

POISSON KERNEL FOR THE ASSOCIATED CONTINUOUS  
 $q$ -ULTRASPHERICAL POLYNOMIALS

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ABSTRACT. The bilinear generating function (the Poisson kernel) for the associated continuous  $q$ -ultraspherical polynomials, first studied by Bustoz and Ismail and later generalized by Ismail and Rahman, is obtained, essentially as a product of a  ${}_2\phi_1$ , a  ${}_3\phi_2$ , and a  ${}_4\phi_3$  basic hypergeometric series. Some related generating functions also are given.

1. Introduction

The continuous  $q$ -ultraspherical polynomials,  $C_n(x; \beta | q)$ , introduced by Rogers [18, 19, 20], can be defined by the generating function

$$\sum_{n=0}^{\infty} C_n(x; \beta | q) t^n = \frac{(\beta t e^{i\theta}, \beta t e^{-i\theta}; q)_{\infty}}{(t e^{i\theta}, t e^{-i\theta}; q)_{\infty}} \tag{1.1}$$

where  $0 < q < 1$ ,  $x = \cos \theta$  ( $0 \leq \theta \leq \pi$ ),  $|t| < 1$ , and

$$(a; q)_{\infty} = \prod_{j=0}^{\infty} (1 - a q^j),$$

$$(a_1, a_2, \dots, a_k; q)_{\infty} = \prod_{j=1}^k (a_j; q)_{\infty}; \tag{1.2}$$

see [7]. Setting  $\beta = 0$  in (1.1), one obtains the generating function for the continuous  $q$ -Hermite polynomials  $H_n(x | q) = (q; q)_n C_n(x; 0 | q)$ :

$$\sum_{n=0}^{\infty} \frac{H_n(x | q)}{(q; q)_n} t^n = \frac{1}{(t e^{i\theta}, t e^{-i\theta}; q)_{\infty}} \tag{1.3}$$

where the shifted factorial  $(a; q)_n$  is defined by

$$(a; q)_0 = 1,$$

$$(a; q)_n = \frac{(a; q)_{\infty}}{(a q^n; q)_{\infty}} = \prod_{j=1}^n (1 - a q^{j-1}) \quad n = 1, 2, \dots \tag{1.4}$$

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As a  $q$ -extension of Mehler's famous bilinear generating function for the classical Hermite polynomials,

$$\sum_{n=0}^{\infty} \frac{H_n(x)H_n(y)}{n!} \left(\frac{t}{2}\right)^n = (1-t^2)^{-\frac{1}{2}} \exp\left\{\frac{2xyt - (x^2 + y^2)t^2}{1-t^2}\right\}, \quad (1.5)$$

see [21, Problem 23] and [22], Rogers [17] found the following extension of (1.5) for the  $q$ -Hermite polynomials:

$$\sum_{n=0}^{\infty} \frac{H_n(x|q)H_n(y|q)}{(q;q)_n} t^n = \frac{(t^2; q)_{\infty}}{|(te^{i(\theta+\varphi)}, te^{i(-\theta+\varphi)}; q)_{\infty}|^2} \quad (1.6)$$

$x = \cos \theta$ ,  $y = \cos \varphi$ ,  $|t| < 1$ . The extension of (1.5) to  $q$ -ultraspherical polynomials was found by Gasper and Rahman [8]:

$$\begin{aligned} P_t(x, y; \beta | q) &:= \sum_{n=0}^{\infty} h_n C_n(x; \beta | q) C_n(y; \beta | q) t^n \\ &= \frac{(q, \beta q; q)_{\infty}}{2\pi(\beta, \beta t^2; q)_{\infty}} \left| \frac{(\beta t e^{i\theta+i\varphi}, \beta t e^{i\theta-i\varphi}; q)_{\infty}}{(t e^{i\theta+i\varphi}, t e^{i\theta-i\varphi}; q)_{\infty}} \right|^2 \\ &\quad \times {}_8W_7(\beta t^2/q; \beta/q, t e^{i\theta+i\varphi}, t e^{-i\theta-i\varphi}, t e^{i\theta-i\varphi}, t e^{-i\theta+i\varphi}; q, \beta q), \end{aligned} \quad (1.7)$$

$\beta$ ,  $t$  assumed real, where

$$h_n = \frac{(q, \beta^2; q)_{\infty}}{2\pi(\beta, \beta q; q)_{\infty}} \frac{(1 - \beta q^n)(q; q)_n}{(1 - \beta)(\beta^2; q)_n} \quad (1.8)$$

is the normalizing constant in the orthogonality relation

$$\int_0^{\pi} C_n(\cos \theta; \beta | q) C_m(\cos \theta; \beta | q) W_{\beta}(\cos \theta | q) d\theta = \frac{\delta_{m,n}}{h_n(\beta | q)}, \quad (1.9)$$

with

$$W_{\beta}(\cos \theta | q) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_{\infty}}{(\beta e^{2i\theta}, \beta e^{-2i\theta}; q)_{\infty}}; \quad (1.10)$$

see [7] and [3], and  ${}_8W_7$  is a special case of a very-well-poised basic hypergeometric series:

$$\begin{aligned} &{}_{2m}W_{2m-1}(a; a_1, a_2, \dots, a_{2m-3}; q, z) \\ &:= {}_{2m}\phi_{2m-1} \left[ \begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, a_1, a_2, \dots, a_{2m-3} \\ \sqrt{a}, -\sqrt{a}, qa/a_1, qa/a_2, \dots, qa/a_{2m-3} \end{matrix}; q, z \right] \end{aligned} \quad (1.11)$$

where the symbol on the right-hand side is a special case of a basic hypergeometric series defined by

$${}_{r+1}\phi_r \left[ \begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, z \right] := \sum_{k=0}^{\infty} \frac{(a_1 \dots a_{r+1}; q)_k}{(q, b_1, \dots, b_r; q)_k} z^k \quad (1.12)$$

where

$$(a_1, \dots, a_j; q)_k = \prod_{i=1}^j (a_i; q)_k;$$

see [7] for conditions of convergence and other properties of these series. One can see that the Poisson kernel in (1.7) reduces to (1.6) when  $\beta = 0$ , except for a constant factor. Formula (1.7) is admittedly more complicated than (1.6), but it does exhibit one of the basic properties of the Poisson kernel, namely, the positivity in some domain of  $\beta$  and  $t$ . The importance of positivity of these bilinear sums in Fourier analysis was pointed out by many authors, see, for example, Askey [1]. But there is renewed interest in Poisson kernels of orthogonal polynomials for a different reason. Wiener [23] showed in 1933 how the classical topic of Fourier integrals could be approached from the point of view of classical orthogonal polynomials, in particular, by using Mehler's formula (1.5). One can show that the expression on the right-hand side of (1.5) approaches a multiple of  $\exp(ixy)$ , the usual kernel of Fourier transform, as  $t \rightarrow i$  on the unit circle. This was suggested by Askey as a suitable starting point for a systematic  $q$ -extension of Fourier Analysis, which resulted in a number of recent publications [2, 4, 15]. It turns out that the most important piece of information that one needs from the Poisson kernel is the location of poles in the complex  $t$ -plane, as far as the Fourier transformation is concerned. One can see that the poles of the kernel in (1.7) are exactly the same as those of the simpler kernel in (1.6). Calculations with more general systems of polynomials, starting with the 2-parameter continuous  $q$ -Jacobi polynomials [9] up to the 4-parameter Askey-Wilson polynomials [16] reveal that this phenomenon persists through the entire Askey-Wilson family. Let us recall that the Askey-Wilson polynomials are defined by [3]

$$p_n(x; a, b, c, d | q) = {}_4\phi_3 \left[ \begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q \right], \quad (1.13)$$

