Methods and Applications of Analysis
5 (3) 1998, pp. 259–272

ISSN 1073–2772

Solvability in L_p of the Dirichlet problem for a singular nonhomogeneous Sturm-Liouville equation

N. Chernyavskaya and L. Shuster

Abstract. We consider the equation

$$-(r(x)y'(x))' + q(x)y(x) = f(x), \qquad x \in R \tag{$*$}$$

where $f(x) \in L_s(R)$, $s \in [1,\infty]$, $r(x) > 0$, $q(x) \geq 0$ for $x \in R$, $\frac{1}{r(x)} \in L_1^{\text{loc}}(R)$, and $q(x) \in L_1^{\text{loc}}(R)$. The inversion problem for equation $(*)$ is called regular in L_p if equation $(*)$ has a unique solution $y(x) \in L_p(R)$ of the form

$$y(x) = \int_{-\infty}^{\infty} G(x,t)f(t)dt, \qquad x \in R,$$

with $\|y\|_p \leq c\|f\|_p$ uniformly in $p \in [1,\infty]$ for any $f(x) \in L_p(R)$. Here $G(x,t)$ is the Green function corresponding to $(*)$, c is an absolute constant. For a given $s \in [1,\infty]$, we give necessary and sufficient conditions for the following assertions to hold simultaneously:

the inversion problem for $(*)$ is regular in L_p, and

$\lim_{|x|\to\infty} y(x) = 0$ for all $f(x) \in L_s(R)$.

1. Introduction

In this paper, we are interested in the existence and certain properties of the solution of a Dirichlet problem :

$$-(r(x)y'(x))' + q(x)y(x) = f(x), \qquad x \in R, \tag{1.1}$$

$$\lim_{|x|\to\infty} y(x) = 0 \tag{1.2}$$

where $f(x) \in L_s(R)$, $s \in [1,\infty]$, and $r(x)$ and $q(x)$ satisfy the conditions

$$r(x) > 0,\ q(x) \geq 0 \text{ for } x \in R; \quad \frac{1}{r(x)} \in L_1^{\text{loc}}(R),\ q(x) \in L_1^{\text{loc}}(R). \tag{1.3}$$

Our goal is to find all $s \in [1,\infty]$ such that if $f(x) \in L_s(R)$, then (1.1)–(1.2) is solvable. Note that we consider (1.1)–(1.2) only for those equations (1.1) for which the inversion problem is regular in L_p. This means [1] that the following two assertions hold simultaneously:

(R1) (1.1) has a unique solution $y(x) \in L_p(R)$ of the form

$$y(x) = (Gf)(x) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} G(x,t)f(t)dt, \qquad x \in R, \tag{1.4}$$

Received October 7, 1997, revised May 19, 1998.
1991 *Mathematics Subject Classification*: 34B05.
Key words and phrases: second-order ordinary differential equation, the Dirichlet problem, existence and uniqueness of the solution, estimates of the solution.

uniformly in $p \in [1,\infty]$ for any $f(x) \in L_p(R)$. Here $G(x,t)$ is the Green function corresponding to (1.1):

$$G(x,t) = \begin{cases} u(x)v(t), & x \ge t, \\ u(t)v(x), & x \le t, \end{cases} \tag{1.5}$$

and $\{u(x), v(x)\}$ is a special fundamental system of solutions (FSS) of the equation

$$(r(x)z'(x))' = q(x)z(x), \ x \in R. \tag{1.6}$$

(R2) There is an absolute constant $c > 0$ such that for all $p \in [1,\infty]$, one has

$$\|y\|_p \le c\|f\|_p \quad \text{if} \quad f(x) \in L_p(R). \tag{1.7}$$

In [1, 2], necessary and sufficient conditions for (R1)–(R2) to hold were found. The conditions coincide and become a criterion for a wide class of equations (1.1) which we call standard; we denote this class by S (see [1] and §2 below). Now we can give the precise statement of our problem I:

I. Let $(1.1) \in S$, let the inversion problem for (1.1) be regular in L_p, and let $s \in [1,\infty]$ be given. Under what condition, for any $f(x) \in L_s(R)$, would the solution (1.4) of (1.1) satisfy the equality (1.2)?

