

## EIGENFUNCTIONS WITH FEW CRITICAL POINTS

DMITRY JAKOBSON &amp; NIKOLAI NADIRASHVILI

**Abstract**

We construct a sequence of eigenfunctions on  $\mathbf{T}^2$  with a bounded number of critical points.

S. T. Yau raised a question about the number and distribution of critical points of eigenfunctions of the Laplacian on a Riemannian manifold ([4, # 76], [5, # 43]). In [6] he investigated this problem in two dimensions and proved, in particular, that under certain curvature assumptions every eigenfunction has a critical point where the critical value is uniformly bounded. Here we prove

**Theorem 1.** *There exists a metric on the two-dimensional torus and a sequence of eigenfunctions such that the corresponding eigenvalues go to infinity but the number of critical points remains bounded.*

This answers in the negative the question raised in [4]; however, our metric is quite special, and it is possible that for a generic metric the number of critical points increases with the growth of the eigenvalue.

The main idea of our construction is to consider a sequence of eigenfunctions  $f_n(x, y) = \sin(nx + y)$  (on  $\mathbf{T}^2$  with the flat metric) whose critical points lie on a union of two line segments, and then change a metric in such a way that instead of two critical “ridges” we shall have a bounded number of critical points.

We consider a Liouville metric (cf. [3])

$$(1) \quad q(x) (dx^2 + dy^2)$$

---

Received December 7, 1999. The first author was partially supported by NSF grant DMS-9802947, and the second author by NSF grant DMS-9971932.

*Key words and phrases.* Laplacian, eigenfunction, critical point, WKB  
 1991 *Mathematics Subject Classification.* 34E20, 58J50

on the torus  $\mathbf{T}^2 = \{(x, y) : 0 \leq x, y \leq 2\pi\}$ . Here  $q$  is a smooth periodic function whose properties we shall specify later. Joint eigenfunctions of the Laplacian  $\Delta = (1/q(x))(\partial^2/\partial x^2 + \partial^2/\partial y^2)$  and  $\partial/\partial y$  have the form

$$f(x, y) = \varphi(x) e^{imy}, \quad m \in \mathbf{Z},$$

where  $\varphi$  satisfies an equation (cf. [3, (4.3)])

$$\varphi''(x) + (\lambda q(x) - m^2) \varphi(x) = 0.$$

In the rest of the paper we shall choose  $m = 1$ . Accordingly,  $\varphi$  satisfies

$$(2) \quad \varphi''(x) + (\lambda q(x) - 1) \varphi(x) = 0.$$

We choose  $q$  to be a periodic function of period  $\pi/2$  and let  $\varphi$  satisfy (2) on  $[0, \pi/2]$  with boundary conditions

$$(3) \quad \varphi'(0) = \varphi(\pi/2) = 0.$$

Then the function  $\varphi_1$  defined by

$$(4) \quad \varphi_1(x) = \begin{cases} \varphi(x), & x \in [0, \pi/2], \\ -\varphi(\pi - x), & x \in [\pi/2, \pi], \\ -\varphi(x - \pi), & x \in [\pi, 3\pi/2], \\ \varphi(2\pi - x), & x \in [3\pi/2, 2\pi]. \end{cases}$$

and its shift  $\varphi_2$  defined by

$$(5) \quad \varphi_2(x) = \varphi_1(x + \pi/2)$$

are two linearly independent solutions of (2) on  $[0, 2\pi]$  (we are considering  $x \bmod 2\pi$  and using the periodicity of  $q$ ).

We denote the spectrum of (2) on  $[0, \pi/2]$  with boundary conditions (3) by  $0 < \lambda_1 < \lambda_2 < \dots$ . Then every  $\lambda_j$  is an eigenvalue of multiplicity two of the equation (2) on  $[0, 2\pi]$  with periodic boundary conditions (the corresponding eigenfunctions  $\varphi_{1,2}(j)$  are given by (4) and (5)). We next investigate the function  $g_j(x)$  defined by

$$(6) \quad g_j(x) = \varphi_1(j)(x)^2 + \varphi_2(j)(x)^2.$$

**Lemma 2.** *There exists  $C > 0$  such that for  $\lambda_j$  large enough the function  $g_j(x)$  is monotonic outside the union of  $(C/\lambda_j)$ -neighborhoods of the critical points of  $q(x)$ .*

*Proof.* The solutions  $\psi_j(x) = \varphi_1(j)(x) + i \varphi_2(j)(x)$  of the equation (2) can be asymptotically expanded in  $t = \sqrt{\lambda}$  (cf. [1], [2, p. 34]). We can make a change of variable (cf. [2, p. 32])

$$\psi(x) = \exp \left\{ d_j \int_0^x \sum_{k=-1}^{\infty} t^{-k} \alpha_k(s) ds \right\}$$

in the equation

$$\psi'' + (t^2 q - 1) \psi = 0.$$

Here  $\varphi_j(0) = 1$  and

$$d_j = \frac{\psi_j'(0)}{\sum_{k=-1}^{\infty} t_j^{-k} \alpha_k(0)}$$

is the normalization constant.

