complex 4-orbifold with  $c_1(Y) = 0$ , admitting Kähler metrics. Let  $\sigma$  be an antiholomorphic involution on  $Y$ . That is,  $\sigma : Y \rightarrow Y$  is a diffeomorphism satisfying  $\sigma^2 = \mathrm{id}$  and  $\sigma^*(J) = -J$ . Define  $\alpha : \mathbb{C}^4 \rightarrow \mathbb{C}^4$  by

$$(6) \quad \alpha : (z_1, z_2, z_3, z_4) \longmapsto (iz_1, iz_2, iz_3, iz_4).$$

Then  $\alpha^4 = 1$ , so that  $\langle \alpha \rangle \cong \mathbb{Z}_4$ , and  $\mathbb{C}^4/\langle \alpha \rangle$  has an isolated singular point at 0. We require that the singular set of  $Y$  should be  $k$  isolated points  $p_1, \dots, p_k$  for some  $k \geq 1$ , each modelled on  $\mathbb{C}^4/\langle \alpha \rangle$ , and that the fixed set of  $\sigma$  in  $Y$  is exactly  $\{p_1, \dots, p_k\}$ . We also suppose that  $Y \setminus \{p_1, \dots, p_k\}$  is simply-connected, and  $h^{2,0}(Y) = 0$ .

In the rest of the section we assume that Condition 5.1 holds.

**Proposition 5.2.** *There is a  $\sigma$ -invariant metric  $g_Y$  on  $Y$  making  $(Y, J, g_Y)$  into a Calabi–Yau orbifold. We can choose the holomorphic volume form  $\theta_Y$  on  $(Y, J, g_Y)$  such that  $\sigma^*(\theta_Y) = \bar{\theta}_Y$ . Let  $(\Omega_Y, g_Y)$  be the torsion-free Spin(7)-structure on  $Y$  from Proposition 3.6. Then  $(\Omega_Y, g_Y)$  is  $\sigma$ -invariant.*

*Proof.* Let  $g'$  be a Kähler metric on  $Y$ . Then  $\sigma^*(g')$  is also a Kähler metric on  $Y$ , and so  $g'' = g' + \sigma^*(g')$  is a  $\sigma$ -invariant Kähler metric on  $Y$ . Let  $\kappa$  be the Kähler class of  $g''$ . Then  $\kappa$  is  $\sigma$ -invariant, regarded as an equivalence class of metrics on  $Y$ . By Condition 5.1 we know that  $c_1(Y) = 0$  and  $h^{2,0}(Y) = 0$ , and that  $Y \setminus S$  is simply-connected, where  $S = \{p_1, \dots, p_k\}$  is the singular set of  $Y$ . Thus by Proposition 3.5, the Kähler class  $\kappa$  contains a unique metric  $g_Y$  such that  $(Y, J, g_Y)$  is a Calabi–Yau orbifold. As  $\kappa$  is  $\sigma$ -invariant we see that  $g_Y$  is  $\sigma$ -invariant, by uniqueness of  $g_Y$ .

Proposition 3.4 shows that there exists a holomorphic volume form  $\theta$  on  $Y$ . Since  $\sigma$  is antiholomorphic, it is easy to show that  $\sigma^*(\theta) = e^{i\phi}\bar{\theta}$ , for some  $\phi \in [0, 2\pi)$ . Define  $\theta_Y = e^{i\phi/2}\theta$ . Then  $\theta_Y$  is a holomorphic volume form for  $(Y, J, g_Y)$ , and  $\sigma^*(\theta_Y) = \bar{\theta}_Y$ , as we want.

Let  $(\Omega_Y, g_Y)$  be as in Proposition 3.6. Then  $\Omega_Y = \frac{1}{2}\omega_Y \wedge \omega_Y + \text{Re}(\theta_Y)$ , where  $\omega_Y$  is the Kähler form of  $g_Y$ . As  $\sigma^*(g_Y) = g_Y$  and  $\sigma^*(J) = -J$  we have  $\sigma^*(\omega_Y) = -\omega_Y$ , and  $\sigma^*(\text{Re}(\theta_Y)) = \text{Re}(\theta_Y)$  as  $\sigma^*(\theta_Y) = \bar{\theta}_Y$ . Thus  $\Omega_Y$  and  $g_Y$  are both  $\sigma$ -invariant.    q.e.d.

In our next result, if  $Y$  is an orbifold and  $p \in Y$  an orbifold point modelled on  $\mathbb{R}^n/G$ , then we say that the *tangent space*  $T_p Y$  to  $Y$  at  $p$  is  $\mathbb{R}^n/G$ , in the obvious way. The proof looks complicated, but it is really only linear algebra.

**Proposition 5.3.** *For each  $j = 1, \dots, k$  we can identify the tangent space  $T_{p_j} Y$  to  $Y$  at  $p_j$  with  $\mathbb{C}^4/\langle\alpha\rangle$  so that  $g_Y$  is identified with  $|dz_1|^2 + \dots + |dz_4|^2$  at  $p_j$ ,  $\theta_Y$  is identified with  $dz_1 \wedge \dots \wedge dz_4$  at  $p_j$ , and  $d\sigma : T_{p_j} Y \rightarrow T_{p_j} Y$  is identified with the map  $\beta : \mathbb{C}^4/\langle\alpha\rangle \rightarrow \mathbb{C}^4/\langle\alpha\rangle$  given by*

$$(7) \quad \beta : (z_1, \dots, z_4)\langle\alpha\rangle \mapsto (\bar{z}_2, -\bar{z}_1, \bar{z}_4, -\bar{z}_3)\langle\alpha\rangle.$$

*Proof.* Since  $J, g_Y$  and  $\theta_Y$  form a Calabi–Yau structure on  $Y$ , there certainly exists an isomorphism  $\iota : T_{p_j} Y \rightarrow \mathbb{C}^4/\langle\alpha\rangle$  which identifies  $g_Y$  with  $|dz_1|^2 + \dots + |dz_4|^2$  and  $\theta_Y$  with  $dz_1 \wedge \dots \wedge dz_4$ . This  $\iota$  is unique up to the action of  $\text{SU}(4)$  on  $\mathbb{C}^4/\langle\alpha\rangle$ . That is, if  $B \in \text{SU}(4)$ , then

$B \circ \iota : T_{p_j} Y \rightarrow \mathbb{C}^4 / \langle \alpha \rangle$  also identifies  $g_Y$  with  $|dz_1|^2 + \cdots + |dz_4|^2$  and  $\theta_Y$  with  $dz_1 \wedge \cdots \wedge dz_4$ .

Now  $d\sigma : T_{p_j} Y \rightarrow T_{p_j} Y$  is complex antilinear, and so  $\iota$  identifies  $d\sigma$  with the map  $\gamma : \mathbb{C}^4 / \langle \alpha \rangle \rightarrow \mathbb{C}^4 / \langle \alpha \rangle$  given by

$$(8) \quad \gamma : \left\{ i^k \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{pmatrix} : k = 0, 1, 2, 3 \right\} \mapsto \left\{ i^k A \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \\ \bar{z}_3 \\ \bar{z}_4 \end{pmatrix} : k = 0, 1, 2, 3 \right\},$$

for some  $4 \times 4$  complex matrix  $A$ . In fact  $A$  is only defined up to multiplication by a power of  $i$ .

As  $d\sigma$  preserves  $g_Y$  and takes  $\theta_Y$  to  $\bar{\theta}_Y$  on  $T_{p_j} Y$ , it follows that  $\gamma$  preserves  $|dz_1|^2 + \cdots + |dz_4|^2$  and takes  $dz_1 \wedge \cdots \wedge dz_4$  to  $d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_4$  on  $\mathbb{C}^4 / \langle \alpha \rangle$ . These imply that  $A\bar{A}^t = I$  and  $\det(A) = 1$ , and so  $A \in \text{SU}(4)$ . Also,  $\gamma^2 = I$  as  $\sigma^2 = \text{id}$ , and this implies that  $A\bar{A} = i^k I$  for  $k = 0, 1, 2$  or  $3$ . And because  $\sigma$  fixes only  $p_1, \dots, p_k$  in  $Y$ , the only fixed point of  $\gamma$  in  $\mathbb{C}^4 / \langle \alpha \rangle$  is  $0$ .

So  $A$  lies in  $\text{SU}(4)$  and satisfies  $A\bar{A} = i^k I$ . When we replace  $\iota$  by  $B \circ \iota$  for  $B \in \text{SU}(4)$ , the matrix  $A$  is replaced by  $BAB^t$ . We wish to show that we can choose  $B \in \text{SU}(4)$  such that the maps  $\beta$  of (7) and  $\gamma$  of (8) coincide. That is, we must show that there exists  $B \in \text{SU}(4)$  and  $l = 0, 1, 2$  or  $3$  such that

$$(9) \quad i^l BAB^t = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Now  $A\bar{A} = i^k I$  shows that  $A$  and  $\bar{A}$  commute, and so  $A\bar{A} = \bar{A}A = \overline{A\bar{A}}$ . Thus  $i^k I$  is a real matrix, which implies that  $k = 0$  or  $2$ , and  $A\bar{A} = \pm I$ . By studying the eigenvectors of  $A$ , one can prove that there exists  $B \in \text{SU}(4)$  such that  $BAB^t$  is one of

$$I, \quad -I, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad i \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We exclude the first three possibilities because  $\gamma$  fixes  $(1, 0, 0, 0)\langle \alpha \rangle$  in  $\mathbb{C}^4 / \langle \alpha \rangle$ , contradicting the fact that the only fixed point of  $\gamma$  in  $\mathbb{C}^4 / \langle \alpha \rangle$  is  $0$ . Putting  $l = 0$  in the fourth case and  $l = 3$  in the fifth, we see that

(9) holds. Thus  $B \circ \iota$  identifies  $T_{p_j}Y$  with  $\mathbb{C}^4/\langle\alpha\rangle$  and satisfies all the conditions of the proposition, and the proof is complete. q.e.d.

Now §4.1 defined a finite group  $G = \langle\alpha, \beta\rangle$  acting on  $\mathbb{R}^8$ , and the definitions (6) and (7) of  $\alpha$  and  $\beta$  above coincide with (4) in §4.1. Thus the singularities of  $Z = Y/\langle\sigma\rangle$  are all modelled on  $\mathbb{R}^8/G$ , and we easily prove:

**Corollary 5.4.** *Define  $Z = Y/\langle\sigma\rangle$ . Then  $Z$  is a compact, real 8-dimensional orbifold. The Spin(7)-structure  $(\Omega_Y, g_Y)$  on  $Y$  pushes down to give a torsion-free Spin(7)-structure  $(\Omega_Z, g_Z)$  on  $Z$ . The singularities of  $Z$  are  $k$  points  $p_1, \dots, p_k$ . For each  $j = 1, \dots, k$  there is an isomorphism  $\iota_j : \mathbb{R}^8/G \rightarrow T_{p_j}Z$  which identifies the Spin(7)-structures  $(\Omega_0, g_0)$  on  $\mathbb{R}^8/G$  and  $(\Omega_Z, g_Z)$  on  $T_{p_j}Z$ . Here  $G$  and  $(\Omega_0, g_0)$  are defined in §4.1.*

## 5.2 Desingularizing $Z$ to get a compact 8-manifold $M$

So far we have constructed a Spin(7)-orbifold  $(Z, \Omega_Z, g_Z)$  with finitely many singular points  $p_1, \dots, p_k$ , each modelled on the singularity  $\mathbb{R}^8/G$  of §4.1. But in §4.1 and §4.2 we wrote down two ALE Spin(7)-manifolds  $X_1$  and  $X_2$  asymptotic to  $\mathbb{R}^8/G$ . We shall now resolve each singular point  $p_j$  in  $Z$  using either  $X_1$  or  $X_2$  to get a compact 8-manifold  $M$ . We include a parameter  $t \in (0, 1]$  in the construction.

**Definition 5.5.** For each  $j$  let  $\iota_j$  be as in Corollary 5.4, and let  $\exp_{p_j} : T_{p_j}Z \rightarrow Z$  be the *exponential map*, which is well-defined as  $Z$  is complete. Then  $\exp_{p_j} \circ \iota_j$  maps  $\mathbb{R}^8/G$  to  $Z$ . Choose  $\zeta > 0$  small, and let  $B_{2\zeta}(\mathbb{R}^8/G)$  be the open ball of radius  $2\zeta$  about 0 in  $\mathbb{R}^8/G$ . Define  $U_j \subset Z$  by  $U_j = \exp_{p_j} \circ \iota_j(B_{2\zeta}(\mathbb{R}^8/G))$ , and  $\psi_j : B_{2\zeta}(\mathbb{R}^8/G) \rightarrow U_j$  by  $\psi_j = \exp_{p_j} \circ \iota_j$ . Let  $\zeta > 0$  be chosen small enough that  $U_j$  is open in  $Z$  and  $\psi_j : B_{2\zeta}(\mathbb{R}^8/G) \rightarrow U_j$  is a diffeomorphism for  $1 \leq j \leq k$ , and that  $U_i \cap U_j = \emptyset$  when  $i \neq j$ .

