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Solutions of the strong Stieltjes moment problem

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Abstract. The strong Stieltjes moment problem for a bi-infinite sequence $\{c_n,\ n = 0, \pm 1, \pm 2, \dots\}$ can be described as follows: (1) find conditions for the existence of a positive measure μ on $[0, \infty)$ such that $c_n = \int_0^\infty t^n d\mu(t)$ for all n, (2) when there is a solution μ, find conditions for uniqueness of the solution, and (3) when there is more than one solution, describe families of solutions. In this paper, we investigate aspects of question (3). In particular, we discuss solutions and their Stieltjes transforms that are obtained by utilizing a theory of quasi-orthogonal and of pseudo-orthogonal Laurent polynomials.

1. Introduction

The classical Stieltjes moment problem can be defined as follows: given a sequence $\{c_n,\ n = 0, 1, 2, \dots\}$ of real numbers, (1) find conditions for there to exist a (positive) measure μ on $[0, \infty)$ such that $c_n = \int_0^\infty t^n d\mu(t)$ for $n = 0, 1, 2, \dots$, (2) when there is a solution of the existence problem, find conditions for uniqueness of the solution, and (3) when there is more than one solution, describe (subfamilies of) the family of all solutions. The problem is called *determinate* when there exists exactly one solution, and *indeterminate* when there exists more than one solution. The problem was treated first by Stieltjes [34] in 1894 and then by Hamburger [12] in 1920–21 for the case when the support of μ is only required to be contained in $(-\infty, \infty)$ (the classical Hamburger moment problem). These initial works were followed by an extensive development of a theory of moment problems where the connection with the theory of orthogonal polynomials plays a central role, e.g., see [1, 2, 7–9, 21, 22, 32, 33, 35, 36].

The *strong* Stieltjes moment problem (and *strong* Hamburger moment problem) can be formulated in the same way as the classical problems, except that here bi-infinite sequences $\{c_n,\ n = 0, \pm 1, \pm 2, \dots\}$ are involved. These problems were introduced around 1980 by Jones, Thron, and Waadeland [20] for the Stieltjes case, and by Jones and Thron [17, 18] for the Hamburger case. In the following years, a theory of these problems and their connection with a theory of orthogonal Laurent polynomials was developed as far as questions (1) and (2) were concerned, see [3–6, 10, 11, 13–20, 29–31]. In [28], a theory concerning question (3) for the strong Hamburger moment problem was developed. In particular, an analog to the Nevanlinna parametrization of solutions in the classical case was presented (cf. [1, 23, 33, 35]). In this paper, we make a contribution to a study of question (3) for the strong Stieltjes moment problem.

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2. Natural solutions

Let $\{c_n,\ n = 0, \pm 1, \pm 2, \dots\}$ be a given bi-infinite sequence of real numbers. The *Strong Hamburger Moment Problem* (SHMP) for the sequence consists of finding all (if any) positive measures μ on $(-\infty, \infty)$ satisfying

$$c_n = \int_{-\infty}^{\infty} \theta^n d\mu(\theta) \quad \text{for} \quad n = 0, \pm 1, \pm 2, \dots. \tag{2.1}$$

The *Strong Stieltjes Moment Problem* (SSMP) similarly consists of finding all (if any) positive measures μ on $[0, \infty)$ satisfying

$$c_n = \int_{0}^{\infty} \theta^n d\mu(\theta) \quad \text{for} \quad n = 0, \pm 1, \pm 2, \dots. \tag{2.2}$$

For any pair (p, q) of integers with $p \le q$, let $\Lambda_{p,q}$ denote the complex linear space spanned by the functions z^j, $j = p, \dots, q$. We write $\Lambda_{2m} = \Lambda_{-m,m}$ and $\Lambda_{2m+1} = \Lambda_{-(m+1),m}$ for $m = 0, 1, 2, \dots$, and $\Lambda = \bigcup_{n=0}^{\infty} \Lambda_n$. An element of Λ is called a *Laurent polynomial*.

Let M be the linear functional defined on the basic elements z^n by

$$M[z^n] = c_n, \quad n = 0, \pm 1, \pm 2, \dots \tag{2.3}$$

The conditions for a measure μ to solve the SHMP and the SSMP are equivalent to

$$M[L] = \int_{-\infty}^{\infty} L(\theta)\, d\mu(\theta) \quad \text{for all} \quad L \in \Lambda \tag{2.4}$$

and

$$M[L] = \int_{0}^{\infty} L(\theta)\, d\mu(\theta) \quad \text{for all} \quad L \in \Lambda, \tag{2.5}$$

respectively. A necessary and sufficient condition for the SHMP to be solvable is that the functional M is positive on $(-\infty, \infty)$, while a necessary and sufficient condition for the SSMP to be solvable is that M is positive on $(0, \infty)$, see [10, 14, 15, 18]. (That M is positive on an interval I means that $M[L] > 0$ for all $L \in \Lambda$ where $L(z) \not\equiv 0$, $L(z) \ge 0$ when $z \in I$.) A moment problem is called *determinate* when there is exactly one solution, and *indeterminate* when there is more than one solution.

In the following, we always assume that M is positive on $(-\infty, \infty)$.

An inner product $\langle\ ,\ \rangle$ is defined on $\Lambda_R \times \Lambda_R$ (where Λ_R denotes the real space spanned by all z^j, $j = 0, \pm 1, \pm 2, \dots$) by

$$\langle P, Q \rangle = M[P(z) \cdot Q(z)]. \tag{2.6}$$

Let $\{\varphi_n\}$ be orthonormal Laurent polynomials obtained from the base $\{1, z^{-1}, z, z^{-2}, z^2, \dots, z^{-n}, z^n, \dots\}$. They shall be assumed normalized such that they may be written in the form

$$\varphi_{2m}(z) = \frac{q_{2m,-m}}{z^m} + \cdots + q_{2m,m} z^m, \qquad q_{2m,m} > 0, \tag{2.7}$$

$$\varphi_{2m+1}(z) = \frac{q_{2m+1,-(m+1)}}{z^{m+1}} + \cdots + q_{2m+1,m} z^m, \qquad q_{2m+1,-(m+1)} > 0, \tag{2.8}$$

for $m = 0, 1, 2, \dots$. The functional M and the system $\{\varphi_n\}$ are called *regular* if $q_{2m,-m} \ne 0$, $q_{2m+1,m} \ne 0$ for all m. In the following, we shall assume that M is regular. That is always the case if M is positive on $(0, \infty)$, see [10, 13, 14].

The *associated* orthogonal Laurent polynomials $\{\psi_n\}$ are defined by

$$\psi_n(z) = M\left[\frac{\varphi_n(\theta)-\varphi_n(z)}{\theta - z}\right]. \tag{2.9}$$

(The functional is applied to its argument as a function of θ.) We note that $\psi_0 \equiv 0$, $\psi_{2m} \in \Lambda_{-m,m-1}$, $\psi_{2m+1} \in \Lambda_{-(m+1),m-1}$.

The functions $\{\varphi_n\}$ and $\{\psi_n\}$ satisfy the following three-term recurrence relation (where f_n, g_n, h_n are constants):

$$\begin{bmatrix}\varphi_{2m}(z)\\ \psi_{2m}(z)\end{bmatrix} = (g_{2m} + h_{2m}z)\begin{bmatrix}\varphi_{2m-1}(z)\\ \psi_{2m-1}(z)\end{bmatrix} + f_{2m}\begin{bmatrix}\varphi_{2m-2}(z)\\ \psi_{2m-2}(z)\end{bmatrix}, \quad m = 1, 2, \ldots, \tag{2.10}$$

$$\begin{bmatrix}\varphi_{2m+1}(z)\\ \psi_{2m+1}(z)\end{bmatrix} = \left(g_{2m+1} + h_{2m+1}\frac{1}{z}\right)\begin{bmatrix}\varphi_{2m}(z)\\ \psi_{2m}(z)\end{bmatrix} + f_{2m+1}\begin{bmatrix}\varphi_{2m-1}(z)\\ \psi_{2m-1}(z)\end{bmatrix}, \quad m = 0, 1, 2, \ldots, \tag{2.11}$$

and

$$\varphi_0 = \frac{1}{\sqrt{c_0}}, \quad \psi_0 = 0, \quad \varphi_{-1} = 0, \quad \psi_{-1} = -\frac{h_1 c_{-1}}{f_1\sqrt{c_0}z}. \tag{2.12}$$

The functions $\{\varphi_n\}$ and $\{\psi_n\}$ thus are denominators and numerators of a continued fraction. This continued fraction is equivalent, in general, to an APT-fraction when M is regular and positive on $(-\infty, \infty)$. When M is positive on $(0, \infty)$, the continued fraction is equivalent to a positive T-fraction, see [15, 19] (cf. also [24–27] for the situation when M is not necessarily regular).

By considering inner products with suitable $\varphi_k(z)$ on the right- and left-hand sides of the recurrence relations (2.10)–(2.11), the following expressions for the coefficients in terms of the coefficients in $\varphi_n(z)$ can be obtained:

$$f_{2m} = \frac{q_{2m-2,m-1}q_{2m,-m}}{q_{2m-1,m-1}q_{2m-1,-m}}, \tag{2.13}$$

$$f_{2m+1} = \frac{q_{2m-1,-m}q_{2m+1,m}}{q_{2m,-m}q_{2m,m}}, \tag{2.14}$$

$$g_{2m} = \frac{q_{2m,-m}}{q_{2m-1,-m}}, \tag{2.15}$$

$$g_{2m+1} = \frac{q_{2m+1,m}}{q_{2m,m}}, \tag{2.16}$$

$$h_{2m} = \frac{q_{2m,m}}{q_{2m-1,m-1}}, \tag{2.17}$$

$$h_{2m+1} = \frac{q_{2m+1,-(m+1)}}{q_{2m,-m}} \tag{2.18}$$

(also see [16]).

The determinant formula for continued fractions in our situation can be written as

$$z\varphi_{2m}(z)\psi_{2m-1}(z) - z\varphi_{2m-1}(z)\psi_{2m}(z) = \frac{q_{2m,-m}}{q_{2m-1,-m}}, \tag{2.19}$$

$$z\varphi_{2m}(z)\psi_{2m+1}(z) - z\varphi_{2m+1}(z)\psi_{2m}(z) = \frac{q_{2m+1,m}}{q_{2m,m}} \tag{2.20}$$

(see [28, 30]).

We introduce the expression $T_n(z,a,b)$ by

$$T_n(z,a,b) = a\psi_n(z) + b\varphi_n(z), \quad n = 0,1,2,\dots \tag{2.21}$$

With this notation, the following general Christoffel–Darboux formulas are valid for arbitrary complex coefficients a,b,c,d and for $z,\zeta \in C - \{0\}$ (see, e.g., [28, 30]):

$$\begin{aligned} &zT_{2m-1}(z,a,b)T_{2m}(\zeta,c,d) - \zeta T_{2m-1}(\zeta,c,d)T_{2m}(z,a,b) \\ &\qquad = \frac{q_{2m,-m}}{q_{2m-1,-m}}\Big[(ad-bc) + (z-\zeta)\sum_{j=0}^{2m-1} T_j(z,a,b)T_j(\zeta,c,d)\Big], \end{aligned} \tag{2.22}$$

$$\begin{aligned} &zT_{2m+1}(z,a,b)T_{2m}(\zeta,c,d) - \zeta T_{2m+1}(\zeta,c,d)T_{2m}(z,a,b) \\ &\qquad = \frac{q_{2m+1,m}}{q_{2m,m}}\Big[(ad-bc) + (z-\zeta)\sum_{j=0}^{2m} T_j(z,a,b)T_j(\zeta,c,d)\Big]. \end{aligned} \tag{2.23}$$

We shall use these formulas in Section 3.

By setting $a = c = 0$, $b = d = 1$ in (2.22)–(2.23), we obtain special Christoffel–Darboux formulas (involving only the orthogonal Laurent polynomials themselves), which may be written in the form

$$\begin{aligned} &\varphi_{2m}(\zeta)[z\varphi_{2m-1}(z) - \zeta\varphi_{2m-1}(\zeta)] - \zeta\varphi_{2m-1}(\zeta)[\varphi_{2m}(z) - \varphi_{2m}(\zeta)] \\ &\qquad\qquad = \frac{q_{2m,-m}}{q_{2m-1,-m}}\,(z-\zeta)\sum_{j=0}^{2m-1}\varphi_j(z)\varphi_j(\zeta), \end{aligned} \tag{2.24}$$

$$\begin{aligned} &\varphi_{2m}(\zeta)[z\varphi_{2m+1}(z) - \zeta\varphi_{2m+1}(\zeta)] - \zeta\varphi_{2m+1}(\zeta)[\varphi_{2m}(z) - \varphi_{2m}(\zeta)] \\ &\qquad\qquad = \frac{q_{2m+1,m}}{q_{2m,m}}\,(z-\zeta)\sum_{j=0}^{2m}\varphi_j(z)\varphi_j(\zeta). \end{aligned} \tag{2.25}$$

Dividing by $z-\zeta$ and letting $\zeta \to z$, we get

$$\varphi_{2m}(z)\frac{d}{dz}\,[z\varphi_{2m-1}(z)] - z\varphi_{2m-1}(z)\frac{d}{dz}\,[\varphi_{2m}(z)] = \frac{q_{2m,-m}}{q_{2m-1,-m}}\sum_{j=0}^{2m-1}\varphi_j(z)^2, \tag{2.26}$$

$$\varphi_{2m}(z)\frac{d}{dz}\,[z\varphi_{2m+1}(z)] - z\varphi_{2m+1}(z)\frac{d}{dz}\,[\varphi_{2m}(z)] = \frac{q_{2m+1,m}}{q_{2m,m}}\sum_{j=0}^{2m}\varphi_j(z)^2. \tag{2.27}$$

All the zeros of φ_n are real and simple, and if M is positive on $(0,\infty)$, they are all positive. Let $\xi_1^{(n)},\dots,\xi_n^{(n)}$ denote these zeros, ordered by size, $\xi_1^{(n)} < \xi_2^{(n)} < \dots < \xi_n^{(n)}$. Then

$$\prod_{k=1}^{2m}\xi_k^{(2m)} = \frac{q_{2m,-m}}{q_{2m,m}}, \quad \prod_{k=1}^{2m+1}\xi_k^{(2m+1)} = -\frac{q_{2m+1,-(m+1)}}{q_{2m+1,m}}. \tag{2.28}$$

It follows that

$$q_{2m,-m} > 0, \quad q_{2m+1,m} < 0, \tag{2.29}$$

when M is positive on $(0,\infty)$. Consequently,

$$f_{2m} < 0, \quad f_{2m+1} < 0, \tag{2.30}$$

$$g_{2m} > 0, \quad g_{2m+1} < 0, \tag{2.31}$$

$$h_{2m} < 0, \quad h_{2m+1} > 0, \tag{2.32}$$

when M is positive on $(0,\infty)$.

