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Exponentially small asymptotics for internal solitary waves with oscillatory tails in a stratified fluid

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Abstract. The objective of this paper is to develop rigorously a method of exponentially small asymptotics for the solution of the equations governing internal solitary waves with oscillatory tails in a density-stratified fluid.

1. Introduction

A recent remarkable discovery in the mathematical theory of solitons is the so-called generalized solitary wave, which consists of a solitary wave plus an oscillatory tail at infinity, and many new ideas have been developed to deal with mathematical difficulties encountered in proving its existence. A generalized solitary wave was first found in the study of progressive surface waves on a liquid of constant density with small surface tension. Let B be the Bond number, a nondimensional surface tension coefficient, and F, the Froude number, the square of a nondimensional wave speed. For $0 < B < 1/3$ and $F > 1$ but near 1, the eigenvalue problem associated with the linearized governing equations possesses a positive eigenvalue and the corresponding eigenfunction is periodic in the horizontal direction. The effect of the periodic eigenfunction manifests itself at infinity where a solitary wave solution decays to zero. Hence a generalized solitary wave comes into being. The existence proof of a generalized solitary wave was first given independently by Beale [1] and Sun [2] on the basis of different methods. However, there are also some similarities in the underlying ideas used in the two approaches. Essentially one has to deal with functions, which consist of a part decaying to zero at infinity and a part being periodic. The latter part is characterized by its amplitude and phase shift. Other existence proofs based on center manifold theory were given by Iooss and Kirchgässner [3], and Turner [4]. Recently generalized solitary waves have also been found in a density-stratified fluid without surface tension. For example, in a two-layer fluid with free surface a generalized solitary wave appears as an internal wave at the interface [5]. For a continuously density-stratified fluid with free surface supported by a rigid horizontal bottom, one has to deal with two eigenvalue problems. One eigenvalue problem under the long wave approximation with $\nu = F^{-1}$ as the eigenvalue parameter yields infinitely many positive eigenvalues, $\nu_n, n = 0, 1, 2, 3, ...$ A progressive wave solution may appear with a wave speed near $\nu_n^{-1/2}$. For each ν_n, there is another eigenvalue problem corresponding to the linearized governing equations with μ as an eigenvalue parameter. There are at most finitely many positive eigenvalues, the corresponding eigenfunctions of which are periodic in the horizontal direction. It has been shown [6] that for $\nu = \nu_0$ there is

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no positive eigenvalue of μ, and this case is similar to the classical solitary wave on a liquid of constant density without surface tension as studied in Friedrichs and Hyers [7]. For $\nu = \nu_1$ there appears a positive eigenvalue of μ and a generalized solitary wave, which represents an internal wave, emerges. For $\nu = \nu_k$, $k > 1$, there may exist more than one positive eigenvalue of μ. Results for these cases may be found in [6], [8], [9].

The most intriguing question regarding a generalized solitary wave concerns the amplitude of its oscillatory tail at infinity. In the case of a liquid of constant density with small surface tension, Beale [1] proved that the amplitude is asymptotically small beyond all orders of a small parameter, and conjectured that it should be exponentially small. The conjecture has been proved to be correct [10]. Naturally one may ask whether a generalized solitary wave corresponding to ν_1 in a stratified fluid without surface tension may share the same property. We note that a formal asymptotic approach to the problem of generalized solitary waves in a stratified fluid with fixed boundaries was used by Akylas and Grimshaw [11], and the same problem was also treated by Kirchgässner and Lankers by the method of center manifold theory [12]. However, up to now a rigorous derivation of the exponentially small estimate in the case of a stratified fluid is still lacking. The objective of this paper is to develop a rigorous asymptotic method for the solution of a system of nonlinear integro-differential equations in order to obtain an exponentially small estimate for the amplitude of the periodic part of the solution, which is the oscillatory tail of a generalized solitary wave in a stratified fluid. Our main contribution may be stated in the following theorem:

Assume that the fluid motion is steady in reference to a coordinate system moving at a constant speed. Let $\rho(\psi) > 0$ with $\rho'(\psi) < 0$ and $U(\psi) > 0$ be the density and the horizontal velocity of the fluid at equilibrium and $z_1(\eta)$ and $Z(\eta)$ be the eigenfunctions corresponding to the eigenvalues 0, κ^2 of μ, respectively, where $\kappa > 0$, ψ is the stream function, $0 \le \psi \le 1$ with $\psi = 0$ at bottom and $\psi = 1$ on the free surface, and

$$\eta(\psi) = \int_0^\psi q(t)^{-1}dt \equiv \int_0^\psi \rho^{1/2}(t)U(t)^{-1}dt.$$

If $\nu = \nu_1(1 - \delta_1\epsilon)$ with $\delta_1 > 0$, then for small $\epsilon > 0$ there exists a streamline function $f(X, \psi)$, where $\psi = \psi(X, f) =$ constant along a streamline with $-\infty < X < +\infty$, satisfies the exact equations governing the motion of the flow, such that

$$\begin{aligned} f(X,\psi) = \eta(\psi) + \epsilon S(\epsilon^{1/2}x)z_1(\eta(\psi)) + \tilde{A}\cos(\kappa(x - \delta\tanh x))Z(\eta(\psi)) \\ + \epsilon^2 O_1(x,\psi) + \tilde{A}\epsilon O_2((x - \delta\tanh x),\psi). \end{aligned}$$

Here $x = X(1-\sigma\epsilon)^{-1/2}$, σ will be determined as part of the solution with σ bounded, δ is a given phase shift satisfying $\sin\delta\kappa \neq 0$ assumed to be small,

$$S(\xi) = (\delta_1\beta_1/\alpha_1)\text{sech}^2((-\delta_1\beta_1/\gamma_1)^{1/2}\xi/2),$$

is the approximate solution for a solitary wave in a stratified fluid with

$$\gamma_1 = -\int_0^1 q^2(\eta) z_1^2(\eta) d\eta,$$
$$\alpha_1 = \int_0^1 q^2(\eta) z_{1\eta}^3(\eta) d\eta \neq 0,$$
$$\beta_1 = \int_0^1 q^2(\eta) z_{1\eta}^2(\eta) d\eta,$$

$O_1(x, \psi) = O(\exp(-d|x|\epsilon^{1/2}))$ with d a positive constant, $O_2(y, \psi)$ is periodic in y with period $2\pi/\kappa$ and bounded for small ϵ,

$$|\tilde{A}| \leq K\epsilon^{(n+2)/2} \exp(-\ell\kappa\epsilon^{-1/2})$$

for $0 < \pi(-\delta_1\beta_1/\gamma_1)^{-1/2} - \tau < \ell < \pi(-\delta_1\beta_1/\gamma_1)^{-1/2}$ with a fixed arbitrarily small number $\tau > 0$, n is any fixed positive integer, K is a constant independent of ϵ and $\tilde{A}$ is the amplitude of the oscillatory tail. We refer a reader to the survey paper by Boyd [13] for more physical examples of generalized solitary waves, and to Segur, Tanveer and Levine [14] for discussions on solitary waves in the presence of small surface tension by Vanden-Broeck [15] and Beale [16], and on exponential asymptotics by Meyer [17] among others. Here we remark that in the case of constant density and nonzero surface tension some numerical evidence [15] indicates that there might be solitary waves without ripples at infinity for special values of parameters involved. It is certainly an open question whether the amplitude of the ripples could be zero.

This paper is organized as follows. In Section 2, we first formulate the problem in terms of the streamline function. In Section 3, an approximate expression for the solitary waves in a stratified fluid is derived by a formal asymptotic approach. In Section 4, basically we expand a solution of the governing equations in terms of the eigenfunctions corresponding to the zero and the positive eigenvalue of μ and others. Two solvability conditions are prescribed so that the Green's function of the linearized equations can be constructed. Then we transform all differential equations into integro-differential equations via Green's functions and extend the horizontal variable to the complex plane. Furthermore, we consider functions consisting of two parts. One part decays to zero at infinity and the other is periodic in the horizontal direction. Several Banach spaces for these functions are defined for later use. In Section 5, the norms of the integral operators corresponding to the Green's functions are estimated. In Section 6, the integro-differential equations are also decomposed into two parts, one for decaying functions and the other for periodic functions. A priori estimates of their solutions are derived. The important result is the exponentially small estimate of the amplitude of the periodic part. Finally in Section 7, an existence theorem is proved by means of the contraction map theorem, which implies the theorem just stated.

2. Formulation

We consider a two-dimensional wave of permanent type moving with constant velocity $c > 0$ in a layer of inviscid, incompressible fluid of variable density with a free surface over a rigid horizontal bottom. A coordinate system moving with the wave is chosen so that in reference to the coordinate system the fluid flow is steady. The governing equations are the following, in $0 < y^* < \zeta^*(x^*)$,

$$\rho^*(u^* u^*_{x^*} + v^* u^*_{y^*}) = -p^*_{x^*},$$
$$\rho^*(u^* v^*_{x^*} + v^* v^*_{y^*}) = -\rho^* g - p^*_{y^*},$$
$$u^*_{x^*} + v^*_{y^*} = 0,$$
$$u^* \rho^*_{x^*} + v^* \rho^*_{y^*} = 0,$$

at the free surface $y^* = \zeta^*(x^*)$,

$$u^* \zeta^*_{x^*} - v^* = 0,\ p^* = 0,$$

at the rigid bottom $y^* = 0$,

$$v^* = 0,$$

where ρ^* is the density, (u^*, v^*) is the velocity, p^* is the pressure and g is the gravitational acceleration. Following Yih [18], we introduce new dependent variables $\tilde{u}$, $\tilde{v}$ such that

$$\tilde{u} = (\rho^*)^{1/2} u^*, \quad \tilde{v} = (\rho^*)^{1/2} v^*,$$

and the governing equations become

$$\tilde{u}\tilde{u}_{x^*} + \tilde{v}\tilde{u}_{y^*} = -p^*_{x^*}, \tag{1}$$
$$\tilde{u}\tilde{v}_{x^*} + \tilde{v}\tilde{v}_{y^*} = -\rho^* g - p^*_{y^*}, \tag{2}$$
$$\tilde{u}_{x^*} + \tilde{v}_{y^*} = 0, \tag{3}$$
$$\tilde{u}\rho^*_{x^*} + \tilde{v}\rho^*_{y^*} = 0 \tag{4}$$

at the free surface $y^* = \zeta^*$,

$$\tilde{u}\zeta^*_{x^*} - \tilde{v} = 0, \quad p^* = 0; \tag{5}$$

at the bottom $y^* = 0$,

$$\tilde{v} = 0. \tag{6}$$

From (3), a stream function $\psi^*(x^*, y^*)$ may be defined such that

$$\tilde{u} = \psi^*_{y^*}(x^*, y^*), \quad \tilde{v} = -\psi^*_{x^*}(x^*, y^*). \tag{7}$$

Note that the domain for (1) to (6) is unknown since ζ^* is to be determined as part of the solution. However both the free surface and bottom are streamlines. Thus to have a fixed domain, we use x^* and ψ^* as independent variables and the so-called streamline function f^* as the dependent variable such that $\psi^*(x^*, f^*) =$ constant defines a streamline. From (4), it is easy to see that $\rho^* = \rho^*(\psi^*)$ and by (1) and (2) we have

$$\nabla^2 \psi^* + g(d\rho^*/d\psi^*) y^* = dh^*/d\psi^*, \tag{8}$$

where $h^*(\psi^*) = ((\psi^*_{y^*})^2 + (\psi^*_{x^*})^2)/2 + p^* + g\rho^* y^*$. Equation (8) was first formulated by Dubreil-Jacotin in 1935 [19]. We let the bottom $y^* = 0$ be $\psi^* = 0$ and on the free surface $y^* = \zeta^*$, $\psi^* = \psi^*_s$ and

$$((\psi^*_{y^*})^2 + (\psi^*_{x^*})^2)/2 + p^* + g\rho^* \zeta^* = h^*(\psi^*_s). \tag{9}$$