$x = \cos \theta$ , which satisfy an orthogonality relation similar to (1.9) if  $\max(|a|, |b|, |c|, |d|) < 1$ . Note that  $C_n(x; \beta | q)$  is a special case of (1.13) (except for a constant factor) that corresponds to setting  $a = \sqrt{\beta}$ ,  $b = \sqrt{\beta q}$ ,  $c = -\sqrt{\beta}$ ,  $d = -\sqrt{\beta q}$ ; see [3, 5, 7]. In this paper, we wish to generalize (1.7) in a different direction. Recall that  $C_n(x; \beta | q)$  satisfies the three-term recurrence relation

$$2xC_n(x; \beta | q) = \frac{1 - q^{n+1}}{1 - \beta q^n} C_{n+1}(x; \beta | q) + \frac{1 - \beta^2 q^{n-1}}{1 - \beta q^n} C_{n-1}(x; \beta | q) \quad (1.14)$$

with  $C_{-1}(x; \beta | q) = 0$ ,  $C_0(x; \beta | q) = 1$ ,  $-1 \leq x \leq 1$ . Bustoz and Ismail [6] introduced an association parameter  $\alpha$  and studied the properties of the associated  $q$ -ultraspherical polynomial  $C_n^\alpha(x; \beta | q)$  that satisfies the recurrence relation

$$2xC_n^\alpha(x; \beta | q) = \frac{1 - q^{n+\alpha+1}}{1 - \beta q^{n+\alpha}} C_{n+1}^\alpha(x; \beta | q) + \frac{1 - \beta^2 q^{n+\alpha-1}}{1 - \beta q^{n+\alpha}} C_{n-1}^\alpha(x; \beta | q) \quad (1.15)$$

with  $C_{-1}^\alpha(x; \beta | q) = 0$ ,  $C_0^\alpha(x; \beta | q) = 1$ . They found a measure  $d\mu(x)$  with respect to which the orthogonality relation

$$\int_{-1}^1 C_n^\alpha(x; \beta | q) C_m^\alpha(x; \beta | q) d\mu(x) = \frac{1 - \beta q^\alpha}{1 - \beta q^{\alpha+n}} \frac{(\beta^2 q^\alpha; q)_n}{(q^{\alpha+1}; q)_n} q^{\alpha n} \delta_{m,n} \quad (1.16)$$

holds. By a straightforward manipulation of the recurrence relation (1.15), Bustoz and Ismail [6] also found the generating function

$$\begin{aligned} G_t^\alpha(x; \beta | q) &:= \sum_{n=0}^{\infty} C_n^\alpha(x; \beta | q) t^n \\ &= \frac{(1 - q^\alpha)}{(1 - 2xt + t^2)} {}_3\phi_2 \left[ \begin{matrix} \beta t e^{i\theta}, \beta t e^{-i\theta}, q \\ q t e^{i\theta}, q t e^{-i\theta} \end{matrix}; q, q^\alpha \right]. \end{aligned} \quad (1.17)$$

In Section 4, we will find an alternative form of (1.17), as well as a few other generating functions. In Section 5, we will compute some bilinear generating functions. The main result of this paper that follows from (5.1) and (5.5) is

$$\begin{aligned} K_t^\alpha(x, y; \beta | q) &:= \sum_{n=0}^{\infty} \frac{(1 - \beta q^{\alpha+n}) (q^{\alpha+1}; q)_n}{(1 - \beta q^\alpha) (\beta^2 q^\alpha; q)_n} (tq^{-\alpha})^n C_n^\alpha(x; \beta | q) C_n^\alpha(y; \beta | q) \\ &= (1 - \beta q^\alpha)^{-1} \{ L_t^\alpha(x, y) - \beta q^\alpha L_{tq}^\alpha(x, y) \} \end{aligned} \quad (1.18)$$

where

$$\begin{aligned} L_t^\alpha(x, y) &= \frac{(1 - q^\alpha)^2}{(1 - e^{-2i\theta})(1 - 2ytq^{-\alpha}e^{i\theta} + t^2q^{-2\alpha}e^{2i\theta})} \frac{(q^{\alpha+1}; q)_\infty}{(\beta^2 q^\alpha; q)_\infty} \\ &\quad \times {}_2\phi_1 \left[ \begin{matrix} \beta e^{-2i\theta}, \beta \\ q e^{-2i\theta} \end{matrix}; q, q^\alpha \right] \\ &\quad \times {}_3\phi_2 \left[ \begin{matrix} \beta tq^{-\alpha} e^{i(\theta+\varphi)}, \beta tq^{-\alpha} e^{i(\theta-\varphi)}, q \\ tq^{1-\alpha} e^{i(\theta+\varphi)}, tq^{1-\alpha} e^{i(\theta-\varphi)} \end{matrix}; q, q^\alpha \right] \\ &\quad \times {}_4\phi_3 \left[ \begin{matrix} \beta e^{2i\theta}, \beta, tq^{-\alpha} e^{i(\theta+\varphi)}, tq^{-\alpha} e^{i(\theta-\varphi)} \\ q e^{2i\theta}, \beta tq^{-\alpha} e^{i(\theta+\varphi)}, \beta tq^{-\alpha} e^{i(\theta-\varphi)} \end{matrix}; q, q \right] \\ &\quad + (\theta \longleftrightarrow -\theta). \end{aligned} \quad (1.19)$$

Note that  $K_t^\alpha$  is a multiple of the Poisson kernel for  $C_n^\alpha(x; \beta | q)$ . This shows that the poles are only slightly shifted by a factor of  $q^\alpha$ , but otherwise remains the same as in (1.7). Unfortunately, we were unable to simplify (1.19) to a form where the positivity would become obvious as it is in (1.7). It is only in the case when  $\alpha = 0$  that the two  ${}_4\phi_3$  series in (1.19) combine to an  ${}_8\phi_7$  which is what we have in (1.7), but otherwise the computation seems quite difficult.

