

ASYMPTOTIC SOLUTION TO A CLASS OF SINGULARLY PERTURBED VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT. A uniformly valid asymptotic expansion for the solution to a class of singularly perturbed Volterra integral equations displaying exponential boundary layer behavior is established. Certain quasilinear ordinary differential equations are a noted special case, and a model for population growth with attrition is briefly discussed.

1. Introduction

Let $f(x, \epsilon) \in C^\infty([0, 1] \times [0, 1])$ and $k(x, t, \epsilon, y) \in C^\infty([0, 1] \times [0, x] \times [0, 1] \times [a, b])$, where (a, b) contains the derivative $f_\epsilon(0, 0)$, and assume $f(0, 0) = 0$. We are interested in the asymptotic behavior as $\epsilon \rightarrow 0^+$ of the solution to the Volterra integral equation

$$\epsilon y(x, \epsilon) + \int_0^x k(x, t, \epsilon, y(t, \epsilon)) dt = f(x, \epsilon). \quad (1.1)$$

The assumption $f(0, 0) = 0$, which implies $y(0, \epsilon) = f_\epsilon(0, 0) + O(\epsilon)$, is nontrivial unless (1.1) is linear. If $k_{yy}(x, t, \epsilon, y) = 0$, we can get $f(0, 0) = 0$ by changing the unknown (if necessary) to $\epsilon y(x, \epsilon)$.

In the special case

$$k(x, t, \epsilon, y) = p(t, \epsilon)y + (x - t)[q(t, \epsilon, y) - p_t(t, \epsilon)]y, \quad (1.2)$$

$$f(x, \epsilon) = \alpha\epsilon + [\beta\epsilon + \alpha p(0, \epsilon)]x, \quad (1.3)$$

equation (1.1) is equivalent to the singularly perturbed initial-value problem

$$\epsilon y'' + p(x, \epsilon)y' + q(x, \epsilon, y) = 0, \quad (1.4)$$

$$y(0) = \alpha, \quad y'(0) = \beta. \quad (1.5)$$

As is well known [5], if $p(x, 0) > 0$ and the reduced problem

$$p(x, 0)u' + q(x, 0, u) = 0, \quad y(0) = \alpha, \quad (1.6)$$

has a solution for $0 \leq x \leq 1$, then the solution to (1.4)–(1.5) exists for $0 \leq x \leq 1$, and it has a uniformly valid asymptotic expansion of the form

$$y(x, \epsilon) = \sum_{n=0}^{N-1} \epsilon^n [u_n(x) + v_n(x/\epsilon)] + O(\epsilon^N), \quad (1.7)$$

where $u_n(x) \in C^\infty[0, 1]$ and $v_n(X) \in C^\infty[0, \infty)$. Furthermore, $v_n(X) = o(X^{-\infty})$. That is, $v_n(X) = o(X^{-r})$ as $X \rightarrow \infty$ for any r . It can also be seen from the linear

Received June 29, 1994, revised December 5, 1994.

1991 *Mathematics Subject Classification*: 41A60, 45M05.

Key words and phrases: singular perturbation, Volterra integral equation, asymptotic expansion.

Fredholm equation theory in [4] that the solution to (1.1) is expressible in the form (1.7) if $k(x, t, \epsilon, y) = c(x, t)y$ and $c(x, x) > 0$.

The object of this paper is to establish the validity of (1.7), under appropriate conditions, for the full nonlinear problem (1.1). In so doing, we develop a procedure for successively computing the individual terms of (1.7). This procedure is rigorous and more explicit than the one presented in [1] and [2], which is a formal treatment of (1.1) and a variety of related problems including singular integral equations problems. We also briefly discuss a model for population growth with attrition.

Insisting that $v_n(X) = o(X^{-\infty})$ in (1.7) may at first seem too restrictive. For instance, if

$$k(x, t, \epsilon, y) = y^2 - 2ty, \quad f(x, \epsilon) = \epsilon(1 + x) - \frac{1}{3}x^3, \quad (1.8)$$

the solution to (1.1) is $y(x, \epsilon) = x + (1 + x/\epsilon)^{-1}$. However, this example is exceptional. Considering just the first term of (1.7), if we are to have $y(x, \epsilon) = u(x) + v(x/\epsilon) + O(\epsilon)$, then (1.1) implies

$$u(0) + v(X) + \int_0^X k(0, 0, 0, u(0) + v(T))dT = Xf_x(0, 0) + f_\epsilon(0, 0). \quad (1.9)$$

