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Square-summable stability in parabolic Volterra difference equations

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Abstract. We consider some linear and nonlinear parabolic Volterra difference equations of the forms

$$\Delta_2\left(u_{m,n} - \sum_{j=1}^{\infty} q_j u_{m,n-r_j}\right) + \sum_{i=1}^{\infty} p_i u_{m,n-k_i} = R\Delta_1^2 u_{m-1,n+1}$$

and

$$\Delta_2\left[h(u_{m,n}) - \sum_{j=1}^{\infty} q_j g(u_{m,n-r_j})\right] + \sum_{i=1}^{\infty} p_i f(u_{m,n-k_i}) = R\Delta_1^2 F(u_{m-1,n+1})$$

for $m = 0, 1, \ldots, M-1$ and $n = 0, 1, \ldots,$ and we obtain several sufficient conditions for the square-summable stability and ϕ-square-summable stability of the zero solution.

1. Introduction

Consider the linear parabolic Volterra difference equations of neutral type

$$\Delta_2\left(u_{m,n} - \sum_{j=1}^{\infty} q_j u_{m,n-r_j}\right) + \sum_{i=1}^{\infty} p_i u_{m,n-k_i} = R\Delta_1^2 u_{m-1,n+1} \tag{1}$$
$$\text{for} \quad m = 0, 1, \ldots, M-1 \quad \text{and} \quad n = 0, 1, \ldots,$$

with homogeneous von Neumann boundary conditions (NBC):

$$\Delta_1 u_{0,n} = \Delta_1 u_{M,n} = 0 \quad \text{for} \quad n = 0, 1, \ldots, \tag{2}$$

and initial conditions (IC):

$$u_{m,i} = \mu_{m,i} \quad \text{for} \quad m = 0, 1, \ldots, M-1 \quad \text{and} \quad i = \ldots, -2, -1, 0, \tag{3}$$

and nonlinear parabolic Volterra difference equations of neutral type

$$\Delta_2\left[h(u_{m,n}) - \sum_{j=1}^{\infty} q_j g(u_{m,n-r_j})\right] + \sum_{i=1}^{\infty} p_i f(u_{m,n-k_i}) = R\Delta_1^2 F(u_{m-1,n+1}) \tag{4}$$
$$\text{for} \quad m = 0, 1, \ldots, M-1 \quad \text{and} \quad n = 0, 1, \ldots,$$

with IC(3) and NBC:

$$\Delta_1 F(u_{0,n}) = \Delta_1 F(u_{M,n}) = 0 \quad \text{for} \quad n = 0, 1, \ldots, \tag{5}$$

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where Δ_1, Δ_1^2, and Δ_2 are forward partial difference operators (see, for instance, Kelley and Peterson [11]) such that $\Delta_1 u_{m,n} := u_{m+1,n} - u_{m,n}$, $\Delta_1^2 u_{m,n} := \Delta_1(\Delta_1 u_{m,n})$ and $\Delta_2 u_{m,n} := u_{m,n+1} - u_{m,n}$ for $m = 0, 1, \ldots, M-1$, $n = 0, \pm 1, \pm 2, \ldots$; $p_i, q_j \in \mathbf{R} = (-\infty, \infty)$, $k_i, r_j \in \{0, 1, \ldots\}$ for $i, j = 1, 2, \ldots$; $\mu_{m,i} \in \mathbf{R}$ for $m = 0, 1, \ldots, M-1$ and $i = \ldots, -1, 0$; $R \in [0, \infty)$ and $f, g, h, F \in C(\mathbf{R}, \mathbf{R})$ such that $f(0) = g(0) = h(0) = F(0) = 0$. Thus, $u_{m,n} = 0$ for $m = 1, 2, \ldots, M-1$ and $n = 0, \pm 1, \pm 2, \ldots$, is a solution of (4), which we call the zero solution. Throughout this paper, let $P := \sum_{i=1}^{\infty} p_i > 0$, $P^* := \sum_{i=1}^{\infty} |p_i|$, $P' := \sum_{i=1}^{\infty} k_i |p_i|$, $P'' := \sum_{i=1}^{\infty} k_i^2 |p_i|$, $Q^* := \sum_{j=1}^{\infty} |q_j|$ and $Q' := \sum_{j=1}^{\infty} r_j |q_j|$, and suppose that P, P^*, P', P'', Q^*, $Q' < \infty$ and that

$$\|\mu\| := \sup\left\{ |\mu_{m,i}| \quad \Big| \quad m = 0, 1, \ldots, M-1 \quad \text{and} \quad i = \ldots, -1, 0 \right\} < \infty. \tag{6}$$

For the sake of convenience in proving the (unique) existence of solutions of (1) with the initial-boundary conditions (2) and (3), we let $u_{m,i} = 0$ for $m < 0$, $m > M+1$, and $i = 0, \pm 1, \pm 2, \ldots$.

