

FRACTIONAL INTEGRATION FOR LAGUERRE EXPANSIONS

George Gasper, Krzysztof Stempak, and Walter Trebels

1. Introduction

The aim of this note is to provide a fractional integration theorem in the framework of Laguerre expansions. The method of proof consists of establishing an asymptotic estimate for the kernel involved and then applying a method of Hedberg [6]. We combine this result with the sufficient (p, p) multiplier criteria of Stempak and Trebels [13]. The resulting sufficient (p, q) multiplier criteria are comparable with the necessary ones of Gasper and Trebels [3,4].

Our notation is essentially that in [13]. Thus we consider the Lebesgue spaces

$$L^p_{v(\gamma)} = \left\{ f : \|f\|_{L^p_{v(\gamma)}} = \left(\int_0^\infty |f(x)|^p x^\gamma dx \right)^{1/p} < \infty \right\}, \quad 1 \leq p < \infty, \quad \gamma > -1,$$

and define the Laguerre function system $\{l_k^\alpha\}$ by

$$l_k^\alpha(x) = (k!/\Gamma(k + \alpha + 1))^{1/2} e^{-x/2} L_k^\alpha(x), \quad \alpha > -1, \quad k \in \mathbf{N}_0.$$

This system is an orthonormal basis in $L^2(\mathbf{R}_+, x^\alpha dx)$ and, for $\gamma < p(\alpha + 1) - 1$, we can associate to any $f \in L^p_{v(\gamma)}$ the Laguerre series

$$f(x) \sim \sum_{k=0}^\infty a_k l_k^\alpha(x), \quad a_k = \int_0^\infty f(x) l_k^\alpha(x) x^\alpha dx.$$

It is convenient to introduce the vector space

$$E = \{ f(x) = p(x)e^{-x/2} : 0 \leq x < \infty, \quad p(x) \text{ a polynomial} \}$$

which is dense in $L^p_{v(\gamma)}$. We note that $f \in E$ has only finitely many non-zero Fourier-Laguerre coefficients. Analogous to the definition of the Hardy and Littlewood fractional integral operator for Fourier series (see [15, Chap. XII, Sec. 8]), we define a fractional integral operator I_σ , $\sigma > 0$, for Laguerre expansions by

$$I_\sigma f(x) = \sum_{k=0}^\infty (k+1)^{-\sigma} a_k l_k^\alpha(x) \quad f \in E.$$

Observing that the l_k^α are eigenfunctions with eigenvalues λ_k of the differential operator

$$L = - \left(x \frac{d^2}{dx^2} + (\alpha + 1) \frac{d}{dx} - \frac{x}{4} \right), \quad \lambda_k = k + (\alpha + 1)/2$$

Received November 16, 1993, revised July 8, 1994.

1991 *Mathematics Subject Classification*: 33C45, 42A45, 42C10.

Key words and phrases: fractional integration, multiplier conditions, Laguerre polynomials.

Research of the first author was supported in part by NSF Grant DMS-9103177.

Research of the second author was done during a stay at the Fb Mathematik of the TH Darmstadt and was supported by the Deutsche Forschungsgemeinschaft under grant 436POL/115/1/0.

(see [14, (5.1.2)]), one realizes that I_1 is an integral operator essentially inverse to L (see the following Remark 2). As we will see in Section 2, the fractional integral $I_\sigma f$ can be interpreted as a twisted convolution of f with a function

$$g_\sigma(x) \sim \Gamma(\alpha + 1) \sum_{k=0}^{\infty} (k+1)^{-\sigma} L_k^\alpha(x) e^{-x/2}. \quad (1)$$

From Theorem 3.1 in [4], it easily follows that $g_\sigma \in L_{v(\gamma)}^1$ when $\alpha - \gamma < \sigma$, so that, by the convolution theorem of Görlich and Markett [5], I_σ extends to a bounded operator from $L_{v(\gamma)}^p$ to $L_{v(\gamma)}^p$ when $0 \leq \alpha p/2 \leq \gamma \leq \alpha$, $1 \leq p \leq \infty$. Our main result is