Thus, in order to have $v(X) = O(1)$ for $0 \leq X < \infty$, it must be that $v(X)$ approaches an equilibrium point of

$$v' + k(0, 0, 0, u(0) + v) = f_x(0, 0) \quad (1.10)$$

as $X \rightarrow \infty$. Without loss of generality, $v(\infty) = 0$, by choice of $u(0)$; therefore $k_y(0, 0, 0, u(0)) \geq 0$. If $k_y(0, 0, 0, u(0)) > 0$, then $k_y(0, 0, 0, y) > 0$ in a neighborhood of $y = u(0)$; therefore $v(X) \rightarrow 0$ exponentially as $X \rightarrow \infty$. That is, $v(X) = o(X^{-\infty})$ if $k_y(0, 0, 0, u(0)) > 0$. On the other hand, if $k_y(0, 0, 0, u(0)) = 0$, as in (1.8), then $f_x(0, 0)$ must just happen to be an extreme value of $k(0, 0, 0, y)$. Indeed, (1.10) also implies $k(0, 0, 0, u(0)) = f_x(0, 0)$. We shall therefore limit our investigation of (1.1) to solutions having asymptotic expansions of the form (1.7) with $v_n(X) = o(X^{-\infty})$.

2. Integral expansion

Let

$$s_N(t, T, \epsilon) = \sum_{n=0}^{N-1} \epsilon^n [u_n(t) + v_n(T)], \quad (2.1)$$

where $u_n(t) \in C^\infty[0, 1]$, $v_n(T) \in C^\infty[0, \infty)$, and $v_n(T) = o(T^{-\infty})$, but $u_n(t)$, $v_n(T)$ are not yet tied to the solution of (1.1). We begin by determining a uniformly valid expansion for

$$Y_N(x, \epsilon) = \int_0^x k(x, t, \epsilon, y_N(t, \epsilon))dt, \quad (2.2)$$

where $y_N(t, \epsilon) = s_N(t, t/\epsilon, \epsilon)$. Note that we are presuming $a < u_0(t) + v_0(T) < b$ for all $(t, T) \in [0, 1] \times [0, \infty)$, so that $y_N(t, \epsilon) \in (a, b)$ for $0 \leq t \leq 1$ and all sufficiently small $\epsilon > 0$. We will use the following two theorems.

Theorem 2.1. *If $h(t, T, \epsilon) \in C^\infty([0, 1] \times [0, \infty) \times [0, 1])$ and $h(t, T, \epsilon) = o(T^{-\infty})$ as $T \rightarrow \infty$, then*

$$h(t, t/\epsilon, \epsilon) = \sum_{n=0}^{N-1} \epsilon^n h_n(t/\epsilon) + O(\epsilon^N), \quad (2.3)$$

where $h_n(T) \in C^\infty[0, \infty)$ is the coefficient of ϵ^n in the Taylor expansion of $h(\epsilon T, T, \epsilon)$, and $h_n(T) = o(T^{-\infty})$.

Theorem 2.2. *If $\phi(T, \epsilon) \in C^\infty([0, \infty) \times [0, 1])$ and $g(t, \epsilon, y) \in C^\infty([0, 1] \times [0, 1] \times [a, b])$, where $a < \phi(T, \epsilon) < b$ for all $(T, \epsilon) \in [0, \infty) \times [0, 1]$, and if $\phi(T, \epsilon) = o(T^{-\infty})$, then*

$$g(t, \epsilon, \phi(t/\epsilon, \epsilon)) = \sum_{n=0}^{N-1} \epsilon^n [g_n(t) + h_n(t/\epsilon)] + O(\epsilon^N), \quad (2.4)$$

where $g_n(t) \in C^\infty[0, 1]$ is the coefficient of ϵ^n in the Taylor expansion of $g(t, \epsilon, 0)$, $h_n(T) \in C^\infty[0, \infty)$ is the coefficient of ϵ^n in the expansion of $g(\epsilon T, \epsilon, \phi(T, \epsilon)) - g(\epsilon T, \epsilon, 0)$, and $h_n(T) = o(T^{-\infty})$.