Our main Theorem 3.1 gives a precise answer to this question for all $s \in [1,\infty]$. In addition, we show (Theorem 3.2) that for $(1.1) \in S$, Problem I for $s \in [1,\infty)$ is equivalent to the following Problem II:

II. Let $(1.1) \in S$, suppose that the inversion problem for (1.1) is regular in L_p, and let $s \in [1,\infty]$ be given. Under what condition does there exist an absolute constant $c = c(s) > 0$ such that for any $f(x) \in L_s(R)$, the solution (1.4) of (1.1) would satisfy an inequality $\|y\|_{C(R)} \le c(s)\|f\|_s$?

We want to stress that our results are formulated in terms of the same auxiliary functions (local integral averages of $r(x)$ and $q(x)$) as the conditions of regularity of the inversion problem for (1.1) in L_p. This is convenient for applications and, on the other hand, as in [1, 2], our results include Dirichlet problems for "interesting" equations with oscillating coefficients.

Note that in the proofs, we mainly use the results of [1, 2, 5], which we summarize for the reader's convenience in §2. In addition, we would like to point out that auxiliary functions of type $\tilde{d}(x)$ (see §2, Theorem 2.4) were introduced and extensively studied by M.O. Otelbaev. Such functions turned out to be an effective tool in solving various problems related to differential and difference operators. For some of these results, ideas, and original techniques, see [4]. The problem treated in the present paper was not considered by Otelbaev. Our approach is based on combining his methods with the method of studying the Sturm-Liouville operator which we proposed in [1, 2].

2. Preliminaries

In this section, we give some notions and results from [1, 2, 5]. Below we denote by c absolute positive constants which are not important for the exposition and which may differ even within a single chain of calculations. Throughout we assume that $r(x)$ and

$q(x)$ satisfy the following conditions:

$$r(x)>0,\quad q(x)\ge 0\quad \text{for}\quad x\in R;\quad \frac{1}{r(x)}\in L_1^{\mathrm{loc}}(R),\quad q(x)\in L_1^{\mathrm{loc}}(R), \tag{2.1}$$

$$\lim_{|d|\to\infty}\left(\int_{x-d}^{x}\frac{dt}{r(t)}\cdot\int_{x-d}^{x}q(t)dt\right)=\infty,\quad x\in R. \tag{2.2}$$

Theorem 2.1. [2] *Consider the equation*

$$(r(x)z'(x))'=q(x)z(x),\qquad x\in R. \tag{2.3}$$

There exists a FSS $\{u(x),v(x)\}$ *of* (2.3) *such that*

$$\begin{gathered} u(x)>0,\ v(x)>0,\ u'(x)<0,\ v'(x)>0 \text{ for } \quad x\in R;\\ r(x)[v'(x)u(x)-u'(x)v(x)]=1 \text{ for }\quad x\in R;\\ \lim_{x\to-\infty}\frac{v(x)}{u(x)}=\lim_{x\to\infty}\frac{u(x)}{v(x)}=0. \end{gathered} \tag{2.4}$$

The FSS $\{u(x),v(x)\}$ with properties (2.4) is called a basic FSS (BFSS) of (2.3).

Lemma 2.1. [2, 3] *For* $x,t\in R$, u *and* v *have the following representations:*

$$u(x)=\sqrt{\rho(x)}\exp\left(-\frac{1}{2}\int_{x_1}^{x}\frac{d\xi}{r(\xi)\rho(\xi)}\right),\ v(x)=\sqrt{\rho(x)}\exp\left(\frac{1}{2}\int_{x_1}^{x}\frac{d\xi}{r(\xi)\rho(\xi)}\right), \tag{2.5}$$

$$G(x,t)\overset{\text{def}}{=}\begin{cases}u(x)v(t), & x\ge t\\ u(t)v(x), & x\le t\end{cases}=\sqrt{\rho(x)\rho(t)}\exp\left(-\frac{1}{2}\left|\int_x^t\frac{d\xi}{r(\xi)\rho(\xi)}\right|\right) \tag{2.6}$$

where $\rho(x)=u(x)v(x)$ *and* x_1 *is the unique root of the equation* $u(x)=v(x)$.