In deriving the formulas mentioned above, our main tool has been the generating function (1.17) as well as the following moment representation of  $C_n^\alpha(x; \beta | q)$ :

$$C_n^\alpha(x; \beta | q) = \int_{-\infty}^{\infty} \xi^n d\nu(\xi; \beta, x) \quad (1.20)$$

where  $\nu$  is a step function whose jumps occur at the points  $q^k e^{i\theta}$  and  $q^k e^{-i\theta}$ ,  $k = 0, 1, \dots$ . The jumps are

$$\begin{aligned} d\nu(q^k e^{i\theta}; \beta, x) &= \frac{(1 - q^\alpha)}{(1 - e^{-2i\theta})} {}_2\phi_1 \left[ \begin{matrix} \beta e^{-2i\theta}, \beta \\ q e^{-2i\theta} \end{matrix}; q, q^\alpha \right] \\ &\quad \times \frac{(q e^{2i\theta} / \beta, q / \beta; q)_k}{(q, q e^{2i\theta}; q)_k} (\beta^2 q^\alpha)^k, \end{aligned} \quad (1.21)$$

with  $d\nu(q^k e^{-i\theta}; \beta, x)$  the same as (1.21) and with  $\theta$  replaced by  $-\theta$ . We prove this important result in Section 3. The idea of this moment representation of orthogonal polynomials is not new. Karlin [13] used such a representation to study the sign regularity of determinants of classical orthogonal polynomials. Carlitz and Al-Salam used them in the 1960s (see the references in [11] and [12]). Ismail and Stanton [11] constructed several moment representations explicitly with special emphasis on the Al-Salam-Chihara and continuous  $q$ -ultraspherical polynomials. Implicit use of moment representations of the continuous  $q$ -ultraspherical polynomials was also there in formulas (7.4.4) and (7.4.7) of [7]. The derivation of (1.20) and (1.21) was facilitated by an alternate form of the polynomial solution of the three-term recurrence relation for the associated Askey-Wilson polynomials found in [10]. We carry out this alternate derivation in Section 2. In Section 6, we will give the  $q \rightarrow 1$  limit of formulas (1.18) and (1.19), as well as the explicit form of the associated ultraspherical polynomials.

## 2. The associated Askey-Wilson polynomials

In [10], Ismail and Rahman found a representation of the associated Askey-Wilson polynomials,  $p_n^\alpha(x; a, b, c, d)$ , as a linear combination of the two linearly independent solutions of the three-term recurrence relation

$$2xp_n^\alpha(x; a, b, c, d) = A_{n+\alpha}p_{n+1}^\alpha(x; a, b, c, d) + B_{n+\alpha}p_n^\alpha(x; a, b, c, d) + C_{n+\alpha}p_{n-1}^\alpha(x; a, b, c, d), \quad (2.1)$$

$p_{-1}^\alpha(x; a, b, c, d) = 0$  and  $p_0^\alpha(x; a, b, c, d) = 1$ , namely,

$$r_{\alpha+n}(z; a, b, c, d) = \frac{(abq^{\alpha+n}, acq^{\alpha+n}, adq^{\alpha+n}, bcdq^{\alpha+n}/z; q)_\infty}{(bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}, azq^{\alpha+n}; q)_\infty} \left(\frac{a}{z}\right)^{n+\alpha} \times {}_8W_7(bcd/zq; b/z, c/z, d/z, abcdq^{\alpha+n-1}, q^{-\alpha-n}; q, qz/a) \quad (2.2)$$

and

$$s_{\alpha+n}(z; a, b, c, d) = \frac{(abcdq^{2\alpha+2n}, bzq^{\alpha+n+1}, czq^{\alpha+n+1}, dzq^{\alpha+n+1}, bcdzq^{\alpha+n}; q)_\infty}{(bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}, q^{\alpha+n+1}, bcdzq^{2\alpha+2n+1}; q)_\infty} (az)^{n+\alpha} \times {}_8W_7(bcdzq^{2\alpha+2n}; bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}, q^{\alpha+n+1}, qz/a; q, az) \quad (2.3)$$

where  $z = e^{i\theta}$ ,  $x = \cos \theta$ , and

$$A_\alpha = a^{-1} \frac{(1-abq^\alpha)(1-acq^\alpha)(1-adq^\alpha)(1-abcdq^{\alpha-1})}{(1-abcdq^{2\alpha-1})(1-abcdq^{2\alpha})}, \quad (2.4)$$

$$C_\alpha = a \frac{(1-bcq^{\alpha-1})(1-bdq^{\alpha-1})(1-cdq^{\alpha-1})(1-q^\alpha)}{(1-abcdq^{2\alpha-1})(1-abcdq^{2\alpha-2})}, \quad (2.5)$$

$$B_\alpha = a + a^{-1} - A_\alpha - C_\alpha. \quad (2.6)$$

One can show by using Bailey's transformation formula [7, III.(37)] for very-well-poised  ${}_8\phi_7$  series that  $r_{\alpha+n}(z; a, b, c, d)$  is a linear combination of  $s_{\alpha+n}(z; a, b, c, d)$  and  $s_{\alpha+n}(z^{-1}; a, b, c, d)$  with coefficients independent of  $n$ . But, of course,  $s_{\alpha+n}(z; a, b, c, d)$

and  $s_{\alpha+n}(z^{-1}; a, b, c, d)$  are also independent solutions of (2.1) as long as  $0 < \theta < \pi$ , so one might seek a representation of the form

$$p_n^\alpha(x; a, b, c, d) = L_\alpha s_{\alpha+n}(z; a, b, c, d) + M_\alpha s_{\alpha+n}(z^{-1}; a, b, c, d). \quad (2.7)$$

The initial conditions then fix the coefficient  $L_\alpha$  and  $M_\alpha$ :

$$\begin{aligned} p_n^\alpha(x; a, b, c, d) &= \frac{s_{\alpha+n}(z; a, b, c, d)s_{\alpha-1}(z^{-1}; a, b, c, d) - s_{\alpha+n}(z^{-1}; a, b, c, d)s_{\alpha-1}(z; a, b, c, d)}{s_\alpha(z; a, b, c, d)s_{\alpha-1}(z^{-1}; a, b, c, d) - s_\alpha(z^{-1}; a, b, c, d)s_{\alpha-1}(z; a, b, c, d)}. \end{aligned} \quad (2.8)$$