Also by the definition of f^*, we have

$$\psi^*_{y^*} = 1/f^*_{\psi^*}, \quad \psi^*_{x^*} = -f^*_{x^*}/f^*_{\psi^*}.$$

Thus (8) and (9) become

$$\begin{aligned}(1/2)\big((1+(f^*_{x^*})^2)/(f^*_{\psi^*})^2\big)_{\psi^*} - (f^*_{x^*}/f^*_{\psi^*})_{x^*}\\ + g(d\rho^*/d\psi^*)f^* = dh^*/d\psi^*,\end{aligned} \tag{10}$$

$$(1/2)(1+(f^*_{x^*})^2)/(f^*_{\psi^*})^2 + g\rho^* f^* = h^*(\psi^*_s) \text{ at } \psi^* = \psi^*_s, \tag{11}$$

$$f^* = 0 \text{ at } \psi^* = 0, \tag{12}$$

where $p^* = 0$ at $\psi^* = \psi^*_s$ is used. We can make all the variables nondimensional by introducing dimensionless variables

$$X = x^*/H^*, \;\; \rho = \rho^*/\rho_0, \;\; \psi = \psi^*/c_1 H^*, \;\; f = f^*/H^*,$$
$$u = u^*/c, \;\; \psi_s = \psi^*_s/c_1 H^* = 1, \;\; h = h^*/c_1^2, \;\; D = h^*(\psi^*_s)/c_1^2,$$
$$\nu = \rho_0 g H^*/c_1^2 = gH^*/c, \;\; c_1 = \rho_0^{1/2} c,$$

where H^* is the uniform depth of the equilibrium state, ρ_0 is a characteristic density and $D = D(\nu)$ is a parameter depending on ν. Rewrite (10) to (12) as

$$(1/2)\big((1+f_X^2)/(f_\psi)^2\big)_\psi - (f_X/f_\psi)_X + \nu(d\rho/d\psi)f = dh/d\psi, \tag{13}$$

$$(1/2)(1+f_X^2)/f_\psi^2 + \nu\rho(1)f = D(\nu) \text{ at } \psi = 1, \tag{14}$$

$$f = 0 \text{ at } \psi = 0. \tag{15}$$

First let us consider an equilibrium state $f = \eta(\psi)$ as a solution of (13) to (15) independent of X. If the equilibrium velocity is $(u, v) = (U(\psi), 0)$, then

$$\begin{aligned} f_\psi = (\tilde{u})^{-1} = (\rho^{1/2}(\psi)U(\psi))^{-1} = (q(\psi))^{-1}\\ \eta(\psi) = \int_0^\psi q^{-1}(t)dt.\end{aligned} \tag{16}$$

We let $\eta(1) = 1$ by a suitable choice of ρ_0. Also assume $q(\psi) > 0$ for $0 \le \psi \le 1$. Since $\eta(\psi)$ is an equilibrium state for (13) to (15), $h(\psi)$ and $D(\nu)$ must be chosen by

$$h(\psi) = D(\nu) + \int_1^\psi (1/2)(1/(\eta_\psi)^2)_\psi + \nu(d\rho/d\psi)\eta\, d\psi, \tag{17}$$

$$D(\nu) = (1/2)(1/\eta_\psi(1))^2 + \nu\rho(1)\eta(1). \tag{18}$$

From (16), we can represent ψ as a function of η and thus $f(X,\psi) = f(X,\psi(\eta))$. Since $\eta(\psi)$ is the equilibrium state and the solutions of (13) to (15) are only considered as perturbations of the equilibrium solution, we assume

$$f(X,\psi) = \eta(\psi) + w(X,\eta) = \eta(\psi) + w(X,\eta(\psi)), \tag{19}$$

where w is a function of X, η. Therefore by (13) to (15), (17) to (19), $w(X,\eta)$ satisfies

$$\begin{aligned}(1/2)\big(q^2(2w_\eta + w_\eta^2 - w_X^2)/(1+w_\eta)^2\big)_\eta\\ + q^2\big(w_X/(1+w_\eta)\big)_X - \nu\rho_\eta w = 0,\end{aligned} \tag{20}$$

$$(1/2)q^2(1)(2w_\eta + w_\eta^2 - w_X^2)/(1+w_\eta)^2 - \nu\rho(1)w = 0 \text{ at } \eta = 1, \tag{21}$$

$$w = 0 \text{ at } \eta = 0, \tag{22}$$

where q and ρ are considered as functions of η also if no confusion arises. In the following, we shall use (20) to (22) as our basic governing equations.

3. Formal derivation of solitary waves

We rewrite (20) to (22) as

$$L_\nu(w) = (q^2 w_\eta)_\eta - \nu\rho_\eta w = \phi_1(X,\eta), \tag{23}$$

$$w_\eta - \nu\rho(1)q^{-2}(1)w = \phi_2(X) \text{ at } \eta = 1, \tag{24}$$

$$w = 0 \text{ at } \eta = 0, \tag{25}$$

where

$$\begin{aligned}\phi_1(x,\eta) &= -q^2 w_{XX} + (3/2)(q^2 w_\eta^2)_\eta + F_1(w),\\ F_1(w) &= (1/2)\big(q^2(w_X^2 - 4w_\eta^3 - 3w_\eta^4)/(1+w_\eta)^2\big)_\eta\\ &\quad + q^2(w_X w_\eta/(1+w_\eta))_X,\\ \phi_2(x) &= (3/2)w_\eta^2 + F_2(w),\\ F_2(w) &= (1/2)(w_X^2 - 4w_\eta^3 - 3w_\eta^4)/(1+w_\eta)^2.\end{aligned}$$

Assume $\rho_\eta < 0$, $q^2 > 0$ and both are sufficiently smooth. It is known from Sturmian theory (Dunford and Schwartz [20]) that the eigenvalue problem

$$L_\nu(q) = (q^2 z_\eta)_\eta - \nu\rho_\eta z = 0, \tag{26}$$

$$z_\eta - \nu\rho(1)q^{-2}(1)z = 0 \text{ at } \eta = 1, \tag{27}$$

$$z = 0 \text{ at } \eta = 0, \tag{28}$$

possesses simple eigenvalues $0 < \nu_0 < \nu_1 < \cdots$, $\nu_n \to +\infty$, and $z_0(\eta), z_1(\eta), \ldots$ are corresponding real eigenfunctions. The asymptotic forms of ν_n and $z_n(\eta)$ for $n \gg 1$ are

$$\nu_n = n^2\pi^2\left(\int_0^1 Q(t)dt\right)^{-2} + O(1), \tag{29}$$

$$z_n(\eta) = \big(q(\eta)Q^{1/2}(\eta)\big)^{-1}\sin\left(\nu_n^{1/2}\int_0^\eta Q(t)dt\right) + O(1/n), \tag{30}$$

where $Q(\eta) = q^{-1}(\eta)(-\rho_\eta)^{1/2}$. Let ν in (23) to (25) be an eigenvalue ν_n of (26) to (28). Then (23) to (25) is solvable if and only if the solvability condition

$$N(\phi_1,\phi_2) \equiv \int_0^1 z_n(\eta)\phi_1(X,\eta)d\eta - z_n(1)\phi_2(X)q^2(1) = 0, \tag{31}$$

holds.

Now let ν be near ν_n and write $\nu = \nu_n(1-\delta_n\epsilon)$ in (23) to (25) where ϵ is a small positive parameter. Assume $X = \xi\epsilon^{-1/2}$ and

$$w = \epsilon(w_1(\xi,\eta) + \epsilon w_2(\xi,\eta) + \cdots), \tag{32}$$

and substitution of the above formal series expansion in (23) to (25) yields a sequence of approximate equations and boundary conditions for $w_1, w_2, \ldots$. The equations for w_1 are the same as (26) to (28) with $\nu = \nu_n$. Thus

$$w_1(\xi,\eta) = S(\xi)z_n(\eta),$$

where $S(\xi)$ is to be determined. The equations for w_2 are

$$L_{\nu_n}(w_2) = -q^2 w_{1\xi\xi} + (3/2)(q^2 w_{1\eta}^2)_\eta - \nu_n \delta_n \rho_\eta w_1,$$
$$w_{2\eta} - \nu_n \rho(1) q^{-2}(1) w_2 = -\nu_n \delta_n \rho(1) q^{-2}(1) w_1 + (3/2) w_{1\eta}^2 \text{ at } \eta = 1,$$
$$w_2 = 0 \text{ at } \eta = 0.$$

By the condition (31), we have

$$\int_0^1 -q^2 S_{\xi\xi} z_n^2 + (3/2) S^2 z_n (q^2 z_{n,\eta}^2)_\eta - \nu_n \delta_n \rho_\eta S z_n^2 \, d\eta$$
$$- z_n(1)(-S\nu_n \delta_n \rho(1) z_n(1) + (3/2) S^2 z_{n\eta}^2(1) q^2(1)) = 0,$$

and it follows that

$$\gamma_n S_{\xi\xi} - (3/2)\alpha_n S^2 + \delta_n \beta_n S = 0, \tag{33}$$

where

$$\gamma_n = -\int_0^1 q^2(\eta) z_n^2(\eta) d\eta,$$
$$\alpha_n = \int_0^1 q^2(\eta) z_{n\eta}^3(\eta) d\eta,$$
$$\beta_n = \int_0^1 q^2(\eta) z_{n\eta}^2(\eta) d\eta,$$

and z_n satisfies (26) to (28). We note that $\gamma_n < 0$ and $\beta_n > 0$, and obtain

$$S(\xi) = (\delta_n \beta_n / \alpha_n) \operatorname{sech}^2\big((-\delta_n \beta_n / \gamma_n)^{1/2} (\xi - \xi_0)/2\big), \tag{34}$$

provided $\alpha_n \neq 0$ and $\delta_n > 0$, where ξ_0 is constant and will be set equal to zero. Therefore $w_1(X, \eta) = \epsilon S(\epsilon^{1/2} X) z_n(\eta)$ is formally a first order approximation to a solution of (23) to (25). Since the case for $n = 0$ has been studied in [6], here we only consider the case of $n = 1$. In the following, we shall rigorously prove that $w_1(X, \eta)$ with an exponentially small oscillatory tail at infinity is an approximation to a solution of (23) to (25). Needless to say, the formal asymptotic series in powers of ϵ fails to catch the exponentially small term. Furthermore the classical existence proof of a solitary wave can not be carried over because of the appearance of a positive eigenvalue of μ as shown in the next section.

4. Integro-differential equations and Banach spaces

Assume $\nu = \nu_1(1 - \delta_1 \epsilon)$ and rewrite (23) to (25) as

$$(q^2 w_\eta)_\eta + q^2 w_{XX} - \nu_1 \rho_\eta w = (3/2)(q^2 w_\eta^2)_\eta - \nu_1 \delta_1 \epsilon \rho_\eta w + F_1(w), \tag{35}$$
$$w_\eta - \nu_1 \rho(1) q^{-2}(1) w = -\nu_1 \delta_1 \epsilon \rho(1) q^{-2}(1) w + \phi_2(X) \text{ at } \eta = 1, \tag{36}$$
$$w = 0 \text{ at } \eta = 0. \tag{37}$$

We consider the following eigenvalue problem,

$$(q^2 N_\eta)_\eta - \nu_1 \rho_\eta N = \mu q^2 N, \tag{38}$$
$$N_\eta - \nu_1 \rho(1) q^{-2}(1) N = 0 \text{ at } \eta = 1, \tag{39}$$
$$N = 0 \text{ at } \eta = 0. \tag{40}$$

Here $\mu = 0$ is an eigenvalue of (38) to (40) and $z_1(\eta)$ as an eigenfunction of (26) to (28) is also an eigenfunction of (38) to (40). As shown in [18], $z_1(\eta)$ has exactly one zero in $(0,1)$. By usual Sturm oscillation theorem [20], we know that $\mu = 0$ is the second largest eigenvalue for (38) to (40) since $z_1(\eta)$ has only one zero in $(0,1)$. Note that $-\mu$ is the eigenvalue parameter in the Sturm oscillation theorem. Thus (38) to (40) have exactly one positive eigenvalue κ^2 with the corresponding eigenfunction $Z(\eta)$ and $\int_0^1 Z^2(\eta)q^2(\eta)d\eta = 1$. Also there exist infinitely many negative eigenvalues of (38) to (40), denoted by $-k_1^2, -k_2^2, \ldots - k_n^2 \to -\infty$ with the corresponding eigenfunctions $u_n(\eta)$ and

$$\int_0^1 u_n^2(\eta)q^2(\eta)d\eta = 1.$$

By the general Sturm-Liouville theorem, we have that for large n

$$k_n^{1/2} = n\pi + \frac{\pi}{2} + O(1/n), \tag{41}$$

$$u_n(\eta) = q^{-1}(\eta)(\sqrt{2}\sin(k_n^{1/2}\eta) + O(1/n)). \tag{42}$$

Now we let

$$w(X,\eta) = \epsilon\big(a^*(X)Z(\eta) + b^*(X)z_1(\eta) + \theta^*(X,\eta)\big) \tag{43}$$

subject to the orthogonality conditions

$$\begin{aligned} &\int_0^1 Z(\eta)\theta^*(X,\eta)q^2(\eta)d\eta = 0, \\ &\int_0^1 z_1(\eta)\theta^*(X,\eta)q^2(\eta)d\eta = 0, \end{aligned} \tag{44}$$

for all X.Basically (44) are the conditions to separate the components $a^*(X)Z(\eta)$ and $b^*(X)z_1(\eta)$ from $\theta^*(X,\eta)$ in the solution w. Furthermore, they also ensure the existence of a Green's function to be constructed later. Let $X = x/(1-\sigma\epsilon)^{1/2}$ (Beale [1]). We note that the strained horizontal variable X is introduced here so that σ can be considered as an extra unknown to be determined by solvability conditions as seen later. If we let $\theta^* = \theta(x,\eta)$, $a^* = a(x)$ and $b^* = b(x)$, then (35) to (37) become

$$(q^2\theta_\eta)_\eta + q^2\theta_{xx} - \nu_1\rho_\eta\theta = F_3(\sigma,a,b,\theta), \tag{45}$$

$$\theta_\eta - \nu_1\rho(1)q^{-2}(1)\theta = F_4(\sigma,a,b,\theta) \text{ at } \eta = 1, \tag{46}$$

$$\theta = 0 \text{ at } \eta = 0, \tag{47}$$

where

$$\begin{aligned} F_3(\sigma,a,b,\theta) =& \sigma\epsilon q^2(aZ + bz_1 + \theta)_{xx} - \nu_1\delta_1\epsilon\rho_\eta(aZ + bz_1 + \theta) \\ &- q^2 b_{xx}z_1 - q^2(a_{xx} + \kappa^2 a)Z + (3/2)\epsilon(q^2(aZ + bz_1 + \theta)_\eta^2)_\eta \\ &+ \epsilon^{-1}F_1(\epsilon(az + bz_1 + \theta)), \\ F_4(\sigma,a,b,\theta) =& -\nu_1\delta_1\epsilon\rho(1)q^{-2}(1)(az + bz_1 + \theta) \\ &+ (3\epsilon/2)(aZ_\eta + bz_{1\eta} + \theta_\eta)^2 + \epsilon^{-1}F_2(\epsilon(aZ + bz_1 + \theta)). \end{aligned}$$