**Proposition 5.6.** *There is a smooth 3-form  $\sigma_j$  on  $B_{2\zeta}(\mathbb{R}^8/G)$  for  $1 \leq j \leq k$  and a constant  $C_1 > 0$ , such that  $\psi_j^*(\Omega_Z) - \Omega_0 = d\sigma_j$  and  $|\nabla^l \sigma_j| \leq C_1 r^{3-l}$  on  $B_{2\zeta}(\mathbb{R}^8/G)$ , for  $l = 0, 1, 2$ . Here  $|\cdot|$  and  $\nabla$  are defined using the metric  $g_0$  on  $B_{2\zeta}(\mathbb{R}^8/G)$ , and  $r : B_{2\zeta}(\mathbb{R}^8/G) \rightarrow [0, 2\zeta]$  is the radius function.*

*Proof.* The derivative of  $\exp_{p_j}$  at 0 is the identity map on  $T_{p_j}Z$ . Thus the derivative of  $\psi_j$  at 0 is  $\iota_j : \mathbb{R}^8/G \rightarrow T_{p_j}Z$ , and so  $\psi_j^*(\Omega_Z)|_0 = \iota_j^*(\Omega_Z) = \Omega_0|_0$ , since  $\iota_j$  identifies  $\Omega_0$  and  $\Omega_Z$ . Therefore  $\psi_j^*(\Omega_Z) = \Omega_0$  at

0 in  $B_{2\zeta}(\mathbb{R}^8/G)$ . As  $\psi_j^*(\Omega_Z) - \Omega_0$  is a 4-form on a subset of  $\mathbb{R}^8/G$ , we can pull it back to  $\mathbb{R}^8$ , and regard  $\psi_j^*(\Omega_Z) - \Omega_0$  as a 4-form on the ball  $B_{2\zeta}(\mathbb{R}^8)$  of radius  $2\zeta$  in  $\mathbb{R}^8$ .

Then  $\psi_j^*(\Omega_Z) - \Omega_0$  is a smooth  $G$ -invariant 4-form on  $B_{2\zeta}(\mathbb{R}^8)$  which vanishes at 0. But  $G$  contains  $-1 : \mathbb{R}^8 \rightarrow \mathbb{R}^8$ , and any 4-form invariant under this map  $-1$  has zero first derivative at 0. Hence  $\psi_j^*(\Omega_Z) - \Omega_0$  vanishes to first order at 0 in  $B_{2\zeta}(\mathbb{R}^8)$ , and so by Taylor's Theorem we can show that  $|\psi_j^*(\Omega_Z) - \Omega_0| = O(r^2)$  and  $|\nabla \psi_j^*(\Omega_Z)| = O(r)$  on  $B_{2\zeta}(\mathbb{R}^8)$ .

Now  $\Omega_Z$  and  $\Omega_0$  are closed, so that  $\psi_j^*(\Omega_Z) - \Omega_0$  is closed, and as  $B_{2\zeta}(\mathbb{R}^8/G)$  is contractible we can write  $\psi_j^*(\Omega_Z) - \Omega_0 = d\sigma_j$  for some smooth 3-form  $\sigma_j$  on  $B_{2\zeta}(\mathbb{R}^8/G)$ . Since  $\psi_j^*(\Omega_Z) - \Omega_0$  vanishes to first order at 0 we can easily arrange that  $\sigma_j$  vanishes to second order at 0, and therefore  $|\nabla^l \sigma_j| = O(r^{3-l})$  for  $l = 0, 1, 2$ , using Taylor's Theorem as above. Thus there exists  $C_1 > 0$  such that  $|\nabla^l \sigma_j| \leq C_1 r^{3-l}$  on  $B_{2\zeta}(\mathbb{R}^8/G)$ , for  $l = 0, 1, 2$  and  $j = 1, \dots, k$ .    q.e.d.

**Definition 5.7.** Let the ALE Spin(7)-manifolds  $(X_n, \Omega_n, g_n)$  and projections  $\pi_n : X_n \rightarrow \mathbb{R}^8/G$  be as in §4.1 and §4.2 for  $n = 1, 2$ . For each  $t \in (0, 1]$  and  $n = 1, 2$  let  $X_n^t = X_n$ , define a Spin(7)-structure  $(\Omega_n^t, g_n^t)$  on  $X_n^t$  by  $\Omega_n^t = t^4 \Omega_n$  and  $g_n^t = t^2 g_n$ , and define  $\pi_n^t : X_n^t \rightarrow \mathbb{R}^8/G$  by  $\pi_n^t = t\pi_n$ . Then  $(X_n^t, \Omega_n^t, g_n^t)$  is an ALE Spin(7)-manifold asymptotic to  $\mathbb{R}^8/G$ .

Using the ideas of [15] or the explicit formula of Calabi [4, p. 285] we can show that there exist  $C_2 > 0$  and a smooth 3-form  $\tau_n^t$  on  $\mathbb{R}^8/G \setminus B_{t\zeta}(\mathbb{R}^8/G)$ , satisfying

$$(10) \quad (\pi_n^t)_*(\Omega_n^t) = \Omega_0 + d\tau_n^t \quad \text{and} \quad |\nabla^l \tau_n^t| \leq C_2 t^8 r^{-7-l} \quad \text{for } l = 0, 1, 2$$

on  $\mathbb{R}^8/G \setminus B_{t\zeta}(\mathbb{R}^8/G)$ , where  $|\cdot|$  and  $\nabla$  are defined using the metric  $g_0$ .

For  $j = 1, \dots, k$ , choose  $n_j$  to be 1 or 2. There are  $2^k$  ways of defining the  $n_j$ . We shall resolve each singular point  $p_j$  in  $Z$  using  $X_{n_j}^t$  to get a 1-parameter family of resolutions  $(M^t, \pi^t)$  of  $Z$ .

**Definition 5.8.** For each  $j = 1, \dots, k$ , define open subsets  $M_0^t$  in  $Z$  and  $M_j^t$  in  $X_{n_j}^t$  for  $1 \leq j \leq k$  by

$$M_0^t = Z \setminus \bigcup_{j=1}^k \psi_j(\overline{B}_{t^4/5\zeta}(\mathbb{R}^8/G)) \quad \text{and} \quad M_j^t = (\pi_{n_j}^t)^{-1}(B_{2t^4/5\zeta}(\mathbb{R}^8/G)).$$

That is,  $M_0^t$  is the complement in  $Z$  of the closed balls of radius  $t^{4/5}\zeta$  about  $p_j$  for  $1 \leq j \leq k$ , and  $M_j^t$  is the inverse image of  $B_{2t^{4/5}\zeta}(\mathbb{R}^8/G)$  in  $X_{n_j}^t$ .

Define an equivalence relation ' $\sim$ ' on the disjoint union  $\coprod_{j=0}^k M_j^t$  by  $x \sim y$  if either (a)  $x = y$ ,

(b)  $x \in M_j^t$  and  $y \in U_j \cap M_0^t$  and  $\psi_j \circ \pi_{n_j}^t(x) = y$ , for some  $j = 1, \dots, k$ ,  
or

(c)  $y \in M_j^t$  and  $x \in U_j \cap M_0^t$  and  $\psi_j \circ \pi_{n_j}^t(y) = x$ , for some  $j = 1, \dots, k$ .

Define the *resolution*  $M^t$  of  $Z$  to be  $\coprod_{j=0}^k M_j^t / \sim$ . It is easy to see that  $M^t$  is a compact 8-manifold. Define a projection  $\pi^t : M^t \rightarrow Z$  by  $\pi^t([x]) = x$  when  $x \in M_0^t$ , and  $\pi^t([x]) = \psi_j \circ \pi_{n_j}^t(x)$  when  $x \in M_j^t$  for some  $j = 1, \dots, k$ , where  $[x]$  is the equivalence class of  $x$  under  $\sim$ . Then  $\pi^t$  is well-defined, continuous and surjective, and  $\pi^t : M^t \setminus \bigcup_{j=1}^k (\pi^t)^{-1}(p_j) \rightarrow Z \setminus \{p_1, \dots, p_k\}$  is a diffeomorphism.

Since the resolutions  $(M^t, \pi^t)$  of  $Z$  form a smooth connected family, they are all diffeomorphic to the same compact 8-manifold  $M$ . We can regard  $M_j^t$  as an open subset of  $M^t$  for  $j = 0, \dots, k$ , and then the  $M_j^t$  form an *open cover* of  $M^t$ . If  $1 \leq i, j \leq k$  and  $i \neq j$  then  $M_i^t \cap M_j^t = \emptyset$ . The overlap  $M_0^t \cap M_j^t$  is naturally isomorphic to an *annulus* in  $\mathbb{R}^8/G$ , with inner radius  $t^{4/5}\zeta$  and outer radius  $2t^{4/5}\zeta$ . The reason for including the factors  $t^{4/5}$  will be explained shortly.

We now calculate the fundamental group of  $M^t$ .

**Proposition 5.9.** *If  $n_j = 1$  for  $j = 1, \dots, k$  then  $\pi_1(M^t) \cong \mathbb{Z}_2$ . Otherwise,  $M^t$  is simply-connected.*

*Proof.* Since  $Y \setminus \{p_1, \dots, p_k\}$  is simply-connected by Condition 5.1 and  $\sigma$  acts freely on  $Y \setminus \{p_1, \dots, p_k\}$ , we see that the fundamental group of  $Z \setminus \{p_1, \dots, p_k\}$  is  $\mathbb{Z}_2$ . The natural inclusion of  $Z \setminus \{p_1, \dots, p_k\}$  in  $M^t$  induces a homomorphism from  $\pi_1(Z \setminus \{p_1, \dots, p_k\})$  to  $\pi_1(M^t)$ , which is easily shown to be surjective. Also, as  $X_{n_j}^t$  is  $X_1$  or  $X_2$  we have  $\pi_1(X_{n_j}^t) \cong \mathbb{Z}_2$ .

Therefore,  $\pi_1(M^t)$  is  $\mathbb{Z}_2$  if the generator of  $\pi_1(Z \setminus \{p_1, \dots, p_k\})$  projects to the nonzero element of  $\pi_1(X_{n_j}^t)$  for all  $1 \leq j \leq k$ , and  $\pi_1(M^t)$  is trivial otherwise. But calculation shows that the generator of  $\pi_1(Z \setminus \{p_1, \dots, p_k\})$  is nonzero in  $\pi_1(X_{n_j}^t)$  if and only if  $n_j = 1$ .    q.e.d.

This shows that of the  $2^k$  possible ways of choosing the  $n_j$ , one possibility gives  $\pi_1(M^t) = \mathbb{Z}_2$ , and the remaining  $2^k - 1$  possibilities all

give simply-connected  $M^t$ .

### 5.3 A Spin(7)-structure $(\Omega^t, g^t)$ on $M^t$ with small torsion

Each open subset  $M_j^t$  in  $M^t$  carries a torsion-free Spin(7)-structure,  $(\Omega_Z, g_Z)$  for  $j = 0$  and  $(\Omega_{n_j}^t, g_{n_j}^t)$  for  $1 \leq j \leq k$ . We shall join these Spin(7)-structures together with a partition of unity to get a Spin(7)-structure  $(\Omega^t, g^t)$  on  $M^t$  and estimate its torsion.

**Definition 5.10.** Let  $\eta : [0, \infty) \rightarrow [0, 1]$  be a smooth function with  $\eta(x) = 0$  for  $x \leq \zeta$  and  $\eta(x) = 1$  for  $x \geq 2\zeta$ . Define a 4-form  $\xi^t$  on  $M^t$  by  $\xi^t = \Omega_Z$  in  $M_0^t \setminus \bigcup_{j=1}^k M_j^t$ , and  $\xi^t = \Omega_{n_j}^t$  in  $M_j^t \setminus M_0^t$  for  $1 \leq j \leq k$ , and

$$(11) \quad \xi^t = \Omega_0 + d(\eta(t^{-4/5}r)\sigma_j) + d((1 - \eta(t^{-4/5}r))\tau_{n_j}^t) \quad \text{in } M_0^t \cap M_j^t$$

for  $1 \leq j \leq k$ , where we identify  $M_0^t \cap M_j^t$  with an annulus in  $\mathbb{R}^8/G$  in the natural way. Since  $\Omega_Z = \Omega_0 + d\sigma_j$  and  $\Omega_{n_j}^t = \Omega_0 + d\tau_{n_j}^t$  in  $M_0^t \cap M_j^t$ , it follows that  $\xi^t$  is smooth, and as  $\Omega_Z$ ,  $\Omega_{n_j}^t$  and  $\Omega_0$  are closed,  $\xi^t$  is closed.

**Lemma 5.11.** *There exists  $C_3 > 0$  such that for each  $j = 1, \dots, k$  and  $t \in (0, 1]$ , this 4-form  $\xi^t$  satisfies*

$$(12) \quad |\xi^t - \Omega_0| \leq C_3 t^{8/5} \quad \text{and} \quad |\nabla(\xi^t - \Omega_0)| \leq C_3 t^{4/5}$$

in  $M_0^t \cap M_j^t$ , where  $|\cdot|$  and  $\nabla$  are defined using the metric  $g_0$ .