If M is positive on $(0,\infty)$ (hence all the zeros of φ_n are positive), it follows easily from (2.25)–(2.26) that between two consecutive zeros of φ_n, there is exactly one zero of φ_{n-1}. Using (2.19)–(2.20) and the just-mentioned separation property of φ_n and φ_{n-1}, we find that between two consecutive zeros of φ_n there is exactly one zero of ψ_n.

In the general situation (M positive on $(-\infty,\infty)$), there exist positive weights $\lambda_1^{(n)},\ldots,\lambda_n^{(n)}$ such that the following quadrature formula is valid (see [10, 14, 16, 29]):

$$M[L] = \sum_{k=1}^{n} \lambda_k^{(n)} L(\xi_k^{(n)}) \quad \text{for} \quad L \in \Lambda_{2n-1}. \tag{2.33}$$

Since the function $f(\theta) = \dfrac{\varphi_n(\theta)-\varphi_n(z)}{\theta - z}$ belongs to Λ_{2n-1}, it follows from (2.9) and (2.33) that

$$\frac{\psi_n(z)}{\varphi_n(z)} = -\sum_{k=1}^{n} \frac{\lambda_k^{(n)}}{\xi_k^{(n)} - z}. \tag{2.34}$$

Let $\nu^{(n)}$ denote the discrete measure with mass $\lambda_k^{(n)}$ at the point $\xi_k^{(n)}$, $k=1,\ldots,n$. Then, according to (2.33), $\nu^{(n)}$ solves the truncated (Hamburger) moment problem

$$c_k = \int_{-\infty}^{\infty} \theta^k d\nu^{(n)}(\theta), \quad k=-2m,\ldots,2m-1 \text{ for } n=2m, \tag{2.35}$$

$$c_k = \int_{-\infty}^{\infty} \theta^k d\nu^{(n)}(\theta), \quad k=-(2m+1),\ldots,2m \text{ for } n=2m+1, \tag{2.36}$$

and, furthermore,

$$\frac{\psi_n(z)}{\varphi_n(z)} = -\int_{-\infty}^{\infty} \frac{d\nu^{(n)}(\theta)}{\theta - z}. \tag{2.37}$$

If M is positive on $(0,\infty)$, $\nu^{(n)}$ solves the corresponding truncated strong Stieltjes moment problem, and the integral in (2.37) can be taken over $(0,\infty)$, since the support of $\nu^{(n)}$ in this case is contained in $(0,\infty)$.

Since

$$\lambda_1^{(n)} + \lambda_2^{(n)} + \cdots + \lambda_n^{(n)} = M[1] = c_0, \tag{2.38}$$

it follows by Helly's theorems that every subsequence $\{\nu^{(n(k))}\}$ contains a subsequence converging to a measure ν which is a solution of the moment problem (i.e., the SHMP, in general, and the SSMP if M is positive on $(0,\infty)$) and such that the corresponding subsequence of $\{\psi_{n(k)}(z)/\varphi_{n(k)}(z)\}$ converges locally uniformly to $-\int_{-\infty}^{\infty}(\theta-z)^{-1}d\nu(\theta)$ outside $(-\infty,\infty)$ (to $-\int_0^{\infty}(\theta-z)^{-1}d\nu(\theta)$ when M is positive on $(0,\infty)$), see [10, 14, 16, 20]. Solutions that can be obtained in this way are called *natural solutions* of the moment problem (cf. [8] for the classical situation). It is known that when M is positive on $(0,\infty)$, the subsequences $\{\nu^{(2m)}\}$ and $\{\nu^{(2m+1)}\}$ converge to measures $N^{(0)}$ and $N^{(\infty)}$ (see Theorem 5.7 and cf. [19, 20]). It follows that when M is positive on $(0,\infty)$, the SHMP and the SSMP have two natural solutions in general and one natural solution when $N^{(0)} = N^{(\infty)}$. It also is known that the SSMP is determinate if and only if there is only one natural solution, also see Section 5.

The *Stieltjes transform* F_μ of a finite measure μ is defined by the formula

$$F_\mu(z) = \int_{-\infty}^{\infty} \frac{d\mu(\theta)}{\theta - z}. \tag{2.39}$$

The function F_μ is a Nevanlinna function, i.e., it is analytic in the open upper half-plane $U = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ (actually outside the support of μ) and maps U into the closed upper half-plane $\overline{U}$. The discussion above shows that when $\{\nu^{(n(j))}\}$ converges to a (natural) solution ν of the SHMP, then $\{\psi_{n(j)}(z)/\varphi_{n(j)}(z)\}$ converges locally uniformly in U to $-F_\nu(z)$.

For proofs and more detailed treatments of the questions discussed in this section, see [10, 13–20, 28–31].

3. Quasi-natural solutions

In this section, we make the general assumption that M is regular and positive on $(-\infty,\infty)$. We shall here review and complete the theory of quasi-orthogonal Laurent polynomials and their use in moment theory. For supplementary material, we refer to [10, 13, 14, 16, 28–30].

The *quasi-orthogonal Laurent polynomials* $\varphi_n(z,\tau)$ of order n are defined by

$$\varphi_{2m}(z,\tau) = \varphi_{2m}(z) - \tau z \varphi_{2m-1}(z) \tag{3.1}$$

$$\varphi_{2m+1}(z,\tau) = \varphi_{2m+1}(z) - \frac{\tau}{z}\varphi_{2m}(z). \tag{3.2}$$

Here τ is a variable, $\tau \in \widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. (For $\tau = \infty$, $\varphi_n(z,\tau) = -\varphi_{n-1}(z)$.) The *associated* quasi-orthogonal Laurent polynomials $\psi_n(z,\tau)$ of order n are defined by

$$\psi_{2m}(z,\tau) = \psi_{2m}(z) - \tau z \psi_{2m-1}(z), \tag{3.3}$$

$$\psi_{2m+1}(z,\tau) = \psi_{2m+1}(z) - \frac{\tau}{z}\ \psi_{2m}(z). \tag{3.4}$$

For $n = 1, 2, \ldots$, we also may write

$$\psi_n(z,\tau) = M\left[\frac{\varphi_n(\theta,\tau) - \varphi_n(z,\tau)}{\theta - z}\right]. \tag{3.5}$$

(Here, the functional M is applied to its argument as a function of θ.)

The *quasi-approximants* $R_n(z,\tau)$ are defined by

$$R_n(z,\tau) = \frac{\psi_n(z,\tau)}{\varphi_n(z,\tau)}. \tag{3.6}$$

For $\tau = 0$, they reduce to the ordinary approximants $\psi_n(z)/\varphi_n(z)$, and for $\tau = \infty$, they reduce to the ordinary approximants $\psi_{n-1}(z)/\varphi_{n-1}(z)$.

With the notation of Section 2, we may write

$$\varphi_{2m}(z,\tau) = \frac{q_{2m,-m}}{z^m} + \cdots + (q_{2m,m} - \tau q_{2m-1,m-1})z^m, \tag{3.7}$$

$$\varphi_{2m+1}(z,\tau) = \frac{(q_{2m+1,-(m+1)} - \tau q_{2m,-m})}{z^{m+1}} + \cdots + q_{2m+1,m}z^m. \tag{3.8}$$

Thus,

$$\varphi_{2m}(z,\tau)=\frac{B_{2m}(z,\tau)}{z^m},\quad \varphi_{2m+1}(z,\tau)=\frac{B_{2m+1}(z,\tau)}{z^{m+1}}, \tag{3.9}$$

where

$$B_{2m}(z,\tau)=q_{2m,-m}+\cdots+(q_{2m,m}-\tau q_{2m-1,m-1})z^{2m}, \tag{3.10}$$

$$B_{2m+1}(z,\tau)=(q_{2m+1,-(m+1)}-\tau q_{2m,-m})+\cdots+q_{2m+1,m}z^{2m+1}. \tag{3.11}$$

Let $\xi_1^{(n)}(\tau),\ldots,\xi_n^{(n)}(\tau)$ denote the zeros of $B_n(z,\tau)$ (in the case that $\deg B_n(z,\tau)=n$), repeated according to multiplicity. Then

$$\xi_1^{(2m)}(\tau)\cdots\xi_{2m}^{(2m)}(\tau)=\frac{q_{2m,-m}}{q_{2m,m}-\tau q_{2m-1,m-1}}, \tag{3.12}$$

$$\xi_1^{(2m+1)}(\tau)\cdots\xi_{2m+1}^{(2m+1)}(\tau)=-\frac{q_{2m+1,-(m+1)}-\tau q_{2m,-m}}{q_{2m+1,m}}. \tag{3.13}$$

Recall that h_n is one of the coefficients in the recurrence formulas for φ_n (cf. (2.10)–(2.11), (2.17)–(2.18)).

Theorem 3.1. *Assume that M is regular and positive on $(-\infty,\infty)$.*

A. *All the zeros of $B_n(z,\tau)$ are real and simple. The origin is not a zero of $B_n(z,\tau)$, hence the zeros of $\varphi_n(z,\tau)$ are the same as the zeros of $B_n(z,\tau)$.*

B. *If M is positive on $(0,\infty)$, then at least $n-1$ zeros of $\varphi_n(z,\tau)$ are positive.*

C. *If M is positive on $(0,\infty)$, then all the zeros of $\varphi_{2m}(z,\tau)$ are positive if and only if $\tau\in(h_{2m},\infty)$, and all the zeros of $\varphi_{2m+1}(z,\tau)$ are positive if and only if $\tau\in(-\infty,h_{2m+1})$.*

Proof. A. For a proof of this general result, see e.g., [14].

B. Let $\xi_1,\ldots,\xi_\lambda$ be the zeros of $\varphi_n(z,\tau)$ of odd order in $(0,\infty)$. First, let $n=2m$, and assume $\lambda\le 2m-2$. Set $T(z)=(z-\xi_1)\cdots(z-\xi_\lambda)/z^m$. Then $T(z)\in\Lambda_{2m-1}$, $zT(z)\in\Lambda_{2m-2}$, and hence

$$M[\varphi_{2m}(z,\tau)\cdot T(z)]=\langle\varphi_{2m}(z),T(z)\rangle-\tau\langle\varphi_{2m-1}(z),zT(z)\rangle=0.$$

On the other hand, $\varphi_{2m}(z,\tau)\cdot T(z)$ does not change sign in $(0,\infty)$, so $M[\varphi_{2m}(z,\tau)\cdot T(z)]\neq 0$, which is a contradiction. Hence, $\lambda\ge 2m-1$. Next, let $n=2m+1$, and assume $\lambda\le 2m-1$. Set $T(z)=(z-\xi_1)\cdots(z-\xi_\lambda)/z^{m-1}$. Then $T(z)\in\Lambda_{2m}$, $\frac{1}{z}T(z)\in\Lambda_{2m-1}$, and hence

$$M[\varphi_{2m+1}(z,\tau)\cdot T(z)]=\langle\varphi_{2m+1}(z),T(z)\rangle-\tau\langle\varphi_{2m}(z),\frac{1}{z}T(z)\rangle=0.$$

On the other hand, $\varphi_{2m+1}(z,\tau)\cdot T(z)$ does not change sign in $(0,\infty)$, so $M[\varphi_{2m+1}(z,\tau)\cdot T(z)]\neq 0$, which is a contradiction. Hence $\lambda\ge 2m$. It follows from this that $\varphi_n(z,\tau)$ has at least $n-1$ simple zeros in $(0,\infty)$.