We multiply (45) by either $Z(\eta)$ or $z_1(\eta)$, integrate the resulting equation with respect to η from zero to one and simplify it by integration by parts and boundary conditions

(46) and (47). Then by use of (44) we have the following two conditions for F_2 and F_4,

$$\int_0^1 F_3(\sigma, a(x), b(x), \theta(x,\eta))Z(\eta)d\eta - q^2(1)Z(1)F_4\Big|_{\eta=1} = 0, \tag{48}$$

$$\int_0^1 F_3(\sigma, a(x), b(x), \theta(x,\eta))z_1(\eta)d\eta - q^2(1)z_1(1)F_4\Big|_{\eta=1} = 0. \tag{49}$$

Since we need to show that $w_1(X,\eta) = \epsilon S(\epsilon^{1/2}X)z_1(\eta)$ is an approximation to a solution of (35) to (37), we let

$$b(x) = S(\epsilon^{1/2}x) + \omega(x), \tag{50}$$

where $S(\xi)$ is defined in (34) with $n = 1$, $\xi_0 = 0$. By substituting (50) for $b(x)$ in (48) and (49) and making use of the expressions for F_3 and F_4, it is obtained that

$$\omega_{xx} + \delta_1\beta_1\gamma_1^{-1}\epsilon\omega - 3\gamma_1^{-1}\alpha_1\epsilon S(\epsilon^{1/2}x)\omega = g_1(S, \sigma, a, \omega, \theta), \tag{51}$$

$$a_{xx} + \kappa^2 a = g_2(S, \sigma, a, \omega, \theta), \tag{52}$$

where

$$\begin{aligned} g_1(S,\sigma,a,\omega,\theta) =& \gamma^{-1}\Big((3/2)\epsilon\alpha_1\omega^2 \int_0^1 \Big(-\sigma\epsilon q^2(aZ + bz_1 + \theta)_{xx} \\ & + \nu_1\delta_1\epsilon\rho_\eta(aZ+\theta) - (3\epsilon/2)\big(q^2((aZ + bz_1 + \theta)_\eta^2 - (bz)_\eta^2)\big)_\eta \\ & - \epsilon^{-1}F_1\big(\epsilon(aZ + (\omega+S)z_1 + \theta)\big)z_1(\eta)\Big)d\eta \\ & + q^2(1)z_1(1)\Big(- \nu_1\delta_1\epsilon\rho(1)q^{-2}(1)(aZ+\theta) \\ & \qquad + (3\epsilon/2)\big((aZ_\eta + bz_{1\eta} + \theta_\eta)^2 - b(z_{1\eta})^2\big) \\ & \qquad + \epsilon^{-1}F_2\big(\epsilon(aZ + bz_1 + \theta)\big)\Big)\Big|_{\eta=1}\Big), \end{aligned}$$

$$\begin{aligned} g_2(X,\sigma,a,\omega,\theta) =& \int_0^1 \Big(\sigma\epsilon q^2(aZ + bz_1 + \theta)_{xx} - \nu_1\delta_1\epsilon\rho_\eta(aZ + bz_1 + \theta) \\ & + (3/2)\epsilon\big(q^2(aZ + bz_1 + \theta)_\eta^2\big)_\eta \\ & + \epsilon^{-1}F_1\big(\epsilon(aZ + bz_1 + \theta)\big)\Big)Z(\eta)d\eta \\ & - q^2(1)Z(1)\Big(- \nu_1\delta_1\epsilon\rho(1)q^{-2}(1)(aZ + bz_1 + \theta) \\ & + (3\epsilon/2)(aZ_\eta + bz_{1\eta} + \theta_\eta)^2 \\ & + \epsilon^{-1}F_2\big(\epsilon(aZ + bz_1 + \theta)\big)\Big)\Big|_{\eta=1}. \end{aligned}$$

Therefore from (35) to (37), we have the equivalent equations (45) to (47), (51) and (52). Now we transform these equations into integral equations. For (45) to (47), it is straightforward to find the Green's function [8]

$$G(x,\eta;\xi,\zeta) = (-1/2)\sum_{n=1}^{\infty} u_n(\eta)u_n(\zeta)k_n^{-1/2}\exp(-|x-\xi|k_n^{1/2}), \tag{53}$$

by using the eigenfunction expansion of (38) to (40) and (44). Hence the solution of (45) to (47) with (44) can be expressed as

$$\theta(x,\eta) = \int_{-\infty}^{+\infty}\int_0^1 G(x,\eta;\xi,\zeta)F_3(\sigma,a,b,\theta)d\zeta d\xi - \int_{-\infty}^{+\infty} G(x,\eta;\xi,1)F_4(\sigma,a,b,\theta)d\xi. \tag{54}$$

From (51), we see that $\psi_1(x) = dS(\epsilon^{1/2}x)/dx$ as an odd function is a solution of the homogeneous equation. Another linearly independent solution $\psi_2(x)$ as an even function can be easily found. Here $\psi_1(x)$ and $\psi_2(x)$ form a fundamental set of solutions and their Wronskian is one. Thus if $\omega(x)$ and $g_1(x)$ in (51) are even, (51) can be inverted to

$$\begin{aligned}\omega(x) &= \psi_1(x)\int_0^x \psi_2(s)g_1(s)ds + \psi_2(x)\int_x^{+\infty}\psi_1(s)g_1(s)ds \\ &= \int_0^{+\infty} k(x,s)g_1(s)ds\end{aligned} \tag{55}$$

for $x \geq 0$, and

$$\omega(x) = \int_0^{+\infty} k(-x,s)g_1(s)ds$$

for $x < 0$. Finally we can transform (52) into

$$a(x) = \tilde{\mu}\cos\kappa x + (1/\kappa)\int_0^x \sin\kappa(x-s)g_2(s)ds \tag{56}$$

if $a(x)$ and $g_2(x)$ are even, where $\tilde{\mu}$ is a constant to be determined later. Therefore we only need to show that there exists a solution (a,b,θ) of (54) to (56) when ϵ is small. Note that $S(\epsilon^{1/2}x)$ in (34) can be analytically extended to complex z-plane with $|\mathrm{Im}\ z| < \pi(-\delta_1\beta_1\epsilon/\gamma_1)^{-1/2}$. In order to obtain an exponentially small estimate, we also need to extend the domain of solutions to the complex plane. Thus (54) to (56) are rewritten in terms of the complex variable z where $\mathrm{Re}\ z = x$ as

$$\begin{aligned}\theta(z,\eta) &= \left(\int_{-\infty}^{z} + \int_z^{+\infty}\right)\int_0^1 G(z,\eta;\xi,\zeta)F_3(\sigma,a,b,\theta)d\zeta d\xi \\ &\quad - \left(\int_{-\infty}^{z} + \int_z^{+\infty}\right) G(z,\eta;\xi,1)F_4(\sigma,a,b,\theta)d\xi \\ &= \mathcal{G}(F_3(\xi,\zeta),F_4(\xi))(x,\eta),\end{aligned} \tag{57}$$

$$\omega(z) = \int_0^{+\infty} k(z,s)g_1(s)ds = \mathcal{L}(g_1(s))(z), \tag{58}$$

$$a(z) = \tilde{\mu}\cos\kappa z + (1/\kappa)\int_0^z \sin\kappa(z-s)g_2(s)ds = \mathcal{P}(g_2(s))(z), \tag{59}$$

and find even analytic functions $\theta(z,\eta)$, $\omega(z)$ and $a(z)$ as solutions of (57) to (59) in $|\mathrm{Im}\ z| \leq \ell\epsilon^{-1/2}$ for $0 < \ell < \pi(-\delta_1\beta_1/\gamma_1)^{-1/2}$. Also the integrands in (57) to (59) are in $|\mathrm{Im}\ z| \leq \ell\epsilon^{-1/2}$.

In the following, we define some Banach spaces for later use. Let $0 < \lambda < 1$, m, n be nonnegative integers and

$$D_r = \big\{(z,\eta) \bigm| 0 \le \eta \le 1, -\infty < \text{ Re } z < +\infty, |\text{Im } z| < r\big\},$$

and define

$$\begin{aligned}
C^0 f(z,\eta) &= \sup_{(z,\eta)\in D_r} \big(|f(z,\eta)| \exp(d\epsilon^{1/2}|\text{Re } z|)\big),\\
C^+ f(z,\eta) &= \sup_{(z,\eta)\in D_r} \big(|f(z,\eta)| \exp(-\kappa|\text{Im } z|)\big),\\
H^0_\lambda f(z,\eta) &= \sup{}^{(z,\eta)\in D_r}_{|\delta_1|\le 1} \big(|f(x+\delta_1+iy,\eta_1) - f(x+iy,\eta_2)|\\
&\qquad\qquad \times (\delta_1^2 + (\eta_1-\eta_2)^2)^{-\lambda/2} \exp(d\epsilon^{1/2}|x|)\big),\\
H^+_\lambda f(z,\eta) &= \sup{}^{(z,\eta)\in D_r}_{|\delta_1|\le 1} \big(|f(x+\delta_1+iy,\eta_1) - f(x+iy,\eta_2)|\\
&\qquad\qquad \times (\delta_1^2 + (\eta_1-\eta_2)^2)^{-\lambda/2} \exp(-\kappa|\text{Im } z|)\big),
\end{aligned}$$

where $d > 0$ is a constant less than $(-\delta_1\beta_1/\gamma_1)^{1/2}/4$. Then we define Banach spaces B^0_n, B^+_n and B_n as follows:

$$\begin{aligned}
B^0_n = \Big\{ & f(z,\eta) \in C^n(D_r\times[0,1]) \Bigm| f(\cdot,\eta) \text{ is analytic}, f(-z,\cdot) = f(z,\cdot) \text{ and}\\
& f(x,\eta) \text{ is real for } x\in\mathbb{R},\\
& \|f\|_{B^0_n} = \sum_{m=0}^{n}\sum_{k=0}^{m} \epsilon^{-\min(k,n-2)/2} C^0\left(\frac{\partial^m f}{\partial z^k \partial\eta^{m-k}}\right)\\
&\qquad + \sum_{m=0}^{n} \epsilon^{-\min(m,n-2)/2} H^0_\lambda\left(\frac{\partial^n f}{\partial z^m\partial\eta^{n-m}}\right) < +\infty\Big\},\\
B^+_n = \Big\{ & f(z,\eta) \in C^n(D_{r_1}\times[0,1]) \Bigm| f(\cdot,\eta) \text{ is analytic}, f(-z,\cdot) = f(z,\cdot),\\
& f(x,\eta) \text{ is real and } f(x+2\pi/\kappa,\eta) = f(x,\eta) \text{ for } x\in\mathbb{R},\\
& \|f\|_{B^+_n} = \sum_{m=0}^{n}\left(H^+_\lambda\left(\frac{\partial^n f}{\partial z^m\partial\eta^{n-m}}\right) + \sum_{k=0}^{m} C^+\left(\frac{\partial^m f}{\partial z^k\partial\eta^{m-k}}\right)\right) < +\infty\Big\},\\
B_n = \Big\{ & f(z,\eta) \in C^n(D_r\times[0,1]) \Bigm| f(\cdot,\eta) \text{ is analytic and}\\
& f(z,\eta) = f^0(z,\eta) + f^+(z-\delta\tanh hz,\eta) \text{ where } f^0\in B^0_n\\
& \text{ and } f^+\in B^+_n, \|f\|_{B_n} = \|f^0\|_{B^0_n} + \|f^+\|_{B^+_n}\Big\},
\end{aligned}$$

where $h = (1/4)(-\delta_1\beta_1/\gamma_1)^{1/2}$. Note that in the definition of B_n, we need D_{r_1} in B^+_n larger than D_r in B^0_n by letting $r_1 > r$. In the following we let

$$r = \ell\epsilon^{-1/2} + \ell_1, \quad r_1 = \ell\epsilon^{-1/2} + \ell_2, \quad 0 < \ell_1 < \ell_2, \tag{60}$$

and $0 < \ell < \pi(-\delta_1\beta_1/\gamma_1)^{-1/2}$. δ is a given constant, called the phase shift of the periodic part of a function in B_n. We keep δ small so that $f^+(z-\delta\tanh hz,\eta)$ is well-defined in D_r. Also note that ℓ is any number in $(0, \pi(-\delta_1\beta_1/\gamma_1))$. In what follows we may choose ℓ in $0 < \pi(-\delta_1\beta_1/\gamma_1)^{-1/2} - \tau < \ell < \pi(-\delta_1\beta_1/\gamma_1)^{-1/2}$ where $\tau > 0$ is a fixed arbitrarily small number.