Proofs. Let $p_n(t, T)$ be the coefficient of ϵ^n in the Taylor expansion of $h(t, T, \epsilon)$, and let $p_{nk}(T)$ be the coefficient of t^k in the expansion of $p_n(t, T)$. Then $p_{nk}(T) \in C^\infty[0, \infty)$, $p_{nk}(T) = o(T^{-\infty})$, and

$$p_n(t, T) = \sum_{k=0}^{N-n-1} t^k p_{nk}(T) + t^{N-n} r_{N-n}(t, T), \quad (2.5)$$

where $r_{N-n}(t, T) \in C^\infty([0, 1] \times [0, \infty))$ and $r_{N-n}(t, T) = o(T^{-\infty})$. Therefore $t^{N-n} r_{N-n}(t, t/\epsilon) = \epsilon^{N-n} [(t/\epsilon)^{N-n} r_{N-n}(t, t/\epsilon)] = O(\epsilon^{N-n})$. Similarly, $t^k p_{nk}(t/\epsilon) = \epsilon^k P_{nk}(t/\epsilon)$, where $P_{nk}(T) = T^k p_{nk}(T) \in C^\infty[0, \infty)$ and $P_{nk}(T) = o(T^{-\infty})$. Thus, we have (2.3) with

$$h_n(T) = \sum_{k=0}^n P_{k, n-k}(T). \quad (2.6)$$

Theorem 2.2 is obtained by applying Theorem 2.1 to

$$h(t, T, \epsilon) = g(t, \epsilon, \phi(T, \epsilon)) - g(t, \epsilon, 0). \quad (2.7)$$

By applying Theorem 2.2 to

$$g(t, \epsilon, y, x) = k(x, t, \epsilon, \sum_{n=0}^{N-1} \epsilon^n u_n(t) + y) \quad (2.8)$$

with

$$\phi(T, \epsilon) = \sum_{n=0}^{N-1} \epsilon^n v_n(T) \quad (2.9)$$

(and x as an uninvolved parameter), it is apparent that

$$k(x, t, \epsilon, y_N(t, \epsilon)) = \sum_{n=0}^{N-1} \epsilon^n [\phi_n(x, t) + \psi_n(x, t/\epsilon)] + O(\epsilon^N), \quad (2.10)$$

where $\phi_n(x, t) \in C^\infty([0, 1] \times [0, x])$, $\psi_n(x, T) \in C^\infty([0, 1] \times [0, \infty))$, and $\psi_n(x, T) = o(T^{-\infty})$. A little computation also reveals

$$\phi_0(x, t) = k(x, t, 0, u_0(t)), \quad (2.11)$$

$$\phi_1(x, t) = k_y(x, t, 0, u_0(t))u_1(t) + k_\epsilon(x, t, 0, u_0(t)), \quad (2.12)$$

and, if we let

$$h(x, t, \epsilon, y) = k(x, t, \epsilon, u_0(0) + y) - k(x, t, \epsilon, u_0(0)), \quad (2.13)$$

then

$$\psi_0(x, T) = h(x, 0, 0, v_0(T)), \quad (2.14)$$

$$\psi_1(x, T) = k_y(x, 0, 0, u_0(0) + v_0(T))v_1(T) + \widehat{\psi}_1(x, T), \quad (2.15)$$

where $\widehat{\psi}_1(x, T)$ is the combination

$$\widehat{\psi}_1(x, T) = (h_\epsilon + Th_t + u'_0(0)Th_y + u_1(0)h_y)(x, 0, 0, v_0(T)). \quad (2.16)$$

In general, for $1 \leq n \leq N - 1$,

$$\phi_n(x, T) = k_y(x, t, 0, u_0(t))u_n(t) + \widehat{\phi}_n(x, t), \quad (2.17)$$

$$\psi_n(x, T) = k_y(x, 0, 0, u_0(0) + v_0(T))v_n(T) + \widehat{\psi}_n(x, T), \quad (2.18)$$

where $\widehat{\phi}_n(x, t)$ and $\widehat{\psi}_n(x, T) - u_n(0)h_y(x, 0, 0, v_0(T))$ are determined by $u_k(t)$, $v_k(T)$ for $0 \leq k \leq n - 1$.

We need to substitute (2.10) into (2.2). Upon applying Theorem 2.1 to

$$\Psi_n(x, X) = \int_X^\infty \psi_n(x, T)dT, \quad (2.19)$$

we get the desired result. Namely,

$$Y_N(x, \epsilon) = \sum_{n=0}^{N-1} \epsilon^n [U_n(x) + \epsilon V_n(x/\epsilon)] + \epsilon^N \theta_N(x, \epsilon), \quad (2.20)$$

where $U_n(x) \in C^\infty[0, 1]$, $V_n(X) \in C^\infty[0, \infty)$, $V_n(X) = o(X^{-\infty})$ and $\theta_N(x, \epsilon) = O(1)$ for $0 \leq x \leq 1$ as $\epsilon \rightarrow 0^+$. In particular,

$$U_0(x) = \int_0^x k(x, t, 0, u_0(t))dt, \quad V_0(X) = - \int_X^\infty h(0, 0, 0, v_0(T))dT. \quad (2.21)$$