Corollary. *One has the following equalities:*

$$\int_{-\infty}^{0}\frac{d\xi}{r(\xi)\rho(\xi)}=\int_0^{\infty}\frac{d\xi}{r(\xi)\rho(\xi)}=\infty. \tag{2.7}$$

Lemma 2.2. [2] *For every* $x\in R$, *the following equations in* $d\ge 0$ *have unique finite positive solutions:*

$$1=\int_{x-d}^{x}\frac{dt}{r(t)}\cdot\int_{x-d}^{x}q(t)dt,\qquad 1=\int_x^{x+d}\frac{dt}{r(t)}\cdot\int_x^{x+d}q(t)dt. \tag{2.8}$$

Denote by $d_1(x)$, $d_2(x)$ the respective solutions of (2.8) and introduce the functions

$$\varphi(x)=\int_{x-d_1(x)}^{x}\frac{dt}{r(t)},\quad \psi(x)=\int_x^{x+d_2(x)}\frac{dt}{r(t)},\quad h(x)=\frac{\varphi(x)\psi(x)}{\varphi(x)+\psi(x)},\ x\in R. \tag{2.9}$$

Theorem 2.2. [2] *For* $x\in R$, *the following inequalities hold:*

$$2^{-1}h(x)\le\rho(x)\le 2h(x). \tag{2.10}$$

Lemma 2.3. [2] *For every* $x\in R$, *the equation*

$$1=\int_{x-d}^{x+d}\frac{d\xi}{r(\xi)h(\xi)} \tag{2.11}$$

has a unique finite positive solution in d. *Denote this solution by* $d(x)$. *The function* $d(x)$ *is continuous for* $x\in R$.

Lemma 2.4. [2] *For $x \in R$, $t \in [x - d(x), x + d(x)]$, the following inequalities hold:*

$$9^{-1}v(x) \le v(t) \le 9v(x), \qquad 9^{-1}u(x) \le u(t) \le 9u(x), \tag{2.12}$$

$$9^{-1}\rho(x) \le \rho(t) \le 9\rho(x), \qquad (36)^{-1}h(x) \le h(t) \le 36h(x). \tag{2.13}$$

Definition 2.1. [2] We say that a system of segments $\{\Delta\}_{n\in N'}$, $N' = \pm 1, \pm 2, \dots,$ forms an $R(x)$-covering of R if the following conditions hold:

1. $\Delta_n = [\Delta_n^-, \Delta_n^+] \overset{\text{def}}{=} [x_n - d(x_n), x_n + d(x_n)], \quad n \in N',$
2. $\Delta_{n+1}^- = \Delta_n^+$ if $n \ge 1$; $\Delta_{n-1}^+ = \Delta_n^-$ if $n \le -1$,
3. $\Delta_1^- = \Delta_{-1}^+ = x,$
4. $\bigcup_{n\in N'} \Delta_n = R.$

Lemma 2.5. [2] *For every $x \in R$, there exists an $R(x)$-covering of R.*

Definition 2.2. [1] A nonhomogeneous equation (1.1) and the corresponding homogeneous equation (2.3) are called standard if

$$m \overset{\text{def}}{=} \sup_{x\in R}(r(x)|\rho'(x)|) < 1. \tag{2.14}$$

The set of all standard equations is denoted by S. By writing (1.1) $\in S$, (2.3) $\in S$, we mean that (1.1) and (2.3) are standard equations.

Lemma 2.6. [1] *Condition* (2.14) *holds if and only if there is $c \ge 1$ such that*

$$c^{-1}\varphi(x) \le \psi(x) \le c\varphi(x) \quad \text{for } x \in R. \tag{2.15}$$

Definition 2.3. [1] We say that for a given equation (1.1), the inversion problem is regular in L_p if one has assertions (R1)–(R2) from §1.