Theorem 1.1. *Let $\alpha \geq 0$, $1 < p \leq q < \infty$. Assume further that $0 < \sigma < \alpha + 1$, $a < (\alpha + 1)/p'$, $b < (\alpha + 1)/q$, $a + b \geq 0$. Then*

$$\|I_\sigma f\|_{L_{v(\alpha-bq)}^q} \leq C \|f\|_{L_{v(\alpha+ap)}^p}, \quad \frac{1}{q} = \frac{1}{p} - \frac{\sigma - a - b}{\alpha + 1}.$$

Remarks. (1) Observe that the upper bounds for a and b imply lower bounds for b and a , respectively.

(2) By the above, it is clear that the sequence $\{(k+1)^{-\sigma}\}$ which generates the fractional integral can be replaced by any sequence $\{\Omega_\sigma(k)\}$ satisfying

$$\Omega_\sigma(k) = \sum_{j=0}^J c_j (k+1)^{-\sigma-j} + O((k+1)^{-\sigma-J})$$

for sufficiently large J , say $J \geq \alpha + 2$. Thus, in particular, the same result is obtained for the sequence $\{\Gamma(k+1)/\Gamma(\sigma+k+1)\}$, see [9].

(3) A weaker version of Theorem 1.1 (e.g., in the case $a = b = 0$) can easily be deduced by the following argument. By a slight modification of the proof of Theorem 3.1 in [4], we have:

Let $\alpha > -1$ and $N \in \mathbf{N}_0$, $N > (2\alpha + 2)(1/r - 1/2) - 1/3$. If $\{f_k\}$ is a bounded sequence with $\lim_{k \rightarrow \infty} f_k = 0$ and

$$\sum_{k=0}^{\infty} (k+1)^{N+(\alpha+1)/r'} |\Delta^{N+1} f_k| \leq K_r(f) < \infty, \quad 1 \leq r < \infty,$$

then there exists a function $f \in L_{v(\alpha)}^r$ with

$$\|f\|_{L_{v(\alpha)}^r} \leq C K_r(f), \quad f(x) \sim \sum_{k=0}^{\infty} f_k L_k^\alpha(x) e^{-x/2}.$$

This applied to the sequence $\{(k+1)^{-\sigma}\}$ gives, by Young's inequality [5],

$$\|I_\sigma f\|_{L_{v(\alpha)}^q} \leq C \|f\|_{L_{v(\alpha)}^p}, \quad \frac{1}{q} > \frac{1}{p} - \frac{\sigma}{\alpha + 1},$$

where $\alpha \geq 0$, $\sigma > 0$ and $1 \leq p, q \leq \infty$.

Next we indicate how Theorem 1.1 can be used to gain some insight into the structure of $M^{p,q}$ -Laguerre multipliers. For the sake of simplicity, let us restrict ourselves

to the case $\gamma = \alpha$. Consider a sequence $m = \{m_k\}$ of numbers and associate to m the operator

$$T_m f(x) = \sum_{k=0}^{\infty} m_k a_k l_k^\alpha(x), \quad f \in E.$$

The sequence m is called a bounded (p, q) -multiplier, notation $m \in M_{\alpha, \alpha}^{p, q}$, if

$$\|m\|_{M_{\alpha, \alpha}^{p, q}} := \inf \left\{ C : \|T_m f\|_{L_{v(\alpha)}^q} \leq C \|f\|_{L_{v(\alpha)}^p} \text{ for all } f \in E \right\}$$

is finite. If $p \leq 2 \leq q$, then sufficient conditions follow at once in the following way: observe that $M_{\alpha, \alpha}^{2, 2} = l^\infty$, choose $\sigma_0, \sigma_1 \geq 0$ such that $I_{\sigma_0} : L^p \rightarrow L^2$, $I_{\sigma_1} : L^2 \rightarrow L^q$, $\sigma_0 + \sigma_1 = \sigma$, and hence