By a *solution* of (1)–(3) or (4), (5) and (3), we mean a sequence $\{u_{m,n}\}$ which is defined for $m = 0, 1, \ldots, M+1$ and $n = 0, \pm 1, \pm 2, \ldots$ and which satisfies (1) or (4) for $m = 1, 2, \ldots, M-1$ and $n = 0, 1, \ldots$, satisfies NBC(2) or (5) for $n = 0, 1, \ldots$, and satisfies IC(3) for $m = 0, 1, \ldots, M-1$ and $i = \ldots, -1, 0$.

By using the method similar to that in Zhang, Liu, and Cheng [12] or simply by iterative calculation, it is easy to show that (1) or (4) has a unique solution for the given boundary and initial conditions satisfying (6) (see Appendix).

In the sequel, we only consider the solutions of (1) and (4) with the initial conditions satisfying (6).

Recently, the oscillation (see [4–6, 17, 21], also Yu and Cui's survey paper [20]) of delay partial differential equations has been widely studied, while Xie [16] considered the stability of partial differential equations. The oscillation (see [1, 13]) and the stability (see [9, 10, 15], see also Burton's books [2, 3]) for Volterra integrodifferential equations also have been extensively approached, while Gopalsamy and Weng [8] considered the stability of a neutral integrodifferential equation. It is well-known that the behavior of a differential equation and its discrete analogue can be quite different. For example, every solution of the logistic equation

$$x'(t) = rx(t)\left[1 - \frac{x(t)}{K}\right]$$

is monotonic. But its discrete analogue

$$x_{n+1} = m\, x_n(1 - x_n)$$

has a chaotic solution when $m = 4$ (see [11]). In addition, there is a difference between the oscillation of delay differential equations and discrete analogues; for example, see [18]. In the last few years, many mathematicians have been studying difference systems. But only a few studies (see [7, 14, 22]) are devoted to partial difference equations and Volterra difference equations; we [14] considered the stability for neutral Volterra difference equations.

Our aim in this paper is to obtain sufficient conditions, which are "sharp" in some sense, for the square-summable stability and ϕ-square-summable stability in parabolic Volterra difference equations of neutral type. Our results generalize the corresponding results in [14, 19].

We now give some definitions which will be needed in this paper.

Definition 1.1. The zero solution of (1) or (4) is said to be asymptotically stable (AS) if every solution $\{u_{m,n}\}$ of (1) or (4) with IC satisfying (6) has the property

$$\lim_{n\to\infty} u_{m,n} = 0 \qquad \text{for} \quad m = 0, 1, \ldots, M+1. \tag{7}$$

Definition 1.2. The zero solution of (1) or (4) is said to be square-summably stable (SSS) if every solution $\{u_{m,n}\}$ of (1) or (4) with IC satisfying (6) has the property

$$\sum_{n=0}^{\infty} u_{m,n}^2 < \infty \qquad \text{for} \quad m = 0, 1, \ldots, M+1. \tag{8}$$

It is easy to see that SSS implies AS.

Definition 1.3. The zero solution of (1) or (4) is said to be ϕ-square-summably stable (ϕ-SSS) if every solution $\{u_{m,n}\}$ of (1) or (4) with IC satisfying (6) has the property

$$\sum_{n=0}^{\infty} \phi^2(u_{m,n}) < \infty \qquad \text{for} \quad m = 0, 1, \ldots, M+1 \tag{9}$$

where $\phi \in C(\mathbf{R}, \mathbf{R})$ and $\phi \not\equiv 0$.

Note that SSS implies ϕ-SSS, and ϕ-SSS implies SCS if $|\phi(x)| \geq |x|$ for $x \in \mathbf{R}$. It also is obvious that ϕ-SSS implies AS if $\phi(x) = 0$ implies $x = 0$.

2. Equation (1)

For (1), we have the following

Theorem 2.1. *Assume that*

$$Q^* + \frac{1}{2}P + P' < 1. \tag{10}$$

Then the zero solution of (1) is SSS.

Proof. It is easy to show that

$$\sum_{i=1}^{\infty} p_i u_{m,n-k_i} = P u_{m,n+1} - \Delta\left(\sum_{i=1}^{\infty} p_i \sum_{s=n-k_i}^{n} u_{m,s}\right).$$

Hence, we can rewrite (1) as

$$\Delta_2\left(u_{m,n} - \sum_{j=1}^{\infty} q_j u_{m,n-r_j} - \sum_{i=1}^{\infty} p_i \sum_{s=n-k_i}^{n} u_{m,s}\right) = -P u_{m,n+1} + R\Delta_1^2 u_{m-1,n+1}.$$

Define a Liapunov sequence by

$$V_n^{(1)} = \sum_{m=0}^{M+1}\left(u_{m,n} - \sum_{j=1}^{\infty} q_j u_{m,n-r_j} - \sum_{i=1}^{\infty} p_i \sum_{s=n-k_i}^{n} u_{m,s}\right)^2.$$