C. It follows from (3.12)–(3.13) together with the result under B that all the zeros of $\varphi_{2m}(z,\tau)$ are positive if and only if $q_{2m,-m}/(q_{2m,m}-\tau q_{2m-1,m-1})>0$, and all the zeros of $\varphi_{2m+1}(z,\tau)$ are positive if and only if $(q_{2m+1,-(m+1)}-\tau q_{2m,-m})/q_{2m+1,m}<0$. By taking into account (2.7)–(2.8), (2.17)–(2.18), and (3.12)–(3.13), we see that the conditions for all the zeros of $\varphi_{2m}(z,\tau)$ and $\varphi_{2m+1}(z,\tau)$ to be positive are $\tau>h_{2m}$ and $\tau<h_{2m+1}$, respectively. (Note that, by (2.32), $\tau=0$ is among the values for which all the zeros of $\varphi_n(z,\tau)$ are positive, in agreement with the statements in Section 2.) $\square$

For each n, each $k = 1, \ldots, n$ and each $\tau \in \mathbb{R}$, let $L_k^{(n)}(z,\tau)$ be the unique Laurent polynomial that satisfies

$$L_k^{(n)}(z,\tau) \in \Lambda_{n-1}, \quad L_k^{(n)}(\xi_j^{(n)}(\tau),\tau) = \delta_{jk} \tag{3.14}$$

(see, e.g., [10, 14]). Set

$$\lambda_k^{(n)}(\tau) = M[L_k^{(n)}(z,\tau)]. \tag{3.15}$$

Theorem 3.2. *Let M be regular and positive on $(-\infty,\infty)$.*

A. *The quadrature formula*

$$M[F] = \sum_{k=1}^{n} \lambda_k^{(n)}(\tau) F\big(\xi_k^{(n)}(\tau)\big) \tag{3.16}$$

is valid for $F \in \Lambda_{-2m,2m-2}$ when $n = 2m$ and for $F \in \Lambda_{-2m,2m}$ when $n = 2m+1$.

B. *The weights $\lambda_k^{(n)}(\tau)$ are positive.*

Proof. See, e.g., [10, 14] and cf. also the remark after Theorem 4.2. □

In particular, the partial fraction decomposition for the quasi-approximants

$$R_n(z,\tau) = -\sum_{k=1}^{n} \frac{\lambda_k^{(n)}(\tau)}{\xi_k^{(n)}(\tau) - z} \tag{3.17}$$

follows as in the argument for formula (2.34).

Let $\nu_\tau^{(n)}$ denote the discrete measure with mass $\lambda_k^{(n)}(\tau)$ at the point $\xi_k^{(n)}(\tau)$, $k = 1, \ldots, n$. Then, according to (3.16), $\nu_\tau^{(n)}$ solves the truncated strong Hamburger moment problem

$$c_k = \int_{-\infty}^{\infty} \theta^k d\nu_\tau^{(n)}(\theta),\ k = -2m, \ldots, 2m-2,\ \text{for } n = 2m, \tag{3.18}$$

$$c_k = \int_{-\infty}^{\infty} \theta^k d\nu_\tau^{(n)}(\theta),\ k = -2m, \ldots, 2m,\ \text{for } n = 2m+1. \tag{3.19}$$

Furthermore,

$$R_n(z,\tau) = -\int_{-\infty}^{\infty} \frac{d\nu_\tau^{(n)}(\theta)}{\theta - z}. \tag{3.20}$$

If M is positive on $(0,\infty)$, then $\nu_\tau^{(n)}$ solves the corresponding truncated strong Stieltjes moment problem if and only if $\tau > h_{2m}$ when $n = 2m$, or if and only if $\tau < h_{2m+1}$ when $n = 2m+1$. Also, when these conditions are satisfied, the integral in (3.20) can be taken over $(0,\infty)$.

As in the argument leading to the natural solutions of the moment problems, it can be seen that every subsequence $\{\nu_{\tau_{n(k)}}^{(n(k))}\}$ contains a subsequence $\{\nu_{\tau_{n(k(j))}}^{(n(k(j)))}\}$ converging to a measure ν which is a solution of the SHMP and such that the corresponding subsequence $\{R_{n(k(j))}(z,\tau_{n(k(j))})\}$ converges to $-F_\nu(z)$, locally uniformly on $\mathbb{C} - \mathbb{R}$.

Solutions that can be obtained in this way shall be called *quasi-natural solutions.* If M is positive on $(0,\infty)$, then the quasi-natural solutions are also solutions of the SSMP, at least if $\tau_{n(k(j))} > 0$ when $n(k(j))$ is even, or if $\tau_{n(k(j))} < 0$ when $n(k(j))$ is odd, since then all the measures $\nu_{\tau_{n(k(j))}}^{n(k(j))}$ have support in $(0,\infty)$.

For each z in the upper half-plane U and each n, let the mapping $\tau \to w$ be defined by

$$w = w_n = -R_n(z, \tau). \tag{3.21}$$

This linear fractional transformation maps $\widehat{\mathbb{R}}$ onto a circle contained in U. We shall use the notation $\Delta_n(z)$ for the open disk bounded by this circle, $\partial\Delta_n(z)$ for the circle itself, and $\overline{\Delta}_n(z)$ for the closed disk $\Delta_n(z) \cup \partial\Delta_n(z)$. Obviously, the half-plane U (i.e., $\tau \in U$) is mapped onto $\Delta_n(z)$ (i.e., $w \in \Delta_n(z)$) or onto $\widehat{\mathbb{C}} - \overline{\Delta}_n(z)$. If M is positive on $(0, \infty)$, then U is always mapped onto $\Delta_n(z)$. This can be seen from the signs of the recurrence coefficients f_n, h_n in this situation (cf. (2.30), (2.32)) and composition of the transformations

$$\tau \to \frac{f_{2m}}{g_{2m} + h_{2m}z - \tau z}, \quad \text{and} \quad \tau \to \frac{f_{2m+1}}{g_{2m+1} + h_{2m+1}\frac{1}{z} - \frac{\tau}{z}}.$$

(The argument is similar to the proof of Theorem 4.6.) Analogous results are valid when z belongs to the lower half-plane $-U$.

By solving (3.21) with respect to τ, we get

$$\tau = \frac{\varphi_{2m}(z)w_{2m} + \psi_{2m}(z)}{z[\varphi_{2m-1}(z)w_{2m} + \psi_{2m-1}(z)]}, \quad n = 2m, \tag{3.22}$$

$$\tau = \frac{z[\varphi_{2m+1}(z)w_{2m+1} + \psi_{2m+1}(z)]}{\varphi_{2m}(z)w_{2m+1} + \psi_{2m}(z)}, \; n = 2m+1. \tag{3.23}$$

Theorem 3.3. *Let M be regular and positive on $(-\infty, \infty)$.*

A. *The disk $\overline{\Delta}_n(z)$ consists of those w that satisfy the inequality*

$$\sum_{j=0}^{n-1} |\psi_j(z) + w\varphi_j(z)|^2 \le \frac{w - \bar{w}}{z - \bar{z}}. \tag{3.24}$$

B.
$$\overline{\Delta}_m(z) \subset \overline{\Delta}_n(z) \quad \textit{for} \quad m > n \tag{3.25}$$

C. *The radius $r_n(z)$ of the disk $\overline{\Delta}_n(z)$ is given by*

$$r_n(z) = \left[|z - \bar{z}| \sum_{j=0}^{n-1} |\varphi_j(z)|^2 \right]^{-1}. \tag{3.26}$$

Proof. A. The circle $\partial\Delta_n(z)$ is given by $\operatorname{Im}\tau = 0$ in formulas (3.22)–(3.23), i.e., $w \in \partial\Delta_n(z)$ if and only if

$$zT_{2m}(z,w,1)\overline{T_{2m-1}(z,w,1)} - \bar{z}\,\overline{T_{2m}(z,w,1)}T_{2m-1}(z,w,1) = 0$$

and

$$zT_{2m+1}(z,w,1)\overline{T_{2m}(z,w,1)} - \bar{z}\,\overline{T_{2m+1}(z,w,1)}T_{2m}(z,w,1) = 0,$$

respectively. By using formulas (2.22)–(2.23) with $\zeta = \bar{z}$, $a = w$, $b = 1$, $c = \bar{w}$, $d = 1$, we find that $w \in \partial\Delta_n(z)$ if and only if

$$\frac{w - \bar{w}}{z - \bar{z}} = \sum_{j=0}^{n-1} |T_j(z,w,1)|^2,$$

from which (3.24) easily follows.

B. The inclusion $\overline{\Delta}_m(z) \subset \overline{\Delta}_n(z)$ for $m > n$ follows immediately from A.

C. It follows from (3.22)–(3.23) and standard properties of linear fractional transformations that

$$r_{2m}(z) = \left| \frac{z[\psi_{2m}(z)\varphi_{2m-1}(z) - \psi_{2m-1}(z)\varphi_{2m}(z)]}{\bar{z}\varphi_{2m}(z)\overline{\varphi_{2m-1}(z)} - z\varphi_{2m-1}(z)\overline{\varphi_{2m}(z)}} \right|, \tag{3.27}$$

$$r_{2m+1}(z) = \left| \frac{z^{-1}[\psi_{2m}(z)\varphi_{2m+1}(z) - \psi_{2m+1}(z)\varphi_{2m}(z)]}{\bar{z}\varphi_{2m}(z)\overline{\varphi_{2m+1}(z)} - z\varphi_{2m+1}(z)\overline{\varphi_{2m}(z)}} \right|. \tag{3.28}$$

Substitution from (2.19)–(2.20) and from (2.22)–(2.23) (with $a = c = 0$, $b = d = 1$, $\zeta = \bar{z}$) leads to the formula (3.26). For more details, see [28, 30]. □

It follows from (3.1)–(3.4), (3.6) that $-\psi_n(z)/\varphi_n(z)$ belongs to both $\partial\Delta_n(z)$ and $\partial\Delta_{n+1}(z)$. Thus $\partial\Delta_{n+1}(z)$ is tangent from the inside to $\partial\Delta_n(z)$ at the point $-\psi_n(z)/\varphi_n(z)$.

It follows from (3.18)–(3.20) that all the points on $\partial\Delta_{2m}(z)$ are values of the Stieltjes transforms $F_\mu(z)$ where μ solves the truncated strong moment problem

$$c_k = \int_{-\infty}^{\infty} \theta^k d\mu(\theta), \quad k = -2m, \dots, 2m-2, \tag{3.29}$$

and all the points on $\partial\Delta_{2m+1}(z)$ are values of the Stieltjes transforms $F_\mu(z)$ where μ solves the truncated strong moment problem

$$c_k = \int_{-\infty}^{\infty} \theta^k d\mu(\theta), \quad k = -2m, \dots, 2m. \tag{3.30}$$

Let $\Omega_{2m}(z)$ denote the arc of $\partial\Delta_{2m}(z)$ corresponding to $\tau \in (h_{2m}, \infty)$ and let $\Omega_{2m+1}(z)$ denote the arc of $\partial\Delta_{2m+1}(z)$ corresponding to $\tau \in (-\infty, h_{2m+1})$. It follows from Theorem 3.1C and (3.18)–(3.20) that all the points of $\Omega_{2m}(z)$ and $\Omega_{2m+1}(z)$ are values of Stieltjes transforms $F_\mu(z)$ where μ solves the truncated strong Stieltjes moment problem

$$c_k = \int_0^{\infty} \theta^k d\mu(\theta), \quad k = -2m, \dots, 2m-2, \tag{3.31}$$

and

$$c_k = \int_0^{\infty} \theta^k d\mu(\theta), \quad k = -2m, \dots, 2m, \tag{3.32}$$

respectively.

For later use, we introduce the set $O_n(z)$ as the segment of the disk $\Delta_n(z)$ obtained as the convex hull of the arc $\Omega_n(z)$.

We define

$$\overline{\Delta}_\infty(z) = \bigcap_{n=1}^{\infty} \overline{\Delta}_n(z), \tag{3.33}$$

and we let $\Delta_\infty(z)$ and $\partial\Delta_\infty(z)$ denote the interior and the boundary of $\overline{\Delta}_\infty(z)$. It follows from Theorem 3.3B that $\overline{\Delta}_\infty(z)$ is either a single point or a closed disk. The radius $r_\infty(z)$ of $\overline{\Delta}_\infty(z)$ is given by (cf. (3.26))

$$r_\infty(z) = \left[|z - \bar{z}| \sum_{j=0}^{\infty} |\varphi_j(z)|^2 \right]^{-1}. \tag{3.34}$$

Theorem 3.4. *Let M be regular and positive on $(-\infty, \infty)$.*

A. *If $\overline{\Delta}_\infty(z_0)$ is a disk for some $z_0 \in U$, then $\overline{\Delta}_\infty(z)$ is a disk for every $z \in U$* (limit circle case). *If $\overline{\Delta}_\infty(z_0)$ is a point for some $z_0 \in U$, then $\overline{\Delta}_\infty(z)$ is a point for every $z \in U$* (limit point case).

B. *In the limit point case, $\sum_{n=0}^{\infty}|\varphi_n(z)|^2 = \infty$ and $\sum_{n=0}^{\infty}|\psi_n(z)|^2 = \infty$ for all $z \in U$.*

C. *In the limit circle case, the series $\sum_{n=0}^{\infty}|\varphi_n(z)|^2$ and $\sum_{n=0}^{\infty}|\psi_n(z)|^2$ converge locally uniformly in $\mathbb{C} - \{0\}$.*

Proof. See [28, Theorems 3.4–3.5]. The locally uniform convergence is implicitly contained in the proof of Theorem 3.5. □

In Section 5, we shall discuss connections between (subsets of) the sets $\overline{\Delta}_\infty(z)$ and solutions of the strong moment problems.

4. Pseudo-natural solutions

In this section, we make the general assumption that M is positive on $(0, \infty)$. The *pseudo-orthogonal Laurent polynomials* $\Phi_n(z, \tau)$ of order n are defined by

$$\Phi_n(z,\tau) = \varphi_n(z) - \tau\varphi_{n-1}(z), \quad \tau \in \widehat{\mathbb{R}}. \tag{4.1}$$

The *associated* pseudo-orthogonal Laurent polynomials $\Psi_n(z, \tau)$ are defined by

$$\Psi_n(z,\tau) = \psi_n(z) - \tau\psi_{n-1}(z), \quad \tau \in \widehat{\mathbb{R}}. \tag{4.2}$$

They also may be written as

$$\Psi_n(z,\tau) = M\left[\frac{\Phi_n(\theta,\tau) - \Phi_n(z,\tau)}{\theta - z}\right]. \tag{4.3}$$

The *pseudo-approximants* $S_n(z, \tau)$ are defined by

$$S_n(z,\tau) = \frac{\Psi_n(z,\tau)}{\Phi_n(z,\tau)}. \tag{4.4}$$

For $\tau = 0$, they are the ordinary approximants $\psi_n(z)/\varphi_n(z)$, and for $\tau = \infty$, they are the ordinary approximants $\psi_{n-1}(z)/\varphi_{n-1}(z)$.