$$\|m\|_{M_{\alpha, \alpha}^{p, q}} \leq \|k^{-\sigma_0}\|_{M_{\alpha, \alpha}^{p, 2}} \|\{k^\sigma m_k\}\|_{M_{\alpha, \alpha}^{2, 2}} \|k^{-\sigma_1}\|_{M_{\alpha, \alpha}^{2, q}},$$

in particular, $m \in M_{\alpha, \alpha}^{p, q}$ when $\{k^\sigma m_k\} \in l^\infty$. Thus, when $1 < p \leq 2 \leq q < \infty$ and σ is as in Theorem 1.1, $\{(-1)^k (k+1)^{-\sigma}\} \in M_{\alpha, \alpha}^{p, q}$ generates a bounded operator which does not fall under the scope of the above introduced fractional integral operators. To formulate a corollary based on a combination of Theorem 1.1 and the multiplier result in [13], we need the notion of a difference operator Δ^s of fractional order s given by

$$\Delta^s m_k = \sum_{j=0}^{\infty} A_j^{-s-1} m_{k+j}, \quad A_j^t = \frac{\Gamma(j+t+1)}{j! \Gamma(t+1)}, \quad t \in \mathbf{R},$$

whenever the sum converges. In view of the remark concerning the case $p < 2 < q$ and on account of duality, we may restrict ourselves to the case $1 < p < q < 2$.

Corollary 1.2. *If $\alpha \geq 0$, $1 < p < q < 2$, and $s > \max\{(2\alpha + 2)(1/q - 1/2), 1\}$, then for some constant C , independent of the sequence $\{m_k\}$, there holds*

$$\|\{m_k\}\|_{M_{\alpha, \alpha}^{p, q}}^2 \leq C \left(\|\{k^\sigma m_k\}\|_\infty^2 + \sup_N \sum_{k=N}^{2N} |(k+1)^{s+\sigma} \Delta^s m_k|^2 (k+1)^{-1} \right).$$

The proof follows as in [2]. The result itself should be compared with the corresponding necessary condition in [3] which also shows that $\{(-1)^k (k+1)^{-\sigma}\} \notin M_{\alpha, \alpha}^{p, q}$ provided $1 < p < q < 2$ and α is sufficiently large.

The plan of the paper is as follows. In Section 2, we derive an asymptotic estimate of the function g_σ defined above by (1). Then the twisted generalized convolution is used to dominate $I_\sigma f$ by a generalized Euclidean convolution of g_σ with f . The latter's mapping behavior is discussed by a method of Hedberg [6] which uses maximal functions, thus giving Theorem 1.1 in the standard weight case $a = b = 0$. In Section 3, we extend this result to some power weights, modifying an argument in Stein and Weiss [10].

2. Proof of the standard weight case

We start by deriving the required asymptotic estimate and showing

Lemma 2.1. *Let $\alpha > -1$. Then, for $x > 0$ and $0 < \sigma < \alpha + 1$, there holds*

$$|g_\sigma(x)| \leq C x^{\sigma - \alpha - 1}. \quad (2)$$

Proof. First note that by subordination [3, p. 1234] there holds for $N > \alpha + 2$, $N \in \mathbf{N}$,

$$g_\sigma(x) = C \sum_{k=0}^{\infty} (\Delta^N(k+1))^{-\sigma} L_k^{\alpha+N}(x) e^{-x/2}.$$

Then the assertion (2) follows after it is proved that

$$\sum_{k=0}^{\infty} (k+1)^{-\sigma-N} |x^{\alpha+1-\sigma} L_k^{\alpha+N}(x) e^{-x/2}| \leq C \quad (3)$$

uniformly in $x > 0$. With the notation

$$\mathcal{L}_k^\alpha(x) = (k!/\Gamma(k+\alpha+1))^{1/2} x^{\alpha/2} e^{-x/2} L_k^\alpha(x), \quad k \in \mathbf{N}_0,$$

this is equivalent to

$$\sum_{k=0}^{\infty} (k+1)^{(\alpha-N)/2-\sigma} |\mathcal{L}_k^{\alpha+N}(x)| \leq C x^{\sigma+(N-\alpha-2)/2}, \quad (4)$$