Using (2.1) and an argument similar to the one used in [10], we can show that

$$\begin{aligned} &\{s_\alpha(z; a, b, c, d)s_{\alpha-1}(z^{-1}; a, b, c, d) - s_\alpha(z^{-1}; a, b, c, d)s_{\alpha-1}(z; a, b, c, d)\}^{-1} \\ &= \frac{(bcq^{\alpha-1}, bdq^{\alpha-1}, cdq^{\alpha-1}, q^\alpha; q)_\infty}{(abq^\alpha, acq^\alpha, adq^\alpha, abcdq^{\alpha-1}; q)_\infty} \\ &\quad \times \frac{a^{1-2\alpha}}{(1 - abcdq^{2\alpha-2})} \frac{(az, a/z; q)_\infty}{(qz^2, q/z^2; q)_\infty} \frac{1}{(z - z^{-1})}, \end{aligned} \quad (2.9)$$

which is just the Casorati determinant of (2.1). Using the transformation formula [7, III (23)] and substituting (2.9) in (2.8), we obtain

$$\begin{aligned} p_n^\alpha(x; a, b, c, d) &= \frac{(1 - abcdq^{\alpha-2})}{(1 - abcdq^{2\alpha-2})} \\ &\quad \times \frac{(aq^\alpha/z, bq^\alpha/z, cq^\alpha/z, dq^\alpha/z, azq^{\alpha+n+1}, bzq^{\alpha+n+1}; q)_\infty}{(abq^\alpha, acq^\alpha, adq^\alpha, bcq^{\alpha+n}, bdq^{\alpha+n}, cdq^{\alpha+n}; q)_\infty} \\ &\quad \times \frac{(czq^{\alpha+n+1}, dzq^{\alpha+n+1}, abcdq^{\alpha+n-1}; q)_\infty}{(q^{\alpha+n+1}, q^{\alpha+1}/z^2, z^2q^{\alpha+n+2}; q)_\infty} \frac{(az)^n}{(1 - z^{-2})} \\ &\quad \times {}_8W_7(q^\alpha/z^2; q/az, q/bz, q/cz, q/dz, q^\alpha; q, abcdq^{\alpha-2}) \\ &\quad \times {}_8W_7(z^2q^{\alpha+n+1}; zq/a, zq/b, zq/c, zq/d, q^{\alpha+n+1}; q, abcdq^{\alpha+n-1}) \\ &\quad + z \longleftrightarrow z^{-1}. \end{aligned} \quad (2.10)$$

This is the form that will enable us to represent  $C_n^\alpha(x; \beta \mid q)$  as the moment of a probability measure.

### 3. Associated q-ultraspherical polynomials as moments

Using the quadratic transformation formula [7, (3.4.7)], we have

$$\begin{aligned} &{}_8W_7(t^2q^\lambda; tq/a, tq^{\frac{1}{2}}/a, -tq/a, -tq^{\frac{1}{2}}/a, q^\lambda; q, a^4q^{\lambda-1}) \\ &= \frac{(t^2q^{\lambda+1}, a^4q^{2\lambda-1}; q)_\infty}{(a^2q^\lambda, a^2t^2q^{2\lambda}; q)_\infty} {}_2\phi_1 \left[ \begin{matrix} qt^2/a^2, q/a^2 \\ qt^2 \end{matrix}; q, a^4q^{\lambda-1} \right]; \end{aligned} \quad (3.1)$$

see also [14].

Setting  $b = aq^{\frac{1}{2}}$ ,  $c = -a$ ,  $d = -aq^{\frac{1}{2}}$  in (2.10), using (3.1), and simplifying the coefficient, we obtain

$$\begin{aligned}
 p_n^\alpha(x; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}}) &= \frac{1 - a^4 q^{\alpha-1}}{1 - z^{-2}} \frac{(a^4 q^{\alpha+n}; q)_\infty}{(q^{\alpha+n+1}; q)_\infty} (az)^n \\
 &\quad \times {}_2\phi_1 \left[ \begin{matrix} q/a^2 z^2, q/a^2 \\ q/z^2 \end{matrix}; q, a^4 q^{\alpha-1} \right] \\
 &\quad \times {}_2\phi_1 \left[ \begin{matrix} qz^2/a^2, q/a^2 \\ qz^2 \end{matrix}; q, a^4 q^{\alpha+n} \right] \\
 &\quad + z \longleftrightarrow z^{-1}.
 \end{aligned} \tag{3.2}$$

Comparing (1.15) with (2.1) for these special values of parameters  $b, c, d$ , we find that

$$C_n^\alpha(x; a^2 | q) = \frac{(a^4 q^\alpha; q)_n}{(q^{\alpha+1}; q)_n} a^{-n} p_n^\alpha(x; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}}). \tag{3.3}$$

So, from (3.2) and (3.3), we have

$$\begin{aligned}
 C_n^\alpha(x; a^2 | q) &= \frac{(a^4 q^{\alpha-1}; q)_\infty}{(q^{\alpha+1}; q)_\infty} \frac{z^n}{(1 - z^{-2})} {}_2\phi_1 \left[ \begin{matrix} q/a^2 z^2, q/a^2 \\ q/z^2 \end{matrix}; q, a^4 q^{\alpha-1} \right] \\
 &\quad \times {}_2\phi_1 \left[ \begin{matrix} qz^2/a^2, q/a^2 \\ qz^2 \end{matrix}; q, a^4 q^{\alpha+n} \right] \\
 &\quad + z \longleftrightarrow z^{-1}.
 \end{aligned} \tag{3.4}$$

If we use [7, III.(3)] on the first  ${}_2\phi_1$  series of the right-hand side, we get

$$\begin{aligned}
 C_n^\alpha(x; a^2 | q) &= \frac{(1 - q^\alpha)}{(1 - z^{-2})} z^n {}_2\phi_1 \left[ \begin{matrix} a^2/z^2, a^2 \\ q/z^2 \end{matrix}; q, q^\alpha \right] \\
 &\quad \times {}_2\phi_1 \left[ \begin{matrix} qz^2/a^2, q/a^2 \\ qz^2 \end{matrix}; q, a^4 q^{\alpha+n} \right] \\
 &\quad + z \longleftrightarrow z^{-1}.
 \end{aligned} \tag{3.5}$$

Since

$$\lim_{\alpha \rightarrow 0^+} (1 - q^\alpha) {}_2\phi_1 \left[ \begin{matrix} a^2/z^2, a^2 \\ q/z^2 \end{matrix}; q, q^\alpha \right] = \frac{(a^2/z^2, a^2; q)_\infty}{(q, q/z^2; q)_\infty}, \tag{3.6}$$

one can see that (3.5) is a generalization of [7, (7.4.4)] as well as of the formula (5.11) of [11]. Replacing  $a$  by  $\beta^{\frac{1}{2}}$ , we find that (1.20) follows from (3.5).