For $1 \leq n \leq N - 1$,

$$U_n(x) = \int_0^x k_y(x, t, 0, u_n(t))dt + \widehat{U}_n(x), \quad (2.22)$$

$$V_n(X) = - \int_X^\infty k_y(0, 0, 0, u_0(0) + v_0(T))v_n(T)dT + \widehat{V}_n(X), \quad (2.23)$$

where, letting $\lambda_{mn}(X)$ denote the coefficient of x^m in the Taylor expansion of $\Psi_{n-m}(x, X)$,

$$\widehat{U}_n(x) = \int_0^x \widehat{\phi}_n(x, t)dt + \Psi_{n-1}(x, 0), \quad (2.24)$$

$$\widehat{V}_n(X) = - \int_X^\infty \widehat{\psi}_n(0, T)dT - \sum_{m=1}^n X^m \lambda_{mn}(X). \quad (2.25)$$

In particular,

$$\widehat{U}_1(x) = \int_0^x k_\epsilon(x, t, 0, u_0(t)) dt + \int_0^\infty h(x, 0, 0, v_0(T)) dT, \quad (2.26)$$

$$\widehat{V}_1(X) = - \int_X^\infty [\widehat{\psi}_1(0, T) + X h_x(0, 0, 0, v_0(T))] dT. \quad (2.27)$$

We could, of course, have absorbed $V_{N-1}(x/\epsilon)$ into $\theta_N(x, \epsilon)$ in (2.20). However, in its present form, since the expansion need not stop at N terms, $\theta_N(x, \epsilon) = p_N(x) + \epsilon \rho_N(x, \epsilon)$, where $p_N(x) \in C^\infty[0, 1]$ and $\rho_N(x, \epsilon) = O(1)$ as $\epsilon \rightarrow 0^+$. Thus, we have the following key result.

Theorem 2.3. *The derivative of the error term in (2.20), $\theta_{Nx}(x, \epsilon) = O(1)$ for $0 \leq x \leq 1$ as $\epsilon \rightarrow 0^+$.*

Proof. Just as we established (2.20) for $Y_N(x, \epsilon)$, we also have

$$Y_{Nx}(x, \epsilon) = \sum_{n=0}^N \epsilon^n [\widetilde{U}_n(x) + \widetilde{V}_n(x/\epsilon)] + \epsilon^{N+1} \widetilde{\theta}_{N+1}(x, \epsilon), \quad (2.28)$$

where $\widetilde{U}_n(x) \in C^\infty[0, 1]$, $\widetilde{V}_n(X) \in C^\infty[0, \infty)$, $\widetilde{V}_n(X) = o(X^{-\infty})$, and $\widetilde{\theta}_{N+1}(x, \epsilon) = O(1)$ for $0 \leq x \leq 1$ as $\epsilon \rightarrow 0^+$, since

$$Y_{Nx}(x, \epsilon) = k(x, x, \epsilon, y_N(x, \epsilon)) + \int_0^x k_x(x, t, \epsilon, y_N(t, \epsilon)) dt. \quad (2.29)$$

Integrating (2.28) and comparing with (2.20) shows that

$$\rho_N(x, \epsilon) = \int_0^{x/\epsilon} \widetilde{V}_N(T) dT + \int_0^x \widetilde{\theta}_{N+1}(t, \epsilon) dt. \quad (2.30)$$

Therefore $\rho_{Nx}(x, \epsilon) = \epsilon^{-1} \widetilde{V}_N(x/\epsilon) + \widetilde{\theta}_{N+1}(x, \epsilon)$, so $\theta_{Nx}(x, \epsilon) = p'_N(x) + \epsilon \rho_{Nx}(x, \epsilon) = O(1)$.