uniformly in $0 < x < \infty$. We will make use of the pointwise estimates for the Laguerre functions in [1, Sec. 2]:

$$|\mathcal{L}_k^{\alpha+N}(x)| \leq C \begin{cases} (x(k+1))^{(\alpha+N)/2}, & \text{if } 0 \leq x \leq c/(k+1), \\ (x(k+1))^{-1/4}, & \text{if } c/(k+1) \leq x \leq d(k+1), \end{cases} \quad (5)$$

for fixed positive constants c and d . These and two further estimates in [8, (2.5)] imply that

$$\sup_{x>0} |\mathcal{L}_k^{\alpha+N}(x)| \leq C, \quad k \in \mathbf{N}_0, \quad (6)$$

and so it is obvious that (6) implies (4) for $x \geq 1$.

Therefore, decomposing the interval $(0, 1)$ dyadically, it suffices to check that

$$\sum_{k=0}^{\infty} (k+1)^{(\alpha-N)/2-\sigma} |\mathcal{L}_k^{\alpha+N}(x)| \leq C(2^j)^{\sigma+(N-\alpha-2)/2}$$

provided $2^j \leq x \leq 2^{j+1}$, $j < 0$. Using the first line of (5), we get

$$\sum_{k=0}^{2^{-j}} (k+1)^{(\alpha-N)/2-\sigma} |\mathcal{L}_k^{\alpha+N}(x)| \leq C(2^j)^{(\alpha+N)/2} \sum_{k=0}^{2^{-j}} (k+1)^{\alpha-\sigma} \leq C(2^j)^{\sigma+(N-\alpha-2)/2},$$

while the second line of (5) gives

$$\sum_{k=2^{-j}}^{\infty} k^{(\alpha-N)/2-\sigma} |\mathcal{L}_k^{\alpha+N}(x)| \leq C2^{-j/4} \sum_{k=2^{-j}}^{\infty} k^{(\alpha-N)/2-\sigma-1/4} \leq C(2^j)^{\sigma+(N-\alpha-2)/2}.$$

This completes the proof of Lemma 2.1.

As already mentioned, we will apply Hedberg's method, which involves maximal functions. To make use of the corresponding results in [11], we switch to the system

$$\psi_k^\alpha(x) = (2k!/\Gamma(k+\alpha+1))^{1/2} e^{-x^2/2} L_k^\alpha(x^2), \quad k \in \mathbf{N}_0,$$

which is obviously orthonormal on $L^2(\mathbf{R}_+, d\mu_\alpha)$, $d\mu_\alpha(x) = x^{2\alpha+1}dx$, $\alpha > -1$. For the sake of simplicity, we write the norm on $L^p(\mathbf{R}_+, d\mu_\alpha)$ as

$$\|F\|_p = \left(\int_0^\infty |F(x)|^p d\mu_\alpha(x) \right)^{1/p}.$$

We adopt the notion of the twisted generalized convolution on $L^1(\mathbf{R}_+, d\mu_\alpha)$ from [11,12],

$$F \times G(x) = \int_0^\infty \tau_x F(y)G(y)d\mu_\alpha(y),$$

where the twisted generalized translation operator τ_x is given by

$$\tau_x F(y) = \frac{\Gamma(\alpha + 1)}{\pi^{1/2}\Gamma(\alpha + 1/2)} \int_0^\pi F((x, y)_\theta) \mathcal{J}_{\alpha-1/2}(xy \sin \theta) (\sin \theta)^{2\alpha} d\theta,$$

$\mathcal{J}_\beta(x) = \Gamma(\beta + 1)J_\beta(x)/(x/2)^\beta$, J_β denoting the Bessel function of order β , and

$$(x, y)_\theta = (x^2 + y^2 - 2xy \cos \theta)^{1/2}.$$

With respect to the system $\{\psi_k^\alpha\}$, this convolution has the following transform property: if $F \sim \sum c_k \psi_k^\alpha$ and $F \times G \sim \sum c_k d_k \psi_k^\alpha$, then $G(x) \sim \Gamma(\alpha+1) \sum d_k L_k^\alpha(x^2)e^{-x^2/2}$.