#### 4. Some generating functions for $C_n^\alpha(x; \beta | q)$

Formula (3.5) is obviously the most convenient form for deriving generating functions for  $C_n^\alpha(x; \beta | q)$ . For example, it is easy to see that for any  $\lambda$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(\lambda; q)_n}{(q; q)_n} C_n^\alpha(x; \beta | q) t^n &= \frac{(1-q^\alpha)}{(1-z^{-2})} \frac{(\lambda z t; q)_\infty}{(z t; q)_\infty} {}_2\phi_1 \left[ \begin{matrix} \beta/z^2, \beta \\ q/z^2 \end{matrix}; q, q^\alpha \right] \\ &\quad \times {}_3\phi_2 \left[ \begin{matrix} qz^2/\beta, q/\beta, zt \\ qz^2, \lambda zt \end{matrix}; q, \beta^2 q^\alpha \right] \\ &\quad + z \longleftrightarrow z^{-1}. \end{aligned} \quad (4.1)$$

Assume that  $|t| < 1$ . Setting  $\lambda = q$ , we get an alternative form of (1.17):

$$\begin{aligned} G_t^\alpha(x; \beta | q) &= \frac{(1-q^\alpha)}{(1-z^{-2})(1-zt)} {}_2\phi_1 \left[ \begin{matrix} \beta/z^2, \beta \\ q/z^2 \end{matrix}; q, q^\alpha \right] \\ &\quad \times {}_3\phi_2 \left[ \begin{matrix} qz^2/\beta, q/\beta, zt \\ qz^2, qzt \end{matrix}; q, \beta^2 q^\alpha \right] \\ &\quad + z \longleftrightarrow z^{-1}. \end{aligned} \quad (4.2)$$

On the other hand, transforming the second  ${}_2\phi_1$  series on the right-hand side of (3.5) by [7, III.(3)], we get

$$\begin{aligned} C_n^\alpha(x; a^2 | q) &= \frac{1-q^\alpha}{1-z^{-2}} \frac{(q^{\alpha+n+1}; q)_\infty}{(\beta^2 q^{\alpha+n}; q)_\infty} z^n {}_2\phi_1 \left[ \begin{matrix} \beta/z^2, \beta \\ q/z^2 \end{matrix}; q, q^\alpha \right] \\ &\quad \times {}_2\phi_1 \left[ \begin{matrix} \beta z^2, \beta \\ qz^2 \end{matrix}; q, q^{\alpha+n+1} \right] \\ &\quad + z \longleftrightarrow z^{-1}. \end{aligned} \quad (4.3)$$

Hence,

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(\beta^2 q^\alpha; q)_n} C_n^\alpha(x; \beta | q) t^n &= \frac{(1-q^\alpha)}{(1-z^{-2})(1-zt)} \frac{(q^{\alpha+1}; q)_\infty}{(\beta^2 q^\alpha; q)_\infty} \\ &\quad \times {}_2\phi_1 \left[ \begin{matrix} \beta/z^2, \beta \\ q/z^2 \end{matrix}; q, q^\alpha \right] \\ &\quad \times {}_3\phi_2 \left[ \begin{matrix} \beta z^2, \beta, zt \\ qz^2, qzt \end{matrix}; q, q^{\alpha+1} \right] \\ &\quad + z \longleftrightarrow z^{-1}. \end{aligned} \quad (4.4)$$

More generally,

$$\begin{aligned}
 \sum_{n=0}^{\infty} \frac{(\lambda, q^{\alpha+1}; q)_n}{(q, \beta^2 q^\alpha; q)_n} C_n^\alpha(x; \beta | q) t^n &= \frac{(1 - q^\alpha)}{(1 - z^{-2})} \frac{(q^{\alpha+1}; q)_\infty}{(\beta^2 q^\alpha; q)_\infty} \frac{(\lambda z t; q)_\infty}{(z t; q)_\infty} \\
 &\quad \times {}_2\phi_1 \left[ \begin{matrix} \beta/z^2, \beta \\ q/z^2 \end{matrix}; q, q^\alpha \right] \\
 &\quad \times {}_3\phi_2 \left[ \begin{matrix} \beta z^2, \beta, z t \\ q z^2, \lambda z t \end{matrix}; q, q^{\alpha+1} \right] \\
 &\quad + z \longleftrightarrow z^{-1}. \tag{4.5}
 \end{aligned}$$

It also follows from (3.5) that

$$\begin{aligned}
 \sum_{n=0}^{\infty} C_{n+k}^\alpha(x; \beta | q) t^n &= \frac{(1 - q^\alpha)}{(1 - z^{-2})} z^k {}_2\phi_1 \left[ \begin{matrix} \beta/z^2, \beta \\ q/z^2 \end{matrix}; q, q^\alpha \right] \\
 &\quad \times {}_3\phi_2 \left[ \begin{matrix} q z^2/\beta, q/\beta, z t \\ q z^2, q z t \end{matrix}; q, \beta^2 q^{\alpha+1} \right] \\
 &\quad + z \longleftrightarrow z^{-1}, \quad k = 0, 1, \dots \tag{4.6}
 \end{aligned}$$

### 5. Some bilinear generating functions for $C_n^\alpha(x; \beta | q)$

In this section, we shall compute a few bilinear generating functions. First, let us consider the simplest one:

$$L_t^\alpha(x, y) := \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(\beta^2 q^\alpha; q)_n} (t q^{-\alpha})^n C_n^\alpha(x; \beta | q) C_n^\alpha(y; \beta | q), \tag{5.1}$$

which is symmetric in  $x$  and  $y$  where  $x = \cos \theta$ ,  $y = \cos \varphi$ ,  $0 < \theta, \varphi < \pi$ ,  $e^{i\theta} = z_1$ , and  $e^{i\varphi} = z_2$ . By assuming  $|t q^{-\alpha}| < 1$  and using (4.3) in (5.1), we find that

$$\begin{aligned}
 L_t^\alpha(x, y) &= \frac{(1 - q^\alpha)}{(1 - z_1^{-2})} \frac{(q^{\alpha+1}; q)_\infty}{(\beta^2 q^\alpha; q)_\infty} {}_2\phi_1 \left[ \begin{matrix} \beta/z_1^2, \beta \\ q/z_1^2 \end{matrix}; q, q^\alpha \right] \\
 &\quad \times \sum_{k=0}^{\infty} \frac{(\beta z_1^2, \beta; q)_k}{(q, q z_1^2; q)_k} q^{(\alpha+1)k} \sum_{n=0}^{\infty} (z_1 t q^{-\alpha+k})^n C_n^\alpha(y; \beta | q) \\
 &\quad + z_1 \longleftrightarrow z_1^{-1}. \tag{5.2}
 \end{aligned}$$