Then we have

$$\Delta V_n^{(1)} = \sum_{m=0}^{M+1} (-Pu_{m,n+1} + R\Delta_1^2 u_{m-1,n+1}) \Big(u_{m,n+1} + u_{m,n} - \sum_{j=1}^{\infty} q_j u_{m,n+1-r_j} - \sum_{j=1}^{\infty} q_j u_{m,n-r_j} - Pu_{m,n+1} - 2\sum_{i=1}^{\infty} p_i \sum_{s=n+1-k_i}^{n} u_{m,s} - \sum_{i=1}^{\infty} p_i u_{m,n-k_i} \Big).$$

(We define $\sum_{i=m}^{n} * = 0$ if $m > n$).

First, let us consider

$$\begin{aligned}
&-P\sum_{m=0}^{M+1} u_{m,n+1}\Big(u_{m,n+1} + u_{m,n} - \sum_{j=1}^{\infty} q_j\, u_{m,n+1-r_j} - \sum_{j=1}^{\infty} q_j\, u_{m,n-r_j} \\
&\qquad - P\, u_{m,n+1} - 2\sum_{i=1}^{\infty} p_i \sum_{s=n+1-k_i}^{n} u_{m,s} - \sum_{i=1}^{\infty} p_i\, u_{m,n-k_i} \Big) \\
&= -P\sum_{m=0}^{M+1} u_{m,n+1}\Big(u_{m,n+1} + u_{m,n} - \sum_{j=1}^{\infty} q_j\, u_{m,n+1-r_j} - \sum_{j=1}^{\infty} q_j\, u_{m,n-r_j} \\
&\qquad - Pu_{m,n+1} - 2\sum_{i=1}^{\infty} p_i \sum_{s=n+1-k_i}^{n} u_{m,s} + u_{m,n+1} - u_{m,n} \\
&\qquad - \sum_{j=1}^{\infty} q_j\, u_{m,n+1-r_j} + \sum_{j=1}^{\infty} q_j\, u_{m,n-r_j} - R\,\Delta_1^2\, u_{m-1,n+1} \Big) \\
&= \sum_{m=0}^{M+1} \Big(-2P\, u_{m,n+1}^2 + 2P\sum_{j=1}^{\infty} q_j\, u_{m,n+1}\, u_{m,n+1-r_j} + P^2\, u_{m,n+1}^2 \\
&\qquad + 2P\sum_{i=1}^{\infty} \sum_{s=n+1-k_i}^{n} u_{m,n+1}u_{m,s} + PR\, u_{m,n+1}\Delta_1^2\, u_{m-1,n+1} \Big) \\
&\le \sum_{m=0}^{M+1} \Big[-2Pu_{m,n+1}^2 + P\sum_{j=1}^{\infty} |q_j|(u_{m,n+1}^2 + u_{m,n+1-r_j}^2) + P^2\, u_{m,n+1}^2 \\
&\qquad + P\sum_{i=1}^{\infty} |p_i| \sum_{s=n+1-k_i}^{n} (u_{m,n+1}^2 + u_{m,s}^2) + PR\, u_{m,n+1}\Delta_1^2\, u_{m-1,n+1} \Big] \\
&= \sum_{m=0}^{M+1} \Big\{ -2P\left[1 - \frac{1}{2}(Q^* + P + P') \right] u_{m,n+1}^2 + P\sum_{j=1}^{\infty} |q_j| u_{m,n+1-r_j}^2 \\
&\qquad + P\sum_{i=1}^{\infty} |p_i| \sum_{s=n+1-k_i}^{n} u_{m,s}^2 + PR\, u_{m,n+1}\Delta_1^2\, u_{m-1,n+1} \Big\}.
\end{aligned}$$

Let us now consider

$$
\begin{aligned}
&R\sum_{m=0}^{M+1}\Big(u_{m,n+1}+u_{m,n}-\sum_{j=1}^{\infty}q_j\,u_{m,n+1-r_j}-\sum_{j=1}^{\infty}q_j\,u_{m,n-r_j}-P\,u_{m,n+1}\\
&\qquad -2\sum_{i=1}^{\infty}p_i\sum_{s=n+1-k_i}^{n}u_{m,s}-\sum_{i=1}^{\infty}p_i\,u_{m,n-k_i}\Big)\Delta_1^2\,u_{m-1,n+1}\\
&\quad =R\sum_{m=0}^{M+1}\Big(u_{m,n+1}+u_{m,n}-\sum_{j=1}^{\infty}q_j\,u_{m,n+1-r_j}-\sum_{j=1}^{\infty}q_j\,u_{m,n-r_j}-P\,u_{m,n+1}\\
&\qquad -2\sum_{i=1}^{\infty}p_i\sum_{s=n+1-k_i}^{n}u_{m,s}+u_{m,n+1}-u_{m,n}-\sum_{j=1}^{\infty}q_j\,u_{m,n+1-r_j}\\
&\qquad +\sum_{j=1}^{\infty}q_j\,u_{m,n-r_j}-R\,\Delta_1^2\,u_{m-1,n+1}\Big)\Delta_1^2\,u_{m-1,n+1}
\end{aligned}
$$