If we set $f(y^2) = F(y)$, $g_\sigma(y^2) = G_\sigma(y)$, we see that

$$|I_\sigma f(x^2)| = |F \times G_\sigma(x)|.$$

Apart from Lemma 2.1, the proof of Theorem 1.1 will be based on the fact that for $\alpha \geq 0$ and suitable F and G ,

$$|F \times G| \leq |F| * |G|,$$

which follows at once from the definition of the generalized Euclidean $*$ -convolution

$$F * G(x) = \int_0^\infty \tau_x^E F(y)G(y)d\mu_\alpha(y)$$

with associated generalized Euclidean translation

$$\tau_x^E F(y) = \int_0^\pi F((x, y)_\theta) d\nu_\alpha(\theta), \quad d\nu_\alpha(\theta) = \frac{\Gamma(\alpha + 1)}{\pi^{1/2}\Gamma(\alpha + 1/2)} (\sin \theta)^{2\alpha} d\theta.$$

Therefore we restrict ourselves to fractional integrals defined via the generalized Euclidean convolution.

Theorem 2.2. *Let $1 < p < q < \infty$, $\alpha > -1/2$, $K_\sigma(x) = x^{2(\sigma-\alpha-1)}$. Then*

$$\|F * K_\sigma\|_q \leq C \|F\|_p, \quad \frac{1}{q} = \frac{1}{p} - \frac{\sigma}{\alpha + 1}.$$

By the above, it is clear that Theorem 1.1 for the case $a = b = 0$ follows from Theorem 2.2.

Proof. Following Hedberg [6], we want to estimate $F * K_\sigma(x)$ pointwise by a suitable maximal function which in this setting turns out to be (see Stempak [11, p. 138])

$$F^*(x) = \sup_{\varepsilon > 0} \varepsilon^{-(2\alpha+2)} \int_0^\varepsilon \tau_x^E(|F|)(y)d\mu_\alpha(y)$$

with the usual boundedness property $\|F^*\|_r \leq C\|F\|_r$, $1 < r \leq \infty$. Now there holds

$$|F * K_\sigma(x)| \leq C \left(\int_0^\delta + \int_\delta^\infty \right) \tau_x^E(|F|)(y) y^{2(\sigma-\alpha-1)} d\mu_\alpha(y) = J_1 + J_2,$$

where $\delta > 0$ will be chosen later appropriately. Clearly,

$$\begin{aligned} J_1 &= \sum_{k=0}^{\infty} \int_{2^{-k-1}\delta}^{2^{-k}\delta} \dots \leq C \sum_{k=0}^{\infty} (2^{-k}\delta)^{2\sigma} (2^{-k}\delta)^{-2\alpha-2} \int_{2^{-k-1}\delta}^{2^{-k}\delta} \tau_x^E(|F|)(s) s^{2\alpha+1} ds \\ &\leq C \delta^{2\sigma} \sum_{k=0}^{\infty} 2^{-k2\sigma} (2^{-k}\delta)^{-2\alpha-2} \int_0^{\delta 2^{-k}} \tau_x^E(|F|)(s) s^{2\alpha+1} ds \leq C \delta^{2\sigma} F^*(x). \end{aligned}$$

On the other hand, by Hölder's inequality,

$$\begin{aligned} J_2 &= \int_\delta^\infty \tau_x^E(|F|)(s) s^{2(\sigma-\alpha-1)} s^{2\alpha+1} ds \leq \|\tau_x^E(|F|)\|_p \left(\int_\delta^\infty s^{2(\sigma-\alpha-1)p'} s^{2\alpha+1} ds \right)^{1/p'} \\ &\leq C \delta^{2\sigma-(2\alpha+2)/p} \|F\|_p, \end{aligned}$$

since τ_x^E are contractions on $L^p(\mathbf{R}_+, d\mu_\alpha)$. Hence

$$|F * K_\sigma(x)| \leq C(\delta^{2\sigma} F^*(x) + \delta^{2\sigma-(2\alpha+2)/p} \|F\|_p).$$

Choosing $\delta = (F^*(x)/\|F\|_p)^{-p/(2\alpha+2)}$ (where $\|F\|_p \neq 0$ is assumed), we obtain

$$|F * K_\sigma(x)| \leq C F^*(x)^{1-\sigma p/(\alpha+1)} \|F\|_p^{\sigma p/(\alpha+1)}$$

and then $\|F * K_\sigma\|_q \leq C\|F\|_p$ due to the inequality for the maximal function with $r = q(1 - \sigma p/(\alpha+1)) = p > 1$. This finishes the proof of Theorem 2.2.