5. Estimates of the integral operators

First let us consider the integral operator $\mathcal{G}$ in (57). By the definition of (57), for $z = x + iy \in D_r$ and $f_1(z, \eta)$, $f_2(z)$ in B_n we have

$$\begin{aligned}
\mathcal{G}(f_1, f_2) &= \left(\int_{-\infty}^{z} + \int_{z}^{+\infty}\right)\int_0^1 G(z, \eta; \xi, \zeta) f_1(\xi, \zeta) d\eta d\zeta \\
&\quad - \left(\int_{-\infty}^{z} + \int_{z}^{+\infty}\right) G(z, \eta; \xi, 1) f_2(\zeta) d\zeta \\
&= \left(\int_{-\infty+iy}^{x+iy} + \int_{x+iy}^{+\infty+iy}\right)\int_0^1 G(x+iy, \eta; \xi, \zeta) f_1(\xi, \zeta) d\zeta d\xi \\
&\quad - \left(\int_{-\infty+iy}^{x+iy} + \int_{x+iy}^{+\infty+iy}\right) G(x+iy, \eta; \xi, \zeta) f_2(\xi) d\xi \\
&= \left(\int_{-\infty}^{x} + \int_{x}^{+\infty}\right)\int_0^1 G(x, \eta; \xi, \zeta) f_1(\xi + iy, \zeta) d\zeta d\xi \\
&\quad - \left(\int_{-\infty}^{x} + \int_{x}^{+\infty}\right) G(x, \eta; \xi, 1) f_2(\xi + iy) d\xi \\
&= \int_{-\infty}^{+\infty}\int_0^1 G(x, \eta; \xi, \zeta) f_1(\xi + iy, \zeta) d\zeta d\xi \\
&\quad - \int_{-\infty}^{+\infty} G(x, \eta; \xi, 1) f_2(\xi + iy) d\xi \\
&= \mathcal{G}_1 f_1 - \mathcal{G}_2 f_2,
\end{aligned} \tag{61}$$

where (53) has been used. The integrals in $\mathcal{G}(f_1, f_2)$ transformed to real cases were studied in [8]. Obviously $\mathcal{G}(f_1, f_2)(z, \eta)$ is analytic in z. From the estimates in real x [8], we have

Lemma 1. (1) *If $f_1(z, \eta) \in B_n^+$ and $f_2(z) \in B_{n+1}^+$ for $n \geq 0$, then $\mathcal{G}(f_1, f_2)(z, \eta) \in B_{n+2}^+$ and*

$$\|\mathcal{G}(f_1, f_2)\|_{B_{n+2}^+} \leq K\big(\|f_1\|_{B_n^+} + \|f_2\|_{B_{n+1}^+}\big).$$

(2) *If $f_1(z, \eta) \in B_n^0$ and $f_2(z) \in B_{n+1}^0$ for $n \geq 0$, then*

$$\mathcal{G}_1(f_1, f_2)(z, \eta) \in B_{n+2}^+ \text{ and } \mathcal{G}(f_1, f_2) = \mathcal{G}_1 f_1 - \mathcal{G}_2 f_2$$

with

$$C^0\left(\frac{\partial^m \mathcal{G}_1 f_1}{\partial z^i \partial \eta^{m-i}}\right) \leq K\left(\sum_{\ell=i}^{m}\sum_{j=0}^{m-\ell} C^0\left(\frac{\partial^{\ell+j} f_1}{\partial z^\ell \partial \eta^j}\right)\right)$$

for $0 \leq m \leq n,\ 0 \leq i \leq m$,

$$C^0\left(\frac{\partial^m \mathcal{G}_2 f_2}{\partial z^i \partial \eta^{m-i}}\right) \leq K\left(\sum_{\ell=i}^{m} C^0\left(\frac{\partial^\ell f_2}{\partial z^\ell}\right)\right)$$

for $0 \le m \le n+1,\ 0 \le i \le m,$

$$C^0\left(\frac{\partial^{n+k}\mathcal{G}_1 f_1}{\partial z^i \partial \eta^{n+k-i}}\right) + H^0_\lambda\left(\frac{\partial^{n+2}\mathcal{G}_1 f_1}{\partial z^i \partial \eta^{n+2-i}}\right) \le K\left(\sum_{j=n_i}^{n}\left(\sum_{p=0}^{n-j} C^0\left(\frac{\partial^{j+p} f_1}{\partial z^j \partial \eta^p}\right) + H^0_\lambda\left(\frac{\partial^n f_1}{\partial z^j \partial \eta^{n-j}}\right)\right)\right)$$

for $k = 1, 2,\ 0 \le i \le n+k,\ n_1 = \min(n, i),$ *and*

$$C^0\left(\frac{\partial^{n+2}\mathcal{G}_2 f_2}{\partial z^i \partial \eta^{n+2-i}}\right) + H^0_\lambda\left(\frac{\partial^{n+2}\mathcal{G}_2 f_2}{\partial z^i \partial \eta^{n+2-i}}\right) \le K\left(\sum_{\ell=i}^{n+1}\left(C^0\left(\frac{\partial^\ell f_2}{\partial z^\ell}\right)\right) + H^0_\lambda\left(\frac{\partial^{n+1} f_2}{\partial z^{n+1}}\right)\right)$$

for $0 \le i \le n+2$, *where* K *is constant.*

The proof of Lemma 1 is only based on the asymptotic behavior of the eigenvalues and eigenfunctions in (41) and (42). The Green's function $G(x, \eta; \xi, \zeta)$ behaves like $\ell n((x-\xi)^2 + (\eta-\zeta)^2)$ for $(x-\xi)^2 + (\eta-\zeta)^2 \le 4$ and like $\exp(-k_1^{1/2}|x-\xi|)$ for $|x-\xi|^2 \ge 4$. We use these approximations for $G(x, \eta; \xi, \zeta)$ and usual arguments for obtaining a priori estimates of solutions of elliptic equations to prove Lemma 1. Since the proof is similar to the one in [8], we omit it here.

Now let us study the operator $\mathcal{L}$ defined in (58). By the definition, we have

$$\mathcal{L}(f(s))(z) = \psi_1(z)\int_0^z \psi_2(s) f(s)ds + \psi_2(z)\int_z^{+\infty} \psi_1(s) f(s)ds, \tag{62}$$

where $f(z) \in B_n$. Since $f(z)$ and $\psi_2(z)$ are even and $\psi_1(z)$ is odd, (62) is well-defined and $\mathcal{L}f$ is even. To have estimates of $\mathcal{L}f(z)$, we need only consider Re $z \ge 0$. Let $f(z) \in B_n^+$ and

$$\mathcal{L}^+ f(z) = -(1/2)(-\gamma_1/\delta_1\beta_1\epsilon)^{1/2}\Big(\int_{-\infty}^z \exp\big(-(-\epsilon\beta_1\delta_1/\gamma_1)^{1/2}(z-s)\big) \times f(s)ds - \int_z^{+\infty} \exp\big((-\epsilon\beta_1\delta_1/\gamma_1)^{1/2}(z-s)\big) f(s)ds\Big). \tag{63}$$

By the asymptotic forms of $\psi_1(z)$ and $\psi_2(z)$ for large Re z, we have

$$\mathcal{L}f(z) = \mathcal{L}^+ f(z) + (Tf)(z), \tag{64}$$

where $(Tf)(z)$ goes to zero with order of $O(\exp(-d\epsilon^{1/2}|\text{Re } z|))$ for large Re z. Since a similar integral operator as (62) was discussed in [8] and the only difference is the coefficients, which will not change the estimates, we state the results without proof.

Lemma 2. (1) *For* $f(z) \in B_n^0$,*we have* $\mathcal{L}f(z) \in B_{n+z}^0$,

$$C^0\left(\frac{\partial^m \mathcal{L}f(z)}{\partial z^m}\right) \le K\epsilon^{(m-2)/2}\|f(z)\|_{B_0^0}$$

for $m = 0, 1, 2,$

$$C^0\left(\frac{\partial^m \mathcal{L}f(z)}{\partial z^m}\right) \le K\epsilon^{(m_1-2)/2}\|f(z)\|_{B_{m-2}^0}$$

for $2 \leq m \leq n+2, m_1 = \min(m,n)$, *and*

$$H^0_\lambda\left(\frac{\partial^{n+2}\mathcal{L}f(z)}{\partial z^{n+2}}\right) \leq K\epsilon^{(n-2)/2}\|f(z)\|_{B^0_n}.$$

(2) *For* $f(z) \in B^+_n$ *with* $n \geq 0$, *we have* $\mathcal{L}^+ f \in B^+_{n+2}$, $Tf \in B^0_{n+2}$,

$$C^+\left(\frac{\partial^m\mathcal{L}^+f(z)}{\partial z^m}\right) + \exp(-\ell\kappa\epsilon^{-1/2})C^0\left(\frac{\partial^m Tf(z)}{\partial z^m}\right) \leq K\epsilon^{(m-2)/2}\|f\|_{B^+_0}$$

for $m = 0, 1, 2$,

$$C^+\left(\frac{\partial^m\mathcal{L}^+f(z)}{\partial z^m}\right) + \exp(-\ell\kappa\epsilon^{-1/2})C^0\left(\frac{\partial^m Tf(z)}{\partial z^m}\right) \leq K\|f\|_{B^+_{m-2}}$$

for $2 \leq m \leq n+2$, *and*

$$H^+_\lambda\left(\frac{\partial^{n+2}\mathcal{L}^+f(z)}{\partial z^{n+2}}\right) + \exp(-\ell\kappa\epsilon^{-1/2})H^0_\lambda\left(\frac{\partial^{n+2}Tf(z)}{\partial z^{n+2}}\right) \leq K\|f\|_{B^+_n}.$$

A similar proof of this Lemma can be found in [8].