3. Asymptotic solution

In this section, we establish the preliminary result that, under appropriate conditions, $y_N(x, \epsilon) = s_N(x, x/\epsilon, \epsilon)$ satisfies (1.1) asymptotically, in the sense that

$$\epsilon y_N(x, \epsilon) + Y_N(x, \epsilon) = f(x, \epsilon) - \epsilon^N \phi_N(x, \epsilon), \quad (3.1)$$

where $\phi_N(x, \epsilon) = O(1)$ uniformly in x for $0 \leq x \leq 1$ as $\epsilon \rightarrow 0^+$. From (2.1) and (2.20) it is clear that for (3.1) to hold, we must have

$$u_{n-1}(x) + U_n(x) = f_n(x), \quad v_n(X) + V_n(X) = 0, \quad (3.2)$$

for $0 \leq n \leq N-1$, where $f_n(x)$ is the coefficient of ϵ^n in the Taylor expansion of $f(x, \epsilon)$ and $u_{-1}(x) = 0$. Furthermore, (3.1) implies $u_n(0) + v_n(0) = f_{n+1}(0)$ and

$$\phi_N(x, \epsilon) = \theta_N(x, \epsilon) + u_{N-1}(x) - \epsilon^{-N} \left[f(x, \epsilon) - \sum_{n=0}^{N-1} \epsilon^n f_n(x) \right], \quad (3.3)$$

where $\theta_N(x, \epsilon)$ is the error term in (2.20). Thus, in light of Theorem 2.3, $\phi_{Nx}(x, \epsilon) = O(1)$ as $\epsilon \rightarrow 0^+$. Also, $\theta_N(0, \epsilon) = -V_{N-1}(0) = v_{N-1}(0)$ by (2.20) and (3.2), and thus $\phi_N(0, \epsilon) = v_{N-1}(0) + u_{N-1}(0) - [f_N(0) + 0(\epsilon)] = O(\epsilon)$.

Of course, we need to show that (3.2) is consistent with the conditions imposed on $u_n(x)$ and $v_n(X)$ in deriving expansion (2.20). This requires some additional hypotheses, which we list below as H2 and H3. We have been assuming H1 all along.

- H1. The function $f(x, \epsilon) \in C^\infty([0, 1] \times [0, 1])$, and $k(x, t, \epsilon, y) \in C^\infty([0, 1] \times [0, x] \times [0, 1] \times [a, b])$. Also, $f_0(0) = 0$ and $f_1(0) \in (a, b)$.
- H2. The integral equation $U_0(x) = f_0(x)$ has a solution $u_0(x) \in C^\infty[0, 1]$ with $a + |f_1(0) - u_0(0)| < u_0(x) < b - |f_1(0) - u_0(0)|$.
- H3. There exists $\kappa > 0$ such that $k_y(x, x, 0, u_0(x)) \geq \kappa$ for all $x \in [0, 1]$, and $y = 0$ is the only root of $h(0, 0, 0, y)$ in an interval containing $f_1(0) - u_0(0)$.

Condition H3 ensures $v_0(X) = o(X^{-\infty})$. Indeed, $k_y(0, 0, 0, u_0(0)) \geq \kappa$ means $v_0 = 0$ is an attractor for $v'_0 + h(0, 0, 0, v_0) = 0$, as noted in Section 1, and the $h(0, 0, 0, y) \neq 0$ assumption means $v_0(0) = f_1(0) - u_0(0)$ is in the domain of attraction. Furthermore, H3 implies $v_0(X) \rightarrow 0$ monotonically, hence $a < u_0(x) + v_0(X) < b$.

For $1 \leq n \leq N - 1$, the equations for $u_n(x)$ and $v_n(X)$ are linear. Indeed, (3.2), (2.22), and (2.23) imply

$$\int_0^x k_y(x, t, 0, u_0(t))u_n(t)dt = f_n(x) - u_{n-1}(x) - \widehat{U}_n(x), \quad (3.4)$$

$$v'_n + k_y(0, 0, 0, u_0(0) + v_0(X))v_n = -\widehat{V}'_n(X), \quad (3.5)$$

for $1 \leq n \leq N - 1$. Thus, by induction, $\widehat{U}_n(x) \in C^\infty[0, 1]$, so, since $k_y(x, x, 0, u_0(x)) > 0$, $u_n(x) \in C^\infty[0, 1]$. Also, $\widehat{V}_n(X) \in C^\infty[0, \infty)$ with $\widehat{V}_n(X) = o(X^{-\infty})$, so $v_n(X) \in C^\infty[0, \infty)$ and $v_n(X) = o(X^{-\infty})$, since $k_y(0, 0, 0, u_0(0) + v_0(X)) \geq \kappa$ for X sufficiently large.

Of course, regarding H2, $U_0(x) = f_0(x)$ may have no solution. For example, there is no solution if $k(x, t, 0, y) = y^2$ and $f(x, 0) = -x$. On the other hand, there can be at most one solution satisfying $k_y(x, x, 0, u_0(x)) \geq \kappa$. Indeed, there can be only one solution to (1.1) of the form (1.7).