However, by (1.17),

$$\begin{aligned}
 \sum_{n=0}^{\infty} (z_1 t q^{-\alpha+k})^n C_n^\alpha(y; \beta | q) &= \frac{(1 - q^\alpha)}{(1 - 2y t z_1 q^{k-\alpha} + z_1^2 t^2 q^{2k-2\alpha})} \\
 &\quad \times {}_3\phi_2 \left[ \begin{matrix} q, \beta t z_1 q^{k-\alpha} e^{i\varphi}, \beta t z_1 q^{k-\alpha} e^{-i\varphi} \\ t z_1 q^{k-\alpha+1} e^{i\varphi}, t z_1 q^{k-\alpha+1} e^{-i\varphi} \end{matrix}; q, q^\alpha \right] \tag{5.3}
 \end{aligned}$$

and so the sum over  $k$  in (5.2) is

$$\begin{aligned}
& (1 - q^\alpha) \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\beta z_1^2, \beta; q)_k}{(q, qz_1^2; q)_k} q^{(\alpha+1)k+\alpha j} \\
& \quad \times \frac{(\beta tz_1 q^{-\alpha} e^{i\varphi}, \beta tz_1 q^{-\alpha} e^{-i\varphi}; q)_{j+k}}{(tz_1 q^{-\alpha} e^{i\varphi}, tz_1 q^{-\alpha} e^{-i\varphi}; q)_{j+k+1}} \frac{(tz_1 q^{-\alpha} e^{i\varphi}, tz_1 q^{-\alpha} e^{-i\varphi}; q)_k}{(\beta tz_1 q^{-\alpha} e^{i\varphi}, \beta tz_1 q^{-\alpha} e^{-i\varphi}; q)_k} \\
& = \frac{(1 - q^\alpha)}{(1 - 2ytz_1 q^{-\alpha} + z_1^2 t^2 q^{-2\alpha})} {}_3\phi_2 \left[ \begin{matrix} q, \beta tz_1 q^{-\alpha} e^{i\varphi}, \beta tz_1 q^{-\alpha} e^{-i\varphi} \\ tz_1 q^{-\alpha+1} e^{i\varphi}, tz_1 q^{-\alpha+1} e^{-i\varphi} \end{matrix}; q, q^\alpha \right] \\
& \quad \times {}_4\phi_3 \left[ \begin{matrix} \beta, \beta z_1^2, tz_1 q^{-\alpha} e^{i\varphi}, tz_1 q^{-\alpha} e^{-i\varphi} \\ qz_1^2, \beta tz_1 q^{-\alpha} e^{i\varphi}, \beta tz_1 q^{-\alpha} e^{-i\varphi} \end{matrix}; q, q \right]. \tag{5.4}
\end{aligned}$$

Hence,

$$\begin{aligned}
L_t^\alpha(x, y) &= \frac{(1 - q^\alpha)^2}{(1 - z^{-2})(1 - 2ytz_1 q^{-\alpha} + z_1^2 t^2 q^{-2\alpha})} \frac{(q^{\alpha+1}; q)_\infty}{(\beta^2 q^\alpha; q)_\infty} \\
& \quad \times {}_2\phi_1 \left[ \begin{matrix} \beta/z_1^2, \beta \\ q/z_1^2 \end{matrix}; q, q^\alpha \right] \\
& \quad \times {}_3\phi_2 \left[ \begin{matrix} q, \beta tz_1 q^{-\alpha} e^{i\varphi}, \beta tz_1 q^{-\alpha} e^{-i\varphi} \\ tz_1 q^{-\alpha+1} e^{i\varphi}, tz_1 q^{-\alpha+1} e^{-i\varphi} \end{matrix}; q, q^\alpha \right] \\
& \quad \times {}_4\phi_3 \left[ \begin{matrix} \beta, \beta z_1^2, tz_1 q^{-\alpha} e^{i\varphi}, tz_1 q^{-\alpha} e^{-i\varphi} \\ qz_1^2, \beta tz_1 q^{-\alpha} e^{i\varphi}, \beta tz_1 q^{-\alpha} e^{-i\varphi} \end{matrix}; q, q \right] \\
& \quad + z_1 \longleftrightarrow z_1^{-1}. \tag{5.5}
\end{aligned}$$

This leads to the expression (1.18) for the Poisson kernel for  $C_n^\alpha(y; \beta | q)$ . Note that the moment representation of  $C_n^\alpha(y; \beta | q)$  also enables us to compute the non-symmetric kernel

$$\begin{aligned}
M_t^\alpha(x, y; \beta, \beta_1) &:= \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(\beta^2 q^\alpha; q)_n} (tq^{-\alpha})^n C_n^\alpha(x; \beta | q) C_n^\alpha(y; \beta_1 | q) \\
&= \frac{(1 - q^\alpha)^2}{(1 - z^{-2})(1 - 2ytz_1 q^{-\alpha} + z_1^2 t^2 q^{-2\alpha})} \frac{(q^{\alpha+1}; q)_\infty}{(\beta^2 q^\alpha; q)_\infty} \\
& \quad \times {}_2\phi_1 \left[ \begin{matrix} \beta/z_1^2, \beta \\ q/z_1^2 \end{matrix}; q, q^\alpha \right] \\
& \quad \times {}_3\phi_2 \left[ \begin{matrix} q, \beta_1 tz_1 q^{-\alpha} e^{i\varphi}, \beta_1 tz_1 q^{-\alpha} e^{-i\varphi} \\ tz_1 q^{-\alpha+1} e^{i\varphi}, tz_1 q^{-\alpha+1} e^{-i\varphi} \end{matrix}; q, q^\alpha \right] \\
& \quad \times {}_4\phi_3 \left[ \begin{matrix} \beta, \beta z_1^2, tz_1 e^{i\varphi} q^{-\alpha}, tz_1 e^{-i\varphi} q^{-\alpha} \\ qz_1^2, \beta_1 tz_1 e^{i\varphi} q^{-\alpha}, \beta_1 tz_1 e^{-i\varphi} q^{-\alpha} \end{matrix}; q, q \right] \\
& \quad + z_1 \longleftrightarrow z_1^{-1}. \tag{5.6}
\end{aligned}$$

Observe that the  ${}_4\phi_3$  series in (5.5) is balanced, but the one in (5.6) is not unless  $\beta_1^2 = \beta^2$ . (5.3) may be seen as an extension of (5.5) of [11].