*Proof.* Expanding (11) we find that

$$\begin{aligned} \xi^t - \Omega_0 &= \eta(t^{-4/5}r)d\sigma_j + (1 - \eta(t^{-4/5}r))d\tau_{n_j}^t \\ &\quad + t^{-4/5}\eta'(t^{-4/5}r)dr \wedge (\sigma_j - \tau_{n_j}^t) \end{aligned}$$

in  $M_0^t \cap M_j^t$ . Since  $t^{4/5}\zeta \leq r \leq 2t^{4/5}\zeta$ , Proposition 5.6 and (10) show that

$$\begin{aligned} |\sigma_j| &\leq 8C_1\zeta^3 t^{12/5}, & |d\sigma_j| &\leq 4C_1\zeta^2 t^{8/5}, & |\nabla d\sigma_j| &\leq 2C_1\zeta t^{4/5}, \\ |\tau_{n_j}^t| &\leq C_2\zeta^{-7} t^{12/5}, & |d\tau_{n_j}^t| &\leq C_2\zeta^{-8} t^{8/5} \quad \text{and} \quad |\nabla d\tau_{n_j}^t| &\leq C_2\zeta^{-9} t^{4/5}. \end{aligned}$$

Combining these with the previous equation and using the facts that  $|dr| = 1$  and  $\eta'$  is bounded independently of  $t$ , we soon prove (12).

q.e.d.

We can now explain why we chose the power  $t^{4/5}$  in Definition 5.8. Suppose we had defined  $M^t$  and  $\xi^t$  using  $t^\alpha$  in place of  $t^{4/5}$ , for some  $\alpha \in [0, 1]$ . Then in the calculation above the  $\sigma_j$  and  $\tau_{n_j}^t$  terms would contribute  $O(t^{2\alpha})$  and  $O(t^{8-8\alpha})$  to  $\xi^t - \Omega_0$  respectively, and so  $\xi^t - \Omega_0$  would be  $O(t^{2\alpha}) + O(t^{8-8\alpha})$ . This is smallest when  $2\alpha = 8 - 8\alpha$ , that is, when  $\alpha = 4/5$ . So the power  $t^{4/5}$  minimizes the size of  $\xi^t - \Omega_0$ .

Now we can define the Spin(7)-structures  $(\Omega^t, g^t)$  on  $M^t$ .

**Definition 5.12.** Let  $\rho$  be as in Proposition 2.5, and choose  $\epsilon \in (0, 1]$  such that  $C_3\epsilon^{8/5} \leq \rho$ . Suppose  $t \in (0, \epsilon]$ . Then

$$|\xi^t - \Omega_0| \leq C_3 t^{8/5} \leq \rho$$

in  $M_0^t \cap M_j^t$  for  $1 \leq j \leq k$  by (12), and so  $\xi^t$  lies in  $\mathcal{T}M^t$  on  $M_0^t \cap M_j^t$  by part (i) of Proposition 2.5. But  $\xi^t$  is  $\Omega_Z$  or  $\Omega_{n_j}^t$  outside the overlaps  $M_0^t \cap M_j^t$ , and thus  $\xi^t \in C^\infty(\mathcal{T}M^t)$ . For each  $t \in (0, \epsilon]$  define  $\Omega^t = \Theta(\xi^t)$ , where  $\Theta$  is given in Proposition 2.5. Then  $\Omega^t \in C^\infty(\mathcal{A}M^t)$ , and so  $\Omega^t$  extends to a Spin(7)-structure  $(\Omega^t, g^t)$  on  $M^t$ . Define a 4-form  $\phi^t$  on  $M^t$  by  $\phi^t = \xi^t - \Omega^t$ . Then  $d\Omega^t + d\phi^t = 0$ , as  $d\xi^t = 0$  on  $M^t$ .

Here  $\xi^t$  is a 4-form which does not lie in  $\mathcal{A}M^t$ , but is close to  $\mathcal{A}M^t$  for small  $t$ , and  $\Omega^t$  is the section of  $\mathcal{A}M^t$  closest to  $\xi^t$ . What is really happening is that the Spin(7)-structure  $(\Omega^t, g^t)$  is equal to  $(\Omega_{n_j}^t, g_{n_j}^t)$  in  $M_j^t \setminus M_0^t$  and to  $(\Omega_Z, g_Z)$  outside  $M_j^t$  for  $j = 1, \dots, k$ , and  $(\Omega^t, g^t)$  interpolates smoothly between these two possibilities on the annulus  $M_j^t \cap M_0^t$ .

#### 5.4 Existence of torsion-free Spin(7)-structures on $M$

Next we shall show that  $(\Omega^t, g^t)$  can be deformed to a torsion-free Spin(7)-structure on  $M$  when  $t$  is small.

**Theorem 5.13.** *In the situation above, there exist constants  $\lambda, \mu, \nu > 0$  such that for all  $t \in (0, \epsilon]$  we have*

- (i)  $\|\phi^t\|_{L^2} \leq \lambda t^{24/5}$  and  $\|d\phi^t\|_{L^{10}} \leq \lambda t^{36/25}$ ;
- (ii) the injectivity radius  $\delta(g^t)$  satisfies  $\delta(g^t) \geq \mu t$ ; and
- (ii) the Riemann curvature  $R(g^t)$  satisfies  $\|R(g^t)\|_{C^0} \leq \nu t^{-2}$ .

Here all norms are calculated using the metric  $g^t$  on  $M^t$ .

*Proof.* Outside the overlaps  $M_0^t \cap M_j^t$  for  $1 \leq j \leq k$  we either have  $\xi^t = \Omega^t = \Omega_Z$  or  $\xi^t = \Omega^t = \Omega_{n_j}^t$ . In both cases  $\phi^t = \xi^t - \Omega^t = 0$ , and

so  $\phi^t$  is zero outside the  $M_0^t \cap M_j^t$ . In  $M_0^t \cap M_j^t$  we apply part (ii) of Proposition 2.5 with  $\Omega = \Omega_0$  and  $\xi = \xi^t$ , to get

$$|\phi^t|_{g^t} \leq |\xi^t - \Omega_0|_{g_0} \quad \text{and} \quad |\nabla^{g^t} \phi^t|_{g^t} \leq C |\nabla^{g_0} (\xi^t - \Omega_0)|_{g_0}.$$

Combining this with (12) gives

$$|\phi^t|_{g^t} \leq C_3 t^{8/5} \quad \text{and} \quad |d\phi^t|_{g^t} \leq |\nabla^{g^t} \phi^t|_{g^t} \leq CC_3 t^{4/5}.$$

Now each  $M_0^t \cap M_j^t$  is an annulus in  $\mathbb{R}^8/G$  with inner radius  $t^{4/5}\zeta$  and outer radius  $2t^{4/5}\zeta$ , and the metric  $g^t$  on  $M_0^t \cap M_j^t$  is close to the flat metric  $g_0$  on  $\mathbb{R}^8/G$ . Therefore we can find  $C_4 > 0$  independent of  $t$  such that  $\sum_{j=1}^k \text{vol}(M_0^t \cap M_j^t) \leq C_4 t^{32/5}$ . Hence

$$\begin{aligned} \int_{M^t} |\phi^t|^2 dV &\leq (C_3 t^{8/5})^2 C_4 t^{32/5} \quad \text{and} \\ \int_{M^t} |d\phi^t|^{10} dV &\leq (CC_3 t^{4/5})^{10} C_4 t^{32/5}. \end{aligned}$$

Taking roots gives part (i) of the theorem, with  $\lambda = C_3 \max(C_4^{1/2}, CC_4^{1/10})$ .

Parts (ii) and (iii) are elementary. The metric  $g_{n_j}^t$  is made by scaling  $g_{n_j}$  by a factor  $t$ . Thus  $\delta(g_{n_j}^t) = t\delta(g_{n_j})$  and  $\|R(g_{n_j}^t)\|_{C^0} = t^{-2}\|R(g_{n_j})\|_{C^0}$ . We make  $g^t$  by gluing together the  $g_{n_j}^t$  on the patches  $M_j^t$  for  $j = 1, \dots, k$  and  $g_z$  on  $M_0^t$ . It is clear that for small  $t$ , the dominant contributions to  $\delta(g^t)$  and  $\|R(g^t)\|_{C^0}$  come from  $\delta(g_{n_j}^t)$  and  $\|R(g_{n_j}^t)\|_{C^0}$  for some  $j$ , and these are proportional to  $t$  and  $t^{-2}$ . This proves (ii) and (iii) for some  $\mu, \nu > 0$ , and the theorem is complete.

q.e.d.

Finally we can prove our main result.

**Theorem 5.14.** *Suppose Condition 5.1 holds, and let  $M$  be the compact 8-manifold defined in Definition 5.8. Then there exist torsion-free Spin(7)-structures  $(\tilde{\Omega}, \tilde{g})$  on  $M$ . If  $\pi_1(M) = \{1\}$  then  $\text{Hol}(\tilde{g}) = \text{Spin}(7)$ , and if  $\pi_1(M) = \mathbb{Z}_2$  then  $\text{Hol}(\tilde{g}) = \mathbb{Z}_2 \times \text{SU}(4)$ .*

*Proof.* Let  $\lambda, \mu, \nu$  be as in Theorem 5.13. Then Theorem 2.6 gives a constant  $\kappa > 0$ . Choose  $t > 0$  with  $t \leq \epsilon \leq 1$  and  $t \leq \kappa$ . Let  $(\Omega, g)$  be the Spin(7)-structure  $(\Omega^t, g^t)$  on  $M = M^t$ , and  $\phi$  the 4-form  $\phi^t$ . Then  $d\Omega + d\phi = 0$  by Definition 5.12, and parts (i)–(iii) of Theorem 5.13 imply (i)–(iii) of Theorem 2.6, as  $t \leq 1$ .

Therefore all the hypotheses of Theorem 2.6 hold, and the theorem shows that there exists a torsion-free Spin(7)-structure  $(\tilde{\Omega}, \tilde{g})$  on  $M$ .

It remains to identify the holonomy group  $\text{Hol}(\tilde{g})$  of  $\tilde{g}$ . Now we can regard the Spin(7)-orbifold  $(Z, \Omega_Z, g_Z)$  as the limit as  $t \rightarrow 0$  of the Spin(7)-manifolds  $(M, \tilde{\Omega}, \tilde{g})$ . Because of this, it is not difficult to show that  $\text{Hol}(g_Z) \subseteq \text{Hol}(\tilde{g})$ .

Now  $\text{Hol}(g_Z) = \mathbb{Z}_2 \times \text{SU}(4)$ , and thus

$$\mathbb{Z}_2 \times \text{SU}(4) \subseteq \text{Hol}(\tilde{g}) \subseteq \text{Spin}(7).$$

If  $\pi_1(M) = \{1\}$  then  $\text{Hol}(\tilde{g})$  is connected. But the only connected Lie subgroup of Spin(7) containing  $\mathbb{Z}_2 \times \text{SU}(4)$  is Spin(7), so  $\text{Hol}(\tilde{g}) = \text{Spin}(7)$ . If  $\pi_1(M) = \mathbb{Z}_2$  then  $\text{Hol}(\tilde{g}) \neq \text{Spin}(7)$  by Theorem 2.3. This forces  $\text{Hol}^0(\tilde{g}) = \text{SU}(4)$ , and it is then easy to see that  $\text{Hol}(\tilde{g}) = \mathbb{Z}_2 \times \text{SU}(4)$ . q.e.d.

Since by Proposition 5.9 we can always choose the  $n_j$  so that  $M$  is simply-connected, we can always arrange for  $\tilde{g}$  to have holonomy Spin(7). When  $\pi_1(M) = \mathbb{Z}_2$ , the complex orbifold  $Y$  has a crepant resolution  $\tilde{Y}$ , which admits Kähler metrics  $\tilde{g}$  with holonomy SU(4), making it into a Calabi–Yau manifold. The action of  $\sigma$  on  $Y$  lifts to a *free* action of  $\sigma$  on  $\tilde{Y}$ , and so  $M = \tilde{Y}/\langle\sigma\rangle$  is a compact 8-manifold. If we choose  $\tilde{g}$  to be  $\sigma$ -invariant then it pushes down to  $M$ , and has holonomy  $\mathbb{Z}_2 \times \text{SU}(4)$ .

## 6. How to apply the construction

We now explain ways of finding orbifolds  $Y$  and involutions  $\sigma : Y \rightarrow Y$  satisfying Condition 5.1, and how to calculate the Betti numbers of the resulting 8-manifolds  $M$  with holonomy Spin(7).

### 6.1 Finding suitable Calabi–Yau 4-orbifolds $Y$

To apply the construction of §5 we need a source of compact Kähler 4-orbifolds  $Y$  with  $c_1(Y) = 0$  and isolated singularities modelled on  $\mathbb{C}^4/\mathbb{Z}_4$ . Fortunately, physicists and algebraic geometers have been studying Calabi–Yau manifolds for many years, mainly in complex dimension 3. Several powerful methods have been developed for constructing Calabi–Yau manifolds, and we will adapt some of these to our problem.

The main idea we shall use is borrowed from Candelas, Lynker and Schrimmrigk [5], who constructed a large number of Calabi–Yau 3-folds as crepant resolutions of hypersurfaces in weighted projective spaces  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_4}^4$ . We shall explain their methods, beginning with *weighted projective spaces*, which are an important class of complex orbifolds.