If $k(x, t, \epsilon, y) = y^3 - y$ and $f(x, \epsilon) = \epsilon(c + \epsilon)$, for example, then $U_0(x) = 0$, since $f_0(x) = 0$. Hence, referring to (2.21), there are three possibilities, namely, $u_0(x) = \pm 1$ and $u_0(x) = 0$. However, if $u_0(x) = 0$, then $v'_0 + v_0^3 - v_0 = 0$ and $v_0(0) = c$. Thus, $v_0(X) \rightarrow \pm 1$ as $X \rightarrow \infty$, unless $c = 0$. But if $c = 0$, condition H3 fails to hold; indeed, the solution to (1.1) in this case, $y(x, \epsilon) = \epsilon / [(1 - \epsilon^2)e^{-2x/\epsilon} + \epsilon^2]$, is not expressible in the form (1.7). If $u_0(x) = 1$, then $h(0, 0, 0, v_0) = v_0^3 + 3v_0^2 + 2v_0$ and $v_0(0) = c - 1$, so $v_0(X) = o(X^{-\infty})$ only if $c > 0$. The remaining possibility, $u_0(x) = -1$, is the proper solution if $c < 0$.

We conclude this section with a brief discussion of the population growth problem modeled by (1.1) when $f(x, \epsilon) = \epsilon s(x)$ and $k(x, t, \epsilon, y) = -s(x - t)y(1 - y/c)$. In this model, the survival function $s(x)$ gives the fraction of the initial population which is still alive at time x (so $s(0) = 1$), $y(x, \epsilon)$ is the (relative) total population size at time x , and $\epsilon^{-1}y(1 - y/c)$ is the (rapid) rate of reproduction. Here we again have $U_0(x) = 0$ and, thus, $u_0(x) = 0$ or $u_0(x) = c$. To satisfy H3, we must take $u_0(x) = c$ and $c > 0$. It follows that $v'_0 = -v_0(1 + v_0/c)$, $v_0(0) = 1 - c$. Therefore

$$u_0(x) + v_0(X) = \frac{c}{1 + (c - 1)e^{-X}}, \quad (3.6)$$

which is notably independent of $s(x)$. In fact, $y_0(x, \epsilon) = u_0(x) + v_0(x/\epsilon)$ is the well-known s-shaped exact solution to the differential equation form of this model which exists when $s(x) = 1$.

With $y_0(x, \epsilon)$ determined, it follows from (3.4) and (2.26) that

$$\int_0^x s(x-t)u_1(t)dt = -c[1-s(x)]. \quad (3.7)$$

Hence $u_1(0) = cs'(0)$; therefore, from (3.5) and (2.27),

$$v_1' + [1 + (2/c)v_0(x)]v_1 = -s'(0)v_0(x), \quad v_1(0) = -cs'(0). \quad (3.8)$$

It is apparent from (3.7) that $u_1(x) < 0$ for $x > 0$. Thus, we see that the effect of attrition is that the population size never reaches the saturation level $y = c$. For instance, if $s(x) = (1 + \alpha x)e^{-\alpha x}$, then

$$u_1(x) = -(\alpha c/2)(1 - e^{-2\alpha x}), \quad v_1(X) = 0. \quad (3.9)$$

In addition to fertility being high, lifetimes are short in this example if α is large. Such problems are also studied in [3], but with the two rates connected. Obviously, here we would at least need $\alpha = o(1/\epsilon)$.

To calculate $u_2(x)$ and $v_2(X)$, we would need to determine $\phi_2(x, t)$ and $\psi_2(x, T)$ in (2.10) and then use the fact that

$$U_2(x) = \int_0^x \phi_2(x, t)dt + \int_0^\infty \psi_1(x, T)dT, \quad (3.10)$$

$$V_2(X) = - \int_X^\infty \left[\psi_2(0, T) + X\psi_{1x}(0, T) + \frac{1}{2}X^2\psi_{0xx}(0, T) \right] dT. \quad (3.11)$$

4. Confirmation

We are in position now to prove that, under assumptions H1–H3, for all $\epsilon > 0$ sufficiently small, (1.1) has a solution for $0 \leq x \leq 1$ with an asymptotic expansion given by (1.7), where $u_n(x)$ and $v_n(X)$ are the functions determined in Sections 2 and 3. We shall follow a procedure patterned after the proof of comparable results for singularly perturbed ordinary differential equations. For example, see [6, pp. 197-205].