Theorem 2.3. [1] *For the inversion problem for* (1.1) *to be regular in L_p, it is necessary, and for* (1.1) $\in S$ *it is sufficient, that*

$$B \overset{\text{def}}{=} \sup_{x\in R}(h(x)d(x)) < \infty. \tag{2.16}$$

Corollary. *Let* (1.1) $\in S$ *and $B < \infty$. Then*

$$H \overset{\text{def}}{=} \sup_{x\in R}\left(\int_{-\infty}^{\infty} G(x,t)dt\right) \le cB. \tag{2.17}$$

Theorem 2.4. [1, 2] *Suppose there is a $\gamma \ge 1$ such that*

$$\gamma^{-1} \le r(x) \le \gamma \quad \text{for} \quad x \in R. \tag{2.18}$$

For every $x \in R$, consider the equation in $d \ge 0$:

$$2 = d\int_{x-d}^{x+d} q(t)dt. \tag{2.19}$$

Equation (2.19) *has a unique positive continuous solution $\tilde{d}(x)$ and, moreover,*

$$c^{-1}\tilde{d}(x) \le d(x) \le c\tilde{d}(x), \quad x \in R, \tag{2.20}$$

$$(2\gamma + 2)^{-1}\tilde{d}(x) \le \rho(x) \le \frac{2\gamma + 1}{2}\tilde{d}(x), \quad x \in R. \tag{2.21}$$

Theorem 2.5. [1] *Under condition* (2.18), *the inversion problem for* (1.1) *is regular in* L_p *if and only if*

$$\mathcal{K} \overset{\text{def}}{=} \sup_{x\in R}(\tilde{d}(x)) < \infty. \tag{2.22}$$

Lemma 2.7. [5] *One has* $\tilde{d}(x) \to 0$ *as* $|x| \to \infty$ *if and only if for any* $a > 0$, *one has*

$$\lim_{|x|\to\infty} \int_{x-a}^{x+a} q(t)dt = \infty. \tag{2.23}$$

3. Statement of results

In this section, we formulate the main results of the paper. Throughout, we use the following notation: $f(x)$ is an arbitrary function from $L_s(R)$, $s \in [1,\infty]$, $y(x) = (Gf)(x)$ (cf. (1.4)). Problem I (see §1) is solved by the following theorem.

Theorem 3.1. *Suppose that the inversion problem for* (1.1) *is regular in* L_p. *If* $s \in [1,\infty)$, *then to have the equality*

$$\lim_{|x|\to\infty} y(x) = 0, \tag{3.1}$$

it is necessary, and under condition (1.1) $\in S$ *it is sufficient, that*

$$\mathcal{D}_s \overset{\text{def}}{=} \sup_{x\in R}\left(h(x)d(x)^{1/s'}\right) < \infty, \qquad \frac{1}{s} + \frac{1}{s'} = 1. \tag{3.2}$$

If $s = \infty$, *then to have* (3.1), *it is necessary, and under the condition* (1.1) $\in S$ *it is sufficient, that* $h(x)d(x)$ *would have a limit as* $|x| \to \infty$ *and*

$$\mathcal{D}_\infty \overset{\text{def}}{=} \lim_{|x|\to\infty}(h(x)d(x)) = 0. \tag{3.2$'$}$$

The following assertion gives a solution of Problem II (see §1).

Theorem 3.2. *Suppose that the inversion problem for* (1.1) *is regular in* L_p, *then*

(a) *If for some* $s \in [1,\infty]$, *one has*

$$\|y\|_{C(R)} \le c(s)\|f\|_s \tag{3.3}$$

where $c(s)$ *is a positive constant depending only on* s, *then* $\mathcal{D}_s < \infty$.

(b) *Let* (1.1) $\in S$, $s \in [1,\infty)$. *Then* (3.3) *holds if and only if* (3.1) *holds.*

(c) *Let* (1.1) $\in S$ *and* $s = \infty$. *Then* (3.3) *holds if and only if* $B < \infty$ *(see* (2.16)*).*

In the following assertion, we show that under the additional assumption (2.18), one has equality (3.1) regardless of s and of the condition (1.1) $\in S$.

Theorem 3.3. *Suppose that* (2.18) *holds and* $\mathcal{K} < \infty$ *(see* (2.22)*). Then for any* $s \in [1,\infty)$, *one has* (3.1). *For* $s = \infty$, *one has* (3.1) *if and only if* (2.23) *holds.*

Remark. For $r(x) \equiv 1$ and $1 \le q(x) \in L_1^{\text{loc}}(R)$, Theorem 3.3 was proved in [5].