**Definition 6.1.** Let  $m \geq 1$  be an integer, and  $a_0, a_1, \dots, a_m$  positive integers with highest common factor 1. Let  $\mathbb{C}^{m+1}$  have complex coordinates on  $(z_0, \dots, z_m)$ , and define an action of the complex Lie group  $\mathbb{C}^*$  on  $\mathbb{C}^{m+1}$  by

$$(13) \quad (z_0, \dots, z_m) \xrightarrow{u} (u^{a_0} z_0, \dots, u^{a_m} z_m), \quad \text{for } u \in \mathbb{C}^*.$$

Define the *weighted projective space*  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_m}^m$  to be  $(\mathbb{C}^{m+1} \setminus \{0\})/\mathbb{C}^*$ , where  $\mathbb{C}^*$  acts on  $\mathbb{C}^{m+1} \setminus \{0\}$  with the action (13). Then  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_m}^m$  is compact and Hausdorff, and has the structure of a *complex orbifold*.

Let  $[z_0, \dots, z_m]$  be a point in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_m}^m$ , and let  $k$  be the highest common factor of the set of those  $a_j$  for which  $z_j \neq 0$ . If  $k = 1$  then  $[z_0, \dots, z_m]$  is a nonsingular point of  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_m}^m$ , and if  $k > 1$  then  $[z_0, \dots, z_m]$  is an orbifold point with orbifold group  $\mathbb{Z}_k$ .

We call a polynomial  $f(z_0, \dots, z_m)$  *weighted homogeneous of degree  $d$*  if

$$f(u^{a_0} z_0, \dots, u^{a_m} z_m) = u^d f(z_0, \dots, z_m) \quad \text{for all } u, z_0, \dots, z_m \in \mathbb{C}.$$

Let  $f$  be such a polynomial, and define a hypersurface  $Y$  in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_m}^m$  by

$$Y = \{[z_0, \dots, z_m] \in \mathbb{C}\mathbb{P}_{a_0, \dots, a_m}^m : f(z_0, \dots, z_m) = 0\}.$$

Then we call  $Y$  a *hypersurface of degree  $d$*  in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_m}^m$ .

We say that  $f$  is *transverse* if  $f(z_0, \dots, z_m) = 0$  and  $df(z_0, \dots, z_m) = 0$  have no common solutions in  $\mathbb{C}^{m+1} \setminus \{0\}$ . If  $f$  is transverse then the only singular points of  $Y$  are also singular points of  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_m}^m$ , and  $Y$  is an *orbifold*, all of whose orbifold groups are cyclic. Note that for given weights  $a_0, \dots, a_m$  and degree  $d$ , there may not exist any transverse polynomials  $f$ .

So let  $Y$  be a hypersurface of degree  $d$  in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_m}^m$ , defined by a transverse polynomial. Using the *adjunction formula*, we find that  $c_1(Y) = 0$  if and only if  $d = a_0 + \dots + a_m$ . In this case it is easy to show that  $Y$  is a *Calabi–Yau orbifold*. Candelas et al. [5] considered the case  $m = 4$ , and used a computer to search for Calabi–Yau 3-orbifolds of this kind, finding some 6000 examples. They then resolved the singularities of each to get a Calabi–Yau 3-manifold.

As we are interested in Calabi–Yau 4-orbifolds, we shall consider hypersurfaces  $Y$  in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$ . Here is a simple class of such  $Y$ .

**Example 6.2.** Let  $a_0, \dots, a_5$  be positive integers with highest common factor 1, and let  $d = a_0 + \dots + a_5$ . Usually we order the  $a_j$  with  $a_0 \leq a_1 \leq \dots \leq a_5$ . Suppose that  $a_j$  divides  $d$  for  $j = 0, \dots, 5$ , and define  $k_j = d/a_j$ . Define a hypersurface  $Y$  in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$  by

$$Y = \{[z_0, \dots, z_5] \in \mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5 : z_0^{k_0} + \dots + z_5^{k_5} = 0\}.$$

Since  $a_j k_j = d$  we see that  $z_0^{k_0} + \dots + z_5^{k_5}$  is a weighted homogeneous polynomial of degree  $d$ , and it is also transverse.

Therefore  $Y$  is a complex orbifold, with singularities only at the intersection of  $Y$  with the singular set of  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$ . Since the degree  $d$  of  $Y$  satisfies  $d = a_0 + \dots + a_5$ , we have  $c_1(Y) = 0$ . Also  $Y$  admits Kähler metrics, as  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$  is Kähler. So  $Y$  is a compact complex orbifold with  $c_1(Y) = 0$ , admitting Kähler metrics.

Now to apply the construction of §5, the singular points of  $Y$  must satisfy Condition 5.1. This is a strong restriction on  $a_0, \dots, a_5$ , which admits only a few solutions. However, we can get many other suitable orbifolds  $Y$  by generalizing our construction a bit. Here are four ways to do this.

- **Defining  $Y$  by a different polynomial.** We could define  $Y$  using some more general transverse weighted homogenous polynomial of degree  $d$  in  $z_0, \dots, z_5$ , instead of  $z_0^{k_0} + \dots + z_5^{k_5}$ . The requirement that  $a_j$  divides  $d$  for  $j = 0, \dots, 5$  is then replaced by some other condition on the  $a_j$  and  $d$ .
- **Dividing by a finite group.** Let  $W$  be a Calabi–Yau hypersurface in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$ , and  $G$  a finite group acting on  $W$  preserving its Calabi–Yau structure. Then  $Y = W/G$  is a Calabi–Yau orbifold.
- **Partial crepant resolutions.** Let  $W$  be a Calabi–Yau hypersurface in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$  which has some singularities of the kind we want, together with other singularities that we don’t want. We let  $Y$  be a partial crepant resolution of  $W$ , which resolves the singularities that we don’t want, leaving those that we do.
- **Complete intersections in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^m$ .** Rather than a hypersurface in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$ , we take  $Y$  to be a *complete intersection* of  $m - 4$  hypersurfaces in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_m}^m$ , for some  $m > 5$ .

We can also use combinations of these four techniques — for instance, we can take  $Y$  to be a partial crepant resolution of  $W/G$ , where  $W$  is a hypersurface in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$ , and  $G$  a finite group acting on  $W$ .

## 6.2 Antiholomorphic maps $\sigma : Y \rightarrow Y$

Suppose we have chosen an orbifold  $Y$  as above, with isolated singular points  $p_1, \dots, p_k$ . The next ingredient in our construction is an antiholomorphic involution  $\sigma : Y \rightarrow Y$ , which should fix only  $p_1, \dots, p_k$ . For example, suppose  $Y$  is a hypersurface in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$ . Then to find  $\sigma$  we would look for an antiholomorphic involution  $\sigma : \mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5 \rightarrow \mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$  with  $\sigma(Y) = Y$ , and restrict  $\sigma$  to  $Y$ .

The most obvious such  $\sigma$  maps  $[z_0, \dots, z_5] \mapsto [\bar{z}_0, \dots, \bar{z}_5]$ . But this will not do, as its fixed points are not isolated in  $Y$ . To get isolated fixed points we need to try something more subtle. Here is an example of the kind of thing we mean.

**Example 6.3.** In the situation of Example 6.2, suppose that  $a_0, \dots, a_3$  are odd and  $a_4, a_5$  even with  $a_0 = a_1$ ,  $a_2 = a_3$  and  $a_4 = a_5$ . Define  $\sigma : \mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5 \rightarrow \mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$  by

$$\sigma : [z_0, \dots, z_5] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_5, \bar{z}_4].$$

As  $\sigma$  swaps the pairs  $z_0, z_1$  and  $z_2, z_3$  and  $z_4, z_5$ , we need  $a_0 = a_1$ ,  $a_2 = a_3$  and  $a_4 = a_5$  for  $\sigma$  to be well-defined. Clearly  $\sigma$  is antiholomorphic, and  $\sigma(Y) = Y$ .

Now  $\sigma^2$  acts by

$$\sigma^2 : [z_0, \dots, z_5] \mapsto [-z_0, -z_1, -z_2, -z_3, z_4, z_5].$$

But putting  $u = -1$  in (13) gives  $[-z_0, -z_1, -z_2, -z_3, z_4, z_5] = [z_0, \dots, z_5]$ , as  $a_0, \dots, a_3$  are odd and  $a_4, a_5$  even. Thus  $\sigma^2 = 1$ , and  $\sigma : Y \rightarrow Y$  is an antiholomorphic involution.

It is not difficult to show that the fixed points of  $\sigma$  in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$  are

$$\{[0, 0, 0, 0, 1, e^{i\theta}] \in \mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5 : \theta \in [0, 2\pi)\}.$$

Now  $[0, 0, 0, 0, 1, e^{i\theta}]$  lies in  $Y$  if  $1 + e^{k_5 i\theta} = 0$ . The solutions to this equation are  $\text{hcf}(k_4, k_5)$  isolated points in  $Y$ .

Observe the trick we have used here: if  $a_j = a_{j+1}$  then we can choose  $\sigma$  to act on the coordinates  $z_j, z_{j+1}$  by  $(z_j, z_{j+1}) \mapsto (\bar{z}_{j+1}, -\bar{z}_j)$ . All the fixed points of  $\sigma$  will then satisfy  $z_j = z_{j+1} = 0$ . By doing this with two pairs of coordinates, say  $z_0, z_1$  and  $z_2, z_3$ , the fixed points of  $\sigma$  satisfy  $z_0 = z_1 = z_2 = z_3 = 0$ . Thus they will be of complex codimension 4 in  $Y$ , and will be *isolated*, as we want.

This trick can also be adapted to more general situations, in which  $Y$  is a quotient by a finite group, or a partial crepant resolution, and so on. Note that as  $\sigma^2$  maps  $(z_j, z_{j+1}) \mapsto (-z_j, -z_{j+1})$ , care must be taken to ensure that  $\sigma^2 = 1$ .

### 6.3 Calculating the Euler characteristic of $Y$

To determine the Betti numbers of the 8-manifold  $M$  that we construct, we will need to know the *Euler characteristic* of  $Y$ . Now there are two different notions of the Euler characteristic of an orbifold, defined by Satake [19, §3.3]. The version we are interested in is the *ordinary Euler characteristic*  $\chi(Y)$ , which is an integer and satisfies  $\chi(Y) = \sum_{j=0}^{2n} (-1)^j b^j(Y)$ . There is also the *orbifold Euler characteristic*  $\chi_V(Y)$ , which is a rational number that crops up naturally in problems involving characteristic classes.

In the next example we explain an elementary and fairly crude method for finding  $\chi(Y)$  in the case that  $Y$  is a hypersurface in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_m}^m$ , of the kind considered in Example 6.2. It is also possible to calculate  $\chi_V(Y)$  using Chern classes and get  $\chi(Y)$  by adding on contributions from the singular set (see for instance Hosono et al. [9, §2]), but we will not discuss this.

**Example 6.4.** Let  $a_0, \dots, a_m, k_0, \dots, k_m$  and  $d$  be positive integers with  $a_j k_j = d$  for  $j = 0, \dots, m$ . For each  $j = 0, \dots, m$ , define  $Y_j \subset \mathbb{C}\mathbb{P}_{a_0, \dots, a_j}^j$  by

$$Y_j = \{[z_0, \dots, z_j] \in \mathbb{C}\mathbb{P}_{a_0, \dots, a_j}^j : z_0^{k_0} + \dots + z_j^{k_j} = 0\},$$

and define  $\pi_j : Y_j \rightarrow \mathbb{C}\mathbb{P}_{a_0, \dots, a_{j-1}}^{j-1}$  by  $\pi_j : [z_0, \dots, z_j] \mapsto [z_0, \dots, z_{j-1}]$ .

Suppose for simplicity that  $a_i$  divides  $a_j$  for  $0 \leq i < j \leq m$ . Then for each  $j$ ,  $\pi_j$  is a  $k_j$ -fold branched cover of  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_{j-1}}^{j-1}$ , branched over  $Y_{j-1}$ . That is, if  $p \in \mathbb{C}\mathbb{P}_{a_0, \dots, a_{j-1}}^{j-1}$  then  $\pi_j^{-1}(p)$  is one point when  $p \in Y_{j-1}$  and  $k_j$  points when  $p \notin Y_{j-1}$ . It follows that

$$\begin{aligned} \chi(Y_j) &= k_j \cdot \chi(\mathbb{C}\mathbb{P}_{a_0, \dots, a_{j-1}}^{j-1}) + (1 - k_j)\chi(Y_{j-1}) \\ (14) \quad &= k_j j + (1 - k_j)\chi(Y_{j-1}), \end{aligned}$$

since  $\chi(\mathbb{C}\mathbb{P}_{a_0, \dots, a_{j-1}}^{j-1}) = j$ . This equation gives  $\chi(Y_j)$  in terms of  $\chi(Y_{j-1})$ . Hence by induction we can write  $\chi(Y_m)$  in terms of  $\chi(Y_0)$ . But  $Y_0 = \emptyset$  so that  $\chi(Y_0) = 0$ , and thus we determine  $\chi(Y_m)$ .