Finally we consider the operator $\mathcal{P}$ defined in (59). Let $f(z) \in B_n$ with $f(z) = f^0(z) + f^+(z - \delta \tanh hz)$ and

$$\mathcal{P}(f(s))(z) = \tilde{\mu}\cos\kappa z + (1/\kappa)\int_0^z \sin\kappa(z-s)f(s)ds. \tag{65}$$

Since $\mathcal{P}f(z)$ is even, we need only obtain the estimates for Re $z \geq 0$. From (65), we have

$$\begin{aligned}\mathcal{P}f(z) &= \tilde{\mu}\cos\kappa z + (1/\kappa)\int_0^z \sin\kappa(z-s)f^0(s)ds \\ &\qquad + (1/\kappa)\int_0^z \sin\kappa(z-s)f^+(s-\delta\tanh hs)ds \\ &= \tilde{\mu}\cos\kappa z + (1/\kappa)\int_0^z \sin\kappa(z-s)f^+(s-\delta)ds \\ &\qquad + (1/\kappa)\int_0^{+\infty} \sin\kappa(z-s)g^0(s)ds - (1/\kappa)\int_z^{+\infty} \sin\kappa(z-s)g^0(s)ds,\end{aligned}$$

where $g^0(s) = f^0(s) + f^+(s - \delta \tanh hs) - f^+(s-\delta)$. Let

$$\begin{aligned}(\mathcal{P}^+f(s))(z) &= \tilde{\mu}\cos\kappa z + (1/\kappa)\int_0^z \sin\kappa(z-s)f^+(s-\delta)ds \\ &\qquad + (1/\kappa)\sin\kappa z\int_0^{+\infty}\cos\kappa s g^0(s)ds \\ &\qquad - (1/\kappa)\cos\kappa z\int_0^{+\infty}\sin\kappa s g^0(s)ds \\ &= p\cos\kappa z + q\sin\kappa z + (1/\kappa)\int_0^z \sin\kappa(z-s)f^+(s-\delta)ds,\end{aligned} \tag{66}$$

where

$$p = \tilde{\mu} - (1/\kappa) \int_0^{+\infty} \sin \kappa s g^0(s) ds,$$
$$q = (1/\kappa) \int_0^{+\infty} \cos \kappa s g^0(s) ds.$$

Replace z in (66) by $z + \delta$ to have

$$\begin{aligned}\mathcal{P}^+ f(z+\delta) &= (p \cos \kappa\delta + q \sin \kappa\delta) \cos \kappa z + (q \cos \kappa\delta - p \sin \kappa\delta) \sin \kappa z \\ &\quad + (1/\kappa) \int_{-\delta}^0 \sin \kappa(z-s) f^+(s) ds + (1/\kappa) \int_0^z \sin \kappa(z-s) f^+(s) ds \\ &= \tilde{A} \cos \kappa z + (1/\kappa) \int_0^z \sin \kappa(z-s) f^+(s) ds \\ &= (M^+ f(s))(z),\end{aligned} \tag{67}$$

provided

$$p \cos \kappa\delta + q \sin \kappa\delta - (1/\kappa) \int_{-\delta}^0 \sin \kappa s f^+(s) ds = \tilde{A}, \tag{68}$$

$$q \cos \kappa\delta - p \sin \kappa\delta + (1/\kappa) \int_{-\delta}^0 \cos \kappa s f^+(s) ds = 0. \tag{69}$$

From (68) and (69), we know that $\tilde{\mu}$ must be chosen by

$$\tilde{\mu} = \tilde{A} \cos \kappa\delta + (1/\kappa) \int_{-\delta}^0 \sin \kappa(s+\delta) f^+(s) ds + (1/\kappa) \int_0^{+\infty} \sin \kappa s g^0(s) ds, \tag{70}$$

and $\tilde{A}$ satisfies

$$\tilde{A} \sin \delta\kappa = (1/\kappa) \int_0^{+\infty} \cos \kappa s g^0(s) ds + (1/\kappa) \int_{-\delta}^0 \cos \kappa(s+\delta) f^+(s) ds. \tag{71}$$

Thus $M^+ f(z)$ is even. For bounded $M^+ f$, the following conditions must be imposed

$$\int_0^{2\pi/\kappa} \sin \kappa s f^+(s) ds = 0, \quad \int_0^{2\pi/\kappa} \cos \kappa s f^+(s) ds = 0.$$

The former is automatically satisfied since $f^+(s)$ is even and periodic in x with period $2\pi/\kappa$. Therefore we only need

$$\int_0^{2\pi/\kappa} \cos \kappa s f^+(s) ds = 0. \tag{72}$$

An integral path in (67) is chosen as follows: let k_1 be an integer so that $|x - 2\pi k_1/\kappa| < 2\pi/\kappa$ with $z = x + iy$ and

$$\begin{aligned}P_a = \big\{ z_1 = x_1 + iy_1 \;\big|\; &(0 \le x_1 \le 2k_1\pi/\kappa, y_1 = 0), \\ &\text{a smooth curve joining } (2k_1\pi/\kappa, 0) \text{ with } x + iy \text{ in } D_r, \\ &\text{and having length less than } |y| + 4\pi/\kappa \big\}.\end{aligned}$$

By using the path p_a and the properties of $\sin \kappa(z-s)$, we have

Lemma 3. *If $f^+(z) \in B^+_{n+1}$ and satisfies* (72)*, then*

$$\int_0^z \sin\kappa(z-s) f^+(s)ds \in B^+_{n+3},$$

and

$$\Big\| \int_0^z \sin\kappa(z-s) f^+(s)ds \Big\|_{B^+_{n+3}} \le K\epsilon^{-1/2} \|f^+\|_{B^+_n}.$$

By definition of $\mathcal{P}f$, we have

$$\begin{aligned} \mathcal{P}f(z) &= \mathcal{P}^+ f - (1/\kappa)\int_z^{+\infty} \sin\kappa(z-s) g^0(s)ds \\ &= M^+ f(z-\delta) - (1/\kappa)\int_z^{+\infty} \sin\kappa(z-s) g^0(s)ds \\ &= M^+ f(z-\delta \tanh hz) + (\mathcal{P}^0 f(s))(z), \end{aligned} \tag{73}$$

where

$$(\mathcal{P}^0 f(s))(z) = M^+ f(z-\delta) - M^+ f(z-\delta\tanh hz) - (1/\kappa)\int_z^{+\infty} \sin\kappa(z-s) g^0(s)ds.$$

Since $\mathcal{P}f$ and $M^+ f(z-\delta \tanh hz)$ are even, $\mathcal{P}^0 f$ is even. So only estimates of $\mathcal{P}^0 f$ for Re $z \ge 0$ are needed. By using Lemma 3, the estimates for $\int_z^{+\infty} \sin\kappa(z-s) g^0(s)ds$ and the definition of $g^0(s)$, we have

Lemma 4. *For $f(z) = f^0(z) + f^+(z-\delta \tanh hz) \in B_n$, $\mathcal{P}^0 f \in B^0_{n+2}$ and*

$$C^0\left(\frac{\partial^m \mathcal{P}^0 f}{\partial z^m}\right) \le K\Bigg((\|\tilde{A}\cos\kappa z\|_{B^+_{n+1}} + \epsilon^{-1/2}\|f^+\|_{B^+_{n+1}}) \exp(\kappa\ell\epsilon^{-1/2}) + \epsilon^{-1/2} C^0\left(\frac{\partial^m f^0}{\partial z^m}\right)\Bigg) \text{ for } 0 \le m \le n,$$

$$\sum_{k=1}^{2} C^0\left(\frac{\partial^{n+k}\mathcal{P}^0 f}{\partial z^{n+k}}\right) + H^0_\lambda\left(\frac{\partial^{n+2}\mathcal{P}^0 f}{\partial z^{n+2}}\right) \le K\Bigg((|\tilde{A}| + \epsilon^{-1/2}\|f^+\|_{B^+_{n+1}}) \exp(\kappa\ell\epsilon^{-1/2}) + \epsilon^{-1/2}\left(C^0\left(\frac{\partial^n f^0}{\partial z^n}\right) + H^0_\lambda\left(\frac{\partial^n f^0}{\partial z^n}\right)\right)\Bigg).$$

A similar proof can also be found in [8]. Now we have all the estimates of the integral operators we need.

6. Decomposition of the equations and estimates

In (57), (58), and (59), we let

$$\begin{gathered} a(z) = a^0(z) + a^+(z-\delta\tanh hz), \;\; \omega(z) = \omega^0(z) + \omega^+(z-\delta\tanh hz), \\ \theta(z,\eta) = \theta^0(z,\eta) + \theta^+(z-\delta\tanh hz, \eta), \end{gathered} \tag{74}$$

where a^0, ω^0 and θ^0 belong to B^0_{n+2} and a^+, ω^+ and θ^+ belong to B^+_{n+3} for $n \geq 0$. We further assume

$$\begin{gathered} a^0(z) = \epsilon^{1/2} q^0_1(z),\ \omega^0(z) = \epsilon^{1/2} q^0_2(z),\ \theta^0(z,\eta) = \epsilon^{1/2} q^0_3(z,\eta), \\ a^+(z) = \tilde{A} v_1(z) = \tilde{A}(\cos\kappa z + \epsilon^{1/2} p^+_1(z)), \\ \omega^+(z) = \tilde{A} v_2(z) = \tilde{A}\epsilon^{1/2} p^+_2(z),\ \theta^+(z,\eta) = \tilde{A} v_3(z,\eta) = \tilde{A}\epsilon^{1/2} p^+_3(z,\eta), \end{gathered} \tag{75}$$

where we express $\tilde{A}$ the amplitude of the oscillatory tail as

$$\tilde{A} = A\epsilon^{(n+1)/2} \exp(-\ell\kappa\epsilon^{-1/2}) = A_1 \exp(-\ell\kappa\epsilon^{-1/2}), \tag{76}$$

and A is some constant to be estimated later for the application of the contraction mapping theorem.

Now let us consider (57) in the domain D_r. Since the Green's function $G(z,\eta;\xi,\zeta)$ is orthogonal to $Z(\zeta)$ and $z_1(\zeta)$ we only have to estimate

$$\begin{aligned} H_1(\sigma,S,a,\omega,\theta)(z,\eta) &\equiv \sigma\epsilon q^2\theta_{xx} - \nu_1\delta_1\epsilon\rho_\eta(aZ + (S+\omega)z_1 + \theta) \\ &+ (3/2)\epsilon\big(q^2((S+\omega)z_1 + aZ + \theta)^2_\eta\big)_\eta + \epsilon^{-1}F_1\big(\epsilon(aZ + (S+\omega)z_1 + \theta)\big), \end{aligned} \tag{77}$$

$$\begin{aligned} H_2(\sigma,S,a,\omega,\theta)(z) &\equiv -\nu_1\delta_1\epsilon\rho(1)q^{-2}(1)(aZ + (S+\omega)z_1 + \theta) \\ &+ (3\epsilon/2)(aZ_\eta + (S+\omega)z_{1\eta} + \theta_\eta)^2 + \epsilon^{-1}F_2(\epsilon(aZ + (S+\omega)z_1 + \theta)), \end{aligned} \tag{78}$$

instead of F_3 and F_4 in (57). Let

$$\begin{aligned} H^+_1(\sigma,v_1,v_2,v_3,\tilde{A})(z,\eta) &= \tilde{A}^{-1}H_1\big(\sigma,0,\tilde{A}(\cos\kappa z + v_1),\tilde{A}v_2,\tilde{A}v_3\big)(z,\eta), \\ H^+_2(\sigma,v_1,v_2,v_3,\tilde{A})(z) &= \tilde{A}^{-1}H_2\big(\sigma,0,\tilde{A}(\cos\kappa z + v_1),\tilde{A}v_2,\tilde{A}v_3\big)(z), \\ H^0_1(\sigma,S,a,\omega,\theta)(z,\eta) &= H_1(\sigma,S,a,\omega,\theta)(z,\eta) - \tilde{A}H^+_1(z - \delta\ \tanh\ hz,\eta), \\ H^0_2(\sigma,S,a,\omega,\theta)(z) &= H_2(\sigma,S,a,\omega,\theta)(z) - \tilde{A}H^+_2(z - \delta\ \tanh\ hz). \end{aligned} \tag{79}$$

By checking terms in H^+_1 and H^+_2, for $|\sigma| + \sum_{i=1}^3 \|v_i\|_{B^+_{n+3}} \leq K$ we can obtain that $H^+_1 \in B^+_{n+1}$ and $H^+_2 \in B^+_{n+2}$ with

$$\|H^+_1\|_{B^+_{n+1}} \leq K\epsilon \text{ and } \|H^+_2\|_{B^+_{n+2}} \leq K\epsilon.$$

Therefore by Lemma 1, it follows that $\mathcal{G}(H^+_1, H^+_2) \in B^+_{n+3}$ and

$$\|\mathcal{G}(H^+_1, H^+_2)(z,\eta)\|_{B^+_{n+3}} \leq K\epsilon.$$

Also by checking terms in H^0_1 and H^0_2, we have that if

$$\epsilon^{1/2}|\sigma| + \|a^0\|_{B^0_{n+2}} + \|\omega^0\|_{B^0_{n+2}} + \|\theta^0\|_{B^0_{n+2}} \leq K\epsilon^{1/2},$$

and

$$\|v_1 - \cos\kappa z\|_{B^+_{n+3}} + \sum_{i=2}^{3} \|v_i\|_{B^+_{n+3}} \leq K\epsilon^{1/2}, \tag{80}$$

then $H^0_1 \in B^0_n$ and $H^0_2 \in B^0_{n+1}$. By Lemma 1 and the definitions of Banach spaces, $\mathcal{G}(H^0_1, H^0_2) \in B^0_{n+2}$ and for $n \geq 0$,

$$\|\mathcal{G}(H^0_1, H^0_2)\|_{B^0_{n+2}} \leq K\epsilon(K_1 + A_1\epsilon^{-n/2}), \tag{81}$$

where K_1 and K are constants independent of ϵ. By (57), (74) and (79), we have

$$\begin{aligned}\theta^0(z,\eta) &+ \theta^+(z-\delta\ \tanh\ hz,\eta)\\ &= \mathcal{G}\big(H_1^0 + \tilde{A}H_1^+(z-\delta\ \tanh\ hz,\eta), H_2^0 + \tilde{A}H_2^+(z-\delta\ \tanh\ hz,\eta)\big).\end{aligned}$$