With the notation of Section 2, we may write

$$\Phi_{2m}(z,\tau) = \frac{(q_{2m,-m} - \tau q_{2m-1,-m})}{z^m} + \cdots + q_{2m,m}z^m, \tag{4.5}$$

$$\Phi_{2m+1}(z,\tau) = \frac{q_{2m+1,-(m+1)}}{z^{m+1}} + \cdots + (q_{2m+1,m} - \tau q_{2m,m})z^m. \tag{4.6}$$

Thus,

$$\Phi_{2m}(z,\tau) = \frac{C_{2m}(z,\tau)}{z^m}, \quad \Phi_{2m+1}(z,\tau) = \frac{C_{2m+1}(z,\tau)}{z^{m+1}}, \tag{4.7}$$

where

$$C_{2m}(z,\tau) = (q_{2m,-m} - \tau q_{2m-1,m}) + \cdots + q_{2m,m}z^{2m}, \tag{4.8}$$

$$C_{2m+1}(z,\tau) = q_{2m+1,-(m+1)} + \cdots + (q_{2m+1,m} - \tau q_{2m,m})z^{2m+1}. \tag{4.9}$$

Let $\zeta_1^{(n)}(\tau), \dots, \zeta_n^{(n)}(\tau)$ denote the zeros of $C_n(z,\tau)$ (in the case that $\deg C_n(z,\tau) = n$), repeated if necessary. Then,

$$\prod_{k=1}^{2m} \xi_k^{(2m)} = \frac{q_{2m,-m} - \tau q_{2m-1,-m}}{q_{2m,m}}, \tag{4.10}$$

$$\prod_{k=1}^{2m+1} \xi_k^{(2m+1)} = -\frac{q_{2m+1,-(m+1)}}{q_{2m+1,m} - \tau q_{2m,m}}. \tag{4.11}$$

Recall that g_n is one of the coefficients in the recurrence formula for φ_n (cf. (2.10)–(2.11), (2.15)–(2.16)).

Theorem 4.1. *Assume that M is positive on $(0,\infty)$.*

A. *All the zeros of $\Phi_n(z,\tau)$ are real and simple, and at least $n-1$ of them are positive.*

B. *All the zeros of $\Phi_{2m}(z,\tau)$ are positive if and only if $\tau \in (-\infty, g_{2m})$, and all the zeros of $\Phi_{2m+1}(z,\tau)$ are positive if and only if $\tau \in (g_{2m+1}, \infty)$.*

Proof. A. Let $\zeta_1, \dots, \zeta_\lambda$ be the zeros of $\Phi_n(z,\tau)$ of odd order in $(0,\infty)$. First, let $n = 2m$ and assume $\lambda \le 2m-1$. Set $T(z) = (z-\zeta_1)\cdots(z-\zeta_\lambda)/z^{m-1}$. Then $T(z) \in \Lambda_{2m-2}$ and hence

$$M[\Phi_{2m}(z,\tau)T(z)] = \langle \varphi_{2m}(z), T(z)\rangle - \tau\langle \varphi_{2m-1}(z), T(z)\rangle = 0.$$

On the other hand, $\Phi_{2m}(z,\tau)T(z)$ does not change sign in $(0,\infty)$, so $M[\Phi_{2m}(z,\tau)\cdot T(z)] \ne 0$, which is a contradiction. Thus, $\lambda \ge 2m-1$. Next, let $n = 2m+1$ and assume $\lambda \le 2m$. Set $(z-\zeta_1)\cdots(z-\zeta_\lambda)/z^m$. Then $T(z) \in \Lambda_{2m-1}$, and hence

$$M[\Phi_{2m+1}(z,\tau)\cdot T(z)] = \langle \Phi_{2m+1}(z), T(z)\rangle - \tau\langle \Phi_{2m}(z), T(z)\rangle = 0.$$

On the other hand, $\Phi_{2m+1}(z,\tau)\cdot T(z)$ does not change sign in $(0,\infty)$, so $M[\Phi_{2m+1}(z,\tau)\cdot T(z)] \ne 0$, which is a contradiction. Thus, $\lambda \ge 2m$. It follows that $\Phi_n(z,\tau)$ has at least $n-1$ simple zeros in $(0,\infty)$. The last zero is then real also.

B. It follows from (4.10)–(4.11) together with the results under A that all the zeros of $\Phi_{2m}(z,\tau)$ are positive if and only if $(q_{2m,-m} - \tau q_{2m-1,-m})/q_{2m,m} > 0$ and all the zeros of $\Phi_{2m+1}(z,\tau)$ are positive if and only if $(q_{2m+1,-(m+1)})/(q_{2m+1,m} - \tau q_{2m,m}) < 0$. Taking into account (2.7)–(2.8), (2.15)–(2.16), and (4.10)–(4.11), we see that the conditions for all the zeros of $\Phi_{2m}(z,\tau)$ and $\Phi_{2m+1}(z,\tau)$ to be positive are that $\tau < g_{2m}$ and $\tau > g_{2m+1}$, respectively. (Note that by (2.31), $\tau = 0$ is among the values for which all the zeros of $\Phi_n(z,\tau)$ are positive. This agrees with the statements in Section 2.) □

For each n, each $k = 1, \dots, n$, and each $\tau \in \mathbb{R}$, let $K_k^{(n)}(z,\tau)$ be the unique Laurent polynomial that satisfies

$$K_k^{(n)}(z,\tau) \in \Lambda_{n-1}, \quad K_k^{(n)}\big(\zeta_j^{(n)}(\tau), \tau\big) = \delta_{j,k}. \tag{4.12}$$

(For the existence of such a Laurent polynomial, see [10, 14].) Set

$$\kappa_k^{(n)}(\tau) = M[K_k^{(n)}(z,\tau)]. \tag{4.13}$$

Theorem 4.2. *Let M be positive on $(0,\infty)$. Then the quadrature formula*

$$M[F] = \sum_{k=1}^{n} \kappa_k^{(n)}(\tau) F\big(\zeta_k^{(n)}(\tau)\big) \tag{4.14}$$

is valid for $F \in \Lambda_{-(2m-1),2m-1}$ *when* $n = 2m$ *and for* $F \in \Lambda_{-(2m+1),2m-1}$ *when* $n = 2m+1$.

Proof. Let $F \in \Lambda$. For each n and τ, we set

$$F_n(z,\tau) = \sum_{k=1}^{n} F\big(\zeta_k^{(n)}(\tau)\big) K_k^{(n)}(z,\tau). \tag{4.15}$$

Note that

$$F_n\big(\zeta_j^{(n)}(\tau),\tau\big) = F\big(\zeta_j^{(n)}(\tau)\big), \quad j = 1,\dots,n. \tag{4.16}$$

Assume that $F \in \Lambda_{-(2m-1),2m-1}$. Then $f(z) = F(z) - F_{2m}(z,\tau) \in \Lambda_{-(2m-1),2m-1}$ and $f(\zeta_k^{(2m)}(\tau)) = 0$ for $k = 1,\dots,2m$. Hence, we may write

$$f(z) = \frac{p_{2m-2}(z)}{z^{m-1}}\ \Phi_{2m}(z,\tau), \tag{4.17}$$

where $p_{2m-2}(z) \in \Pi_{2m-2}$, and thus, $\frac{p_{2m-2}(z)}{z^{m-1}} \in \Lambda_{2m-2}$. It follows by orthogonality that

$$M[f] = \Big\langle \frac{p_{2m-2}(z)}{z^{m-1}}, \Phi_{2m}(z,\tau) \Big\rangle = 0, \tag{4.18}$$

which means

$$M[F] = M[F_n(z,\tau)] = \sum_{k=1}^{n} M[K_k^{(2m)}(z,\tau)] F\big(\zeta_k^{(2m)}(\tau)\big). \tag{4.19}$$

Next, assume that $F \in \Lambda_{-(2m+1),2m-1}$. Then

$$g(z) = F(z) - F_{2m+1}(z,\tau) \in \Lambda_{-(2m+1),2m-1}$$

and $g\big(\zeta_k^{(2m+1)}(\tau)\big) = 0$ for $k = 1,\dots,2m+1$. Hence, we may write

$$g(z) = \frac{s_{2m-1}(z)}{z^{m}}\ \Phi_{2m+1}(z,\tau), \tag{4.20}$$

where $s_{2m-1}(z) \in \Pi_{2m-1}$, and thus, $\frac{s_{2m-1}(z)}{z^{m}} \in \Lambda_{2m-1}$. It follows by orthogonality that

$$M[g] = \Big\langle \frac{s_{2m-1}(z)}{z^{m}}, \quad \Phi_{2m+1}(z,\tau) \Big\rangle = 0, \tag{4.21}$$

which means

$$M[F] = M[F_{2m+1}(z,\tau)] = \sum_{k=1}^{2m+1} M[K_k^{(2m+1)}(z,\tau)] F\big(\zeta_k^{(2m+1)}(\tau),\tau\big). \tag{4.22}$$

The result now follows by (4.13), (4.15), (4.16), and (4.22). □

Remark. The positivity of the weights $\lambda_k^{(n)}(\tau)$ in Theorem 3.2 can be established as follows. Since $(L_k^{(2m)}(z,\tau))^2 \in \Lambda_{-2m,2m-2}$ and $(L_k^{(2m+1)}(z,\tau))^2 \in \Lambda_{-2m,2m}$, the quadrature formula (3.16) may be applied to $(L_k^{(n)}(z,\tau))^2$. It then follows by (3.14)–(3.16) that

$$\lambda_j^{(n)}(\tau) = \sum_{k=1}^{n} \lambda_k^{(n)}(\tau)\big(L_k^{(n)}(\zeta_j^{(n)}(\tau),\tau)\big)^2 = M\big[\big(L_k^{(n)}(z,\tau)\big)^2\big] > 0. \tag{4.23}$$

This argument cannot be used to establish positivity of the weights $\kappa_k^{(n)}(\tau)$ since $(K_k^{(n)}(z,\tau))^2$ does not belong to the domain of validity of the appropriate quadrature formula.

The partial fraction decomposition for the pseudo-approximants

$$S_n(z,\tau) = -\sum_{k=1}^{n} \frac{\kappa_k^{(n)}(\tau)}{\zeta_k^{(n)}(\tau) - z} \tag{4.24}$$

is obtained as in the argument for formula (2.34).

We shall show that the weights $\kappa_k^{(n)}$ are positive when $\tau \in (-\infty, g_{2m})$ in case $n = 2m$ and when $\tau \in (g_{2m+1}, \infty)$ in case $n = 2m+1$. We shall use separation properties of zeros of $\Phi_n(z,\tau)$ and $\Psi_n(z,\tau)$ to establish this.

Proposition 4.3. *Assume that M is positive on $(0,\infty)$. Let τ be such that all the zeros of $\Phi_n(z,\tau)$ are positive. Then between two consecutive zeros of $\Phi_n(z,\tau)$ there is exactly one zero of $\Psi_n(z,\tau)$.*

Proof. Let τ be given, and assume that there is a common zero z_0 for $\Phi_n(z,\tau)$ and $\Psi_n(z,\tau)$, i.e., such that $\varphi_n(z_0) - \tau\varphi_{n-1}(z_0) = 0$ and $\psi_n(z_0) - \tau\psi_{n-1}(z_0) = 0$. Then, $\psi_n(z_0)\varphi_{n-1}(z_0) - \psi_{n-1}(z_0)\varphi_n(z_0) = 0$, which contradicts the determinant formulas (2.19)–(2.20). (Recall that z_0 would have to be positive.) Hence, $\Phi_n(z,\tau)$ and $\Psi_n(z,\tau)$ have no common zeros.