If  $a_i$  does not divide  $a_j$  for some  $0 \leq i < j \leq m$ , then  $\pi_j$  is also branched over other parts of  $\mathbb{CP}_{a_0, \dots, a_{j-1}}^{j-1}$ . Let  $p = [z_0, \dots, z_{j-1}]$  be in  $\mathbb{CP}_{a_0, \dots, a_{j-1}}^{j-1} \setminus Y_{j-1}$ , and let  $I$  be the set of  $i$  in  $\{0, \dots, j-1\}$  for which  $z_i \neq 0$ . Define  $l = \text{hcf}(a_i : i \in I)$  and  $m = \text{hcf}(l, a_j)$ . Then it turns out that  $\pi_j^{-1}(p)$  is  $k_j m/l$  points in  $Y_j$ . Clearly  $k_j m/l = k_j$  if  $l = m$ , that is, if  $l$  divides  $a_j$ .

Thus  $\pi_j$  is also branched over subsets of  $\mathbb{CP}_{a_0, \dots, a_{j-1}}^{j-1} \setminus Y_{j-1}$  corresponding to subsets  $I \subseteq \{0, \dots, j-1\}$  for which  $l = \text{hcf}(a_i : i \in I)$  does not divide  $a_j$ . To calculate  $\chi(Y_j)$  in this case we must modify (14) by adding in contributions from each such  $I$ . We will explain this when we meet it in examples later.

#### 6.4 How to find topological invariants of $Y$ , $Z$ and $M$

To calculate the cohomology and fundamental group of our complex orbifolds  $Y$  we will need the following result, a form of the *Lefschetz Hyperplane Theorem*. It is proved in Griffiths and Harris [8, p. 156] and Goresky and MacPherson [6, p. 153].

**Theorem 6.5.** *Let  $M$  be a compact,  $m$ -dimensional complex manifold,  $N$  a nonsingular hypersurface in  $M$ , and  $L$  the holomorphic line bundle over  $M$  associated to the divisor  $N$ . Suppose  $L$  is positive. Then:*

- (a) *the map  $H^k(M, \mathbb{C}) \rightarrow H^k(N, \mathbb{C})$  induced by the inclusion  $N \hookrightarrow M$  is an isomorphism for  $0 \leq k \leq m-2$  and injective for  $k = m-1$ , and*
- (b) *the map of homotopy groups  $\pi_k(N) \rightarrow \pi_k(M)$  induced by the inclusion  $N \hookrightarrow M$  is an isomorphism for  $0 \leq k \leq m-2$  and surjective for  $k = m-1$ .*

*The result also holds if  $M$  and  $N$  are orbifolds instead of manifolds, and  $N$  is a nonsingular hypersurface in the orbifold sense.*

Here is a procedure for calculating the fundamental group and Betti numbers of  $Y$ ,  $Z$  and  $M$ . The most difficult part is finding the Euler characteristic  $\chi(Y)$ , which we have already explained above.

- (a) Calculate  $\pi_1(Y)$ ,  $H^2(Y, \mathbb{C})$  and  $H^3(Y, \mathbb{C})$  explicitly. This can usually be done using Theorem 6.5. If  $Y$  is a hypersurface in  $\mathbb{CP}_{a_0, \dots, a_5}^5$  then  $\pi_1(Y) = \{1\}$ ,  $H^2(Y, \mathbb{C}) = \mathbb{C}$  and  $H^3(Y, \mathbb{C}) = 0$ . Verify that  $\pi_1(Y \setminus \{p_1, \dots, p_k\}) = \{1\}$  and  $h^{2,0}(Y) = 0$ , as in Condition 5.1.

- (b) Compute the Euler characteristic  $\chi(Y)$  of  $Y$ , as in §6.3.
- (c) Calculate  $H^2(Z, \mathbb{C})$  and  $H^3(Z, \mathbb{C})$  from  $H^2(Y, \mathbb{C})$  and  $H^3(Y, \mathbb{C})$ . Note that  $H^j(Z, \mathbb{C})$  is the  $\sigma$ -invariant part of  $H^j(Y, \mathbb{C})$ . Since  $\sigma$  swaps  $H^{p,q}(Y)$  and  $H^{q,p}(Y)$ , it follows that  $b^3(Z) = \frac{1}{2}b^3(Y)$ .
- (d) Compute the Euler characteristic  $\chi(Z)$  of  $Z$ . If  $\sigma$  fixes  $k$  points in  $Y$ , then this is given by  $\chi(Z) = \frac{1}{2}(\chi(Y) + k)$ .
- (e) From (c) we know  $b^2(Z)$  and  $b^3(Z)$ , and  $b^1(Z) = 0$  as  $\pi_1(Z)$  is finite. Thus we can calculate  $b^4(Z)$  using the formula  $b^4(Z) = \chi(Z) - 2 - 2b^2(Z) + 2b^3(Z)$ .
- (f) Now  $M$  was constructed in §5 by gluing  $X_{n_1}, \dots, X_{n_k}$  into  $Z$ , where  $n_j = 1$  or  $2$  and  $X_1, X_2$  are defined in §4. It is easy to show that the Betti numbers of  $X_1$  and  $X_2$  are  $b^1 = b^2 = b^3 = 0$  and  $b^4 = 1$ . Therefore the Betti numbers  $b^j(M)$  satisfy

$$(15) \quad b^j(M) = b^j(Z) \text{ for } j = 1, 2, 3, \text{ and } b^4(M) = b^4(Z) + k.$$

Also, Proposition 5.9 gives  $\pi_1(M)$ .

- (g) As  $M$  has metrics with holonomy Spin(7) or  $\mathbb{Z}_2 \times \text{SU}(4)$  by Theorem 5.14, we know that  $\hat{A}(M) = 1$ . Thus (2) gives

$$b^2(M) - b^3(M) - b_+^4(M) + 2b_-^4(M) + 25 = 0.$$

So we can calculate  $b_{\pm}^4(M)$  using the equations

$$(16) \quad \begin{aligned} b_+^4(M) &= \frac{1}{3}(b^2(M) - b^3(M) + 2b^4(M) + 25), \\ b_-^4(M) &= \frac{1}{3}(-b^2(M) + b^3(M) + b^4(M) - 25). \end{aligned}$$

## 6.5 A way of checking the answers

If you make a mistake at some stage in these calculations, which is quite easy to do, then you are likely not to notice unless your values for  $b_{\pm}^4(M)$  are not integers. Thus it is desirable to have some method for checking the answers. Here is a way of doing this. All of our examples have been checked for consistency in this way and others, but for brevity we will leave out the calculations.

Suppose we can compute the Hodge number  $h^{3,1}(Y)$ , using complex geometry. Then we can compute  $b_-^4(Z)$  using the formula

$$b_-^4(Z) = h^{3,1}(Y) + b^2(Y) - b^2(Z) - 1.$$

But as  $X_1$  and  $X_2$  have  $b_-^4 = 1$ , as in (15) we have  $b_-^4(M) = b_-^4(Z) + k$ . This gives an independent way of finding  $b_-^4(M)$ , which can be compared with your answer in part (g) above.

Now there is a complicated method for computing  $h^{3,1}(Y)$  involving spectral sequences, and also a much simpler method called the ‘polynomial deformation method’ which *does not always give the right answer*. Both are discussed by Green and Hübsch [7]. Here is a sketch of the polynomial deformation method.

For simplicity suppose that  $Y$  is a hypersurface of degree  $d$  in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$ . As  $Y$  is a Calabi-Yau orbifold,  $h^{3,1}(Y)$  is the dimension of the moduli space of complex structures on  $Y$ . We assume (this is *not* necessarily true) that every small deformation of  $Y$  is also a hypersurface of degree  $d$  in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$ , and that two nearby isomorphic hypersurfaces  $Y, Y'$  of degree  $d$  are related by an automorphism of  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$ .

If these assumptions hold, then  $h^{3,1}(Y) = m - n$ , where  $m$  is the dimension of the space of hypersurfaces of degree  $d$  in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$ , and  $n$  is the dimension of the automorphism group of  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$ . Both  $m$  and  $n$  are readily computed from  $a_0, \dots, a_5$  and  $d$ .

## 7. A simple example

Let  $Y$  be the hypersurface of degree 12 in  $\mathbb{C}\mathbb{P}_{1,1,1,1,4,4}^5$  given by

$$Y = \{[z_0, \dots, z_5] \in \mathbb{C}\mathbb{P}_{1,1,1,1,4,4}^5 : z_0^{12} + z_1^{12} + z_2^{12} + z_3^{12} + z_4^3 + z_5^3 = 0\}.$$

Then  $c_1(Y) = 0$ , as  $12 = 1 + 1 + 1 + 1 + 4 + 4$ , and  $Y$  is Kähler as  $\mathbb{C}\mathbb{P}_{1, \dots, 4}^5$  is Kähler. Calculation shows that  $Y$  has three singular points  $p_1 = [0, 0, 0, 0, 1, -1]$ ,  $p_2 = [0, 0, 0, 0, 1, e^{\pi i/3}]$  and  $p_3 = [0, 0, 0, 0, 1, e^{-\pi i/3}]$ , satisfying Condition 5.1.

We use the method of §6.3 to calculate the Euler characteristic  $\chi(Y)$ .

**Proposition 7.1.** *The orbifold  $Y$  defined above has  $\chi(Y) = 4887$ .*

*Proof.* Define  $Y_j$  and  $\pi_j$  as in §6.3, where  $Y_5 = Y$ . Then  $Y_1$  is the set of 12 points  $[z_0, z_1]$  in  $\mathbb{C}\mathbb{P}^1$  with  $z_0^{12} + z_1^{12} = 0$ , and so  $\chi(Y_1) = 12$ . Now  $\pi_2 : Y_2 \rightarrow \mathbb{C}\mathbb{P}^1$  is a 12-fold branched cover branched over  $Y_1$ , so by (14) we have

$$\chi(Y_2) = 12\chi(\mathbb{C}\mathbb{P}^1) - 11\chi(Y_1) = 12 \cdot 2 - 11 \cdot 12 = -108.$$

Similarly,  $\pi_3 : Y_3 \rightarrow \mathbb{C}\mathbb{P}^2$  is a 12-fold branched cover branched over  $Y_2$ , so that

$$\chi(Y_3) = 12\chi(\mathbb{C}\mathbb{P}^2) - 11\chi(Y_2) = 12 \cdot 2 - 11 \cdot (-108) = 1224.$$

And  $\pi_4 : Y_4 \rightarrow \mathbb{C}\mathbb{P}^3$  is a 3-fold branched cover of  $\mathbb{C}\mathbb{P}^3$  branched over  $Y_3$ , giving

$$\chi(Y_4) = 3\chi(\mathbb{C}\mathbb{P}^3) - 2\chi(Y_3) = 3 \cdot 4 - 2 \cdot 1224 = -2436.$$

Finally,  $\pi_5 : Y \rightarrow \mathbb{C}\mathbb{P}_{1,1,1,1,4}^4$  is a 3-fold branched cover of  $\mathbb{C}\mathbb{P}_{1,1,1,1,4}^4$  branched over  $Y_4$ , and so

$$\chi(Y) = 3\chi(\mathbb{C}\mathbb{P}_{1,1,1,1,4}^4) - 2\chi(Y_4) = 3 \cdot 5 - 2 \cdot (-2436) = 4887,$$

as we want. q.e.d.

**Proposition 7.2.** *The Betti numbers of  $Y$  are*

$$b^0(Y) = 1, \quad b^1(Y) = 0, \quad b^2(Y) = 1, \quad b^3(Y) = 0 \quad \text{and} \quad b^4(Y) = 4883.$$

Also  $Y \setminus \{p_1, p_2, p_3\}$  is simply-connected and  $h^{2,0}(Y) = 0$ .

*Proof.* Theorem 6.5 shows that  $H^k(Y, \mathbb{C}) \cong H^k(\mathbb{C}\mathbb{P}_{1,\dots,4}^5, \mathbb{C})$  for  $0 \leq k \leq 3$ . Since  $b^k(\mathbb{C}\mathbb{P}_{1,\dots,4}^5)$  is 1 for  $k$  even with  $0 \leq k \leq 10$  and 0 otherwise, this shows that  $b^0(Y) = b^2(Y) = 1$  and  $b^1(Y) = b^3(Y) = 0$ , and so  $b^4(Y) = 4883$  as  $\chi(Y) = 4887$ .

Theorem 6.5 also gives  $\pi_1(Y) \cong \pi_1(\mathbb{C}\mathbb{P}_{1,\dots,4}^5)$ , so  $Y$  is simply-connected. As the nonsingular set of  $\mathbb{C}\mathbb{P}_{1,\dots,4}^5$  is simply-connected, we can strengthen this to show that  $Y \setminus \{p_1, p_2, p_3\}$  is simply-connected. The isomorphism  $H^k(Y, \mathbb{C}) \cong H^k(\mathbb{C}\mathbb{P}_{1,\dots,4}^5, \mathbb{C})$  above identifies  $H^{p,q}(Y)$  with  $H^{p,q}(\mathbb{C}\mathbb{P}_{1,\dots,4}^5)$ , and so  $h^{p,q}(Y) = h^{p,q}(\mathbb{C}\mathbb{P}_{1,\dots,4}^5)$  for  $p+q \leq 3$ . Hence  $h^{2,0}(Y) = 0$ .

q.e.d.