3. Extension to power weights

The proof of Theorem 1.1 in the general case follows along the lines in Section 2 from

Theorem 3.1. *Let $\alpha > -1/2$, $0 < \sigma < \alpha + 1$ and $a < (\alpha + 1)/p'$, $b < (\alpha + 1)/q$, with $a + b \geq 0$. If $1 < p \leq q < \infty$, then*

$$\|K_\sigma * F(x)x^{-2b}\|_q \leq C\|F(x)x^{2a}\|_p, \quad \frac{1}{q} = \frac{1}{p} - \frac{\sigma - a - b}{\alpha + 1}.$$

Proof. An equivalent version of the inequality above is

$$\left(\int_0^\infty |Sf(x)|^q d\mu_\alpha(x) \right)^{1/q} \leq C \left(\int_0^\infty |f(x)|^p d\mu_\alpha(x) \right)^{1/p}$$

where

$$f(x) = x^{2a} F(x), \quad Sf(x) = \int_0^\infty K(x, y) f(y) d\mu_\alpha(y),$$

and

$$K(x, y) = x^{-2b} \left(\int_0^\pi (x, y)_\theta^{2(\sigma-\alpha-1)} d\nu_\alpha(\theta) \right) y^{-2a}.$$

We first consider the case $p = q$. Then $\sigma = a + b$ and, therefore, the kernel $K(x, y)$ is homogeneous of degree $-(2\alpha + 2)$: $K(rx, ry) = r^{-(2\alpha+2)} K(x, y)$, $r > 0$. It now suffices to check that (cf. Section 2 of [13])

$$\int_0^\infty K(1, y) y^{-(2\alpha+2)/p} d\mu_\alpha(y) < \infty.$$

We first note that the function $K(1, y)$ has at most an integrable singularity at $y = 1$, since for $1/2 \leq y \leq 2$ we have

$$\int_0^\pi \frac{(\sin \theta)^{2\alpha} d\theta}{((1-y)^2 + 4y \sin^2(\theta/2))^{\alpha+1-\sigma}} \leq C(1 + |1-y|^{\sigma-1}).$$

To deal with the singularity at 0, we note that for $y < 1/2$ we have $(1, y)_\theta \approx 1$, and therefore $\int_0^\pi (1, y)_\theta^{2(\sigma-\alpha-1)} d\nu_\alpha(\theta) \approx 1$ as well. Hence $K(1, y) \leq Cy^{-2a}$ and thus

$$\int_0^{1/2} K(1, y) y^{-(2\alpha+2)/p} d\mu_\alpha(y) < \infty, \quad a < (\alpha+1)/p'.$$

For $y > 2$, we have $(1, y)_\theta \approx y$, $\int_0^\pi (1, y)_\theta^{2(\sigma-\alpha-1)} d\nu_\alpha(\theta) \approx y^{2(\sigma-\alpha-1)}$, and hence

$$\int_2^\infty K(1, y) y^{-(2\alpha+2)/p} d\mu_\alpha(y) < \infty, \quad b < (\alpha+1)/q,$$

since $\sigma < (\alpha+1)/p$.