The zeros of $\Phi_n(z,\tau)$ and $\Psi_n(z,\tau)$ separate each other for $\tau = 0$ by the discussion following formula (2.32). Since $\Phi_n(z,\tau)$ and $\Psi_n(z,\tau)$ have no common zeros for τ in the interval in question, it follows from the continuity of the zeros with respect to τ that the zeros of $\Phi_n(z,\tau)$ and $\Psi_n(z,\tau)$ separate each other as stated. □

Proposition 4.4. *Assume that M is positive on $(0,\infty)$. Let τ be such that all the zeros of $\Phi_{2m}(z,\tau)$ and $\Phi_{2m+1}(z,\tau)$ are positive. Then,*

$$\lim_{x\to\infty} \Phi_{2m}(x,\tau) = \infty, \qquad \lim_{x\to\infty} \Psi_{2m}(x,\tau) = \infty, \tag{4.25}$$

$$\lim_{x\to\infty} \Phi_{2m+1}(x,\tau) = -\infty, \qquad \lim_{x\to\infty} \Psi_{2m+1}(x,\tau) = -\infty. \tag{4.26}$$

Proof. From (4.3), (4.5), (4.6), it follows that we may write

$$\Psi_{2m}(z,\tau) = \frac{r_{2m,-m}}{z^m} + \cdots + q_{2m,m}c_0 z^{m-1}, \tag{4.27}$$

$$\Psi_{2m+1}(z,\tau) = \frac{r_{2m+1,-(m+1)}}{z^{m+1}} + \cdots + (q_{2m+1,m} - \tau q_{2m,m})c_0 z^{m-1}. \tag{4.28}$$

The limiting values (4.25)–(4.26) now follow from (2.7)–(2.8), (2.29), (4.5)–(4.6). □

Theorem 4.5. *Assume that M is positive on $(0,\infty)$. Let τ be such that all the zeros of $\Phi_n(z,\tau)$ are positive. Then all the weights $\kappa_k^{(n)}(\tau)$ in the quadrature formula (4.14) are positive.*

Proof. It follows from (4.24) that

$$\kappa_j^{(n)}(\tau) = \lim_{z\to\zeta_j^{(n)}(\tau)} \frac{\Psi_n(z,\tau)}{\frac{\Phi_n(z,\tau)-\Phi_n(\zeta_j^{(n)}(\tau),\tau)}{z-\zeta_j^{(n)}(\tau)}} = \frac{\Psi_n(\zeta_j^{(n)}(\tau),\tau)}{\Phi_n'(\zeta_j^{(n)}(\tau),\tau)}. \tag{4.29}$$

By combining the separation result of Proposition 4.3 with the limiting behavior results given in Proposition 4.4, we find that $\Psi_n(z,\tau)/\Phi_n'(z,\tau)$ is positive for all the zeros ζ of $\Phi_n(z,\tau)$. □

Remark. It can be verified by the same kind of reasoning that when τ is such that one of the zeros ζ_k of $\Phi_n(z,\tau)$ is negative, then the weight $\kappa_k^{(n)}(\tau)$ at this zero is negative and the weights at the other zeros are positive.

Let τ be such that all the zeros of $\Phi_n(z,\tau)$ are positive. Let $\sigma_\tau^{(n)}$ denote the discrete measure with mass $\kappa_k^{(n)}(\tau)$ at the point $\zeta_k^{(n)}(\tau)$, $k=1,\ldots,n$. According to Theorem 4.2 and Theorem 4.5, $\sigma_\tau^{(n)}$ solves the truncated strong Stieltjes moment problem

$$c_k = \int_0^\infty \theta^k d\sigma_\tau^{(2m)}(\theta), \quad k=-(2m-1),\ldots,(2m-1) \text{ for } n=2m, \tag{4.30}$$

$$c_k = \int_0^\infty \theta^k d\sigma_\tau^{(2m+1)}(\theta), \quad k=-(2m+1),\ldots,2m-1 \text{ for } n=2m+1. \tag{4.31}$$

In addition,

$$S_n(z,\tau) = -\int_0^\infty \frac{d\sigma_\tau^{(n)}(\theta)}{\theta - z}. \tag{4.32}$$

As in the argument leading to the natural and the quasi-natural solutions of the moment problem, it can be seen that every subsequence $\{\sigma_{\tau_{n(k)}}^{(n(k))}\}$ contains a subsequence $\{\sigma_{\tau_{n(k(j))}}^{(n(k(j)))}\}$ converging to a measure σ which is a solution of the SSMP and such that the corresponding subsequence $\{S_{n(k(j))}(z,\tau_{n(k(j))})\}$ converges to $-F_\sigma(z)$, locally uniformly on $\mathbb{C}-\{0\}$.

Solutions that can be obtained in this way shall be called *pseudo-natural solutions*. We stress the fact that pseudo-natural solutions of the SHMP are defined only when M is positive on $(0,\infty)$ and they are automatically solutions of the SSMP.

For each z in U and each n, let the mapping $\tau \to \omega$ be defined by

$$\omega = \omega_n = -S_n(z,\tau). \tag{4.33}$$

This linear fractional transformation maps $\widehat{\mathbb{R}}$ onto a circle. We shall denote by $D_n(z)$ the open disk bounded by this circle, by $\partial D_n(z)$ the circle itself, and by $\overline{D}_n(z)$ the closed disk $D_n(z) \cup \partial D_n(z)$.

We define the linear fractional transformation $s_n(z,\tau)$ (for a fixed $z \in U$ and fixed n) by

$$s_{2m}(z,\tau) = \frac{-f_{2m}}{g_{2m} + h_{2m}z - \tau}, \quad m = 1, 2, \ldots, \tag{4.34}$$

$$s_{2m+1}(z,\tau) = \frac{-f_{2m+1}}{g_{2m+1} + h_{2m+1}/z - \tau}, \quad m = 1, 2, \ldots, \tag{4.35}$$

$$s_1(z,\tau) = \frac{-f_1/z}{g_1 + h_1/z - \tau}. \tag{4.36}$$

We then have

$$S_n(z,\tau) = -s_1 \circ s_2 \circ \cdots \circ s_n(z,\tau). \tag{4.37}$$

Theorem 4.6. *Assume that M is positive on $(0,\infty)$; then*

$$D_m(z) \subset D_n(z) \tag{4.38}$$

when $m > n$.

Proof. By taking into account the sign of f_n, g_n, and h_n given in (2.30)–(2.32), we verify that $s_n(z,\tau) \in U$ when $\tau \in U$, i.e., $s_n(z,U) \subset U$, for $n = 2, 3, \ldots$. Since $S_n(z,\tau) = S_{n-1}(z, s_n(z,\tau))$, this means that $S_n(z,U) = S_{n-1}(z, s_n(z,U)) \subset S_{n-1}(z,U)$. We also observe that $s_n(z,U)$ does not contain ∞. Hence, $D_n(z) = -S_n(z,U)$, so $D_{n+1}(z) \subset D_n(z)$. ☐

Let $\Gamma_{2m}(z)$ denote the arc of $\partial D_{2m}(z)$ corresponding to $\tau \in (-\infty, g_{2m})$, and let $\Gamma_{2m+1}(z)$ denote the arc of $\partial D_{2m+1}(z)$ corresponding to $\tau \in (g_{2m+1}, \infty)$. It follows from Theorem 4.1B and (4.30)–(4.32) that all the points of $\Gamma_{2m}(z)$ and $\Gamma_{2m+1}(z)$ are values of the Stieltjes transforms $F_\mu(z)$ where μ solves the truncated strong Stieltjes moment problem

$$c_k = \int_0^\infty \theta^k d\mu(\theta), \quad k = -(2m-1), \ldots, (2m-1), \tag{4.39}$$

and

$$c_k = \int_0^\infty \theta^k d\mu(\theta), \quad k = -(2m+1), \ldots, (2m-1), \tag{4.40}$$

respectively.

For later use, we introduce the set $G_n(z)$ as the segment of the disk $D_n(z)$ obtained as the convex hull of the arc $\Gamma_n(z)$.

We define

$$\overline{D}_\infty(z) = \bigcap_{n=1}^{\infty} \overline{D}_n(z), \tag{4.41}$$

and let $D_\infty(z)$ and $\partial D_\infty(z)$ denote the interior and the boundary of $\overline{D}_\infty(z)$. As for $\overline{\Delta}_\infty(z)$, it follows that $\overline{D}_\infty(z)$ is either a single point or a closed disk.

In Section 5, we shall discuss connections between subsets of $\overline{D}_\infty(z)$ and solutions of the SSMP.

5. Stieltjes transforms of solutions

In this section, we make the general assumption that M is regular and positive on $(-\infty, \infty)$. When solutions of the SSMP are involved, M is assumed to be positive on $(0, \infty)$.

We shall study more closely the relationship between Stieltjes transforms of solutions of moment problems and the systems of disks $\{\overline{\Delta}_n(z)\}$, $\{\overline{D}_n(z)\}$, circles $\{\partial\Delta_n(z)\}$, $\{\partial D_n(z)\}$, and segments $\{O_n(z)\}$, $\{G_n(z)\}$.

Before proceeding, we note from the definitions of $\partial\Delta_n(z)$ and $\partial D_n(z)$ that the points $-\psi_n(z)/\varphi_n(z)$ and $-\psi_{n-1}(z)/\varphi_{n-1}(z)$ belong to both of these circles (in fact, to $\Omega_n(z)$ and $\Gamma_n(z)$). Thus, the two circles intersect at these two points.

We recall from Section 3 that $\Omega_n(z)$ consists of the values of the Stieltjes transforms at z of the discrete measures determined by quasi-orthogonal Laurent polynomials of order n and support in $(0, \infty)$. Similarly (from Section 4), $\Gamma_n(z)$ consists of the values of the Stieltjes transforms at z determined by pseudo-orthogonal Laurent polynomials of order n and support in $(0, \infty)$.

Theorem 5.1. *Assume that M is positive on $(0, \infty)$.*

A. $$\Omega_n(z) = \partial\Delta_n(z) \cap \overline{D}_{n-1}(z). \tag{5.1}$$

B. $$\Gamma_n(z) = \partial D_n(z) \cap \overline{\Delta}_{n-1}(z). \tag{5.2}$$

Proof. According to earlier remarks, the point $-\psi_{n-1}(z)/\varphi_{n-1}(z)$ is a common point for the circles $\partial\Delta_{n-1}(z), \partial\Delta_n(z), \partial D_{n-1}(z)$, and $\partial D_n(z)$.

A. We recall that $-R_{2m}(z, \tau) \in \Omega_{2m}(z)$ if and only if $\tau \in (h_{2m}, \infty)$. The value $\tau = \infty$ gives the end point $-\psi_{n-1}(z)/\varphi_{n-1}(z)$. For the other end point of $\Omega_{2m}(z)$, we get

$$\begin{aligned}\varphi_{2m}(z, h_{2m}) &= \varphi_{2m}(z) - h_{2m} z \varphi_{2m-1}(z) \\ &= (g_{2m} + h_{2m} z)\varphi_{2m-1}(z) + f_{2m}\varphi_{2m-2}(z) - h_{2m} z \varphi_{2m-1}(z),\end{aligned}$$

by using the recurrence formula (2.10), hence,

$$\varphi_{2m}(z, h_{2m}) = g_{2m}\Big[\varphi_{2m-1}(z) - \Big(\frac{f_{2m}}{g_{2m}}\Big)\varphi_{2m-2}(z)\Big]. \tag{5.3}$$

Thus,

$$\varphi_{2m}(z, h_{2m}) = g_{2m}\Phi_{2m-1}(z, \tau) \quad \text{for } \tau = -\frac{f_{2m}}{g_{2m}}. \tag{5.4}$$

Similarly, we get

$$\psi_{2m}(z, h_{2m}) = g_{2m}\Psi_{2m-1}(z, \tau) \quad \text{for } \tau = -\frac{f_{2m}}{g_{2m}}. \tag{5.5}$$

This shows that

$$R_{2m}(z, h_{2m}) = S_{2m-1}(z, \tau) \quad \text{for } \tau = -\frac{f_{2m}}{g_{2m}}. \tag{5.6}$$

In the same way, by using the recurrence formula (2.11), we get

$$R_{2m+1}(z, h_{2m+1}) = S_{2m}(z, \tau) \quad \text{for } \tau = -\frac{f_{2m+1}}{g_{2m+1}}. \tag{5.7}$$

This shows that the second end point of $\Omega_n(z)$ lies on $\partial D_{n-1}(z)$. It follows that $\Omega_n(z) = \partial\Delta_n(z) \cap \overline{D}_{n-1}(z)$.

B. We recall that $-S_{2m}(z,\tau) \in \Gamma_{2m}(z)$ if and only if $\tau \in (-\infty, g_{2m})$. The value $\tau = -\infty$ gives the end point $-\psi_{n-1}(z)/\varphi_{n-1}(z)$. For the other end point of $\Gamma_{2m}(z)$, we get

$$\begin{aligned}\Phi_{2m}(z, g_{2m}) &= \varphi_{2m}(z) - g_{2m}\varphi_{2m-1}(z)\\ &= (g_{2m} + h_{2m}z)\varphi_{2m-1}(z) + f_{2m}\varphi_{2m-2}(z) - g_{2m}\varphi_{2m-1}(z),\end{aligned}$$

by using the recurrence formula (2.10), hence,

$$\Phi_{2m}(z, g_{2m}) = h_{2m}z\Big[\varphi_{2m-1}(z) - \Big(-\frac{f_{2m}}{h_{2m}}\Big)\cdot\frac{1}{z}\ \varphi_{2m-2}(z)\Big]. \tag{5.8}$$

Thus,

$$\Phi_{2m}(z, g_{2m}) = h_{2m}z\varphi_{2m-1}(z,\tau) \quad \text{for } \tau = \frac{f_{2m}}{h_{2m}}. \tag{5.9}$$

Similarly, we get

$$\Psi_{2m}(z, g_{2m}) = h_{2m}z\psi_{2m-1}(z,\tau) \quad \text{for } \tau = -\frac{f_{2m}}{h_{2m}}. \tag{5.10}$$

This means that

$$S_{2m}(z, g_{2m}) = R_{2m-1}(z,\tau) \quad \text{for } \tau = -\frac{f_{2m}}{h_{2m}}. \tag{5.11}$$

In the same way, by using the recurrence formula (2.11), we get

$$S_{2m+1}(z, g_{2m+1}) = R_{2m}(z,\tau) \quad \text{for } \tau = -\frac{f_{2m+1}}{h_{2m+1}}. \tag{5.12}$$

This shows that the second end point of $\Gamma_n(z)$ lies on $\partial\Delta_{n-1}(z)$. It follows that $\Gamma_n(z) = \partial D_n(z) \cap \overline{\Delta}_{n-1}(z)$. □

We introduce the notations

$$\Sigma_{p,q}(z) = \Big\{\ F_\mu(z)\ :\ \mu \text{ is a solution of the truncated SHMP } c_k = \int_{-\infty}^{\infty}\theta^k d\mu(\theta),\ k = p,\dots,q\ \Big\}, \tag{5.13}$$

$$\Sigma^+_{p,q}(z) = \Big\{\ F_\mu(z)\ :\ \mu \text{ is a solution of the truncated SSMP } c_k = \int_{0}^{\infty}\theta^k d\mu(\theta),\ k = p,\dots,q\ \Big\}. \tag{5.14}$$

Proposition 5.2. *The sets $\Sigma_{p,q}(z)$ and $\Sigma^+_{p,q}(z)$ are convex.*

Proof. Let μ and ν be two solutions of one of the two moment problems indicated. Then also $\sigma = t\mu + (1-t)\nu$, $0 \le t \le 1$, is a solution of the same problem. So the result follows from the fact that $F_\sigma(z) = tF_\mu(z) + (1-t)F_\nu(z)$. □

Theorem 5.3. *Assume that M is positive on* $(0,\infty)$.