$$
\begin{aligned}
&\le 2R\sum_{m=0}^{M+1}u_{m,n+1}\Delta_1^2\,u_{m-1,n+1}-2R\sum_{j=1}^{\infty}q_j\sum_{m=0}^{M+1}u_{m,n+1-r_j}\Delta_1^2\,u_{m-1,n+1}\\
&\quad -PR\sum_{m=0}^{M+1}u_{m,n+1}\Delta_1^2\,u_{m-1,n+1}-2R\sum_{i=1}^{\infty}p_i\sum_{s=n+1-k_i}^{n}\sum_{m=0}^{M+1}u_{m,s}\Delta_1^2\,u_{m-1,n+1}.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
\Delta V_n^{(1)}\le &-2P\left[1-\frac{1}{2}(Q^*+P+P')\right]\sum_{m=0}^{M+1}u_{m,n+1}^2+P\sum_{m=0}^{M+1}\sum_{j=1}^{\infty}|q_j|u_{m,n+1-r_j}^2\\
&+P\sum_{m=0}^{M+1}\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n}u_{m,s}^2+2R\sum_{m=0}^{M+1}u_{m,n+1}\Delta_1^2u_{m-1,n+1}\\
&-R\sum_{j=1}^{\infty}q_j\sum_{m=0}^{M+1}u_{m,n+1-r_j}\Delta_1^2\,u_{m-1,n+1}\\
&-2R\sum_{i=1}^{\infty}p_i\sum_{s=n+1-k_i}^{n}\sum_{m=0}^{M+1}u_{m,s}\Delta_1^2u_{m-1,n+1}.
\end{aligned}
$$

By using a summation-by-parts formula and NBC(2) (here, we define $\Delta_1 u_{i,n} = 0$ for $i \le 0$ and $i \ge M+1$), we get

$$
\begin{aligned}
2R\sum_{m=0}^{M+1} u_{m,n+1}\Delta_1^2 u_{m-1,n+1} &= -2R\sum_{m=0}^{M+1}(\Delta_1 u_{m,n+1})^2 \\
&\quad - 2R\sum_{j=1}^{\infty} q_j \sum_{m=0}^{M+1} u_{m,n+1-r_j}\Delta_1^2 u_{m-1,n+1} \\
&= 2R\sum_{j=1}^{\infty} q_j \sum_{m=0}^{M+1} \Delta_1 u_{m,n+1-r_j}\Delta_1 u_{m,n+1} \\
&\le R\sum_{m=0}^{M+1}\sum_{j=1}^{\infty}|q_j|\left[(\Delta_1 u_{m,n+1})^2 + (\Delta_1 u_{m,n+1-r_j})^2\right] \\
&= RQ^*\sum_{m=0}^{M+1}(\Delta_1 u_{m,n+1})^2 + R\sum_{m=0}^{M+1}\sum_{j=1}^{\infty}|q_j|(\Delta_1 u_{m,n+1-r_j})^2 \\
&\quad - 2R\sum_{i=1}^{\infty} p_i \sum_{s=n+1-k_i}^{n}\sum_{m=0}^{M+1} u_{m,s}\Delta_1^2 u_{m-1,n+1} \\
&= 2R\sum_{i=1}^{\infty} p_i \sum_{s=n+1-k_i}^{n}\sum_{m=0}^{M+1}\Delta_1 u_{m,s}\Delta_1 u_{m,n+1} \\
&\le R\sum_{m=0}^{M+1}\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n}\left[(\Delta_1 u_{m,s})^2 + (\Delta_1 u_{m,n+1})^2\right] \\
&= RP'\sum_{m=0}^{M+1}(\Delta_1 u_{m,n+1})^2 + R\sum_{m=0}^{M+1}\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n}(\Delta_1 u_{m,s})^2.
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
\Delta V_n^{(1)} &\le -2P\left[1 - \frac{1}{2}(Q^* + P + P')\right]\sum_{m=0}^{M+1} u_{m,n+1}^2 \\
&\quad - 2R\left[1 - \frac{1}{2}(Q^* + P')\right]\sum_{m=0}^{M+1}(\Delta_1 u_{m,n+1})^2 \\
&\quad + P\sum_{m=0}^{M+1}\sum_{j=1}^{\infty}|q_j| u_{m,n+1-r_j}^2 + P\sum_{m=0}^{M+1}\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n} u_{m,s}^2 \\
&\quad + R\sum_{m=0}^{M+1}\sum_{j=1}^{\infty}|q_j|(\Delta_1 u_{m,n+1-r_j})^2 + R\sum_{m=0}^{M+1}\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n}(\Delta_1 u_{m,s})^2.
\end{aligned}
$$