Now define a map  $\sigma : Y \rightarrow Y$  by

$$\sigma : [z_0, \dots, z_5] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_5, \bar{z}_4].$$

As in Example 6.3, we find that  $\sigma$  is an antiholomorphic involution of  $Y$ , and that the fixed points of  $\sigma$  are exactly  $p_1, p_2, p_3$ . Thus Condition 5.1 holds for  $Y$  and  $\sigma$ . So we can apply the construction of §5, and resolve the orbifold  $Z = Y/\langle\sigma\rangle$  to get a compact 8-manifold  $M$ . Choosing  $n_j = 2$  for at least one  $j = 1, 2, 3$ , Proposition 5.9 shows that  $M$  is simply-connected, and Theorem 5.14 shows that  $M$  admits metrics with holonomy Spin(7).

**Theorem 7.3.** *This compact 8-manifold  $M$  has Betti numbers*

$$b^0 = 1, \quad b^1 = b^2 = b^3 = 0, \quad b^4 = 2446, \quad b^4_{\pm} = 1639 \quad \text{and} \quad b^4_{-} = 807.$$

*There exist metrics with holonomy  $\text{Spin}(7)$  on  $M$ , which form a smooth family of dimension 808.*

*Proof.* We first calculate the Betti numbers of  $Z$ . As  $\sigma$  fixes 3 points in  $Y$ , by properties of the Euler characteristic we find that  $\chi(Z) = \frac{1}{2}(\chi(Y) + 3)$ . But  $\chi(Y) = 4887$  by Proposition 7.1, so  $\chi(Z) = 2445$ . As  $H^k(Z, \mathbb{C})$  is the  $\sigma$ -invariant part of  $H^k(Y, \mathbb{C})$  we see from Proposition 7.2 that  $b^0(Z) = 1$  and  $b^1(Z) = b^3(Z) = 0$ . Also  $H^2(Y, \mathbb{C})$  is generated by  $[\omega_Y]$  and  $\sigma^*(\omega_Y) = -\omega_Y$ , so  $\sigma$  acts as  $-1$  on  $H^2(Y, \mathbb{C})$ , and  $H^2(Z, \mathbb{C}) = 0$ .

Thus  $b^0(Z) = 1$ ,  $b^1(Z) = b^2(Z) = b^3(Z) = 0$  and  $\chi(Z) = 2445$ , giving  $b^4(Z) = 2443$ . Equation (15) then gives the Betti numbers of  $M$ , and (16) gives  $b^4_{\pm}$ . Theorem 5.14 shows that there exist torsion-free  $\text{Spin}(7)$ -structures  $(\tilde{\Omega}, \tilde{g})$  on  $M$ , with  $\text{Hol}(\tilde{g}) = \text{Spin}(7)$  as  $M$  is simply-connected. By Theorem 2.4 the moduli space of metrics on  $M$  with holonomy  $\text{Spin}(7)$  is a smooth manifold of dimension  $1 + b^4_{-}(M) = 808$ .  
q.e.d.

## 7.1 A variation on this example

Here is a variation on the above, using the idea of *partial crepant resolution* mentioned in §6.1. Let  $Y$  be as above, but define  $\sigma' : Y \rightarrow Y$  by

$$\sigma' : [z_0, \dots, z_5] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_4, \bar{z}_5].$$

Then  $\sigma'$  is an antiholomorphic involution of  $Y$ , which fixes the point  $p_1 = [0, 0, 0, 0, 1, -1]$  in  $Y$ , and no other points. In particular,  $\sigma'$  swaps over the other two singular points  $p_2, p_3$ .

Thus  $Y$  and  $\sigma'$  do not satisfy Condition 5.1, because the fixed set of  $\sigma'$  is not the same as the singular set  $\{p_1, p_2, p_3\}$  of  $Y$ . To rectify this we resolve the singular points  $p_2, p_3$ . Let  $Y'$  be the blow-up of  $Y$  at  $p_2$  and  $p_3$ . This is a crepant resolution of  $Y$ , and so is also a Calabi–Yau orbifold.

Then  $Y'$  has just the one singular point  $p_1$ . The action of  $\sigma'$  on  $Y$  lifts to  $Y'$ , with sole fixed point  $p_1$ . Thus Condition 5.1 holds for  $Y'$  and  $\sigma'$ . Therefore we can apply the construction of §5 to  $Y'$  and  $\sigma'$ , so that  $Z' = Y'/\langle\sigma'\rangle$  is a compact  $\text{Spin}(7)$ -orbifold with one singular point  $p_1$  modelled on  $\mathbb{R}^8/G$ . Choosing  $n_1 = 2$  we get a resolution  $M'$  of

$Z'$ , which is a compact, simply-connected 8-manifold admitting metrics with holonomy Spin(7).

We shall calculate the topological invariants of  $Y'$  and  $M'$ .

**Proposition 7.4.** *The Betti numbers of  $Y'$  are*

$$b^0 = 1, \quad b^1 = 0, \quad b^2 = 3, \quad b^3 = 0 \quad \text{and} \quad b^4 = 4885, \quad \text{so that} \quad \chi(Y') = 4893.$$

Also,  $Y' \setminus \{p_1\}$  is simply-connected and  $h^{2,0}(Y') = 0$ .

*Proof.* By definition  $Y'$  is the blow-up of  $Y$  at  $p_2, p_3$ . Each blow-up fixes  $b^1$  and  $b^3$  and adds 1 to  $b^2$  and  $b^4$ . So the Betti numbers of  $Y'$  follow from Proposition 7.2. As  $Y \setminus \{p_1, p_2, p_3\}$  is simply-connected and  $h^{2,0}(Y) = 0$ , we see that  $Y' \setminus \{p_1\}$  is simply-connected and  $h^{2,0}(Y') = 0$ .  
q.e.d.

Here is the analogue of Theorem 7.3:

**Theorem 7.5.** *This compact 8-manifold  $M'$  has Betti numbers*

$$b^0 = 1, \quad b^1 = 0, \quad b^2 = 1, \quad b^3 = 0, \quad b^4 = 2444, \quad b^4_+ = 1638 \quad \text{and} \quad b^4_- = 806.$$

*There exist metrics with holonomy Spin(7) on  $M'$ , which form a smooth family of dimension 807.*

*Proof.* As  $\sigma$  fixes 1 point in  $Y'$  we have  $\chi(Z') = \frac{1}{2}(\chi(Y') + 1)$ , so  $\chi(Z') = 2447$  by the previous proposition. Since  $H^k(Z', \mathbb{C})$  is the  $\sigma$ -invariant part of  $H^k(Y', \mathbb{C})$  we have  $b^0(Z') = 1$  and  $b^1(Z') = b^3(Z') = 0$ . Now  $b^2(Y') = 3$ , and  $H^2(Y', \mathbb{C})$  is generated by  $[\omega_{Y'}]$  and the cohomology classes dual to the two exceptional divisors  $\mathbb{C}\mathbb{P}^3$  introduced by blowing up  $p_2$  and  $p_3$ . But  $\sigma'$  swaps  $p_2$  and  $p_3$ , so  $\sigma'_*$  swaps the corresponding classes in  $H^2(Y', \mathbb{C})$ , and  $\sigma'_*(\omega_{Y'}) = -\omega_{Y'}$  by definition. Therefore  $H^2(Y', \mathbb{C}) \cong \mathbb{C} \oplus \mathbb{C}^2$ , where  $\sigma'_*$  acts as 1 on  $\mathbb{C}$  and  $-1$  on  $\mathbb{C}^2$ . Hence  $H^2(Z', \mathbb{C}) \cong \mathbb{C}$ , and  $b^2(Z') = 1$ .

Thus  $b^0(Z') = b^2(Z') = 1$ ,  $b^1(Z') = b^3(Z') = 0$  and  $\chi(Z') = 2447$ , giving  $b^4(Z') = 2443$ . Equation (15) then gives the Betti numbers of  $M$ , and (16) gives  $b^4_{\pm}$ . Theorem 5.14 shows that there exist torsion-free Spin(7)-structures  $(\tilde{\Omega}, \tilde{g})$  on  $M$ , with  $\text{Hol}(\tilde{g}) = \text{Spin}(7)$  as  $M$  is simply-connected. By Theorem 2.4 the moduli space of metrics on  $M$  with holonomy Spin(7) is a smooth manifold of dimension  $1 + b^4_-(M) = 807$ .  
q.e.d.

Observe that the Betti numbers of  $M$  and  $M'$  in Theorems 7.3 and 7.5 are very similar. It is an interesting question whether one can regard  $M$  and  $M'$  as two different resolutions of some singular  $\text{Spin}(7)$ -manifold  $M_0$ , not necessarily an orbifold. We leave this as a research exercise for the reader; the answer is not as simple as it looks.

### 8. Examples from hypersurfaces in $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$

Here are three more examples based on hypersurfaces in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$ .

#### 8.1 A hypersurface of degree 16 in $\mathbb{C}\mathbb{P}_{1,1,1,1,4,8}^5$

Let  $Y$  be the hypersurface of degree 16 in  $\mathbb{C}\mathbb{P}_{1,1,1,1,4,8}^5$  given by

$$Y = \{[z_0, \dots, z_5] \in \mathbb{C}\mathbb{P}_{1,1,1,1,4,8}^5 : z_0^{16} + z_1^{16} + z_2^{16} + z_3^{16} + z_4^4 + z_5^2 = 0\}.$$

Then  $c_1(Y) = 0$ . We find that  $Y$  has two singular points  $p_1 = [0, 0, 0, 0, 1, i]$  and  $p_2 = [0, 0, 0, 0, 1, -i]$ , both satisfying Condition 5.1.

Following Propositions 7.1 and 7.2, we find that  $\chi(Y) = 9498$ , and

**Proposition 8.1.** *The Betti numbers of  $Y$  are*

$$b^0 = 1, \quad b^1 = 0, \quad b^2 = 1, \quad b^3 = 0 \quad \text{and} \quad b^4 = 9494.$$

Also  $Y \setminus \{p_1, p_2\}$  is simply-connected and  $h^{2,0}(Y) = 0$ .

Define an antiholomorphic involution  $\sigma : Y \rightarrow Y$  by

$$\sigma : [z_0, \dots, z_5] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_4, -\bar{z}_5].$$

The fixed points of  $\sigma$  are exactly the singular points  $p_1, p_2$  of  $Y$ . Thus Condition 5.1 holds for  $Y$  and  $\sigma$ , and we can apply the construction of §5. Resolving  $Z = Y/\langle\sigma\rangle$  gives a compact 8-manifold  $M$ . We choose at least one of  $n_1, n_2$  to be 2, so that  $M$  is simply-connected. Then as in Theorem 7.3, we get:

**Theorem 8.2.** *This compact 8-manifold  $M$  has Betti numbers*

$$b^0 = 1, \quad b^1 = b^2 = b^3 = 0, \quad b^4 = 4750, \quad b_+^4 = 3175 \quad \text{and} \quad b_-^4 = 1575.$$

*There exist metrics with holonomy  $\text{Spin}(7)$  on  $M$ , which form a smooth family of dimension 1576.*

## 8.2 A hypersurface of degree 24 in $\mathbb{C}\mathbb{P}_{1,1,1,1,8,12}^5$

Let  $Y$  be the hypersurface of degree 24 in  $\mathbb{C}\mathbb{P}_{1,1,1,1,8,12}^5$  given by

$$Y = \{[z_0, \dots, z_5] \in \mathbb{C}\mathbb{P}_{1,1,1,1,8,12}^5 : z_0^{24} + z_1^{24} + z_2^{24} + z_3^{24} + z_4^3 + z_5^2 = 0\}.$$

Then  $c_1(Y) = 0$ . We find that  $Y$  has one singular point  $p_1 = [0, 0, 0, 0, -1, 1]$ , which satisfies Condition 5.1.

Following Proposition 7.1, we find that  $\chi(Y) = 23\,325$ . Care is needed to get the right answer here. Define  $\pi_5 : Y \rightarrow \mathbb{C}\mathbb{P}_{1,1,1,1,8}^4$  by  $\pi_5 : [z_0, \dots, z_5] \mapsto [z_0, \dots, z_4]$ , and  $Y_4 \subset \mathbb{C}\mathbb{P}_{1,1,1,1,8}^4$  by

$$Y_4 = \{[z_0, \dots, z_4] \in \mathbb{C}\mathbb{P}_{1,1,1,1,8}^4 : z_0^{24} + z_1^{24} + z_2^{24} + z_3^{24} + z_4^3 = 0\}.$$

Then  $\pi_5$  is a double cover of  $\mathbb{C}\mathbb{P}_{1,1,1,1,8}^4$  branched over  $Y_4$  and the point  $[0, 0, 0, 0, 1]$  in  $\mathbb{C}\mathbb{P}_{1,1,1,1,8}^4$ . Hence we get

$$\chi(Y) = 2\chi(\mathbb{C}\mathbb{P}_{1,1,1,1,8}^4) - \chi(Y_4) - \chi([0, 0, 0, 0, 1]) = 9 - \chi(Y_4).$$

If we had not observed that  $\pi_5$  is also branched over  $[0, 0, 0, 0, 1]$ , then we would have got  $\chi(Y) = 23\,326$ , which is incorrect.