A. $$\Sigma_{-2m,2m-2}(z) = \overline{\Delta}_{2m}(z). \tag{5.15}$$

B. $$\Sigma_{-2m,2m}(z) = \overline{\Delta}_{2m+1}(z). \tag{5.16}$$

Proof. It follows from (3.18)–(3.21) that $\partial\Delta_{2m}(z) \subset \Sigma_{-2m,2m-2}(z)$ and $\partial\Delta_{2m+1}(z) \subset \Sigma_{-2m,2m}(z)$. Hence, $\overline{\Delta}_{2m}(z) \subset \Sigma_{-2m,2m-2}(z)$ and $\overline{\Delta}_{2m+1}(z) \subset \Sigma_{-2m,2m}(z)$ by Proposition 5.2.

On the other hand, let $w \in \Sigma_{-2m,2m-2}(z)$ and $w \in \Sigma_{-2m,2m}(z)$. Then there exist measures μ which solve the truncated moment problems (3.18) and (3.19), respectively, such that $w = F_\mu(z)$. The inner products defined by μ and M coincide on $\Lambda_{-2m,2m-2}$ and $\Lambda_{-2m,2m}$. Let the function f_z be defined by $f_z(\theta) = \frac{1}{\theta - z}$. We find that

$$\langle f_z(\theta), f_z(\theta)\rangle = \int_{-\infty}^{\infty} \frac{d\mu(\theta)}{(\theta - z)(\theta - \bar{z})} = \frac{w - \bar{w}}{z - \bar{z}}$$

and $\langle f_z(\theta), \varphi_j(\theta)\rangle = \psi_j(z) + w\varphi_j(z)$ for $j = 0, \ldots, 2m-1$ and $j = 0, \ldots, 2m$, respectively. Bessel's inequality then takes the form (3.24), which shows that $w = \overline{\Delta}_{2m}(z)$ and $w \in \overline{\Delta}_{2m+1}(z)$, respectively. Thus $\Sigma_{-2m,2m-2}(z) \subset \overline{\Delta}_{2m}(z)$ and $\Sigma_{-2m,2m}(z) \subset \overline{\Delta}_{2m+1}(z)$. □

Proposition 5.4. *Let M be positive on $(0, \infty)$; then the following inclusions hold:*

A. $O_{2m}(z) \subset \Sigma^+_{-2m,2m-2}(z), \qquad O_{2m+1}(z) \subset \Sigma^+_{-2m,2m}(z), \qquad (5.17)$

B. $G_{2m}(z) \subset \Sigma^+_{-(2m-1),2m-1}(z), \qquad G_{2m+1}(z) \subset \Sigma^+_{-(2m+1),2m-1}(z). \qquad (5.18)$

Proof. It follows from the definition of $\Omega_n(z)$ and (3.31)–(3.32) that $\Omega_{2m}(z) \subset \Sigma^+_{-2m,2m-2}(z)$ and $\Omega_{2m+1}(z) \subset \Sigma^+_{-2m,2m}(z)$. Similarly, it follows from the definition of $\Gamma_n(z)$ and (4.39)–(4.40) that $\Gamma_{2m}(z) \subset \Sigma^+_{-(2m-1),2m-1}(z)$ and $\Gamma_{2m+1}(z) \subset \Sigma^+_{-(2m+1),2m-1}(z)$. The results now follow from the definitions of $O_n(z)$ and $G_n(z)$ together with Proposition 5.2. □

We introduce the notations

$$\Sigma_\infty(z) = \Big\{ F_\mu(z) \ : \ \mu \text{ is a solution of the SHMP} \quad c_k = \int_{-\infty}^{\infty} \theta^k d\mu(\theta) \quad k = 0, \pm 1, \pm 2, \ldots \Big\}, \tag{5.19}$$

$$\Sigma^+_\infty = \Big\{ F_\mu(z) \ : \ \mu \text{ is a solution of the SSMP} \quad c_k = \int_{0}^{\infty} \theta^k d\mu(\theta) \quad k = 0, \pm 1, \pm 2, \ldots \Big\}, \tag{5.20}$$

$$T_\infty(z) = \Big\{ F_\mu(z) \ : \ \mu \text{ is a quasi-natural solution of the SHMP} \quad c_k = \int_{-\infty}^{\infty} \theta^k d\mu(\theta), \quad k = 0, \pm 1, \pm 2, \ldots \Big\}, \tag{5.21}$$

$$\Omega_\infty(z) = \Big\{ F_\mu(z) \ : \ \mu \text{ is a quasi-natural solution of the SSMP} \quad c_k = \int_{0}^{\infty} \theta^k d\mu(\theta), \quad k = 0, \pm 1, \pm 2, \ldots \Big\}, \tag{5.22}$$

$$\Gamma_\infty(z) = \Big\{ F_\mu(z) \ : \ \mu \text{ is a pseudo-natural solution of the SSMP} \quad c_k = \int_{0}^{\infty} \theta^k d\mu(\theta), \quad k = 0, \pm 1, \pm 2, \ldots \Big\}. \tag{5.23}$$

Furthermore, we denote the convex hull of $\Omega_\infty(z)$ by $O_\infty(z)$ and the convex hull of $\Gamma_\infty(z)$ by $G_\infty(z)$.

Proposition 5.5. *The following equalities hold:*

A. $$\Sigma_\infty(z) = \bigcap_{-p,q=1}^{\infty} \Sigma_{p,q}(z), \tag{5.24}$$

B. $$\Sigma^+_\infty(z) = \bigcap_{-p,q=1}^{\infty} \Sigma^+_{p,q}(z). \tag{5.25}$$

Proof. That $\Sigma_\infty(z), \Sigma^+_\infty(z)$ are contained in the intersection sets is obvious. The reverse inclusions follow by use of Helly's theorems. □

Proposition 5.6. *The sets $\Sigma_\infty(z)$ and $\Sigma^+_\infty(z)$ are both convex.*

Proof. This follows as in the proof of Proposition 5.2. □

We shall make use of the following important result.

Theorem 5.7. *Assume that M is positive on $(0,\infty)$, then*

A. *The sequences*

$$\left\{-\frac{\psi_{2m}(z)}{\varphi_{2m}(z)}\right\} \quad \textit{and} \quad \left\{-\frac{\psi_{2m+1}(z)}{\varphi_{2m+1}(z)}\right\}$$

converge locally uniformly to functions $F^{(0)}(z)$ and $F^{(\infty)}(z)$ for $z \notin [0,\infty)$.

B. *The sequences of discrete measures $\{\nu^{(2m)}\}$ and $\{\nu^{(2m+1)}\}$ converge to natural solutions $N^{(0)}$ and $N^{(\infty)}$ of the* SSMP *and $F_{N^{(0)}}(z) = F^{(0)}(z)$, $F_{N^{(\infty)}}(z) = F^{(\infty)}(z)$.*

C. *For any solution μ of the* SSMP *and any $x \in (-\infty, 0)$, the following inequalities hold:*

$$F^{(0)}(x) \le F_\mu(x) \le F^{(\infty)}(x). \tag{5.26}$$

(This result is proved in [20] in a somewhat different way.)

Proof. From the recurrence relations (2.10)–(2.11), we obtain the formulas

$$\begin{aligned} &\varphi_{2m}(z)\psi_{2m-2}(z) - \psi_{2m}(z)\varphi_{2m-2}(z) \\ &\qquad = (f_{2m} + h_{2m}z)\big[\varphi_{2m-1}(z)\psi_{2m-2}(z) - \psi_{2m-1}(z)\varphi_{2m-2}(z)\big], \end{aligned} \tag{5.27}$$

$$\begin{aligned} &\varphi_{2m+1}(z)\psi_{2m-1}(z) - \psi_{2m+1}(z)\varphi_{2m-1}(z) \\ &\qquad = (g_{2m+1} + h_{2m+1}z^{-1})\big[\varphi_{2m}(z)\psi_{2m-1}(z) - \psi_{2m}(z)\varphi_{2m-1}(z)\big], \end{aligned} \tag{5.28}$$

and hence,

$$\frac{\psi_{2m}(z)}{\varphi_{2m}(z)} - \frac{\psi_{2m-2}(z)}{\varphi_{2m-2}(z)} = \frac{q_{2m-1,m-1}(g_{2m} + h_{2m}z)}{q_{2m-2,m-1}z\varphi_{2m}(z)\varphi_{2m-2}(z)}, \tag{5.29}$$

$$\frac{\psi_{2m+1}(z)}{\varphi_{2m+1}(z)} - \frac{\psi_{2m-1}(z)}{\varphi_{2m-1}(z)} = -\frac{q_{2m,-m}(g_{2m+1} + h_{2m+1}z^{-1})}{q_{2m-1,-m}z\varphi_{2m+1}(z)\varphi_{2m-1}(z)}, \tag{5.30}$$

by use of (2.19)–(2.20).

A. It follows from (2.7), (2.8), (2.29), and the fact that all the zeros of φ_n are positive for all n that $\varphi_n(x)$ and $\varphi_{n+2}(x)$ have opposite sign for $x \in (-\infty, 0]$, $n = 0, 1, 2, \ldots$. It then follows from (2.15)–(2.18) that

$$\frac{\psi_{2m}(x)}{\varphi_{2m}(x)} - \frac{\psi_{2m-2}(x)}{\varphi_{2m-2}(x)} < 0 \quad \text{for } x \in (-\infty, 0), \tag{5.31}$$

$$\frac{\psi_{2m+1}(x)}{\varphi_{2m+1}(x)} - \frac{\psi_{2m-1}(x)}{\varphi_{2m-1}(x)} > 0 \quad \text{for } x \in (-\infty, 0). \tag{5.32}$$

Furthermore, it follows from (2.7)–(2.8) and (4.27)–(4.28) that $\psi_n(x)/\varphi_n(x) < 0$ for $x \in (-\infty, 0]$ for all n. We also obtain from (2.20) that

$$\frac{\psi_{2m}(x)}{\varphi_{2m}(x)} - \frac{\psi_{2m+1}(x)}{\varphi_{2m+1}(x)} > 0 \quad \text{for } x \in (-\infty, 0). \tag{5.33}$$

Thus, we have for every m:

$$\frac{\psi_{2m-1}(x)}{\varphi_{2m-1}(x)} < \frac{\psi_{2m+1}(x)}{\varphi_{2m+1}(x)} < \frac{\psi_{2m}(x)}{\varphi_{2m}(x)} < \frac{\psi_{2m-2}(x)}{\varphi_{2m-2}(x)} < 0 \tag{5.34}$$

for $x \in (-\infty, 0]$. It follows that the sequences $\{\psi_{2m}(x)/\varphi_{2m}(x)\}$ and $\{\psi_{2m+1}(x)/\varphi_{2m+1}(x)\}$ converge to finite negative values for $x \in (-\infty, 0)$. From (2.34), (2.38) and the fact that all zeros $\xi_k^{(n)}$ are positive, we conclude from local boundedness by a normal family argument that $\{\psi_{2m}(z)/\varphi_{2m}(z)\}$ and $\{\psi_{2m+1}(z)/\varphi_{2m+1}(z)\}$ converge locally uniformly on $\mathbb{C} - [0, \infty)$.