Further substitution  $\psi'/\psi = w$  reduces the equation above to the Ricatti equation

$$w' + w^2 + t^2 q(x) - 1 = 0$$

for  $w = \sum_{k=-1}^{\infty} t^{-k} \alpha_k(x)$  from which  $\alpha_k$ -s can be found inductively from the asymptotic expansion in  $t$ .

In particular,  $\alpha_{-1}^2 + q = 0$ . We assume that  $q(x)$  is not identically constant and that

$$q(x) \gg 1,$$

so we can choose

$$\alpha_{-1}(x) = i \sqrt{q(x)}.$$

The next few terms are given by

$$(7) \quad \left\{ \begin{array}{l} \alpha_0 = -q'/(4q), \\ \alpha_1 = (-i) \left( 1/(2q^{1/2}) - 5(q')^2/(32q^{5/2}) + q''/(8q^{3/2}) \right), \\ \alpha_2 = (q''' - 4q')/(16q^2) - 9q'q''/(32q^3) + 15(q')^3/(64q^4), \\ \alpha_3 = \frac{-i}{8q^{3/2}} \left( 1 + \frac{6q'' - q'''}{4q} + \frac{28q'q''' + 19(q'')^2 - 50(q')^2}{16q^2} \right. \\ \quad \left. + \frac{1105(q')^4}{256q^4} - \frac{221(q'')^2q''}{32q^3} \right), \\ \alpha_4 = \frac{q(q'''' - 8q''' + 16q'') + 17q''q''' + 10q'q'''' - 54q'q''}{64q^4} \\ \quad + \frac{3q'(80(q')^2 - 102(q'')^2 - 75q'q''')}{256q^5} \\ \quad + \frac{1695(q')^3(2qq'' - (q')^2)}{1024q^7}. \end{array} \right.$$

Let  $h_j(x)$  be a constant multiple of the logarithmic derivative of the function  $g_j(x) = \psi_j(x) \overline{\psi_j(x)}$ ,

$$h_j(x) = \frac{g_j'(x)}{2d_j g_j(x)}.$$

It has an asymptotic expansion in  $\lambda_j = t_j^2$  given by

$$(8) \quad h_j(x) = \sum_{k=0}^{\infty} \alpha_{2k}(x) \lambda_j^{-k}$$

The error term in the  $n$ -term expansion is  $O(\lambda_j^{-n})$ , uniformly in  $x$  and  $j$  (cf. [1], [2]). The lemma now follows from (7) and (8). q.e.d.

We next investigate the behavior of  $g_j(x)$  in the  $C/\lambda_j$ -neighborhoods of the critical points of  $q(x)$ . We assume that  $q(x) = q(\pi/2 - x)$  and that  $q$  has a unique minimum at 0 and a unique maximum at  $\pi/4$  on  $[0, \pi/2)$ . The Taylor expansion of  $q$  at a critical point  $x_0$  has the form

$$(9) \quad q(x_0 + x) = a_0(1 + a_1x^2 + \sum_{j=2}^{\infty} a_jx^{2j}),$$

where  $a_0 > 0, a_1 > 0$  at  $x_0 = 0$  and  $a_0 > 0, a_1 < 0$  at  $x_0 = \pi/4$ . It follows from the symmetries of  $q$  that  $g_j(x) = g_j(-x)$ ,  $g_j(\pi/4 - x) = g_j(\pi/4 + x)$ .

We next differentiate (8) (cf. [1], [2]) and substitute (9) into the resulting expression to study the asymptotic expansions of  $h'_j(x) = h'(x)$  (in  $x$  and  $\lambda_j = \lambda$ ) in  $C/\lambda_j$ -neighborhoods of  $x_0 = 0$  and  $x_0 = \pi/4$ . We get

$$(10) \quad h'(x_0 + x) = b_1(\lambda) + b_2(\lambda)x^2 + O(|x|^4 + \lambda^{-3}),$$