First, we consider the linear Volterra equation

$$\epsilon z(x, \epsilon) + \int_0^x k_y(x, t, \epsilon, y_N(t, \epsilon))z(t, \epsilon)dt = g(x, \epsilon). \quad (4.1)$$

We expect that if $y(x, \epsilon)$ is the solution to (1.1), then the difference $\epsilon^{-N}[y(x, \epsilon) - y_N(x, \epsilon)]$ nearly satisfies (4.1) when $g(x, \epsilon) = \phi_N(x, \epsilon)$, where $\phi_N(x, \epsilon)$ is the function in (3.1).

Theorem 4.1. *Choose $\epsilon_0 > 0$ so that $a < y_N(x, \epsilon) < b$ on $D = \{(x, \epsilon) : 0 \leq x \leq 1, 0 < \epsilon \leq \epsilon_0\}$. Assume $g(x, \epsilon)$ is continuously differentiable with respect to x on D , and assume $g(0, \epsilon) = O(\epsilon)$, $g_x(x, \epsilon) = O(1)$. Then the solution to (4.1) is uniformly bounded on D .*

Proof. In terms of

$$w(x, \epsilon) = g_x(x, \epsilon) - \int_0^x k_{xy}(x, t, \epsilon, y_N(t, \epsilon))z(t, \epsilon)dt, \quad (4.2)$$

(4.1) is equivalent to

$$\epsilon z' + k_y(x, x, \epsilon, y_N(x, \epsilon))z = w(x, \epsilon), \quad z(0, \epsilon) = \epsilon^{-1}g(0, \epsilon). \quad (4.3)$$

Thus, in terms of

$$A(x, \epsilon) = \int_0^x k_y(s, s, \epsilon, y_N(s, \epsilon))ds, \quad (4.4)$$

if we let

$$G(x, \epsilon) = \epsilon^{-1}g(0, \epsilon)e^{-A(x, \epsilon)/\epsilon} + \epsilon^{-1} \int_0^x e^{-[A(x, \epsilon) - A(t, \epsilon)]/\epsilon} g_x(t, \epsilon) dt, \quad (4.5)$$

$$K(x, t, \epsilon) = \epsilon^{-1} \int_t^x e^{-[A(x, \epsilon) - A(t, \epsilon)]/\epsilon} k_{xy}(s, t, \epsilon, y_N(t, \epsilon)) ds, \quad (4.6)$$

it follows that (4.1) also is equivalent to

$$z(x, \epsilon) + \int_0^x K(x, t, \epsilon)z(t, \epsilon) dt = G(x, \epsilon). \quad (4.7)$$

By an application of Theorem 2.2,

$$k_y(s, s, \epsilon, y_N(s, \epsilon)) = k_y(s, s, 0, u_0(s)) + h_y(0, 0, 0, v_0(s/\epsilon)) + O(\epsilon). \quad (4.8)$$

Therefore

$$A(x, \epsilon) - A(t, \epsilon) = \int_t^x k_y(s, s, 0, u_0(s)) ds + O(\epsilon), \quad (4.9)$$

hence $A(x, \epsilon) - A(t, \epsilon) \geq \kappa(x - t) + O(\epsilon)$. Thus, we have

$$\int_0^x e^{-[A(x, \epsilon) - A(t, \epsilon)]/\epsilon} g_x(t, \epsilon) dt = O(\epsilon), \quad (4.10)$$

hence $G(x, \epsilon) = O(1)$ on D . Similarly, $K(x, t, \epsilon) = O(1)$ for $0 \leq x \leq 1$, $0 \leq t \leq x$, $0 < \epsilon \leq \epsilon_0$. Hence, from (4.7), by an application of Gronwall's inequality, $z(x, \epsilon) = O(1)$ on D .

If we denote the solution to (4.1) by $z(x, \epsilon) = \mathcal{K}g(x, \epsilon)$, then \mathcal{K} is a linear operator whose domain is all $g(x, \epsilon)$ satisfying the conditions of Theorem 4.1. Let \mathbb{B} denote the Banach space of all $f(x) \in C^0[0, 1]$ with $\|f\| = \max |f(x)|$, and let $S = \{f(x) \in \mathbb{B} : \|f\| \leq 2d\}$, where d is chosen so that $d > \|\mathcal{K}\phi_N(x, \epsilon)\|$ for $0 < \epsilon \leq \epsilon_0$. For example, $f(x) = \epsilon x$ is in S for any $\epsilon \leq 2d$. Finally, let

$$E(x, t, \epsilon, y, \rho) = \epsilon^{-2N} [k(x, t, \epsilon, y + \epsilon^N \rho) - k(x, t, \epsilon, y) - \epsilon^N \rho k_y(x, t, \epsilon, y)], \quad (4.11)$$

which is continuously differentiable. In particular, for ϵ_0 sufficiently small, there exists $M \geq 0$ such that both

$$|E(x, t, \epsilon, y_N(t, \epsilon), \rho)| \leq M \quad \text{and} \quad |E_\rho(x, t, \epsilon, y_N(t, \epsilon), \rho)| \leq M, \quad (4.12)$$

for all $(x, t, \epsilon, \rho) \in [0, 1] \times [0, x] \times (0, \epsilon_0) \times [-2d, 2d]$.