B. This follows by standard arguments from A and (2.37).

C. Let μ be an arbitrary solution of the SSMP. It follows by orthogonality properties that

$$\int_0^\infty \left[\frac{\varphi_{2m}(\theta)^2}{\varphi_{2m}(z)^2} - \frac{\varphi_{2m}(\theta)}{\varphi_{2m}(z)}\right] \frac{d\mu(\theta)}{\theta - z} = 0 \tag{5.35}$$

and

$$\int_0^\infty \left[\frac{\theta\varphi_{2m+1}(\theta)^2}{z\varphi_{2m+1}(z)^2} - \frac{\varphi_{2m+1}(\theta)}{\varphi_{2m+1}(z)}\right] \frac{d\mu(\theta)}{\theta - z} = 0. \tag{5.36}$$

Hence, it follows from (2.9) that

$$\frac{\psi_{2m}(z)}{\varphi_{2m}(z)} + \int_0^\infty \frac{d\mu(\theta)}{\theta - z} = \frac{1}{\varphi_{2m}(z)^2} \int_0^\infty \frac{\varphi_{2m}(\theta)^2}{\theta - z}\, d\mu(\theta), \tag{5.37}$$

$$\frac{\psi_{2m+1}(z)}{\varphi_{2m+1}(z)} + \int_0^\infty \frac{d\mu(\theta)}{\theta - z} = \frac{1}{z\varphi_{2m+1}(z)^2} \int_0^\infty \frac{\theta\varphi_{2m+1}(\theta)^2}{\theta - z}\, d\mu(\theta). \tag{5.38}$$

Consequently,

$$-\frac{\psi_{2m}(x)}{\varphi_{2m}(x)} < \int_0^\infty \frac{d\mu(\theta)}{\theta - x} < -\frac{\psi_{2m+1}(x)}{\varphi_{2m+1}(x)} \tag{5.39}$$

for $x \in (-\infty, 0)$. From the result under A we conclude that

$$F^{(0)}(x) \le \int_0^\infty \frac{d\mu(\theta)}{\theta - x} \le F^{(\infty)}(x) \tag{5.40}$$

for $x \in (-\infty, 0)$. See also [11, 19, 20]. □

Theorem 5.8. *Assume that M is positive on $(0,\infty)$; then*

A. $$T_\infty(z) = \partial\Delta_\infty(z), \tag{5.41}$$

B. $$\Omega_\infty(z) = \partial\Delta_\infty(z) \cap \overline{D}_\infty(z), \tag{5.42}$$

C. $$\Gamma_\infty(z) = \partial D_\infty(z) \cap \overline{\Delta}_\infty(z). \tag{5.43}$$

Proof. A. We see from the discussion following formula (3.26) that $T_\infty(z)$ consists of all limit points of sequences $\{-R_n(z,\tau_n)\}$, and since $-R_n(z,\tau) \in \partial\Delta_n(z)$, it follows that these points belong to $\partial\Delta_\infty(z)$. Thus $T_\infty(z) \subset \partial\Delta_\infty(z)$.

On the other hand, let $w_\infty \in \partial\Delta_\infty(z)$; then there exists a sequence $\{w_n\}$ converging to w_∞ with $w_n \in \partial\Delta_n$. For each n there is a τ_n such that $w_n = -R_n(z,\tau_n)$. According to Helly's theorems, there exists a subsequence $\{n(k)\}$ such that $\{\nu^{(n(k))}_{\tau_{n(k)}}\}$ converges to a quasi-natural solution ν of the SHMP and such that $\{F_{\nu^{(n(k))}}(z)\}$ converges to $F_\nu(z)$. Since $w_n = -R_n(z,\tau_n) = F_{\nu^{(n(k))}}(z)$, we conclude that $w_\infty \in T_\infty(z)$. Thus $\partial\Delta_\infty(z) \subset T_\infty(z)$.

B,C. According to Theorem 5.7A, the subarcs of $\Omega_{2m}(z)$ corresponding to $\tau \geq 0$ converge to a subarc $A_\infty(z)$ of $\partial\Delta_\infty(z)$ with end points $F^{(0)}(z)$ and $F^{(\infty)}(z)$. Similarly, the subarcs of $\Omega_{2m+1}(z)$ corresponding to $\tau \leq 0$ converge to the same subarc $A_\infty(z)$ (because $-R_{2m}(z,\tau)$ and $-R_{2m+1}(z,\tau)$ traverse $\partial\Delta_{2m}(z)$ and $\partial\Delta_{2m+1}(z)$ in opposite directions with increasing τ). Also, according to Theorem 5.7A, the subarcs of $\Gamma_{2m}(z)$ corresponding to $\tau \leq 0$ converge to a subarc $B_\infty(z)$ of $\partial D_\infty(z)$ with end points $F^{(0)}(z)$ and $F^{(\infty)}(z)$, and similarly, the subarcs of $\Gamma_{2m+1}(z)$ corresponding to $\tau \geq 0$ converge to the same subarc $B_\infty(z)$ of $\partial D_\infty(z)$. By considering the directions traveled by $R_{2m}(z,\tau)$ and $S_{2m}(z,\tau)$, for example, with increasing τ, we find that $A_\infty(z) = \partial\Delta_\infty(z) \cap \overline{D}_\infty(z)$ and $B_\infty(z) = \partial D_\infty(z) \cap \overline{\Delta}_\infty(z)$. It follows from Theorem 5.1 that the sequences of whole arcs $\{\Omega_n(z)\}$ and $\{\Gamma_n(z)\}$ converge to the same limiting arcs $A_\infty(z) = \partial\Delta_\infty(z) \cap \overline{D}_\infty(z)$ and $\partial D_\infty(z) \cap \overline{\Delta}_\infty(z)$, respectively.

An argument similar to the second part of the proof of A shows that $A_\infty(z) \subset \Omega_\infty(z)$ and $B_\infty(z) \subset \Gamma_\infty(z)$. Since there are no measures $\sigma^{(n)}_\tau$ with support not contained in $(0,\infty)$, we conclude as in the first part of the proof of A that $\Gamma_\infty(z) \subset B_\infty(z)$, and thus, $\Gamma_\infty(z) = B_\infty(z) = \partial D_\infty(z) \cap \overline{\Delta}_\infty(z)$.

It remains to show that $\Omega_\infty(z) \subset A_\infty(z)$, i.e., that a sequence of measures $\{\nu^{(n(k))}_{\tau_{n(k)}}\}$ where the support of each measure is not contained in $(0,\infty)$, cannot give rise to a quasi-natural solution of the SSMP.

It follows from [28, Theorems 3.1,3.2] (see also Section 6) that there is a continuous one-to-one correspondence between the points of $\partial\Delta_\infty(z)$ and the quasi-natural solutions of the SHMP. From Theorem 5.7C, it then follows that there can be no quasi-natural solution μ of the SSMP with Stieltjes transform outside the arc $A_\infty(z)$ bounded by $F^{(0)}(z)$ and $F^{(\infty)}(z)$. (Otherwise, some points of $A_\infty(z)$ would be the Stieltjes transform of at least two quasi-natural solutions having different values at an $x \in (-\infty,0)$, due to continuity of the Stieltjes transforms w.r.t. z and the parametrization of $\partial\Delta_\infty(z)$.) We thus may conclude that $\Omega_\infty(z) \subset A_\infty(z)$. ☐

Corollary 5.9. *The following inclusion holds:*

$$\overline{\Delta}_\infty(z) \cap \overline{D}_\infty(z) \subset \Sigma^+_\infty(z). \tag{5.44}$$

Proof. This follows immediately from Proposition 5.6, Theorem 5.8, and the fact that the convex hull $O_\infty(z) \cup G_\infty(z)$ of $\Omega_\infty(z) \cup \Gamma_\infty(z)$ is $\overline{\Delta}_\infty(z) \cap \overline{D}_\infty(z)$. ☐

Theorem 5.10. *Assume that M is positive on $(0,\infty)$ and that the associated* SHMP *is indeterminate. Then the associated* SSMP *is determinate if and only if the intersection $\overline{\Delta}_\infty(z) \cap \overline{D}_\infty(z)$ reduces to a single point.*

Proof. It follows from Theorem 5.7C that the SSMP is determinate if and only if the two natural solutions $N^{(0)}$ and $N^{(\infty)}$ coincide, which by Theorem 5.8 is the case if and only if $\overline{\Delta}_\infty(z) \cap \overline{D}_\infty(z)$ reduces to a single point. □

6. Nevanlinna parametrization of solutions

The assumptions on M in this section are as in Section 5.

Let x_0 be an arbitrary fixed point on $\mathbb{R} - \{0\}$. We define functions $\alpha_n, \beta_n, \gamma_n, \delta_n$ (depending on the parameter x_0) by

$$\alpha_n(z) = (z - x_0) \sum_{j=1}^{n-1} \psi_j(x_0)\psi_j(z), \tag{6.1}$$

$$\beta_n(z) = -1 + (z - x_0) \sum_{j=1}^{n-1} \psi_j(x_0)\varphi_j(z), \tag{6.2}$$

$$\gamma_n(z) = 1 + (z - x_0) \sum_{j=1}^{n-1} \varphi_j(x_0)\psi_j(z), \tag{6.3}$$

$$\delta_n(z) = (z - x_0) \sum_{j=0}^{n-1} \varphi_j(x_0)\varphi_j(z). \tag{6.4}$$

Since the coefficients in $\varphi_j(z)$ and $\psi_j(z)$ are real, it follows that $\alpha_n(z), \beta_n(z), \gamma_n(z)$, $\delta_n(z)$ are real for real z.

Proposition 6.1. *Assume that M is regular and positive on $(-\infty,\infty)$.*

A. *$\beta_{2m}(z), \delta_{2m}(z)$ are linearly independent quasi-orthogonal Laurent polynomials of order $2m$, and $\alpha_{2m}(z), \gamma_{2m}(z)$ are linearly independent associated quasi-orthogonal Laurent polynomials of order $2m$.*

B. *$z^{-1}\beta_{2m+1}(z)$, $z^{-1}\delta_{2m+1}(z)$ are linearly independent quasi-orthogonal Laurent polynomials of order $2m+1$, and $z^{-1}\alpha_{2m+1}(z)$, $z^{-1}\gamma_{2m+1}(z)$ are linearly independent associated quasi-orthogonal Laurent polynomials of order $2m+1$.*

C. *The equality*

$$\alpha_n(z)\delta_n(z) - \beta_n(z)\gamma_n(z) = 1 \tag{6.5}$$

holds for all n.

Proof. It follows from (5.1)–(5.4) and the Christoffel–Darboux formulas (2.22)–(2.23) that $\beta_{2m}(z)$ and $\delta_{2m}(z)$ can be expressed linearly in terms of $\varphi_{2m}(z)$ and $z\varphi_{2m-1}(z)$, $\alpha_{2m}(z)$ and $\gamma_{2m}(z)$ in terms of $\psi_{2m}(z)$ and $z\psi_{2m-1}(z)$, $z^{-1}\beta_{2m+1}(z)$ and $z^{-1}\delta_{2m+1}(z)$ in terms of $\varphi_{2m+1}(z)$ and $z^{-1}\varphi_{2m}(z)$, and $z^{-1}\alpha_{2m+1}(z)$ and $z^{-1}\gamma_{2m+1}(z)$ in terms of $\psi_{2m+1}(z)$ and $z^{-1}\psi_{2m}(z)$. On the other hand, $\varphi_{2m}(z,\tau)$ can be expressed linearly in terms of $\beta_{2m}(z)$ and $\delta_{2m}(z), \psi_{2m}(z,\tau)$ in terms of $\alpha_{2m}(z)$ and $\gamma_{2m}(z)$, $\varphi_{2m+1}(z,\tau)$ in terms of $z^{-1}\beta_{2m+1}(z)$ and $z^{-1}\delta_{2m+1}(z)$, and $\psi_{2m+1}(z,\tau)$ in terms of $z^{-1}\alpha_{2m+1}(z)$ and $z^{-1}\gamma_{2m+1}(z)$. Formula (6.5) is obtained by substituting for $\varphi_n(z)$, $\psi_n(z)$, $\varphi_{n-1}(z)$, $\psi_{n-1}(z)$ expressed in terms of $\alpha_n(z)$, $\beta_n(z)$, $\gamma_n(z)$, $\delta_n(z)$ in the determinant formulas (2.19)–(2.20). See [28, Section 3]. □

It follows from (6.5) that for an arbitrary complex number t, $\alpha_n(z)t - \gamma_n(z)$ and $\beta_n(z)t - \delta_n(z)$ have no common zeros.

By substituting $\varphi_n(z)$, $\varphi_{n-1}(z)$, $\psi_n(z)$, $\psi_{n-1}(z)$ expressed in terms of $\alpha_n(z)$, $\beta_n(z)$, $\gamma_n(z)$, $\delta_n(z)$ in the definitions (3.3)–(3.4), (3.6), expressions for the quasi-approximants $R_n(z,\tau)$ are given by

$$R_{2m}(z,\tau) = \frac{\alpha_{2m}(z)t_{2m}(\tau) - \gamma_{2m}(z)}{\beta_{2m}(z)t_{2m}(\tau) - \delta_{2m}(z)}, \tag{6.6}$$

$$R_{2m+1}(z,\tau) = \frac{\alpha_{2m+1}(z)t_{2m+1}(\tau) - \gamma_{2m+1}(z)}{\beta_{2m+1}(z)t_{2m+1}(\tau) - \delta_{2m+1}(z)}, \tag{6.7}$$

where

$$t_{2m}(\tau) = \frac{\varphi_{2m}(x_0) - \tau x_0 \varphi_{2m-1}(x_0)}{\psi_{2m}(x_0) - \tau x_0 \psi_{2m-1}(x_0)}, \tag{6.8}$$

$$t_{2m+1}(\tau) = \frac{x_0\varphi_{2m+1}(x_0) - \tau \varphi_{2m}(x_0)}{x_0\psi_{2m+1}(x_0) - \tau \psi_{2m}(x_0)}. \tag{6.9}$$

We note that the linear fractional transformation $\tau \to t = t_n(\tau)$ maps $\widehat{\mathbb{R}}$ biuniquely onto $\widehat{\mathbb{R}}$.

We define

$$T_n(z,t) = \frac{\alpha_n(z)t - \gamma_n(z)}{\beta_n(z)t - \delta_n(z)}. \tag{6.10}$$

We then may write

$$T_n(z,t) = R_n(z,\tau), \tag{6.11}$$

where t is obtained from τ by the transformations (6.8)–(6.9).

The mapping $t \to -T_n(z,t)$ maps U (i.e., $t \in U$) onto $\Delta_n(z)$ (i.e., $w = -T_n(z,t)$) for every $z \in U$ (cf. [28]).