Now, consider the case $1 < p < q < \infty$. We use still another equivalent version of the inequality to be proved, namely

$$\int_0^\infty \int_0^\infty K(x, y) f(y) g(x) d\mu_\alpha(y) d\mu_\alpha(x) \leq C \|f\|_p \|g\|_{q'},$$

assuming for simplicity that f and g are nonnegative. Writing $\mathbf{R}_+ \times \mathbf{R}_+ = D_1 \cup D_2 \cup D_3$ where

$$D_1 = \{y/2 \leq x \leq 2y\}, \quad D_2 = \{2y < x\}, \quad D_3 = \{x < y/2\},$$

it suffices to check for $i = 1, 2, 3$ that

$$I_i = \iint_{D_i} K(x, y) f(y) g(x) d\mu_\alpha(y) d\mu_\alpha(x) \leq C \|f\|_p \|g\|_{q'}.$$

Consider I_1 first. Since $a + b \geq 0$, for $x, y \in D_1$ and $\theta \in (0, \pi)$ we have

$$((x, y)_\theta)^{2(a+b)} \leq Cx^{2(a+b)} \leq Cx^{2b}y^{2a},$$

so

$$I_1 \leq \iint_{D_1} \tilde{K}(x, y) f(y) g(x) d\mu_\alpha(y) d\mu_\alpha(x)$$

with $\tilde{K}(x, y) = \tau_x^E K_{\sigma-a-b}(y)$. Hence we are reduced to showing that

$$\int_0^\infty K_{\sigma-a-b} * f(x) g(x) d\mu_\alpha(x) \leq \|f\|_p \|g\|_{q'},$$

which is implied by Theorem 2.2.

To estimate I_2 and I_3 , we need the following lemma.

Lemma 3.2. *Let $\alpha > -1$ and $V_\delta f(x) = x^{2(\delta-\alpha-1)} \int_0^x f(y) y^{-2\delta} d\mu_\alpha(y)$, $\delta < (\alpha+1)/p'$. Then*

$$\|V_\delta f\|_p \leq C \|f\|_p, \quad |V_\delta f(x)| \leq Cx^{-(2\alpha+2)/p} \|f\|_p.$$

Proof. We have that $V_\delta f(x) = \int_0^\infty L(x, y)f(y)d\mu_\alpha(y)$, where $L(x, y) = x^{2(\delta-\alpha-1)}y^{-2\delta}$ for $y < x$ and equals 0 otherwise. Clearly L is homogeneous of degree $-(2\alpha + 2)$. Therefore the norm inequality is implied by

$$\int_0^\infty L(1, y)y^{-(2\alpha+2)/p}d\mu_\alpha(y) = \int_0^1 y^{-2\delta-(2\alpha+2)/p+2\alpha+1}dy < \infty,$$

which is finite since $\delta < (\alpha + 1)/p'$. For the pointwise estimate, we simply write

$$\begin{aligned} |V_\delta f(x)| &\leq x^{2(\delta-\alpha-1)} \int_0^x |f(y)|y^{-2\delta}d\mu_\alpha(y) \\ &\leq Cx^{2(\delta-\alpha-1)} \left(\int_0^x y^{-2\delta p'}d\mu_\alpha(y) \right)^{1/p'} \|f\|_p \\ &\leq Cx^{-(2\alpha+2)/p} \|f\|_p, \end{aligned}$$

which finishes the proof of the lemma.

Estimating I_2 , we note that for $x, y \in D_2$ and all θ , $0 < \theta < \pi$, there holds $x < 2(x - y) \leq 2((x - y)^2 + 2xy(1 - \cos \theta))^{1/2} = 2(x, y)_\theta$. Hence

$$\begin{aligned} I_2 &\leq C \int_0^\infty x^{2(\sigma-\alpha-1-b)}g(x) \left(\int_0^x f(y)y^{-2a}d\mu_\alpha(y) \right) d\mu_\alpha(x) \\ &= C \int_0^\infty g(x)x^{2(\sigma-a-b)}V_a f(x)d\mu_\alpha(x) \leq C\|g\|_{q'} \|x^{2(\sigma-a-b)}V_a f\|_q. \end{aligned}$$

It now suffices to estimate the latter L^q norm by $\|f\|_p$. We have

$$\int_0^\infty |x^{2(\sigma-a-b)}V_a f(x)|^q d\mu_\alpha(x) = \int_0^\infty |V_a f(x)|^p |V_a f(x)|^{q-p} x^{2(\sigma-a-b)q} d\mu_\alpha(x).$$