Now, define another Liapunov sequence by

$$V_n^{(2)} = \sum_{m=0}^{M+1}\Bigg[P\sum_{j=1}^{\infty}|q_j|\sum_{s=n+1-r_j}^{n}u_{m,s}^2 + P\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n}\sum_{t=s}^{n}u_{m,t}^2 + R\sum_{j=1}^{\infty}|q_j|\sum_{s=n+1-r_j}^{n}(\Delta_1 u_{m,s})^2 + R\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n}\sum_{t=s}^{n}(\Delta_1 u_{m,t})^2\Bigg].$$

Then, we obtain

$$\begin{aligned}\Delta V_2^{(2)} = &\sum_{m=0}^{M+1}\Bigg\{P\sum_{j=1}^{\infty}|q_j|(u_{m,n+1}^2 - u_{m,n+1-r_j}^2) + P\sum_{i=1}^{\infty}|p_i|(k_i u_{m,n+1}^2 - \sum_{s=n+1-k_i}^{n}u_{m,s}^2) + R\sum_{j=1}^{\infty}|q_j|\Big[(\Delta_1 u_{m,n+1})^2 - (\Delta_1 u_{m,n+1-r_j})^2\Big] \\ &+ R\sum_{i=1}^{\infty}|p_i|\Big[k_i(\Delta_1 u_{m,n+1})^2 - \sum_{s=n+1-k_i}^{n}(\Delta_1 u_{m,s})^2\Big]\Bigg\} \\ = &\sum_{m=0}^{M+1}\Bigg[PQ^* u_{m,n+1}^2 + PP' u_{m,n+1}^2 + RQ^*(\Delta_1 u_{m,s})^2 + RP'(\Delta_1 u_{m,n+1})^2 \\ &- P\sum_{j=1}^{\infty}|q_j|u_{m,n+1-r_j}^2 - P\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n}u_{m,s}^2 - R\sum_{j=1}^{\infty}|q_j|(\Delta_1 u_{m,n+1})^2 \\ &- R\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n}(\Delta_1 u_{m,s})^2\Bigg].\end{aligned}$$

Finally, we take the following Liapunov sequence

$$V_n = V_n^{(1)} + V_n^{(2)}.$$

By using (10), we finally get

$$\Delta V_n \le -2P\left(1 - Q^* - \frac{1}{2}P - P'\right)\sum_{m=0}^{M+1}u_{m,n+1}^2. \tag{11}$$

Therefore, $\{V_n\}$ is decreasing and has a nonnegative limit of $\{V_n\}$ because $V_n \ge 0$ for $n = 0, 1, \ldots$. Now, summing the two sides of (11) from $n = 0$ to $n = \infty$, we have

$$2P\left(1 - Q^* - \frac{1}{2}P - P'\right)\sum_{n=0}^{\infty}\sum_{m=0}^{M+1}u_{m,n+1}^2 \le V_0$$

or

$$\sum_{n=0}^{\infty}\sum_{m=0}^{M+1}u_{m,n}^2 \le \sum_{m=0}^{M+1}\mu_{m,0}^2 + \frac{V_0}{2P(1 - Q^* - \frac{1}{2}P - P')} < \infty.$$

The proof is complete. □

Remark 2.1. Let $u_{m,n}$be independent of m and $x_n = u_{m,n}$, $q_j = k_i = 0$ for $i, j = 1, 2, \ldots$ and $R = 0$. Then (1) becomes an ordinary difference equation:

$$\Delta x_n + Px_n = 0 \qquad \text{for} \qquad n = 0, 1, \ldots, \tag{12}$$

and (10) becomes

$$\frac{1}{2}P < 1. \tag{13}$$

One can easily prove that the condition (13) is a necessary and sufficient condition for SSS in (12) (in fact, the absolute summable stability, i.e., its solutions $\{x_n\}$ with IC $x_i = \mu_i$ for $i = \dots, -1, 0$ satisfying

$$\|\mu\| = \sup\{|\mu_i| \quad \text{for} \quad i = \dots, -1, 0\} < \infty$$

has the property: $\sum_{n=0}^{\infty} |x_n| < \infty$). Therefore, in this sense, the condition (10) is a "sharp" condition.

As a special case, we consider a linear parabolic Volterra difference equation of retarded type

$$\Delta_2 u_{m,n} + \sum_{i=1}^{\infty} p_i u_{m,n-k_i} = R\Delta_1^2 u_{m-1,n+1} \tag{14}$$
$$\text{for} \quad m = 0, 1, \dots, M-1 \quad \text{and} \quad n = 0, 1, \dots,$$

with NBC(2) and IC(3). By Theorem 2.1, we have

Corollary 2.1. *If*

$$\frac{1}{2}P + P' < 1, \tag{15}$$

then the zero solution of (14) *is* SSS.