As in Proposition 7.2, we show:

**Proposition 8.3.** *The Betti numbers of  $Y$  are*

$$b^0 = 1, \quad b^1 = 0, \quad b^2 = 1, \quad b^3 = 0 \quad \text{and} \quad b^4 = 23\,231.$$

Also  $Y \setminus \{p_1\}$  is simply-connected and  $h^{2,0}(Y) = 0$ .

Define an antiholomorphic involution  $\sigma : Y \rightarrow Y$  by

$$\sigma : [z_0, \dots, z_5] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_4, \bar{z}_5].$$

The fixed points of  $\sigma$  are exactly the singular point  $p_1$  of  $Y$ . Thus Condition 5.1 holds for  $Y$  and  $\sigma$ , and choosing the simply-connected resolution  $M$  of  $Z = Y/\langle\sigma\rangle$ , in the usual way we get:

**Theorem 8.4.** *This compact 8-manifold  $M$  has Betti numbers*

$$b^0 = 1, \quad b^1 = b^2 = b^3 = 0, \quad b^4 = 11\,662, \quad b_+^4 = 7783 \quad \text{and} \quad b_-^4 = 3879.$$

*There exist metrics with holonomy Spin(7) on  $M$ , which form a smooth family of dimension 3880.*

This is the example with the largest value of  $b^4$  known to the author.

### 8.3 A hypersurface of degree 40 in $\mathbb{C}\mathbb{P}_{1,1,5,5,8,20}^5$

Here is a more complicated example, in which the hypersurface in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_5}^5$  has other singularities which must first be resolved. Let  $W$  be the hypersurface of degree 40 in  $\mathbb{C}\mathbb{P}_{1,1,5,5,8,20}^5$  given by

$$W = \{[z_0, \dots, z_5] \in \mathbb{C}\mathbb{P}_{1,1,5,5,8,20}^5 : z_0^{40} + z_1^{40} + z_2^8 + z_3^8 + z_4^5 + z_5^2 = 0\}.$$

Then  $c_1(W) = 0$ . The singularities of  $W$  are the disjoint union of the single point  $p_1 = [0, 0, 0, 0, -1, 1]$  and the nonsingular curve  $\Sigma$  of genus 3 given by

$$\Sigma = \{[0, 0, z_2, z_3, 0, z_5] \in \mathbb{C}\mathbb{P}_{1,1,5,5,8,20}^5 : z_2^8 + z_3^8 + z_5^2 = 0\}.$$

The singular point at  $p_1$  satisfies Condition 5.1. The singularity at each point of  $\Sigma$  is modelled on  $\mathbb{C} \times \mathbb{C}^3 / \mathbb{Z}_5$ , where the generator  $\beta$  of  $\mathbb{Z}_5$  acts on  $\mathbb{C}^3$  by

$$\beta : (z_0, z_1, z_4) \mapsto (e^{2\pi i/5} z_0, e^{2\pi i/5} z_1, e^{-4\pi i/5} z_4).$$

Now the singularity  $\mathbb{C}^3 / \mathbb{Z}_5$  normal to  $\Sigma$  in  $W$  has a unique crepant resolution  $X$ , which can be described using toric geometry. Let  $Y$  be the partial crepant resolution of  $W$  which resolves the singularities at  $\Sigma$  using  $X$ , but leaves the singular point  $p_1$  unchanged.

**Proposition 8.5.** *The Betti numbers of  $Y$  are*

$$b^0 = 1, \quad b^1 = 0, \quad b^2 = 3, \quad b^3 = 12, \quad \text{and} \quad b^4 = 7453.$$

Also  $Y \setminus \{p_1\}$  is simply-connected and  $h^{2,0}(Y) = 0$ .

*Proof.* Calculating the Betti numbers of  $W$  in the usual way gives

$$(17) \quad b^0(W) = 1, \quad b^1(W) = 0, \quad b^2(W) = 1, \quad b^3(W) = 0, \quad b^4(W) = 7449.$$

As  $W$  is modelled on  $\mathbb{C} \times \mathbb{C}^3 / \mathbb{Z}_5$  at each point of  $\Sigma$ , the resolution  $Y$  is modelled on  $\mathbb{C} \times X$ . Since  $b^2(X) = b^4(X) = 2$ , the Betti numbers of  $Y$  satisfy

$$b^k(Y) = b^k(W) + 2b^{k-2}(\Sigma) + 2b^{k-4}(\Sigma).$$

But  $\Sigma$  has genus 3, and so its Betti numbers are  $b^0(\Sigma) = b^2(\Sigma) = 1$  and  $b^1(\Sigma) = 6$ . Combining this with (17) gives the Betti numbers of  $Y$ . The last part follows as in Proposition 7.2.    q.e.d.

Define  $\sigma : W \rightarrow W$  by

$$\sigma : [z_0, \dots, z_5] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_4, \bar{z}_5].$$

The only fixed point of  $\sigma$  is  $p_1$ . Moreover,  $\sigma$  lifts to the resolution  $Y$  of  $W$ , and  $\sigma : Y \rightarrow Y$  is an antiholomorphic involution which fixes only  $p_1$  in  $Y$ . Thus Condition 5.1 holds for  $Y$  and  $\sigma$ , and we can apply the construction of §5, and resolve  $Z = Y/\langle\sigma\rangle$  to get a simply-connected 8-manifold  $M$ . Proceeding in the usual way, the end result is

**Theorem 8.6.** *This compact 8-manifold  $M$  has Betti numbers*

$$b^0 = 1, \quad b^1 = b^2 = 0, \quad b^3 = 6, \quad b^4 = 3730, \quad b_+^4 = 2493 \quad \text{and} \quad b_-^4 = 1237.$$

*There exist metrics with holonomy Spin(7) on  $M$ , which form a smooth family of dimension 1238.*

Note that  $b^3 > 0$  in this example; this is because the resolution of the singular curve  $\Sigma$  contributes  $H^1(\Sigma, \mathbb{C}) \otimes H^2(X, \mathbb{C}) = \mathbb{C}^6 \otimes \mathbb{C}^2 = \mathbb{C}^{12}$  to  $H^3(Y, \mathbb{C})$ . Half of this  $\mathbb{C}^{12}$  is  $\sigma$ -invariant, and so pushes down to  $H^3(Z, \mathbb{C})$  and lifts to  $H^3(M, \mathbb{C})$ .

### 9. A hypersurface in $\mathbb{CP}_{1,1,1,1,2,2}^5$ over $\mathbb{Z}_2$

Let  $W$  be the hypersurface of degree 8 in  $\mathbb{CP}_{1,1,1,1,2,2}^5$  given by

$$W = \{[z_0, \dots, z_5] \in \mathbb{CP}_{1,1,1,1,2,2}^5 : z_0^8 + z_1^8 + z_2^8 + z_3^8 + z_4^4 + z_5^4 = 0\}.$$

Then  $c_1(W) = 0$ . We find that  $W$  has four singular points  $p_1, \dots, p_4$  modelled on  $\mathbb{C}^4/\{\pm 1\}$ , given by

$$\begin{aligned} [0, 0, 0, 0, 1, e^{\pi i/4}], & \quad [0, 0, 0, 0, 1, e^{3\pi i/4}], \\ [0, 0, 0, 0, 1, e^{5\pi i/4}], & \quad [0, 0, 0, 0, 1, e^{7\pi i/4}]. \end{aligned}$$

Define  $\beta : W \rightarrow W$  by

$$\beta : [z_0, \dots, z_5] \mapsto [iz_0, iz_1, iz_2, iz_3, z_4, z_5].$$

Then  $\beta^2 = 1$ , as  $[z_0, \dots, z_5] = [-z_0, -z_1, -z_2, -z_3, z_4, z_5]$  in  $\mathbb{CP}_{1,1,1,1,2,2}^5$ . The fixed set of  $\beta$  is the four points  $p_1, \dots, p_4$  together with the compact complex surface  $S$  in  $W$ , given by

$$S = \{[z_0, z_1, z_2, z_3, 0, 0] \in \mathbb{CP}_{1,1,1,1,2,2}^5 : z_0^8 + z_1^8 + z_2^8 + z_3^8 = 0\}.$$

Thus  $W/\langle\beta\rangle$  is a compact complex orbifold. Its singular set is the disjoint union of  $p_1, \dots, p_4$  and  $S$ . Each singular point  $p_j$  is modelled on  $\mathbb{C}^4/\mathbb{Z}_4$ , where the generator  $\alpha$  of  $\mathbb{Z}_4$  acts on  $\mathbb{C}^4$  by (6). Each singular point in  $S$  is locally modelled on  $\mathbb{C}^2 \times \mathbb{C}^2/\{\pm 1\}$ .

Let  $Y$  be the blow-up of  $W/\langle\beta\rangle$  along  $S$ . Because the singularities normal to  $S$  are modelled on  $\mathbb{C}^2/\{\pm 1\}$ , this is a *partial crepant resolution*. So  $Y$  is a compact complex orbifold with isolated singular points  $p_1, \dots, p_4$ , modelled on  $\mathbb{C}^4/\langle\alpha\rangle$ . Now  $c_1(W) = 0$ , so  $c_1(W/\langle\beta\rangle) = 0$ , and as  $Y$  is a partial crepant resolution of  $W/\langle\beta\rangle$  we see that  $c_1(Y) = 0$ .

**Proposition 9.1.** *The Betti numbers of  $Y$  are*

$$b^0 = 1, \quad b^1 = 0, \quad b^2 = 2, \quad b^3 = 0 \quad \text{and} \quad b^4 = 1806.$$

Also  $Y \setminus \{p_1, \dots, p_4\}$  is simply-connected and  $h^{2,0}(Y) = 0$ .

*Proof.* As in Proposition 7.1, we find  $\chi(W) = 2708$  and  $\chi(S) = 304$ . Thus

$$\begin{aligned} \chi(W/\langle\beta\rangle) &= \frac{1}{2}(\chi(W) + \chi(4 \text{ points}) + \chi(S)) \\ &= \frac{1}{2}(2708 + 4 + 304) = 1508. \end{aligned}$$

Using Theorem 6.5 we find that  $W$  has  $b^0 = b^2 = 1$  and  $b^1 = b^3 = 0$ , and it soon follows that  $W/\langle\beta\rangle$  also has  $b^0 = b^2 = 1$  and  $b^1 = b^3 = 0$ . Since  $\chi(W/\langle\beta\rangle) = 1508$  we see that  $b^4(W/\langle\beta\rangle) = 1504$ .

Now  $Y$  is the blow-up of  $W/\langle\beta\rangle$  along  $S$ , so that each point of  $S$  is replaced by a copy of  $\mathbb{C}\mathbb{P}^1$ . It can be shown that the Betti numbers of  $Y$  satisfy

$$(18) \quad b^k(Y) = b^k(W/\langle\beta\rangle) + b^{k-2}(S).$$

But  $S$  can be thought of as an octic in  $\mathbb{C}\mathbb{P}^3$ , and by the usual method we find that the Betti numbers of  $S$  are  $b^0 = 1$ ,  $b^1 = 0$ ,  $b^2 = 302$ ,  $b^3 = 0$  and  $b^4 = 1$ . Combining these with (18) and the Betti numbers of  $W/\langle\beta\rangle$  above gives the Betti numbers of  $Y$ . The last part follows as usual. q.e.d.

Define an antiholomorphic involution  $\sigma : W \rightarrow W$  by

$$\sigma : [z_0, \dots, z_5] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_5, \bar{z}_4].$$

The fixed points of  $\sigma$  are exactly the singular points  $p_1, \dots, p_4$  of  $W$ . Also  $\sigma$  commutes with  $\beta$ , and acts freely on  $S$ . Hence  $\sigma$  pushes down to

an antiholomorphic involution of  $W/\langle\beta\rangle$ , and lifts to the blow-up  $Y$ , to give an antiholomorphic involution  $\sigma : Y \rightarrow Y$  with fixed points  $p_1, \dots, p_4$ .

Thus Condition 5.1 holds for  $Y$  and  $\sigma$ , and in the usual way we choose a simply-connected resolution  $M$  of  $Z = Y/\langle\sigma\rangle$  satisfying:

**Theorem 9.2.** *This compact 8-manifold  $M$  has Betti numbers*

$$b^0 = 1, \quad b^1 = b^2 = b^3 = 0, \quad b^4 = 910, \quad b_+^4 = 615 \quad \text{and} \quad b_-^4 = 295.$$

*There exist metrics with holonomy  $\text{Spin}(7)$  on  $M$ , which form a smooth family of dimension 296.*

### 9.1 A variation on this example

We shall use the idea of §7.1 to make a second 8-manifold  $M'$  from the orbifold  $Y$  above. Let  $W$  and  $Y$  be as in §9.1, but define  $\sigma' : W \rightarrow W$  by

$$\sigma' : [z_0, \dots, z_5] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_4, i\bar{z}_5].$$

Then  $\sigma'$  pushes down to  $W/\langle\beta\rangle$  and lifts to  $Y$  as above. However, this time  $\sigma'$  fixes the singular points  $p_1 = [0, 0, 0, 0, 1, e^{\pi i/4}]$  and  $p_2 = [0, 0, 0, 0, 1, e^{5\pi i/4}]$  in  $Y$ , but it swaps round  $p_3 = [0, 0, 0, 0, 1, e^{3\pi i/4}]$  and  $p_4 = [0, 0, 0, 0, 1, e^{7\pi i/4}]$ .