Since $a < (\alpha + 1)/p'$, by the pointwise estimate in the above lemma

$$|V_a f(x)| \leq Cx^{-(2\alpha+2)/p} \|f\|_p,$$

hence

$$|V_a f(x)|^{q-p} x^{2(\sigma-a-b)q} \leq C\|f\|_p^{q-p}$$

due to the identity from assumptions. Hence

$$\int_0^\infty |x^{2(\sigma-a-b)}V_a f(x)|^q d\mu_\alpha(x) \leq C\|f\|_p^{q-p} \int_0^\infty |V_a f(x)|^p d\mu_\alpha(x) \leq C\|f\|_p^q,$$

by the norm inequality from Lemma 3.2. The estimate of I_3 is similar and we omit it. The proof of the theorem is complete.

After submitting this paper the authors became acquainted with the following result of Kanjin and Sato [7]:

$$\|I_\sigma f\|_{L^q_{v(\alpha q/2)}} \leq C\|f\|_{L^p_{v(\alpha p/2)}}, \quad 0 < \frac{1}{q} = \frac{1}{p} - \sigma, \quad p > 1, \quad \alpha \geq 0,$$

whereas for $-1 < \alpha < 0$, there occurs $(1 + \frac{\alpha}{2})^{-1} < p$, $q < -\frac{2}{\alpha}$ as an additional restriction. Combining the above Theorem 1.1 in the case $a = b = \alpha = 0$ with Kanjin's transplantation theorem (cf. [13]) one at once recovers the above stated result of Kanjin and Sato.

References

1. R. Askey and S. Wainger, *Mean convergence of expansions in Laguerre and Hermite series*, Amer. J. Math. **87** (1965), 695–708.
2. G. Gasper and W. Trebels, *Jacobi and Hankel multipliers of type (p, q) , $1 < p < q < \infty$* , Math. Ann. **237** (1978), 243–251.
3. ———, *On necessary multiplier conditions for Laguerre expansions*, Canad. J. Math. **43** (1991), 1228–1242.
4. ———, *On necessary multiplier conditions for Laguerre expansions II*, SIAM J. Math. Anal. **25** (1994), 384–391.
5. E. Görlich and C. Markett, *A convolution structure for Laguerre series*, Indag. Math. **44** (1982), 161–171.
6. L. I. Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc. **36** (1972), 505–510.
7. Kanjin and Sato, *The Hardy-Littlewood theorem on fractional integration for Laguerre series*, Proc. Amer. Math. Soc. (to appear).
8. B. Muckenhoupt, *Mean convergence of Hermite and Laguerre series II*, Trans. Amer. Math. Soc. **147** (1970), 433–460.
9. ———, *Transplantation theorems and multiplier theorems for Jacobi series*, Mem. Amer. Math. Soc. **64** (1986), No. 356.
10. E. M. Stein and G. Weiss, *Fractional integrals on n -dimensional Euclidean space*, J. Math. Mech. **7** (1958), 503–514.
11. K. Stempak, *Almost everywhere summability of Laguerre series*, Studia Math. **100** (1991), 129–147.
12. ———, *Almost everywhere summability of Laguerre series II*, Studia Math. **103** (1992), 317–327.
13. K. Stempak and W. Trebels, *On weighted transplantation and multipliers for Laguerre expansions*, Math. Ann. **300** (1994), 203–219.
14. G. Szegő, *Orthogonal Polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ. 23, Providence, R.I., 1975.
15. A. Zygmund, *Trigonometric Series*, 2nd ed., Cambridge Univ. Press, Cambridge 1968.

DEPT. OF MATHEMATICS, NORTHWESTERN UNIVERSITY, EVANSTON, IL 60208, U.S.A.

INSTITUTE OF MATHEMATICS, UNIVERSITY OF WROCLAW, PL. GRUNWALDZKI 2/4, 50–384 WROCLAW,
POLAND

FB. MATHEMATIK, TH DARMSTADT, SCHLOSSGARTENSTR. 7, D–64289 DARMSTADT, GERMANY