3. Equation (4)

For (4), we have the following

Theorem 3.1. *Let* (10) *be true. And suppose that*

$$f(x)h(x) \ge \max\Big\{f^2(x), g^2(x)\Big\}, \qquad x \in \mathbf{R}, \tag{16}$$

and

$$\begin{aligned}[h(y) - h(x)][F(y) - F(x)] \ge \\ \max\big\{[F(y) - F(x)]^2, [g(y) - g(x)]^2, [f(y) - f(x)]^2\big\}\end{aligned} \tag{17}$$

for y, $x \in \mathbf{R}$. *Then, the zero solution of* (4) *is* f-SSS *and* g-SSS.

Proof. It is easy to show that

$$\Delta_2\Big[h(u_{m,n}) - \sum_{j=1}^{\infty} q_j g(u_{m,n-r_j}) - \sum_{i=1}^{\infty} p_i \sum_{s=n-k_i}^{n} f(u_{m,s})\Big]$$
$$= -Pf(u_{m,n+1}) + R\Delta_1^2 F(u_{m-1,n+1}).$$

One can define a Liapunov sequence as follows

$$V_n^{(1)} = \sum_{m=0}^{M+1}\Big[h(u_{m,n}) - \sum_{j=1}^{\infty} q_j g(u_{m,n-r_j}) - \sum_{i=1}^{\infty} p_i \sum_{s=n-k_i}^{n} f(u_{m,s})\Big]^2.$$

As in the proof of Theorem 2.1, one gets

$$\begin{aligned}\Delta V_n^{(1)} \le & -2P\sum_{m=0}^{M+1} f(u_{m,n+1})h(u_{m,n+1}) + P(Q^*+P+P')\sum_{m=0}^{M+1} f^2(u_{m,n+1})\\ & -2R\sum_{m=0}^{M+1}\Delta_1 h(u_{m,n+1})\Delta_1 F(u_{m,n+1}) + R(Q^*+P')\sum_{m=0}^{M+1}[\Delta_1 F(u_{m,n+1})]^2\\ & + P\sum_{m=0}^{M+1}\sum_{j=1}^{\infty}|q_j|g^2(u_{m,n+1-r_j}) + P\sum_{m=0}^{M+1}\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n} f^2(u_{m,s})\\ & + R\sum_{m=0}^{M+1}\sum_{j=1}^{\infty}|q_j|[\Delta_1 g(u_{m,n+1-r_j})]^2 + R\sum_{m=0}^{M+1}\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n}[\Delta_1 f(u_{m,s})]^2.\end{aligned}$$

Then, one can take another Liapunov sequence as follows

$$\begin{aligned}V_n^{(2)} = \sum_{m=0}^{M+1}\Bigg\{ & P\sum_{j=1}^{\infty}|q_j|\sum_{s=n+1-r_j}^{n} g^2(u_{m,s}) + P\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n}\sum_{t=s}^{n} f^2(u_{m,t})\\ & + R\sum_{j=1}^{\infty}|q_j|\sum_{s=n+1-r_j}^{n}[\Delta_1 g(u_{m,s})]^2\\ & + R\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n}\sum_{t=s}^{n}[\Delta_1 f(u_{m,t})]^2\Bigg\}.\end{aligned}$$

It follows that

$$\begin{aligned}\Delta V_n^{(2)} = \sum_{m=0}^{M+1}\Bigg\{ & PQ^* g^2(u_{m,n+1}) + PP' f^2(u_{m,n+1}) + RQ^*[\Delta_1 g(u_{m,n+1})]^2\\ & + RP'[\Delta_1 f(u_{m,n+1})]^2 - P\sum_{j=1}^{\infty}|q_j|g^2(u_{m,n+1-r_j})\\ & - P\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n} f^2(u_{m,s}) - R\sum_{j=1}^{\infty}|q_j|[\Delta_1 g(u_{m,n+1-r_j})]^2\\ & - R\sum_{i=1}^{\infty}|p_i|\sum_{s=n+1-k_i}^{n}[\Delta_1 f(u_{m,s})]^2\Bigg\}.\end{aligned}$$