4. Estimates of auxiliary functions.

Below we use the following fact:

Theorem 4.1. [2] *Suppose that under assumptions* (2.1) *there exist functions* $r_1(x)$, $r_2(x)$, $q_1(x)$, $q_2(x)$ *such that*

(1) $r(x) = r_1(x) + r_2(x)$, $q(x) = q_1(x) + q_2(x)$, $x \in R$, *where* $r_1(x), q_1(x)$ *are continuous and positive for* $x \in R$;

(2) *for some constants* a, b $(b \geq 3a > 0)$ *and* $|x| \gg 1$, *one has inequalities*

$$a^{-1} \leq \frac{r_1(t)}{r_1(x)} \leq a, \ a^{-1} \leq \frac{q_1(t)}{q_1(x)} \leq a \ \text{ for } |t-x| \leq b\hat{d}(x), \quad \hat{d}(x) = \sqrt{\frac{r_1(x)}{q_1(x)}}; \tag{4.1}$$

(3) *there is* $\delta \in (0,1]$ *such that* $r(x) \geq \delta r_1(x)$ *for* $x \in R$; *and*

(4) $\lim_{|x|\to\infty} \varkappa_1(x) = \lim_{|x|\to\infty} \varkappa_2(x) = 0$ *where*

$$\varkappa_1(x) = \frac{1}{\sqrt{r_1(x)q_1(x)}} \sup_{|z|\leq b\hat{d}(x)} \left| \int_x^{x+z} q_2(t)dt \right|,$$

$$\varkappa_2(x) = \sqrt{r_1(x)q_1(x)} \sup_{|z|\leq b\hat{d}(x)} \left| \int_x^{x+z} \frac{r_2(t)}{r_1(t)^2} dt \right|.$$

Then the following assertions hold:

(A) *One has* (2.2). *Equations* (2.8) *have unique finite positive solutions* $d_1(x)$ *and* $d_2(x)$ *for every* $x \in R$. *In addition,* (2.3) $\in S$ *and*

$$c^{-1}\frac{1}{\sqrt{r_1(x)q_1(x)}} \leq h(x) \leq \frac{c}{\sqrt{r_1(x)q_1(x)}}, \quad x \in R. \tag{4.2}$$

(B) *If, in addition,* $b \geq 160a^3\delta^{-2}$, *then (see* (4.1)*)*

$$c^{-1}\hat{d}(x) \leq d(x) \leq c\hat{d}(x), \qquad x \in R. \tag{4.3}$$

Corollary. [1] *Let* $r(x)$ *and* $q(x)$ *satisfy the hypotheses of Theorem* 4.1, *then the inversion problem for* (1.1) *is regular in* L_p *if and only if* $\inf_{x\in R}\ q_1(x) > 0$.

Example. Consider (1.1) with the following coefficients:

$$r(x) = \exp(-(2\alpha-1)|x|) + \frac{1}{2}\exp(-(2\alpha-1)|x|)\sin(\exp(2\alpha|x|)), \ \alpha \geq 1, \ x \in R,$$
$$q(x) = \exp(|x|) + \exp(|x|)\sin(\exp(2\alpha|x|)), \qquad x \in R.$$

According to Theorem 4.1, set $r_1(x) = \exp(-(2\alpha-1)|x|)$, $q_1(x) = \exp(|x|)$, $x \in R$. Clearly, $\delta = 1/2$, $\hat{d}(x) = \exp(-\alpha(x))$. Since all the functions are even, it suffices to give the needed estimates for $x \gg 1$. Since $\hat{d}(x) = \exp(-\alpha x)$, it is easy to see that inequalities (4.1) are true, say, for $a = 2$, $b = 160a^3\delta^{-2} = 320$. To estimate $\varkappa_1(x)$ and $\varkappa_2(x)$ in this case, one can apply the second mean value theorem ([6], Ch.12, §3):

$$\begin{aligned}\varkappa_1(x) &= \exp((\alpha-1)x) \sup_{|z|\leq b\hat{d}(x)} \left| \int_x^{x+z} \exp(-(2\alpha-1)t)\cdot\exp(2\alpha t)\sin(\exp(2\alpha t))dt \right| \\ &\leq c\exp((\alpha-1)x) \sup_{|z|\leq b\hat{d}(x)} \exp(-(2\alpha-1)x) \left| \int_x^{x+z} \exp(2\alpha t)\sin(\exp(2\alpha t))dt \right| \\ &\leq c\exp(-\alpha x),\end{aligned}$$