Let the oscillatory part

$$\theta^+(z,\eta) = \tilde{A}v_3(z,\eta) = \tilde{A}\mathcal{G}(H_1^+, H_2^+)(z,\eta) = \tilde{A}V_3(z,\eta), \tag{82}$$

and it follows that

$$\begin{aligned}\theta^0(z,\eta) &= \mathcal{G}(H_1^0, H_2^0) + \tilde{A}\mathcal{G}\big(H_1^+(\xi-\delta\ \tanh\ h\xi,\zeta), H_2^+(\xi-\delta\ \tanh\ h\xi)\big)(z,\eta)\\ &\qquad - \tilde{A}\mathcal{G}(H_1^+(\xi,\zeta), H_2^+(\xi))(z-\delta\ \tanh\ hz,\eta)\\ &= \mathrm{I} + \tilde{A}\mathrm{I\!I} \equiv \Theta^0(z,\eta),\end{aligned} \tag{83}$$

where ξ and ζ are the integration variables. However

$$\begin{aligned}\mathrm{I\!I} &= \mathcal{G}\big(H_1^+(\xi-\delta\ \tanh\ h\xi,\zeta) - H_1^+(\xi-\delta,\zeta), H_2^+(\xi-\delta\ \tanh\ \xi) - H_2^+(\xi-\delta)\big)\\ &\qquad - \big(\mathcal{G}(H_1^+, H_2^+)(z-\delta\ \tanh\ hz,\eta) - \mathcal{G}(H_1^+, H_2^+)(z-\delta,\eta)\big)\\ &\equiv \mathrm{I\!I\!I}_1 + \mathrm{I\!I\!I}_2.\end{aligned}$$

By the evenness of $\mathrm{I\!I}$ in z, we only need estimates for $\mathrm{Re}\ z \geq 0$. If we consider the functions in B_n^0 defined for $\mathrm{Re}\ z \geq 0$, by Lemma 1, $\mathrm{I\!I\!I}_1 \in B_{n+2}^0$ with

$$\|\mathrm{I\!I\!I}_1\|_{B_{n+2}^0} \leq K\epsilon^{1-(n/2)} \exp(\ell\kappa\epsilon^{-1/2}),$$

and by (75), $\mathrm{I\!I\!I}_2 \in B_{n+2}^0$ with

$$\|\mathrm{I\!I\!I}_2\|_{B_{n+2}^0} \leq K\epsilon^{1-(n/2)} \exp(\ell\kappa\epsilon^{-1/2}).$$

By (79), (80), (81) and (83), we have $\Theta^0(z,\eta) \in B_{n+2}^0$ and

$$\|\Theta^0(z,\eta)\|_{B_{n+2}^0} \leq K\epsilon(K_1 + A_1\epsilon^{-n/2}).$$

We state the above results as

Theorem 1. *Let*

$$\begin{aligned}\epsilon^{1/2}|\sigma| &+ \|a^0\|_{B_{n+2}^0} + \|\omega^0\|_{B_{n+2}^0} + \|\theta^0\|_{B_{n+2}^0}\\ &+ \|v_1 - \cos\kappa z\|_{B_{n+3}^+} + \sum_{i=2}^{3}\|v_i\|_{B_{n+3}^+} \leq K\epsilon^{1/2},\end{aligned}$$

and $|A_1| \leq K$ *for* $n \geq 0$. *Then* $V_3(z,\eta) \in B_{n+3}^+$ *and* $\Theta^0(z,\eta) \in B_{n+2}^0$,

$$\|V_3(z,\eta)\|_{B_{n+3}^+} \leq K\epsilon \text{ and } \|\Theta^0(z,\eta)\|_{B_{n+2}^0} \leq K\epsilon(K_1 + A_1\epsilon^{-n/2}),$$

where K *and* K_1 *are constants independent of* ϵ.

Now we take up (58). Let

$$g_1^+(\sigma, v_1, v_2, v_3, A_1)(z) = \tilde{A}^{-1}g_1(0,\sigma,\tilde{A}v_1,\tilde{A}v_2,\tilde{A}v_3)(z), \tag{84}$$

$$g_1^0(\sigma, S, a, \omega, \theta)(z) = g_1(S,\sigma,a,\omega,\theta)(z) - \tilde{A}^{-1}g_1^+(z-\delta\ \tanh\ hz). \tag{85}$$

By checking g_1^+ term by term, from

$$\|v_1 - \cos\kappa z\|_{B_{n+3}^+} + \epsilon^{1/2}\|v_2\|_{B_{n+3}^+} + \|v_3\|_{B_{n+3}^+} + \epsilon|\sigma| + |A_1| \leq K\epsilon, \tag{86}$$

it is shown that $g_1^+ \in B_{n+1}^+$ and

$$\|g_1^+ - \epsilon K_1^* \cos \kappa z - K_2^* \epsilon v_{2zz}\|_{B_{n+1}^+} \leq K\epsilon^2,$$

where K_1^* and K_2^* are coefficients of $\epsilon \cos \kappa z$ and ϵv_{2zz} in g_1^+ respectively. By (63), (64) and Lemma 2, we have $\mathcal{L}^+ g_1^+ \in B_{n+3}^+$ and $(Tg_1^+)(z) \in B_{n+2}^0$ such that

$$\|\mathcal{L}^+(g_1^+ - \epsilon K_1^* \cos \kappa z - \epsilon K_2^* v_{2zz}\|_{B_{n+3}^+} \leq K\epsilon,$$

$$\|T(g_1^+ - \epsilon K_1^* \cos \kappa z - \epsilon K_2^* v_{2zz})\|_{B_{n+2}^0} \leq K\epsilon^{1-(n/2)} \exp(\ell\kappa\epsilon^{-1/2}).$$

Then using integration by parts twice and Lemma 3 again, we obtain

$$\|\mathcal{L}^+(\epsilon K_1^* \cos \kappa z + \epsilon K_2^* v_{2zz})\|_{B_{n+3}^+} \leq K\epsilon,$$

$$\|T(\epsilon K_1^* \cos \kappa z + \epsilon K_2^* v_{2zz})\|_{B_{n+2}^0} \leq K\epsilon^{1-(n/2)} \exp(\ell\kappa\epsilon^{-1/2}).$$

Therefore we define

$$V_{21}(z) \equiv \mathcal{L}^+(g_1^+)(z) = v_2(z), \tag{87}$$

and it follows $V_{21}(z) \in B_{n+3}^+$ and

$$\|V_{21}(z)\|_{B_{n+3}^+} \leq K\epsilon. \tag{88}$$

Also by checking terms in $g_1^0(z)$, we find that (86) together with

$$\|a^0\|_{B_{n+2}^0} + \epsilon^{1/2}\|\omega^0\|_{B_{n+2}^0} + \|\theta^0\|_{B_{n+2}^0} \leq K\epsilon,$$

implies $g_1^0 \in B_n^0$ and

$$C^0\left(\frac{\partial^i g_{11}^0}{\partial z^i}\right) \leq K\epsilon(K_1\epsilon^{1+i/2} + \tilde{A}\epsilon^{1/2}) \text{ and } 0 \leq i \leq n,$$
$$H_\lambda^0\left(\frac{\partial^n g_{11}^0}{\partial z^i}\right) \leq K\epsilon(K_1\epsilon^{1+n/2} + \tilde{A}\epsilon^{1/2}),$$

where $g_{11}^0 = g_1^0 - K_3^*\omega_{zz}^0\epsilon - \tilde{A}\cos\kappa(z - \delta \tanh hz)K_4^*(S,\sigma)$, K_3^* is the constant coefficient of $\omega_{zz}^0\epsilon$ and $K_4^*(S,\sigma)$ is the coefficient of $\tilde{A}\cos\kappa(z - \delta \tanh hz)$ depending upon $S(z)$ and σ only. Thus by Lemma 2, $\mathcal{L}g_{11}^0 \in B_{n+2}^0$ and

$$\|\mathcal{L}g_{11}^0\|_{B_{n+2}^0} \leq K(K_1\epsilon + A_1\epsilon^{-(n-1)/2}).$$

Integration by parts twice yields

$$\left\|\mathcal{L}\Big(K_3^*\omega_{zz}^0 + \tilde{A}\cos\kappa(z - \delta \tanh hz)K_4^*(S,\sigma)\Big)\right\|_{B_{n+2}^0} \leq K(K_1 + A_1\epsilon^{-(n-1)/2}).$$

Therefore

$$\|\mathcal{L}g_1^0(z)\|_{B_{n+2}^0} \leq K(K_1\epsilon + A_1\epsilon^{-(n-1)/2}). \tag{89}$$

From (58), (74), (75) and (87), we have

$$v_2(z) = V_{21}(z),$$

and

$$\begin{aligned}\omega^0(z) &= \mathcal{L}(g_1(s))(z) - \tilde{A}V_{21}(z-\delta\tanh hz)\\ &= \mathcal{L}\big(g_1^0(s) + \tilde{A}g_1^+(s-\delta\tanh hs)\big)(z)\\ &\qquad - \tilde{A}\mathcal{L}^+(g_1^+(s))(z-\delta\tanh hz)\\ &= \mathcal{L}(g_1^0(s))(z) + \tilde{A}\mathrm{III}(z)\\ &\equiv \Omega_1^0(z). \end{aligned} \tag{90}$$

By the evenness of $\mathrm{III}(z)$, we only need the estimates for $\mathrm{Re}\ z \geq 0$. For $\mathrm{Re}\ z \geq 0$,

$$\begin{aligned}\mathrm{III}(z) = \mathcal{L}\big(&g_1^+(s-\delta\tanh hs) - g_1^+(s-\delta)\big)(z)\\ &+ \mathcal{L}^+(g_1^+(s))(z-\delta) + T(g_1^+(s-\delta))(z) - \mathcal{L}^+(g^+(s))(z-\delta\tanh hz).\end{aligned}$$

By Lemma 2, the estimates for $g_1^+(s)$ and integration by parts twice we find that

$$\big\|\mathcal{L}\big(g_1^+(s-\delta\tanh hs) - g^+(s-\delta)\big)(z)\big\|_{B^0_{n+2}} \leq K\epsilon^{-(n-1)/2}\exp(-\ell\kappa\epsilon^{-1/2}).$$

Here we only consider the functions in B^0_{n+2} for $\mathrm{Re}\ z \geq 0$. Also by (88) and the definition of B^0_{n+2} (only for $\mathrm{Re}\ z \geq 0$), we have

$$\big\|\mathcal{L}^+(g^+(s))(z-\delta) - \mathcal{L}^+(g^+(s))(z-\delta\tanh hz)\big\|_{B^0_{n+2}} \leq K\epsilon^{-(n-2)/2}\exp(-\ell\kappa\epsilon^{-1/2}).$$

Thus by combining the estimate for $T(g^+(s-\delta))(z)$, it is obtained that

$$\|\mathrm{III}(z)\|_{B^0_{n+2}} \leq K\epsilon^{-(n-1)/2}\exp(-\ell\kappa\epsilon^{-1/2}).$$

Therefore from (89) and (90), we have that $\Omega_1^0(z) \in B^0_{n+2}$ and

$$\|\Omega_1^0(z)\|_{B^0_{n+2}} \leq K(K_1\epsilon + A_1\epsilon^{-(n-1)/2}).$$

We summarize the results as follows.

Theorem 2. *Assume that* $\|v_1 - \cos\kappa z\|_{B^+_{n+3}} + \|v_3\|_{B^+_{n+3}} + |A_1| + \|a^0\|_{B^0_{n+2}} + \|\theta^0\|_{B^0_{n+2}} \leq K\epsilon$ *and* $\|v_2\|_{B^+_{n+3}} + \|\omega^0\|_{B^0_{n+2}} + \epsilon^{1/2}|\sigma| \leq K\epsilon^{1/2}$. *Then*

$$\|V_{21}(\sigma, v_1, v_2, v_3, A_1)(z)\|_{B^+_{n+3}} \leq K\epsilon,$$

$$\|\Omega_1^0(\sigma, v_1, v_2, v_3, a^0, \omega^0, \theta^0, A_1)(z)\| \leq K(K_1\epsilon + A_1\epsilon^{-(n-1)/2}),$$

where K, K_1 *are constants independent of* ϵ.