uniformly in  $j$ ; here

$$\left\{ \begin{array}{l} b_1 = \frac{-a_1}{2} + \frac{1}{\lambda a_0} \left[ \frac{3a_2 - a_1}{2} - \frac{9a_1^2}{8} \right] \\ \quad + \frac{1}{(\lambda a_0)^2} \left[ 3a_2 - \frac{2a_1 + 45a_3}{4} + \frac{a_1(324a_2 - 54a_1 - 153a_1^2)}{16} \right], \\ \frac{b_2}{3} = \frac{a_1^2}{2} - a_2 + \frac{1}{\lambda a_0} \left[ a_1^2 - a_2 - 12a_1a_2 + \frac{30a_3 - 21a_1^3}{4} \right] \\ \quad + \frac{1}{(\lambda a_0)^2} \left[ 15(a_3 - 7a_4) - a_2 + \frac{3a_1^2}{2} + 21a_1^3 + \frac{3225a_1^4}{32} \right. \\ \quad \left. + \frac{3}{4} (122a_2^2 - 48a_1a_2 - 399a_1^2a_2 + 280a_1a_3) \right]. \end{array} \right.$$

The function  $q(x)$  was chosen so that  $a_1 \neq 0$  in (9). It follows that in  $C/\lambda_j$ -neighborhoods of the critical points

$$(11) \quad h'_j(x_0 + x) = \frac{-a_1}{2} + O(1/\lambda_j).$$

If  $g_j$  had two or more critical points in a  $C/\lambda_j$ -neighborhood of a critical point of  $q$ , then  $h_j$  would have at least two zeros there and  $h'_j$  would vanish, contradicting (11) for large enough  $\lambda_j$ . Therefore  $g_j$  has at most one critical point in every such neighborhood for large  $\lambda_j$ . Together with Lemma 2 this proves

**Lemma 3.** *The number of critical points of  $g_j(x)$  is uniformly bounded above.*

We are now ready to prove the theorem. Let

$$(12) \quad f_j(x, y) = \varphi_1(j)(x) \sin y + \varphi_2(j)(x) \cos y,$$

where  $\varphi_{1,2}(j)(x)$  are defined by (4) and (5). The function  $f_j(x, y)$  is equal to

$$(g_j(x))^{1/2} \sin(\Phi_j(x) + y),$$

where  $\Phi_j(x)$  is a continuous monotone function defined by

$$\cos(\Phi_j(x)) = \varphi_1(j)(x)/(g_j(x))^{1/2}, \quad \sin(\Phi_j(x)) = \varphi_2(j)(x)/(g_j(x))^{1/2}$$

( $\Phi$  is monotone since a nonzero linear combination of  $\varphi_1$  and  $\varphi_2$  cannot have a second order zero).

At a critical point  $(x_0, y_0)$  of  $f_j$  we have

$$\frac{\partial f_j}{\partial y} = (g_j(x))^{1/2} \cos(\Phi_j(x) + y) = 0,$$

so

$$(13) \quad y + \Phi_j(x) = \pi/2 + \pi k, k \in \mathbf{Z}.$$

Also,

$$\frac{\partial f_j}{\partial x} = \frac{g'_j(x)}{2(g_j(x))^{1/2}} \sin(\Phi_j(x) + y) = 0$$

(we have used the equality  $\cos(\Phi_j(x) + y) = 0$ ). Accordingly, by Lemma 3,  $x$  can take a bounded number of values. Together with (13) this shows that the number of critical points of  $f_j(x, y)$  is uniformly bounded above, and the proof is finished. q.e.d.

**Remark.** One can show that for large  $\lambda_j$  the eigenfunctions that were constructed have exactly 16 critical points.

## References

- [1] M. Evgrafov & M. Fedoryuk, *Asymptotic behavior of solutions of the equation  $w''(z) - p(z, \lambda)w(z) = 0$  as  $\lambda \rightarrow \infty$  in the complex  $z$ -plane*, Russian Math. Surveys **21** (1966) 1–48.
- [2] M. Fedoryuk, *Asymptotic analysis: linear ordinary differential equations*, Springer, Berlin, 1993.
- [3] D. Kosygin, A. Minusov & Y. Sinai, *Statistical properties of the spectra of Laplace-Beltrami operators on Liouville surfaces*, Russian Math. Surveys **48** (1993) 1–142.
- [4] S. T. Yau, *Problem section*, Seminar on Differential Geom., (ed. by S. T. Yau) Ann. of Math. Stud. **102** (1982) 669–706.
- [5] ———, *Open problems in geometry*, Differential geometry: partial differential equations on manifolds, Proc. Symp. Pure Math. **54** (1993) 1–28.
- [6] ———, *A note on the distribution of critical points of eigenfunctions*, Tsing Hua Lectures in Geometry and Analysis 315–317, Internat. Press, 1997.

UNIVERSITY OF CHICAGO