Finally, use of (3.5) and (1.17) gives

$$\begin{aligned}
 \sum_{n=0}^{\infty} C_n^\alpha(x; \beta | q) C_n^\alpha(y; \beta_1 | q) t^n &= \frac{(1 - q^\alpha)^2}{(1 - z^{-2})(1 - 2ytz_1 + z_1^2 t^2)} \\
 &\quad \times {}_2\phi_1 \left[ \begin{matrix} \beta/z_1^2, \beta \\ q/z_1^2 \end{matrix} ; q, q^\alpha \right] \\
 &\quad \times {}_3\phi_2 \left[ \begin{matrix} q, \beta_1 t z_1 e^{i\varphi}, \beta_1 t z_1 e^{-i\varphi} \\ q t z_1 e^{i\varphi}, q t z_1 e^{-i\varphi} \end{matrix} ; q, q^\alpha \right] \\
 &\quad \times {}_4\phi_3 \left[ \begin{matrix} q z_1^2 / \beta, q / \beta, t z_1 e^{i\varphi}, t z_1 e^{-i\varphi} \\ q z_1^2, \beta_1 t z_1 e^{i\varphi}, \beta_1 t z_1 e^{-i\varphi} \end{matrix} ; q, q \right] \\
 &\quad + z_1 \longleftrightarrow z_1^{-1}. \tag{5.7}
 \end{aligned}$$

## 6. Some limiting formulas

We will compute the  $q \rightarrow 1$  limits of formulas (3.4) and (5.5). We believe that these formulas are new.

It is clear from the expressions on the right-hand side of (3.4) and (5.5) that one cannot take the term-by-term limits. One has to do some transformations of the  $q$ -series involved in these formulas. Using (III. 1) and (III. 3) of [7], in that order, we find that

$${}_2\phi_1 \left[ \begin{matrix} q/\beta z^2, q/\beta \\ q/z^2 \end{matrix} ; q, \beta^2 q^{\alpha-1} \right] = \frac{(\beta q^\alpha, \beta/z^2; q)_\infty}{(\beta^2 q^{\alpha-1}, q/z^2; q)_\infty} {}_2\phi_1 \left[ \begin{matrix} q^\alpha, q/\beta \\ \beta q^\alpha \end{matrix} ; q, \beta/z^2 \right] \tag{6.1}$$

and

$${}_2\phi_1 \left[ \begin{matrix} q z^2 / \beta, q / \beta \\ q z^2 \end{matrix} ; q, \beta^2 q^{\alpha+n} \right] = \frac{(\beta q^{\alpha+n+1}, \beta z^2; q)_\infty}{(\beta^2 q^{\alpha+n}, q z^2; q)_\infty} {}_2\phi_1 \left[ \begin{matrix} q^{\alpha+n+1}, q / \beta \\ \beta q^{\alpha+n+1} \end{matrix} ; q, \beta z^2 \right] \tag{6.2}$$

These two formulas along with the definition of the  $q$ -gamma function [7, (1.10.1)] give

$$\begin{aligned}
 C_n^\alpha(x; q^\lambda | q) &= \frac{\Gamma_q(\alpha+1)\Gamma_q(2\lambda+\alpha+n)}{\Gamma_q(\alpha+\lambda)\Gamma_q(\alpha+n+\lambda+1)} \frac{(z^2 q^\lambda, q^\lambda/z^2; q)_\infty}{(q z^2, q/z^2; q)_\infty} \frac{z^n}{(1-z^{-2})} \\
 &\quad \times {}_2\phi_1 \left[ \begin{matrix} q^\alpha, q^{1-\lambda} \\ q^{\alpha+\lambda} \end{matrix} ; q, q^\lambda/z^2 \right] {}_2\phi_1 \left[ \begin{matrix} q^{\alpha+n+1}, q^{1-\lambda} \\ q^{\alpha+n+1+\lambda} \end{matrix} ; q, z^2 q^\lambda \right] \\
 &\quad + z \longleftrightarrow z^{-1}, \tag{6.3}
 \end{aligned}$$

where it is assumed that  $\operatorname{Re}(\lambda) > 0$  and  $\alpha \geq 0$ .

So the associated ultraspherical polynomials have the representation:

$$\begin{aligned}
 C_n^{\lambda, \alpha}(x) &= \frac{\Gamma(\alpha+1)\Gamma(2\lambda+\alpha+n)}{\Gamma(\alpha+\lambda)\Gamma(\alpha+n+\lambda+1)} \frac{2^{2-2\lambda} e^{i(n+1)\theta}}{2i(\sin\theta)^{2\lambda-1}} \\
 &\quad \times {}_2F_1 \left[ \begin{matrix} \alpha, 1-\lambda \\ \alpha+\lambda \end{matrix} ; e^{-2i\theta} \right] {}_2F_1 \left[ \begin{matrix} \alpha+n+1, 1-\lambda \\ \alpha+n+1+\lambda \end{matrix} ; e^{2i\theta} \right] \\
 &\quad + \theta \longleftrightarrow -\theta. \tag{6.4}
 \end{aligned}$$

For convergence on the unit circle of the two Gaussian series on the right-hand side we need the stronger condition  $\operatorname{Re}(\lambda) > \frac{1}{2}$ . One can rewrite (6.4) in the form

$$C_n^{\lambda, \alpha}(x) = \frac{\Gamma(\alpha+1)\Gamma(2\lambda+\alpha+n)}{\Gamma(\alpha+\lambda)\Gamma(\alpha+n+\lambda+1)} \frac{2^{2-2\lambda}}{(\sin\theta)^{2\lambda-1}} \\ \times \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\alpha)_j (1-\lambda)_j (\alpha+n+1)_k (1-\lambda)_k}{j! k! (\alpha+\lambda)_j (\alpha+n+1+\lambda)_k} \sin(n+1-2j+2k)\theta, \quad (6.5)$$

which is a generalization of [21, (4.9.22)] to which it reduces when  $\alpha = 0$ . In order to find the  $q \rightarrow 1$  limit of  $L_t^\alpha(x, y)$ , we do not use the final formula (5.5); rather we use the intermediate formula (5.2). First, by [7, (III.1)]

$${}_2\phi_1 \left[ \begin{matrix} \beta/z_1^2, \beta \\ q/z_1^2 \end{matrix}; q, q^\alpha \right] = \frac{(\beta q^\alpha, \beta/z_1^2; q)_\infty}{(q^\alpha, q/z_1^2; q)_\infty} {}_2\phi_1 \left[ \begin{matrix} q^\alpha, q/\beta \\ \beta q^\alpha \end{matrix}; q, \beta/z_1^2 \right].$$

Now, the series over  $k$  in (5.2) is

$$\sum_{n=0}^{\infty} (z_1 t q^{-\alpha})^n C_n^\alpha(y; \beta | q) {}_2\phi_1 \left[ \begin{matrix} \beta z_1^2, \beta \\ q z_1^2 \end{matrix}; q, q^{\alpha+n+1} \right] \\ = \frac{(\beta q^{\alpha+1}, \beta z_1^2; q)_\infty}{(q^{\alpha+1}, q z_1^2; q)_\infty} \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(\beta q^{\alpha+1}; q)_n} (z_1 t q^{-\alpha})^n C_n^\alpha(y; \beta | q) \\ \times {}_2\phi_1 \left[ \begin{matrix} q^{\alpha+n+1}, q/\beta \\ \beta q^{\alpha+n+1} \end{matrix}; q, \beta z_1^2 \right]. \quad (6.6)$$