Theorem 4.2. *There exists $\epsilon_0 > 0$ such that the nonlinear problem*

$$\epsilon z(x, \epsilon) + \int_0^x F(x, t, \epsilon, z(t, \epsilon)) dt = \phi_N(x, \epsilon), \quad (4.13)$$

where

$$F(x, t, \epsilon, z) = k_y(x, t, \epsilon, y_N(t, \epsilon))z + \epsilon^N E(x, t, \epsilon, y_N(t, \epsilon), z), \quad (4.14)$$

has a solution in S , provided $0 < \epsilon \leq \epsilon_0$.

Proof. Problem (4.13) is equivalent to $z(x, \epsilon) = \mathcal{N}z(x, \epsilon)$, where, for any $z(x, \epsilon) \in S$, the operator \mathcal{N} is defined by

$$\mathcal{N}z(x, \epsilon) = \mathcal{K}[\phi_N(x, \epsilon) - \epsilon^N \int_0^x E(x, t, \epsilon, y_N(t, \epsilon), z(t, \epsilon)) dt]. \quad (4.15)$$

There exists $m > 0$ such that $\|\mathcal{K}g(x, \epsilon)\| \leq m\|g(x, \epsilon)\|$ for any $g(x, \epsilon)$ in the domain of \mathcal{K} , hence $\|\mathcal{N}z(x, \epsilon)\| \leq d + \epsilon^N mM$ for any $z(x, \epsilon) \in S$. Thus, there exists $\epsilon_0 > 0$ such that, if $z(x, \epsilon) \in S$, then $\mathcal{N}z(x, \epsilon) \in S$ for $0 < \epsilon \leq \epsilon_0$. Furthermore, since (4.12) also implies

$$\begin{aligned} |E(x, t, \epsilon, y_N(t, \epsilon), z(t, \epsilon)) - E(x, t, \epsilon, y_N(t, \epsilon), w(t, \epsilon))| \\ \leq M|z(t, \epsilon) - w(t, \epsilon)| \end{aligned} \quad (4.16)$$

whenever $z(t, \epsilon), w(t, \epsilon) \in S$, we know

$$\|\mathcal{N}z(x, \epsilon) - \mathcal{N}w(x, \epsilon)\| \leq mM\epsilon^N \|z(x, \epsilon) - w(x, \epsilon)\|. \quad (4.17)$$

Thus, \mathcal{N} is contracting on S for $0 < \epsilon \leq \epsilon_0 < (mM)^{-1/N}$.

Finally, we can state our main result, and its proof is now a simple computation.

Theorem 4.3. *Let $R_N(x, \epsilon)$ be the solution to (4.13) determined by Theorem 4.2. Then $y(x, \epsilon) = y_N(x, \epsilon) + \epsilon^N R_N(x, \epsilon)$ satisfies (1.1) for $0 \leq x \leq 1$, $0 < \epsilon \leq \epsilon_0$.*

Proof. For $0 \leq x \leq 1$, $0 \leq t \leq x$, $0 < \epsilon \leq \epsilon_0$, we have

$$k(x, t, \epsilon, y_N(t, \epsilon) + \epsilon^N R_N(t, \epsilon)) = k(x, t, \epsilon, y_N(t, \epsilon)) + \epsilon^N F(x, t, \epsilon, R_N(t, \epsilon)), \quad (4.18)$$

and $y_N(x, \epsilon)$ satisfies (3.1). Therefore

$$\begin{aligned} \int_0^x k(x, t, \epsilon, y_N(t, \epsilon) + \epsilon^N R_N(t, \epsilon)) dt \\ = f(x, \epsilon) - \epsilon^N [\phi_N(x, \epsilon) - \epsilon y_N(x, \epsilon)] + \epsilon^N [\phi_N(x, \epsilon) - \epsilon R_N(x, \epsilon)]. \end{aligned} \quad (4.19)$$

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