Finally, one takes the Liapunov sequence as follows:

$$V_n = V_n^{(1)} + V_n^{(2)}.$$

Then, one obtains from the above and (17)

$$\begin{aligned}
\Delta V_n \le \sum_{m=0}^{M+1} \Big\{ & -2Pf(u_{m,n+1})h(u_{m,n+1}) + P(Q^* + P + P')f^2(u_{m,n+1}) \\
& + PQ^* g^2(u_{m,n+1}) + PP'f^2(u_{m,n+1}) + RQ^*\Big[\Delta_1 g(u_{m,n+1})\Big]^2 \\
& - 2R\Delta_1 h(u_{m,n+1})\Delta_1 F(u_{m,n+1}) + R(Q^* + P')\Big[\Delta_1 F(u_{m,n+1})\Big]^2 \\
& + RP'\Big[\Delta_1 f(u_{m,n+1})\Big]^2 \Big\} \\
\le \sum_{m=0}^{M+1} \Big[& -2Pf(u_{m,n+1})h(u_{m,n+1}) \\
& + P(Q^* + P + 2P')f^2(u_{m,n+1}) + PQ^* g^2(u_{m,n+1}) \Big] \\
= -2P \sum_{m=0}^{M+1} \Big[& \alpha f(u_{m,n+1})h(u_{m,n+1}) - \left(\frac{Q^*}{2} + \frac{P}{2} + P'\right) f^2(u_{m,n+1}) \\
& + (1-\alpha)f(u_{m,n+1})h(u_{m,n+1}) - \frac{Q^*}{2} g^2(u_{m,n+1}) \Big] \\
\le -2P \sum_{m=0}^{M+1} & \left(\alpha - \frac{Q^*}{2} - \frac{P}{2} - P'\right) f^2(u_{m,n+1}) \\
& - 2P \sum_{m=0}^{M+1} \left(1 - \alpha - \frac{Q^*}{2}\right) g^2(u_{m,n+1}).
\end{aligned}$$

Then,

$$\Delta V_n \le -2P\left(1 - Q^* - \frac{P}{2} - P'\right) \sum_{m=0}^{M+1} f^2(u_{m,n+1}) \quad \text{for} \quad \alpha = 1 - \frac{Q^*}{2}$$

and

$$\Delta V_n \le -2P\left(1 - Q^* - \frac{P}{2} - P'\right) \sum_{m=0}^{M+1} g^2(u_{m,n+1}) \quad \text{for} \quad \alpha = \frac{Q^*}{2} + \frac{P}{2} + P'.$$

Since $f(x)$ and $g(x)$ are continuous functions and $\|\mu\| < \infty$, one has

$$\sum_{n=0}^{\infty} \sum_{m=0}^{M+1} \phi^2(u_{m,n}) \le \sum_{m=0}^{M+1} \phi^2(\mu_{m,0}) + \frac{V_0}{2P(1 - Q^* - \frac{1}{2}P - P')} < \infty,$$

where $\phi(x) = f(x)$ or $\phi(x) = g(x)$. This completes the proof. □

We now give two corollaries.

Consider the nonlinear parabolic Volterra difference equation of neutral type:

$$\Delta_2\Big[u_{m,n} - \sum_{j=1}^{\infty} q_j g(u_{m,n-r_j})\Big] + \sum_{i=1}^{\infty} p_i f(u_{m,n-k_i}) = R\Delta_1^2 u_{m-1,n+1} \tag{18}$$
$$\text{for } m = 0, 1, \ldots, M-1 \quad \text{and} \quad n = 0, 1, \ldots,$$

with NBC(2) and IC(3). By using Theorem 3.1, we obtain the following

Corollary 3.1. *Let* (10) *be true, and assume that*

$$xf(x) \ge \max\left\{f^2(x),\ g^2(x)\right\}, \qquad x \in \mathbf{R}, \tag{19}$$

and that

$$(y-x)^2 \ge \max\left\{[f(y)-f(x)]^2,\ [g(y)-g(x)]^2\right\}, \qquad y, x \in \mathbf{R}. \tag{20}$$

Then, the zero solution of (18) *is* f-SSS *and* g-SSS.

More specifically, we consider the nonlinear parabolic Volterra difference equation of retarded type:

$$\Delta_2\, u_{m,n} + \sum_{i=1}^{\infty} p_i f(u_{m,n-k_i}) = R\Delta_1^2\, u_{m-1,n+1} \tag{21}$$
$$\text{for } m = 0, 1, \ldots, M-1 \quad \text{and} \quad n = 0, 1, \ldots,$$

with NBC(2) and IC(3). Again, by Theorem 3.1, we get

Corollary 3.2. *Let* (15) *be true, and assume that*

$$xf(x) \ge f^2(x), \qquad x \in \mathbf{R}, \tag{22}$$

and that

$$(y-x)^2 \ge [f(y)-f(x)]^2, \qquad y, x \in \mathbf{R}. \tag{23}$$

Then, the zero solution of (21) *is* f-SSS.

4. An example

Consider the generalized first equation of Open Problem 6.8.1 in Kocic and Ladas [12]:

$$\Delta_2 u_{m,n} + \sum_{i=1}^{\infty} p_i \left[\exp(u_{m,n-k_i}) - 1\right] = R\Delta_1^2 u_{m-1,n+1} \tag{24}$$
$$\text{for} \quad m = 0, 1, \ldots, M-1 \quad \text{and} \quad n = 0, 1, \ldots$$

and

$$\Delta_2 u_{m,n} + \sum_{i=1}^{\infty} p_i \left[1 - \exp(-u_{m,n-k_i})\right] = R\Delta_1^2 u_{m-1,n+1} \tag{25}$$
$$\text{for} \quad m = 0, 1, \ldots, M-1 \quad \text{and} \quad n = 0, 1, \ldots,$$

with NBC(2) and IC(3), respectively.