We denote by $\mu_t^{(n)}$ the discrete measure determined by the quadrature formula associated with $\beta_n(z)t - \delta_n(z)$. Then

$$\mu_t^{(n)} = \nu_\tau^{(n)}, \quad \text{where} \quad t = t_n(\tau). \tag{6.12}$$

It follows by (3.17), (3.20), (6.10)–(6.11) that

$$F_\mu(z) = -\frac{\alpha_n(z)t - \gamma_n(z)}{\beta_n(z)t - \delta_n(z)} \quad \text{when} \quad \mu = \mu_t^{(n)}. \tag{6.13}$$

We shall use the following convergence result for the functions $\alpha_n, \beta_n, \gamma_n, \delta_n$.

Theorem 6.2. *Assume that M is regular and positive on $(-\infty,\infty)$, and assume that the associated* SHMP *is indeterminate. Then the functions $\alpha_n(z), \beta_n(z), \gamma_n(z), \delta_n(z)$ converge locally uniformly on $\mathbb{C} - \{0\}$ to analytic functions $\alpha(z), \beta(z), \gamma(z), \delta(z)$ given by*

$$\alpha(z) = (z - x_0)\sum_{j=1}^{\infty} \psi_j(x_0)\psi_j(z), \tag{6.14}$$

$$\beta(z) = -1 + (z - x_0)\sum_{j=1}^{\infty} \psi_j(x_0)\varphi_j(z), \tag{6.15}$$

$$\gamma(z) = 1 + (z - x_0) \sum_{j=1}^{\infty} \varphi_j(x_0)\psi_j(z), \tag{6.16}$$

$$\delta(z) = (z - x_0) \sum_{j=0}^{\infty} \varphi_j(x_0)\varphi_j(z). \tag{6.17}$$

Proof. See [28]. □

We note that the mapping

$$t \to -\frac{\alpha(z)t - \gamma(z)}{\beta(z)t - \delta(z)}$$

maps U onto $\Delta_\infty(z)$ and $\widehat{\mathbb{R}}$ onto $\partial\Delta_\infty(z)$ for every $z \in U$.

It follows easily from (6.13) and Theorem 6.2 that for each $t \in \widehat{\mathbb{R}}$, the discrete measures $\mu_t^{(n)}$ converge to a (quasi-natural) solution μ_t of the SHMP such that

$$\int_{-\infty}^{\infty} \frac{d\mu_t(\theta)}{\theta - z} = -\frac{\alpha(z)t - \gamma(z)}{\beta(z)t - \delta(z)}. \tag{6.18}$$

A straightforward argument shows that all quasi-natural solutions of the SHMP are of the form μ_t (cf. [28]).

Theorem 6.3. *Assume that M is regular and positive on $(-\infty, \infty)$. The mapping $t \to \mu_t$ establishes a one-to-one correspondence between $\widehat{\mathbb{R}}$ and the family of all quasi-natural solutions of the SHMP.*

Proof. Different values of t give different functions $-(\alpha(z)t - \gamma(z))/(\beta(z)t - \delta(z))$. Hence, the mapping $t \to \mu_t$ is clearly one-to-one from $\widehat{\mathbb{R}}$ onto the family of all solutions of the form μ_t and these constitute exactly all the quasi-natural solutions. □

Recall that the function $f(z)$ is a Nevanlinna function if it is analytic in U and maps U into $\overline{U}$. The extended Nevanlinna class $\mathcal{N}^*$ consists of all Nevanlinna functions and the constant ∞.

The Nevanlinna parametrization for the solutions of the SHMP can be stated as follows.

Theorem 6.4. *Assume that M is regular and positive on $(-\infty, \infty)$ and that the associated* SHMP *is indeterminate. Then there exists a one-to-one correspondence between the functions φ in the extended Nevanlinna class $\mathcal{N}^*$ and the measures μ in the class $\mathcal{M}$ of solutions of the* SHMP*. The correspondence is given by*

$$\int_{-\infty}^{\infty} \frac{d\mu(\theta)}{\theta - z} = -\frac{\alpha(z)\varphi(z) - \gamma(z)}{\beta(z)\varphi(z) - \delta(z)}. \tag{6.19}$$

Proof. See [28, Section 4]. □

A function in $\mathcal{N}^*$ is either a constant in $\widehat{\mathbb{R}}$ or a function that maps U into U. In the latter case, $F_\mu(z) \in \Delta_\infty(z)$ for all $z \in U$ (cf. the equation $\frac{w-\bar{w}}{z-\bar{z}} = \Sigma_{j=0}^{n-1}|T_j(z, w, 1)|^2$ for $\partial\Delta_\infty(z)$). Thus for any solution μ of the SHMP, either $F_\mu(z) \in \Delta_\infty(z)$ for all $z \in U$ or $F_\mu(z) \in \partial\Delta_\infty(z)$ for all $z \in U$. The latter situation occurs if and only if the function φ is a constant in $\widehat{\mathbb{R}}$, i.e., if and only if the solution μ is quasi-natural.

A solution μ of the SHMP with the property $F_\mu(z) \in \partial\Delta_\infty(z)$ for all $z \in U$ is called *N-extremal.* Thus the N-extremal solutions are exactly the quasi-natural solutions.

A solution of the SHMP is called a *canonical solution* if the corresponding function φ in $\mathcal{N}^*$ is of the form $\varphi(z) = P(z)/Q(z)$, where P and Q are polynomials with real coefficients. Note that all constants in $\widehat{\mathbb{R}}$ are among these solutions. The canonical solution is said to be of order r if $\max(\deg P, \deg Q) = r$, where P and Q have no common factors. For the sake of explicitness, we restate Theorem 6.3 as follows.

Theorem 6.5. *Assume that M is regular and positive on $(-\infty, \infty)$ and that the associated* SHMP *is indeterminate. Then the quasi-natural solutions are exactly the canonical solutions of order* 0.

Proof. Immediate from Theorem 6.3 and the definition of canonical solutions. □

Theorem 6.6. *Assume that M is positive on $(0, \infty)$ and that the associated* SHMP *is indeterminate. Assume that $x_0 \in (-\infty, 0)$. Then the sequences*

$$\left\{\frac{\varphi_{2m}(z)}{\psi_{2m}(x_0)}\right\}, \ \left\{\frac{\psi_{2m}(z)}{\psi_{2m}(x_0)}\right\}, \ \left\{\frac{\varphi_{2m+1}(z)}{\psi_{2m+1}(x_0)}\right\}, \ \left\{\frac{\psi_{2m+1}(z)}{\psi_{2m+1}(x_0)}\right\},$$

converge locally uniformly in $\mathbb{C} - \{0\}$ to analytic functions $f^{(0)}(z)$, $g^{(0)}(z)$, $f^{(\infty)}(z)$, $g^{(\infty)}(z)$, respectively.

Proof. φ_n, ψ_n can be expressed in terms of $\alpha_n, \beta_n, \gamma_n, \delta_n$ as follows (see [28, Section 3]):

$$\varphi_{2m}(z) = \psi_{2m}(x_0)\Big[\delta_{2m}(z) - \frac{\varphi_{2m}(x_0)}{\psi_{2m}(x_0)}\ \beta_{2m}(z)\Big], \tag{6.20}$$

$$\psi_{2m}(z) = \psi_{2m}(x_0)\Big[\gamma_{2m}(z) - \frac{\varphi_{2m}(x_0)}{\psi_{2m}(x_0)}\ \alpha_{2m}(z)\Big], \tag{6.21}$$

$$\varphi_{2m+1}(z) = \frac{x_0}{z}\ \psi_{2m+1}(x_0)\Big[\delta_{2m+1}(z) - \frac{\varphi_{2m+1}(x_0)}{\psi_{2m+1}(x_0)}\ \beta_{2m+1}(z)\Big], \tag{6.22}$$

$$\psi_{2m+1}(z) = \frac{x_0}{z}\ \psi_{2m+1}(x_0)\Big[\gamma_{2m+1}(z) - \frac{\varphi_{2m+1}(x_0)}{\psi_{2m+1}(x_0)}\ \alpha_{2m}(z)\Big]. \tag{6.23}$$

According to Theorem 5.7, $\{\varphi_{2m}(x_0)/\psi_{2m}(x_0)\}$ and $\{\varphi_{2m+1}(x_0)/\psi_{2m+1}(x_0)\}$ converge to values in $\widehat{\mathbb{R}} - \{0\}$. According to Theorem 6.2, $\alpha_n(z)$, $\beta_n(z)$, $\gamma_n(z)$, $\delta_n(z)$ converge locally uniformly in $\mathbb{C}$. Hence the result follows from (6.20)–(6.23). □

Proposition 6.7. *The pseudo-approximants $S_n(z, \tau)$ may be expressed as*

$$S_{2m}(z, \tau) = \frac{\alpha_{2m}(z)t_{2m}(\tau, z) - \gamma_{2m}(z)}{\beta_{2m}(z)t_{2m}(\tau, z) - \delta_{2m}(z)}, \tag{6.24}$$

$$S_{2m+1}(z, \tau) = \frac{\alpha_{2m+1}(z)t_{2m+1}(\tau, z) - \gamma_{2m+1}(z)}{\beta_{2m+1}(z)t_{2m+1}(\tau, z) - \delta_{2m+1}(z)}, \tag{6.25}$$

where

$$t_{2m}(\tau, z) = \frac{z\varphi_{2m}(x_0) - \tau x_0\varphi_{2m-1}(x_0)}{z\psi_{2m}(x_0) - \tau x_0\psi_{2m-1}(x_0)}, \tag{6.26}$$

$$t_{2m+1}(\tau, z) = \frac{x_0\varphi_{2m+1}(x_0) - \tau z\varphi_{2m}(x_0)}{x_0\psi_{2m+1}(x_0) - \tau z\psi_{2m}(x_0)}. \tag{6.27}$$

Proof. This follows from (3.1)–(3.6), (4.1)–(4.4), and (6.6)–(6.9). □

We recall that $\sigma_\tau^{(n)}$ is the discrete measure determined by the quadrature formula associated with the pseudo-orthogonal Laurent polynomial $\Phi_n(z,\tau)$. We shall write

$$\rho_t^{(n)} = \sigma_\tau^{(n)}, \qquad t = t_n(\tau), \tag{6.28}$$

where $t_n(\tau)$ is given by (6.8)–(6.9). We then may write

$$\int_0^\infty \frac{d\rho_t^{(n)}(\theta)}{\theta - z} = -\frac{\alpha_n(z)t_n(z,\tau) - \gamma_n(z)}{\beta_n(z)t_n(z,\tau) - \delta_n(z)}. \tag{6.29}$$

It follows from the proof of Theorem 6.6 that $\{t_{2m}(0)\}$ $(= \{t_{2m+1}(\infty)\})$ and $\{t_{2m+1}(0)\}$ $(= \{t_{2m}(\infty)\})$ converge to values $t^{(0)}$ and $t^{(\infty)}$. By using Helly's theorems and the fact that $\{\alpha_n(z)\}$, $\{\beta_n(z)\}$, $\{\gamma_n(z)\}$, $\{\delta_n(z)\}$ converge, we conclude that for every t in the interval determined by $t^{(0)}$ and $t^{(\infty)}$, $\{\rho_t^{(n)}\}$ converges to a (pseudo-natural) solution of the SSMP. These solutions have value of the Stieltjes transform exactly on the arc $\Gamma_\infty = \partial D_\infty(z) \cap \overline{\Delta}_\infty(z)$.

Theorem 6.8. *Assume that M is positive on $(0,\infty)$ and that the associated* SHMP *is indeterminate. Then the pseudo-natural solutions of the* SSMP *are canonical solutions of order* 1.

Proof. Let t be fixed, and let τ_n be the corresponding τ-value determined by the formulas (6.8)–(6.9). Since $\int_0^\infty (\theta - z)^{-1} d\rho_t^{(n)}(\theta)$ converges to $\int_0^\infty (\theta - z)^{-1} d\rho_t(\theta)$ and $\{\alpha_n(z)\}$, $\{\beta_n(z)\}$, $\{\gamma_n(z)\}$, $\{\delta_n(z)\}$ converge to $\alpha(z)$, $\beta(z)$, $\gamma(z)$, $\delta(z)$, it follows from formula (6.29) that $\{t_n(z,\tau_n)\}$ converges.

Theorem 6.6 and formulas (6.26)–(6.27) then show that $t_n(z,\tau_n)$ converges to a function $t(z)$ of the form

$$t(z) = \frac{az + b}{cz + d}. \tag{6.30}$$

Since

$$\int_0^\infty \frac{d\rho_t(\theta)}{\theta - z} = -\frac{\alpha(z)t(z) - \gamma(z)}{\beta(z)t(z) - \delta(z)}, \tag{6.31}$$

we conclude that $t(z)$ is the function of the extended Nevanlinna class $\mathcal{N}^*$ corresponding to ρ_t, see Theorem 6.4. Thus ρ_t is a canonical solution of order 1. □

7. Structure of solutions

The quasi-orthogonal and pseudo-orthogonal solutions of an indeterminate SHMP are essentially discrete measures (the origin belongs to the spectrum as a non-isolated point). This follows as a special case of a general structure theorem for canonical solutions. For the sake of completeness, we state this theorem. A proof can be found in [28, Section 5].

Theorem 7.1. *Assume that M is regular and positive on $(-\infty,\infty)$ and that the associated* SHMP *is indeterminate. Let μ be a canonical solution, with $\varphi(z) = P(z)/Q(z)$ in the Nevanlinna parametrization, P and Q without common factors. Then the spectrum of μ consists of the zeros $\{z_k\}$ of the function $\beta(z)P(z) - \delta(z)Q(z)$ and, in addition, the origin. The measure μ has a mass of magnitude $-\rho_k$ at z_k, where ρ_k is the residue of the Stieltjes transform $F_\mu(z)$ at z_k. The origin is a point of continuity.*

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