We now substitute (6.5) and (6.6) into (5.2) and replace  $\beta$  by  $q^\lambda$  to get

$$L_t^\alpha(x, y) = \frac{\Gamma_q(\alpha+1)\Gamma_q(2\lambda+\alpha)}{\Gamma_q(\alpha+\lambda)\Gamma_q(\alpha+\lambda+1)} \frac{(z_1^2 q^\lambda, q^\lambda/z_1^2; q)_\infty}{(q z_1^2, q/z_1^2; q)_\infty} (1-z_1^{-2})^{-1} \\ \times {}_2\phi_1 \left[ \begin{matrix} q^\alpha, q^{1-\lambda} \\ q^{\alpha+\lambda} \end{matrix}; q, q^\lambda/z_1^2 \right] \\ \times \sum_{n=0}^{\infty} \frac{(q^{\alpha+1}; q)_n}{(q^{\alpha+\lambda+1}; q)_n} (z_1 t q^{-\alpha})^n C_n^\alpha(y; q^\lambda | q) {}_2\phi_1 \left[ \begin{matrix} q^{\alpha+n+1}, q^{1-\lambda} \\ q^{\alpha+\lambda+n+1} \end{matrix}; q, z_1^2 q^\lambda \right] \\ + z_1 \longleftrightarrow z_1^{-1}, \quad (6.7)$$

for  $\operatorname{Re}(\lambda) > 0$ ,  $\operatorname{Re}(\alpha) \geq 0$ . We now take the limit to obtain

$$\lim_{q \rightarrow 1^-} L_t^\alpha(x, y) = \frac{\Gamma(\alpha+1)\Gamma(2\lambda+\alpha)}{\Gamma(\alpha+\lambda)\Gamma(\alpha+\lambda+1)} \frac{2^{2-2\lambda}}{(\sin\theta)^{2\lambda-1}} \frac{e^{i\theta}}{2i} {}_2F_1 \left[ \begin{matrix} \alpha, 1-\lambda \\ \alpha+\lambda \end{matrix}; e^{-2i\theta} \right] \\ \times \sum_{n=0}^{\infty} \frac{(\alpha+1)_n}{(\alpha+\lambda+1)_n} (te^{i\theta})^n C_n^{\lambda, \alpha}(y) {}_2F_1 \left[ \begin{matrix} \alpha+n+1, 1-\lambda \\ \alpha+n+1+\lambda \end{matrix}; e^{2i\theta} \right] \\ + \theta \longleftrightarrow -\theta. \quad (6.8)$$

To ensure convergence of the two  ${}_2F_1$  series on the right-hand side, we assume, as before, that  $\operatorname{Re}(\lambda) > \frac{1}{2}$ .

We now shall express the series over  $n$  on the right-hand side of (6.8) as a double integral. First, by Euler's integral representation [7, (1.11.10)], we have

$$\begin{aligned} & \frac{\Gamma(\alpha + n + 1)\Gamma(\lambda)}{\Gamma(\alpha + 1 + n + \lambda)} {}_2F_1 \left[ \begin{matrix} \alpha + n + 1, 1 - \lambda \\ \alpha + n + 1 + \lambda \end{matrix} ; e^{2i\theta} \right] \\ &= \int_0^1 u^{\alpha+n} (1-u)^{\lambda-1} (1-ue^{2i\theta})^{\lambda-1} du, \quad 0 < \theta < \pi, \quad n = 0, 1, \dots \end{aligned} \quad (6.9)$$

Since

$$\frac{\alpha}{\alpha + k} = \alpha \int_0^1 v^{\alpha+k-1} dv, \quad \operatorname{Re}(\alpha) > 0, \quad (6.10)$$

and, by [6, (3.13)],

$$C_n^{\lambda, \alpha}(y) = \sum_{k=0}^n \frac{\alpha}{\alpha + k} C_{n-k}^{\lambda}(y) C_k^{1-\lambda}(y), \quad (6.11)$$

we find that

$$\begin{aligned} & \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1 + \lambda)} \sum_{n=0}^{\infty} \frac{(\alpha + 1)_n}{(\alpha + \lambda + 1)_n} (te^{i\theta})^n C_n^{\lambda, \alpha}(y) {}_2F_1 \left[ \begin{matrix} \alpha + n + 1, 1 - \lambda \\ \alpha + n + 1 + \lambda \end{matrix} ; e^{2i\theta} \right] \\ &= \frac{\alpha}{\Gamma(\lambda)} \int_0^1 \int_0^1 u^{\alpha} (1-u)^{\lambda-1} (1-ue^{2i\theta})^{\lambda-1} v^{\alpha-1} \\ & \quad \times \left\{ \sum_{n=0}^{\infty} (ute^{i\theta})^n \sum_{k=0}^n C_{n-k}^{\lambda}(y) C_k^{1-\lambda}(y) v^k \right\} du dv \\ &= \frac{\alpha}{\Gamma(\lambda)} \int_0^1 \int_0^1 u^{\alpha} (1-u)^{\lambda-1} (1-ue^{2i\theta})^{\lambda-1} v^{\alpha-1} (1-2utye^{i\theta} + u^2t^2e^{2i\theta})^{-\lambda} \\ & \quad \times (1-2uvtye^{i\theta} + u^2v^2t^2e^{2i\theta})^{\lambda-1} dudv, \end{aligned} \quad (6.12)$$

where the last line has been obtained by using the well-known generating function of the ultraspherical polynomials. Thus, we get

$$\begin{aligned} \lim_{q \rightarrow 1^-} L_t^{\alpha}(x, y) &= \frac{\alpha \Gamma(2\lambda + \alpha)}{\Gamma(\alpha + \lambda) \Gamma(\lambda)} \frac{2^{2-2\lambda}}{(\sin \theta)^{2\lambda-1}} \frac{e^{i\theta}}{2i} {}_2F_1 \left[ \begin{matrix} \alpha, 1 - \lambda \\ \alpha + \lambda \end{matrix} ; e^{-2i\theta} \right] \\ & \quad \times \int_0^1 \int_0^1 u^{\alpha} (1-u)^{\lambda-1} (1-ue^{2i\theta})^{\lambda-1} v^{\alpha-1} (1-2utye^{i\theta} + u^2t^2e^{2i\theta})^{-\lambda} \\ & \quad \times (1-2uvtye^{i\theta} + u^2v^2t^2e^{2i\theta})^{\lambda-1} dudv \\ & \quad + \theta \longleftrightarrow -\theta, \end{aligned} \quad (6.13)$$

with  $|t| < 1$ . Using (6.13), one can easily find the limit of  $K_t^{\alpha}(x, y)$ .

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