Thus, Condition 5.1 does not hold for  $Y$  and  $\sigma'$ , as the fixed set  $\{p_1, p_2\}$  of  $\sigma'$  does not coincide with the singular set  $\{p_1, \dots, p_4\}$  of  $Y$ . So let  $Y'$  be the blow-up of  $Y$  at  $p_3$  and  $p_4$ . Then  $Y'$  is a partial crepant resolution of  $Y$ , as the singularities at  $p_3, p_4$  are modelled on  $\mathbb{C}^4/\mathbb{Z}_4$ . The singularities of  $Y'$  are  $p_1, p_2$ , and  $\sigma'$  lifts to an antiholomorphic involution of  $Y'$  fixing only  $p_1$  and  $p_2$ .

We find the Betti numbers of  $Y'$  by adding contributions to those of  $Y$ , as in §7.1. Applying the construction of §5 to  $Y'$  and  $\sigma'$  gives a simply-connected 8-manifold  $M'$ , such that

**Theorem 9.3.** *This compact 8-manifold  $M'$  has Betti numbers*

$$b^0 = 1, \quad b^1 = 0, \quad b^2 = 1, \quad b^3 = 0, \quad b^4 = 908, \quad b_+^4 = 614 \quad \text{and} \quad b_-^4 = 294.$$

*There exist metrics with holonomy  $\text{Spin}(7)$  on  $M'$ , which form a smooth family of dimension 295.*

### 10. Complete intersections in $\mathbb{C}\mathbb{P}_{a_0, \dots, a_6}^6$

We now try starting with the intersection of two hypersurfaces in  $\mathbb{C}\mathbb{P}_{a_0, \dots, a_6}^6$ .

#### 10.1 The intersection of two octics in $\mathbb{C}\mathbb{P}_{1,1,1,1,4,4,4}^6$

Let  $Y$  be the complete intersection of two octics in  $\mathbb{C}\mathbb{P}_{1,1,1,1,4,4,4}^6$  given by

$$Y = \{[z_0, \dots, z_5] \in \mathbb{C}\mathbb{P}_{1,1,1,1,4,4,4}^6 : \begin{aligned} z_0^8 + z_1^8 + 2iz_2^8 - 2iz_3^8 + z_4^2 - z_5^2 &= 0, \\ 2iz_0^8 - 2iz_1^8 + z_2^8 + z_3^8 + z_4^2 - z_5^2 &= 0 \end{aligned}\}.$$

Then  $c_1(Y) = 0$ . We find that  $Y$  has 4 singular points

$$\begin{aligned} p_1 &= [0, 0, 0, 0, 1, 1, 1], & p_2 &= [0, 0, 0, 0, 1, -1, -1], \\ p_3 &= [0, 0, 0, 0, 1, 1, -1] & \text{and } p_4 &= [0, 0, 0, 0, 1, -1, 1], \end{aligned}$$

satisfying Condition 5.1.

By adapting the method of §6.3 we can show that  $\chi(Y) = 2580$ , and applying Theorem 6.5 twice we find that  $b^k(Y) = b^k(\mathbb{C}\mathbb{P}_{1, \dots, 4}^6)$  for  $0 \leq k \leq 3$ . Thus we prove:

**Proposition 10.1.** *The Betti numbers of  $Y$  are*

$$b^0 = 1, \quad b^1 = 0, \quad b^2 = 1, \quad b^3 = 0 \quad \text{and} \quad b^4 = 2576,$$

Also  $Y \setminus \{p_1, \dots, p_4\}$  is simply-connected and  $h^{2,0}(Y) = 0$ .

Define an antiholomorphic involution  $\sigma : Y \rightarrow Y$  by

$$\sigma : [z_0, \dots, z_6] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_4, \bar{z}_5, \bar{z}_6].$$

The fixed points of  $\sigma$  are exactly the singular points  $p_1, \dots, p_4$  of  $Y$ , and Condition 5.1 holds for  $Y$  and  $\sigma$ . Proceeding in the usual way, we set  $Z = Y/\langle\sigma\rangle$  and resolve  $Z$  to get a simply-connected 8-manifold  $M$ , which satisfies:

**Theorem 10.2.** *This compact 8-manifold  $M$  has Betti numbers*

$$b^0 = 1, \quad b^1 = b^2 = b^3 = 0, \quad b^4 = 1294, \quad b_+^4 = 871 \quad \text{and} \quad b_-^4 = 423.$$

*There exist metrics with holonomy  $\text{Spin}(7)$  on  $M$ , which form a smooth family of dimension 424.*

## 10.2 A variation on this example

Now let  $Y$  be as in §10.1, but define  $\sigma' : Y \rightarrow Y$  by

$$\sigma' : [z_0, \dots, z_6] \mapsto [\bar{z}_3, -\bar{z}_2, \bar{z}_1, -\bar{z}_0, \bar{z}_4, \bar{z}_6, \bar{z}_5].$$

Then  $\sigma'$  is an antiholomorphic involution, with fixed points  $p_1$  and  $p_2$ , which swaps round  $p_3$  and  $p_4$ . Following the method of §7.1, define  $Y'$  to be the blow-up of  $Y$  at  $p_3$  and  $p_4$ . Then  $Y'$  is a Calabi–Yau orbifold,  $\sigma'$  lifts to  $Y'$ , and Condition 5.1 holds for  $Y'$  and  $\sigma'$ .

As usual we set  $Z' = Y'/\langle \sigma' \rangle$  and resolve  $Z'$  to get a simply-connected 8-manifold  $M'$ , such that we have

**Theorem 10.3.** *This compact 8-manifold  $M'$  has Betti numbers*

$$b^0 = 1, \quad b^1 = 0, \quad b^2 = 1, \quad b^3 = 0, \quad b^4 = 1292, \quad b_+^4 = 870 \quad \text{and} \quad b_-^4 = 422.$$

*There exist metrics with holonomy  $\text{Spin}(7)$  on  $M'$ , which form a smooth family of dimension 423.*

## 10.3 The intersection of two 12-tics in $\mathbb{CP}_{3,3,3,3,4,4,4}^6$

Let  $P(z_4, z_5, z_6)$  and  $Q(z_4, z_5, z_6)$  be generic homogeneous cubic polynomials with real coefficients, and define  $W$  to be the complete intersection of two 12-tics in  $\mathbb{CP}_{3,3,3,3,4,4,4}^6$  given by

$$W = \{[z_0, \dots, z_5] \in \mathbb{CP}_{3,3,3,3,4,4,4}^6 : z_0^4 + z_1^4 + z_2^4 + z_3^4 + P(z_4, z_5, z_6) = 0, \\ iz_0^4 - iz_1^4 + 2iz_2^4 - 2iz_3^4 + Q(z_4, z_5, z_6) = 0\}.$$

Then  $c_1(W) = 0$ . As  $P$  and  $Q$  are generic, the singular set of  $W$  is the disjoint union of the 9 points  $p_1, \dots, p_9$  given by

$$\{[0, 0, 0, 0, z_4, z_5, z_6] \in \mathbb{CP}_{3,3,3,3,4,4,4}^6 : P(z_4, z_5, z_6) = Q(z_4, z_5, z_6) = 0\},$$

and the curve  $\Sigma$  of genus 33 given by

$$\Sigma = \{[z_0, z_1, z_2, z_3, 0, 0, 0] \in \mathbb{CP}_{3,3,3,3,4,4,4}^6 : z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0, \\ iz_0^4 - iz_1^4 + 2iz_2^4 - 2iz_3^4 = 0\}.$$

Each point  $p_j$  satisfies Condition 5.1, and each point of  $\Sigma$  is modelled on  $\mathbb{C} \times \mathbb{C}^3/\mathbb{Z}_3$ , where the action of  $\mathbb{Z}_3$  on  $\mathbb{C}^3$  is generated by

$$\beta : (z_4, z_5, z_6) \mapsto (e^{2\pi i/3} z_4, e^{2\pi i/3} z_5, e^{2\pi i/3} z_6).$$

Define an antiholomorphic involution  $\sigma : W \rightarrow W$  by

$$\sigma : [z_0, \dots, z_6] \mapsto [\bar{z}_1, -\bar{z}_0, \bar{z}_3, -\bar{z}_2, \bar{z}_4, \bar{z}_5, \bar{z}_6].$$

Then the fixed points of  $\sigma$  are some subset of  $\{p_1, \dots, p_9\}$ . Exactly which subset depends on the choice of  $P$  and  $Q$ , but  $\sigma$  must fix an odd number of the  $p_j$ , as the remaining  $p_j$  are swapped in pairs.

So let  $\sigma$  fix  $2k+1$  of the  $p_j$ , for some  $k = 0, \dots, 4$ , and number the  $p_j$  such that  $\sigma$  fixes  $p_1, \dots, p_{2k+1}$  and swaps  $p_{2k+2}, \dots, p_9$  in pairs. Define  $Y_k$  to be the blow-up of  $W$  along  $\Sigma$  and at the points  $p_{2k+2}, \dots, p_9$ . Then  $Y_k$  is a partial crepant resolution of  $W$ . Thus  $Y_k$  is a Calabi–Yau orbifold, with singular points  $p_1, \dots, p_{2k+1}$ . Also  $\sigma$  lifts to  $Y_k$  to give an antiholomorphic involution  $\sigma : Y_k \rightarrow Y_k$  with fixed points  $p_1, \dots, p_{2k+1}$ .

It can be shown that we can choose  $P$  and  $Q$  so that  $k$  takes any value in  $\{0, 1, 2, 3, 4\}$ . For example, if  $P = z_4^3 - z_5^3$  and  $Q = z_4^3 - z_6^3$  then  $\sigma$  fixes only  $p_1 = [0, 0, 0, 0, 1, 1, 1]$ , so that  $k = 0$ , but if  $P = z_4^2 z_5 - z_5^3$  and  $Q = z_4^2 z_6 - z_6^3$  then  $\sigma$  fixes the 9 points  $[0, 0, 0, 0, 1, z_5, z_6]$  for  $z_5, z_6 \in \{1, 0, -1\}$ , and  $k = 4$ .

Combining the methods used to prove Propositions 8.5 and 10.1, we get

**Proposition 10.4.** *The Betti numbers of  $Y_k$  are  $b^0 = 1$ ,  $b^1 = 0$ ,  $b^2 = 10 - 2k$ ,  $b^3 = 66$ ,  $b^4 = 395 - 2k$ ,  $b_+^4 = 262$  and  $b_-^4 = 133 - 2k$ . Also  $Y_k \setminus \{p_1, \dots, p_{2k+1}\}$  is simply-connected, and  $h^{2,0}(Y_k) = 0$ .*

In the usual way we resolve  $Z_k = Y_k / \langle \sigma \rangle$  to get  $M_k$ , which satisfies

**Theorem 10.5.** *For each  $k = 0, \dots, 4$  there is a compact 8-manifold  $M_k$  with Betti numbers  $b^0 = 1$ ,  $b^1 = 0$ ,  $b^2 = 4 - k$ ,  $b^3 = 33$ ,  $b^4 = 200 + 2k$ ,  $b_+^4 = 132 + k$  and  $b_-^4 = 68 + k$ . There exist metrics with holonomy  $\text{Spin}(7)$  on  $M_k$ , which form a smooth family of dimension  $69 + k$ .*

These examples have the largest value of  $b^3$  and the smallest values of  $b^4$  that the author has found using this construction.

## 11. Conclusions

In Table 1 we give the Betti numbers  $(b^2, b^3, b^4)$  of the compact 8-manifolds with holonomy  $\text{Spin}(7)$  that we constructed in §7–§10. There are 14 sets of Betti numbers, none of which coincide with any in [10], so we have found at least 14 topologically distinct new examples of compact 8-manifolds with holonomy  $\text{Spin}(7)$ .

Table 1: Betti numbers  $(b^2, b^3, b^4)$  of compact Spin(7)-manifolds

(4, 33, 200)	(3, 33, 202)	(2, 33, 204)	(1, 33, 206)	(0, 33, 208)
(1, 0, 908)	(0, 0, 910)	(1, 0, 1292)	(0, 0, 1294)	(1, 0, 2444)
(0, 0, 2446)	(0, 6, 3730)	(0, 0, 4750)	(0, 0, 11 662)	

The examples of §7–§10 are by no means all the manifolds that can be produced using the methods of this paper, but only a selection chosen for their simplicity and to illustrate certain techniques. Readers are invited to look for other examples themselves; the author would be particularly interested in examples which have especially large or small values of  $b^4$ .

We have also chosen to restrict our attention in §5–§10 to orbifolds  $Y$  all of whose singularities are modelled on  $\mathbb{C}^4/\mathbb{Z}_4$ , where the generator  $\alpha$  of  $\mathbb{Z}_4$  acts as in (6). This is not a necessary restriction, and there are other types of singularities for  $Y$  and  $Z$  for which the construction would work, such as the  $\mathbb{R}^8/G^n$  considered in §4.3, and which occur in suitable orbifolds  $Y$ . However, the author has not found many such  $Y$ ; the  $\mathbb{C}^4/\mathbb{Z}_4$  singularities do seem to be the easiest to construct.

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