Finally we consider (59). Let

$$g_2^+(\sigma, v_1, v_2, v_3, A_1)(z) = \tilde{A}^{-1}g_2(0, \sigma, \tilde{A}v_1, \tilde{A}v_2, \tilde{A}v_3)(z), \tag{91}$$

$$g_2^0(S, \sigma, a, \omega, \theta, A_1)(z) = g_2(z) - \tilde{A}g_2^+(z-\delta\tanh hz). \tag{92}$$

For $\mathrm{Re}\ z \geq 0$, let

$$E^0(z) = g_2^0(z) + \tilde{A}g_2^+(z-\delta\tanh hz) - \tilde{A}g_2^+(z-\delta). \tag{93}$$

By (65), (66) with $g^0(s) = E^0(s)$, (70) and (71), we choose

$$\begin{aligned}\mu &= \tilde{A}\cos\delta\kappa + (\tilde{A}/\kappa)\int_{-\delta}^{0}\sin\kappa(s+\delta)g_2^+(s)ds + (\epsilon/\kappa)\int_0^{+\infty}\sin\kappa s E^0(s)ds,\\ \tilde{A}\sin\delta\kappa &= (1/\kappa)\int_0^{+\infty}\cos\kappa s E^0(s)ds + (\tilde{A}/\kappa)\int_{-\delta}^{0}\cos\kappa(s+\delta)g_2^+(s)ds,\end{aligned} \tag{94}$$

and by (67),

$$v_1(z) = \cos\kappa z + (1/\kappa)\int_0^z \sin\kappa(z-s)g_2^+(s)ds \equiv V_1(z). \tag{95}$$

To have $v_1(z)$ bounded, from (72) we need

$$\int_0^{2\pi}\cos\kappa s g_2^+(s)ds = 0. \tag{96}$$

By the definition of $g_2^+(z)$ and (75), (96) becomes

$$\begin{aligned}0 = &\int_0^{2\pi}\cos\kappa s\Big(\sigma\epsilon\big(\cos\kappa s + \epsilon^{1/2}p_1^+(s)\big)_{ss} - \nu_1\delta_1\epsilon\big(\cos\kappa s + \epsilon^{1/2}p_1^+(s)\big)\\ &\times\Big(\int_0^1\rho_\eta Z^2(\eta)d\eta - \rho(1)Z^2(1)\Big) + \int_0^1\Big(\sigma\epsilon q^2 v_{3ss} - \nu_1\delta_1\epsilon\rho_\eta(v_2z_1+v_3)\\ &+ (3\tilde{A}/2)\epsilon\big(q^2(v_1Z+v_2z_1+v_3)^2_\eta\big)_\eta + (\epsilon\tilde{A})^{-1}F_1(\tilde{A}\epsilon(v_1Z+v_2z_1+v_3))\Big)\\ &\times Z(\eta)d\eta - q^2(1)Z(1)\Big(-\nu_1\delta_1\epsilon\rho(1)q^{-2}(1)(v_2z_1+v_3)\\ &+ (3\tilde{A}\epsilon/2)(v_1Z_\eta + v_2z_{1\eta} + v_{3\eta})^2\\ &+ (\tilde{A}\epsilon)^{-1}F_2\big(\tilde{A}\epsilon(v_1Z+v_2z_1+v_3)\big)\Big)\Big|_{\eta=1}\Big)ds\\ = &\int_0^{2\pi}\cos\kappa s\Big(-\sigma\epsilon\kappa^2\cos\kappa s - \nu_1\delta\epsilon\cos\kappa s\Big(\int_0^1\rho_\eta Z^2(\eta)d\eta - \rho(1)Z^2(1)\Big)\Big)ds\\ &+ \epsilon\tilde{Y}_1(\sigma, p_1^+(s), v_2(s), v_3(s,\eta), \tilde{A}).\end{aligned} \tag{97}$$

Thus

$$\begin{aligned}\sigma &= (-\nu_1\delta/\kappa^2)\Big(\int_0^1\rho_\eta Z^2(\eta)d\eta - \rho(1)Z^2(1)\Big) + \tilde{Y}_1/(\pi\kappa^2)\\ &\equiv \chi + Y_1(\sigma, p_1^+, v_2, v_3, \tilde{A}).\end{aligned} \tag{98}$$

Also by checking the terms in Y_1, it is found that under the condition $\epsilon^{1/2}(|\sigma| + \|p_1^+\|_{B_2^+}) + \sum_{i=2}^3\|v_i\|_{B_2^+} + \epsilon^{-1/2}|A_1| \le K\epsilon^{1/2}$,

$$|Y_1(\sigma, p_1^+, v_2, v_3, \tilde{A}| \le K\epsilon^{1/2}. \tag{99}$$

We rewrite $g_2^+(z)$ in (91) by using (98),

$$\begin{aligned}g_2^+(\sigma, v_1, v_2, v_3, A_1)(z) &= -\sigma\epsilon\kappa^2\cos\kappa z + g_{21}^+(z)\\ &= -(\chi + Y_1(\sigma, p_1^+, v_2, v_3, \tilde{A}))\epsilon\kappa^2\cos\kappa z + g_{21}^+(z).\end{aligned} \tag{100}$$

Note that $g_2^+(\sigma, v_1, v_2, v_3, A)(z)$ is still used to denote the right hand side of (100) if no confusion arises. It follows that (96) will be satisfied by (98). From the expression of $g_2^+(z)$, it is easy to check that under the condition

$$\epsilon^{1/2}(|\sigma| + \|p_1^+\|_{B_{n+3}^+}) + \sum_{i=2}^{3} \|v_i\|_{B_{n+3}^+} + \epsilon^{-1/2}|A_1| \le K\epsilon^{1/2},$$

we can have $g_2^+(z) \in B_{n+1}^+$ and $\|g_2^+(z)\|_{B_{n+1}^+} \le K\epsilon^{3/2}$. By Lemma 3,

$$\Big\| \int_0^z \sin\kappa(z-s) g_2^+(s) ds \Big\|_{B_{n+3}^+} \le K\epsilon.$$

Again by checking terms in $g_2^0(z)$, if

$$\epsilon^{1/2}|\sigma| + \|a^0\|_{B_{n+2}^0} + \|\omega^0\|_{B_{n+2}^0} + \|\theta^0\|_{B_{n+2}^0} + \|v_1 - \cos\kappa z\|_{B_{n+3}^+} + \sum_{i=2}^{3} \|v_i\|_{B_{n+3}^+} + \epsilon^{-1/2}|A_1| \le K\epsilon^{1/2} \quad \text{for } n \ge 0,$$

we obtain $g_2^0(z) \in B_n^0$, and if $g_{21}^0(z) = g_2^0(z) - g_2^0(S, \sigma, 0, 0, 0, 0)(z)$,

$$C^0\left(\frac{\partial^i g_{21}^0(z)}{\partial z^i}\right) \le K\epsilon(\epsilon^{(1+i)/2} + A_1) \quad \text{for } 0 \le i \le n,$$

$$H_\lambda^0\left(\frac{\partial^n g_{21}^0(z)}{\partial z^n}\right) \le K\epsilon(\epsilon^{(1+n)/2} + A_1).$$

By integration by parts several times, it is easy to show that

$$C^0 \frac{\partial^j}{\partial z^j} \int_z^{+\infty} \sin\kappa(z-s) g_2^0(S, \sigma, 0, 0, 0, 0)(s) ds \le K\epsilon^{j/2},$$

for $0 \le j \le n+2$. From (59), (74), (75), (95) and (73), we have

$$\begin{aligned} a^0(z) &= \mathcal{P}(g_2(s))(z) - \tilde{A}V_1(z - \delta \tanh hz) \\ &= \tilde{A}\big(V_1(z-\delta) - V_1(z - \delta \tanh hz)\big) - (1/\kappa)\int_z^{+\infty} \sin\kappa(z-s) E^0(s) ds \\ &= \mathcal{P}^0(g_2(s))(z) \equiv \Delta_1^0(\sigma, v_1, v_2, v_3, a^0, \omega^0, \theta^0, A_1)(z). \end{aligned} \tag{101}$$

Using the above estimates for $g_2^0(z)$ and $g_2^+(z)$ and Lemma 4, we have $\Delta_1^0(z) \in B_{n+2}^0$ and

$$\|\Delta_1^0(z)\|_{B_{n+2}^0} \le K\Big(\big(\|\tilde{A}\cos\kappa z\|_{B_{n+1}^+} + K_1|\tilde{A}_1|\epsilon^{1/2}\big) \times \epsilon^{-n/2} \exp(\ell\kappa\epsilon^{-1/2}) + K_2\epsilon\Big). \tag{102}$$

Note that the term $\tilde{A}\cos\kappa z$ in (102) is exclusively the first term of $\tilde{A}V_1(z)$ in (95) and a term of σ in $g_2^+(s)$ of (95) has been replaced by (100). We state the above results as

Theorem 3. *From the condition*

$$\epsilon^{1/2}|\sigma| + \|a^0\|_{B_{n+2}^0} + \|\omega^0\|_{B_{n+2}^0} + \|\theta^0\|_{B_{n+2}^0} + \|v_1 - \cos\kappa z\|_{B_{n+3}^+} + \sum_{i=2}^{3} \|v_i\|_{B_{n+3}^+} + |A_1|\epsilon^{-1/2} \le K\epsilon^{1/2},$$

it follows that $V_1(z) \in B^0_{n+3}$ *and* $\Delta^0_1 \in B^0_{n+2}$ *with*

$$\|V_1 - \cos\kappa z\|_{B^0_{n+3}} \leq K\epsilon,$$
$$\|\Delta^0_1(z)\|_{B^0_{n+2}} \leq K\big(\|\tilde{A}\cos\kappa z\|_{B^+_{n+1}} + K_1|\tilde{A}|\epsilon^{1/2}\big)\epsilon^{-n/2}\exp(\ell\kappa\epsilon^{-1/2}),$$

where $\tilde{A}$ *is determined by* (94) *and* K, K_1 *and* K_2 *are constants independent of* ϵ.

We study (94) for $\tilde{A}$. Choose δ small and $\sin\delta\kappa \neq 0$. From (93), write

$$\begin{aligned} E^0(z) &= E^0_1(z) + g^0_2(z) - E^0_1(z) + \tilde{A}g^+_2(z-\delta\tanh hz) - \tilde{A}g^+_2(z-\delta) \\ &= E^0_1(z) + \tilde{A}E^0_2(z), \end{aligned}$$

where $E^0_1(z) = g_2(S,\sigma,a^0,\omega^0,\theta^0)(z)$ and each term in $g^0_2(z) - E^0_1(z)$ has at least a factor of either a^+, ω^+ or θ^+ which introduces a factor $\tilde{A}$. It is straightforward to show that for $|\sigma| + \|a^0\|_{B^0_2} + \|\omega^0\|_{B^0_2} + \|\theta^0\|_{B^0_2} + \sum_{i=1}^3 \|v_i\|_{B^+_2} + |A| \leq K$,

$$\Big|(\epsilon/\kappa)\int_0^{+\infty}\cos\kappa s(\tilde{A}E^0_2(s))ds\Big| \leq K\tilde{A}\epsilon^{1/2}.$$

Thus (94) becomes

$$\begin{aligned} \tilde{A} &= (1/\kappa\sin\delta\kappa)\int_0^{+\infty}\cos\kappa s E^0_1(s)ds + (\tilde{A}/\kappa\sin\delta\kappa) \\ &\quad\times\Big(\int_0^{+\infty}\cos\kappa s E^0_2(s)ds + \int_{-\delta}^0\cos\kappa(s+\delta)g^+_2(s)ds\Big) \\ &= \mathrm{I} + \tilde{A}\mathrm{II} = \Lambda_1(\sigma, v_1, v_2, v_3, a^0, \omega^0, \theta^0, A_1), \end{aligned} \tag{103}$$

and $|\mathrm{II}| \leq K\epsilon^{1/2}$. By letting $D^0_1(z) = g_2(S,\sigma,0,0,0)$, checking the terms in $E^0_1(z)$ and using the evenness of $E^0_1(z)$ and integration by parts n-times, we have

$$\begin{aligned} \mathrm{I} &= (\kappa\sin\delta\kappa)^{-1}\int_0^{+\infty}\cos(\kappa s + \frac{n\pi}{2})(\partial^n E^0_1(s)/\partial s^n)\kappa^{-n}ds \\ &= (2\kappa\sin\delta\kappa)^{-1}\int_{-\infty}^{+\infty}\cos(\kappa s + \frac{n\pi}{2})(\partial^n E^0_1(s)/\partial s^n)\kappa^{-n}ds \\ &= (2\kappa\sin\delta\kappa)^{-1}\int_{-\infty+i\ell\epsilon^{-1/2}}^{+\infty+i\ell\epsilon^{-1/2}}\exp(i(\kappa s + \frac{n\pi}{2}))(\partial^n E^0_1(s)/\partial s^n)\kappa^{-n}ds \\ &= (2\kappa\sin\delta\kappa)^{-1}\exp(-\ell\epsilon^{-1/2}\kappa)\Big(\int_{-\infty}^{+\infty}\exp(i(\kappa t + \frac{n\pi}{2})) \\ &\qquad\times\big(\partial^n(E^0_1(s) - D^0_1(s))/\partial s^n\big)\Big|_{s=t+\ell\epsilon^{-1/2}i}\kappa^{-n}dt \\ &\quad+\int_{-\infty}^{+\infty}\exp(i(\kappa t + \frac{n\pi}{2}))(\partial^n D^0_1(s)/\partial s^n)\Big|_{s=t+\ell\epsilon^{-1/2}i}\kappa^{-n}dt\Big) \\ &= (2\kappa\sin\kappa\delta)^{-1}\exp(-\ell\epsilon^{-1/2}\kappa)(\mathrm{I}_1 + \mathrm{I}_2). \end{aligned} \tag{104}$$

By the definition of B^0_{n+2}, it is easy to show that

$$|\mathrm{I}_1| \leq K\epsilon^{1/2}\epsilon^{n/2}\big(\|a^0\|_{B^0_{n+2}} + \|\omega^0\|_{B^0_{n+2}} + \|\theta^0\|_{B^0_{n+2}}\big).$$

Using integration by parts several times, we have

$$|\mathrm{I}_2| \leq K\epsilon^{(n+3)/2}.$$

Therefore let $\|a^0\|_{B^0_{n+2}} + \|\omega^0\|_{B^0_{n+2}} + \|\theta^0\|_{B^0_{n+2}} \le K\epsilon^{1/2}$, and we have

$$|\mathrm{I}| \le K\epsilon^{(n+2)/2}\exp(-\ell\epsilon^{-1/2}\kappa). \tag{105}$$

Note that K also depends upon $(\sin\kappa\delta)^{-1}$, where δ is a small fixed constant and must be chosen so that $\sin\kappa\delta$ is not zero. Now we have all the estimates for the equations.