For (24) (resp. (25)), we know that $f(x) = e^x - 1$ (resp. $f(x) = 1 - e^{-x}$). It is easy to prove that (22) (resp. (23)) is satisfied for $x < 0$ (resp. $x > 0$). Hence, we can choose $\phi(x) = e^x - 1$ (resp. $\phi(x) = 1 - e^{-x}$) which satisfies

$$\phi(x) = 0 \quad \text{implies} \quad x = 0.$$

Hence, if (15) is true, then every negative (or positive) solution which satisfies (6) must satisfy (7) where a negative (or positive) solution means $\mu_{m,i}$, $u_{m,n} <$ (or $>$) 0 for $m = 0, 1, \dots, M-1$, $i = \dots, -1, 0$, and $n = 0, 1, \dots$.

Combining (24) with (25), we consider the following equation:

$$\Delta_2 u_{m,n} + \sum_{i=1}^{\infty} p_i \Big[1 - \exp(-u_{m,n-k_i} \operatorname{sgn} u_{m,n-k_i})\Big] \operatorname{sgn} u_{m,n-k_i} = R\Delta_1^2 u_{m-1,n+1}$$
$$\text{for} \quad m = 0, 1, \dots, M-1 \quad \text{and} \quad n = 0, 1, \dots, \tag{26}$$

with NBC(2) and IC(3) and obtain the following result.

Theorem 4.1. *If* (15) *holds, then the zero solution of* (26) *is* AS.

Appendix

On the (unique) existence of solutions of initial-boundary value problem (1), (2), and (3): rewrite (1) as

$$\begin{aligned} u_{m,n+1} - u_{m,n} - \sum_{j=1}^{\infty} q_j\, u_{m,n+1-r_j} + \sum_{j=1}^{\infty} q_j\, u_{m,n-r_j} + \sum_{i=1}^{\infty} p_i\, u_{m,n-k_i} \\ = R\Delta_1\, u_{m,n+1} - R\Delta_1\, u_{m-1,n+1} \\ = R(u_{m+1,n+1} - 2u_{m,n+1} + u_{m-1,n+1}). \end{aligned}$$

We may assume that $R \neq 0$ and $r_j \in \{1, 2, \dots\}$. Then, for $m = 0$ and $n = 0$, we have $\Delta_1\, u_{0,1} = 0$, so

$$u_{0,1} - u_{0,0} - \sum_{j=1}^{\infty} q_j\, u_{0,1-r_j} + \sum_{j=1}^{\infty} q_j\, u_{0,-r_j} + \sum_{i=0}^{\infty} p_i\, u_{0,-k_i} = -R\, u_{0,1}.$$

It follows that

$$u_{0,1} = \frac{1}{1+R}\left(u_{0,0} + \sum_{j=1}^{\infty} q_j\, u_{0,1-r_j} - \sum_{j=1}^{\infty} q_j\, u_{0,-r_j} - \sum_{i=0}^{\infty} p_i\, u_{0,-k_i}\right)$$

and

$$u_{1,1} = u_{0,1}.$$

For $m = 1$ and $n = 0$, we have $\Delta_1\, u_{0,1} = 0$, so

$$u_{2,1} = \frac{1}{R}\left(u_{1,1} - u_{1,0} - \sum_{j=1}^{\infty} q_j\, u_{1,1-r_j} + \sum_{j=1}^{\infty} q_j\, u_{1,-r_j} + \sum_{i=1}^{\infty} p_i\, u_{1,-k_i}\right) + u_{1,1}.$$

For $m = M-1$ and $n = 0$, we have

$$\begin{aligned} u_{M,1} =& \frac{1}{R}\left(u_{M-1,1} - u_{M-1,0} - \sum_{j=1}^{\infty} q_j\, u_{M-1,1-r_j} + \sum_{j=1}^{\infty} q_j\, u_{M-1,-r_j}\right. \\ & \left. + \sum_{i=1}^{\infty} p_i\, u_{M-1,-k_i}\right) + 2u_{M-1,1} - u_{M-2,1}. \end{aligned}$$

Finally, for $m = M$ and $n = 0$, we have $\Delta_1\, u_{M,1} = 0$, so we have

$$u_{M+1,1} = u_{M,1}.$$

In this way, we can successively calculate

$$u_{0,2},\ \ldots,\ u_{M,2},\ u_{M+1,2},\ u_{0,3},\ \ldots,\ u_{M,3},\ u_{M+1,3},\ldots\ .$$

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