7. Existence Proof

We use the notations introduced in (75) and (76). There are now eight unknowns

$$U = (\sigma, p_1^+(z), p_2^+(z), p_3^+(z,\eta), q_1^0(z), q_2^0(z), q_3^0(z,\eta), A), \tag{106}$$

and eight equations (82), (83), (87), (90), (95), (98), (101) and (103). We define a closed convex set in the Banach space $\mathbb{R}\times(B^+_{n+3})^3\times(B^0_{n+2})^3\times\mathbb{R}$,

$$\mathcal{S}_b = \Big\{U \in \mathbb{R}\times(B^+_{n+3})^3\times(B^0_{n+2})^3\times\mathbb{R} \;\Big|\; \|V\| = \sum_{i=1}^{3}\big(\|p_i^+\|_{B^+_{n+3}} + \|q_i^0\|_{B^0_{n+2}}\big) \\ + |\sigma| + |A| < +\infty,\ \|U\| - |\sigma| + |\sigma-\chi| \le b\Big\},$$

where χ is defined in (98) and b is a small positive fixed number.

For $U \in \mathcal{S}_b$, by the conditions of Theorems 1, 2 and 3 and the procedure to transform the differential equations into integro-differential equations, we need to transform (82), (83), (89), (95), (98) and (103) into

$$\begin{aligned}\sigma &= \chi + Y_1\big(\sigma, p_1^+, \epsilon^{1/2}p_2^+, \epsilon^{1/2}p_3^+, A\epsilon^{(n+1)/2}\exp(-\ell\kappa\epsilon^{-1/2})\big)\\ &\equiv \chi + Y(\sigma, p_1^+, p_2^+, p_3^+, A), &(107)\\ p_1^+(z) &= \epsilon^{-1/2}V_1(z) \equiv P_1(\sigma, p_1^+, p_2^+, p_3^+, A)(z), &(108)\\ p_2^+(z) &= \epsilon^{-1/2}V_{21}(\sigma, v_1, v_2, v_3, A_1),\\ &= \epsilon^{-1/2}V_{21}(\sigma, \cos\kappa z + \epsilon^{1/2}P_1, \epsilon^{1/2}p_2^+, \epsilon^{1/2}P_3, A\epsilon^{(n+1)/2})\\ &\equiv P_2(\sigma, p_1^+, p_2^+, p_3^+, A), &(109)\\ p_3^+(z,\eta) &= \epsilon^{-1/2}V_3(z,\eta) \equiv P_3(\sigma, p_1^+, p_2^+, p_3^+, A)(z), &(110)\\ A &= \epsilon^{-(n+1)/2}\exp(\ell\kappa\epsilon^{-1/2})\Lambda_1(\sigma, \cos\kappa z + \epsilon^{1/2}p_1^+,\\ &\qquad \epsilon^{1/2}p_2^+, \epsilon^{1/2}p_3^+, \epsilon^{1/2}q_1^0, \epsilon^{1/2}q_2^0, \epsilon^{1/2}q_3^0, A\epsilon^{(n+1)/2})\\ &\equiv \Lambda(\sigma, p_1^+, p_2^+, p_3^+, q_1^0, q_2^0, q_3^0, A), &(111)\\ q_3^0(z,\eta) &= \epsilon^{-1/2}\Theta^0(z,\eta) = Q_3(\sigma, p_1^+, p_2^+, p_3^+, q_1^0, q_2^0, q_3^0, A). &(112)\end{aligned}$$

We use (111) to replace A in the term

$$A\epsilon^{(n+1)/2}\exp(-\kappa\ell\epsilon^{-1/2})\big(\cos(z-\delta) - \cos(z - \delta\,\tanh hz)\big),$$

in (101). After this substitution, (101) becomes

$$a^0(z) = \Delta^0(\sigma, v_1, v_2, v_3, a^0, \omega^0, \theta^0, A_1)(z).$$

We rewrite it as

$$\begin{aligned}q_1^0(z) &= \epsilon^{-1/2}\Delta^0(\sigma, \cos\kappa z + \epsilon^{1/2}p_1^+, \epsilon^{1/2}p_2^+, \epsilon^{1/2}p_3^+, \epsilon^{1/2}q_1^0, \epsilon^{1/2}q_2^0, \epsilon^{1/2}q_3^0, A\epsilon^{(n+1)/2})\\ &= Q_1(\sigma, p_1^+, p_2^+, p_3^+, q_1^0, q_2^0, q_3^0, A). &(113)\end{aligned}$$

Finally we express (90) as

$$\begin{aligned}q_2^0(z) &= \epsilon^{-1/2}\Omega_1^0(\sigma, v_1, v_2, v_3, a^0, \omega^0, \theta^0, A_1)(z)\\ &= \epsilon^{-1/2}\Omega_1^0(\sigma, \cos\kappa z + \epsilon^{1/2}P_1, \epsilon^{1/2}p_2^+, \epsilon^{1/2}P_3, \epsilon^{1/2}Q_1, \epsilon^{1/2}q_2^0, \epsilon^{1/2}Q_3, A\epsilon^{(n+1)/2})\\ &= Q_3(\sigma, p_1^+, p_2^+, p_3^+, q_1^0, q_2^0, q_3^0, A). \qquad (114)\end{aligned}$$

Obviously (107) to (114) are equivalent to (82), (83), (87), (95), (98), (103), (101) and (90). Let

$$\mathcal{T}(U) = (\chi + Y, P_1, P_2, P_3, Q_1, Q_2, Q_3, \Lambda)(U).$$

By Theorem 1, 2 and 3, estimates (99), (103) and (105), and the construction of P_2, Λ, Q_1, Q_2, we obtain that for $U \in \mathcal{S}_b$,

$$|Y| + \sum_{i=1}^{3} \|P_i\|_{B_{n+3}^+} + \sum_{i=1}^{3} \|Q_i\|_{B_{n+2}^0} + |\Lambda| \le K\epsilon^{1/2}, \qquad (115)$$

where K is independent of ϵ. Thus if ϵ is small enough such that $K\epsilon^{1/2} < b$, $\mathcal{T}$ maps $\mathcal{S}_b$ into itself. We also have the following

Theorem 4. *For $U^{(1)}, U^{(2)} \in \mathcal{S}_b$,*

$$\|\!|\mathcal{T}(U^{(1)}) - \mathcal{T}(U^{(2)})\|\!| \le K\epsilon^{1/2}\|\!|U^{(1)} - U^2\|\!|.$$

The proof of this theorem is same as the one of (115) if we follow the same steps to obtain Theorems 1, 2 and 3 from Lemmas 1, 2 and 3. Therefore we omit it here.

By Theorem 4, if $K\epsilon^{1/2} \le 1/2$ for small ϵ, then $\mathcal{T}$ is a contraction in $\mathcal{S}_b$. By the contraction mapping theorem, $\mathcal{T}$ has a fixed point in $\mathcal{S}_b$. Thus (107) to (114) have a solution $(\sigma, p_1^+, p_2^+, p_3^+, q_1^0, q_2^0, q_3^0, A)$ and by (115)

$$|\sigma - \chi| + \sum_{i=1}^{3} \|p_i^+\|_{B_{n+3}^+} + \sum_{i=1}^{3} \|q_i^0\|_{B_{n+2}^0} + |A| \le K\epsilon^{1/2}.$$

Finally we go back to the original variables and obtain a solution (a, ω, θ) for the integro-differential equations (57) to (59) by choosing appropriate constants. Therefore we have a solution for (45) to (47), (51) and (52). Now we summarize the existence result as follows:

Theorem 5. *There exists a small $\epsilon_0 > 0$ such that for $0 < \epsilon \le \epsilon_0$ the equations* (35) *to* (37) *possess a solution $w(X, \eta)$ with*

$$w(X, \eta) = \epsilon\big(S(\epsilon^{1/2}x)z_1(\eta) + a(x)Z(\eta) + \omega(x)z_1(\eta) + \theta(x, \eta)\big),$$

where $x = (1 - \sigma\epsilon)^{1/2}X$, $S(\epsilon^{1/2}x)$ is defined in (34),

$$\omega(x) = \omega^0(x) + \tilde{A}v_2(x - \delta\ \tanh\ hx), a(x) = a^0(x) + \tilde{A}v_1(x - \delta\ \tanh\ hx),$$

$$\theta(x, \eta) = \theta^0(x, \eta) + \tilde{A}v_3(x - \delta\ \tanh\ hx, \eta),$$

with the following properties: if x is replaced by the complex variable z,

$$a^0, \omega^0, \theta^0 \in B_{n+2}^0 \text{ and } v_1, v_2, v_3 \in B_{n+3}^+,$$

$$\|a^0\|_{B_{n+2}^0} + \|\omega^0\|_{B_{n+2}^0} + \|\theta^0\|_{B_{n+2}^0} + \|v_1 - \cos\kappa z\|_{B_{n+3}^+} + \sum_{i=2}^{3} \|v_i\|_{B_{n+3}^+} \le K\epsilon, \quad |\sigma - \chi| \le K\epsilon^{1/2},$$

and $|\tilde{A}| \le K\epsilon^{(n+2)/2}\exp(-\ell\kappa\epsilon^{-1/2})$*, where* χ *is defined in* (98)*,* δ *is a fixed small constant with* $\sin\delta\kappa \neq 0$*, and* K *is independent of* ϵ*, but may depend on* ℓ *and other parameters in the definitions of* B^0_{n+2} *and* B^+_{n+3}*.*

Therefore, assume that $\nu = \nu_1(1-\delta_1\epsilon)$ with $\delta_1 > 0$ and $q(t) = \rho^{1/2}(t)U(t)$. For small ϵ, there exists a constant σ depending upon ϵ such that (13) to (15) have a solution

$$\begin{aligned} f(X,\psi) = &\int_0^\psi q^{-1}(t)dt + \epsilon S(\epsilon^{1/2}x)z_1\Big(\int_0^\psi q^{-1}(t)dt\Big) \\ &+ \tilde{A}\Big(\cos(\kappa(x-\delta\tanh hx))Z\Big(\int_0^\psi q^{-1}(t)dt\Big)\Big) \\ &+ \epsilon^2 O_1(x,\psi) + \tilde{A}\epsilon O_2(x-\delta\tanh hx,\psi), \end{aligned}$$

where $x = (1-\sigma\epsilon)^{1/2}X$, $O_1(x,\psi)$ is of order $O(\exp(-d|x|\epsilon^{1/2})$ for $x\in\mathbb{R}, \psi\in[0,1]$, $O_2(x,\psi)$ is periodic and bounded for small ϵ and $|\tilde{A}| \le K\epsilon^{(n+2)/2}\exp(-\ell\kappa\epsilon^{-1/2})$ for $0 < \pi(-\delta_1\beta_1/\gamma_1)^{-1/2} - \tau < \ell < \pi(-\delta_1\beta_1/\gamma_1)^{-1/2}$ and any fixed positive integer n. Thus an approximation of $f(X,\psi)$ up to first order is

$$\int_0^\psi q^{-1}(t)dt + \epsilon S(\epsilon^{1/2}X)z_1\Big(\int_0^\psi q^{-1}(t)dt\Big),$$

and the amplitude of the oscillatory tail is exponentially small.

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