$$\begin{aligned}\varkappa_2(x) &= \exp(-(\alpha-1)x)\\ &\quad\times \sup_{|z|\le b\hat{d}(x)}\left|\frac{1}{2}\int_x^{x+z}(\sin\exp(2\alpha t)\exp(-2(\alpha-1)t)[\exp(2(\alpha-1)t)]^2dt\right|\\ &= \exp(-(\alpha-1)x\sup_{|z|\le b\hat{d}(x)}\left|\frac{1}{2}\int_x^{x+z}\exp(-t)[\exp(2\alpha t)\sin(\exp 2\alpha t)]dt\right|\\ &\le c\exp(-(\alpha-1)x)\sup_{|z|\le b\hat{d}(x)}\exp(-x)\left|\int_x^{x+z}\exp(2\alpha t)\sin(\exp(2\alpha t))dt\right|\\ &\le c\exp(-\alpha x).\end{aligned}$$

It follows that all the hypotheses of Theorem 4.1 are satisfied, and thus

$$c^{-1}\exp((\alpha-1)|x|)\le h(x)\le c\exp((\alpha-1)|x|),\quad x\in R, \tag{4.4}$$

$$c^{-1}\exp(-\alpha|x|)\le d(x)\le c\exp(-\alpha|x|),\quad x\in R. \tag{4.5}$$

Since $\inf_{x\in R} q_1(x)=\inf_{x\in R}\exp(|x|)=1$, by the corollary of Theorem 4.1, we conclude that in this case the inversion problem for (1.1) is regular in L_p and, in addition, by Theorem 4.1, (1.1) $\in S$. Let $s\in[1,\infty)$. By (4.4) and (4.5), one has inequalities

$$c^{-1}\exp\left(\frac{\alpha}{s}-1\right)\le h(x)d(x)^{1/s'}\le c\exp\left(\frac{\alpha}{s}-1\right)$$

and, therefore, $\mathcal{D}_s<\infty$ if $s\ge\alpha$, and $\mathcal{D}_s=\infty$ if $s<\alpha$.

By Theorem 3.1, we conclude that if $1\le s<\infty$, then the Dirichlet problem for (1.1) with the above coefficients is not solvable in $L_s(R)$ for $s<\alpha$ (i.e., there exist $f(x)\in L_s(R)$, $s<\alpha$ such that (1.1)–(1.2) has no solutions), and, conversely, for $s\ge\alpha$ the Dirichlet problem is solvable in $L_s(R)$ (i.e., for any $f(x)\in L_s(R)$, problem (1.1)–(1.2) has a unique solution $y\in L_s(R)$ with $\|y\|_s\le c\|f\|_s$, $\|y\|_{C(R)}\le c(s)\|f\|_s$). For $s=\infty$, we have $\mathcal{D}_\infty=0$ by (4.4)–(4.5), i.e., for any $f(x)\in L_\infty(R)$, the Dirichlet problem has a unique solution $y\in L_\infty(R)$ with $\|y\|_{C(R)}\le c\|f\|_\infty$.

5. Auxiliary results

To prove Theorems 3.1–3.3, we need the following assertions.

Lemma 5.1. *Let $\varepsilon\in[0,1]$, $t\in[x-\varepsilon d(x),\ x+\varepsilon d(x)]$, $x\in R$. Then*

$$(1-\varepsilon)d(x)\le d(t)\le(1+\varepsilon)d(x). \tag{5.1}$$

For the function $d(x)$, one has the relations

$$\overline{\lim_{x\to-\infty}}(x+d(x))=-\infty,\qquad \underline{\lim_{x\to\infty}}(x-d(x))=\infty. \tag{5.2}$$

Proof. Let us verify that for $x\in R$, one has the inequality

$$|d(x+\eta)-d(x)|\le|\eta|\quad\text{for}\quad|\eta|\le d(x). \tag{5.3}$$

Indeed, suppose, for example, that $0\le\eta\le d(x)$. Then

$$1=\int_{x-d(x)}^{x+d(x)}\frac{d\xi}{r(\xi)h(\xi)}\le\int_{x+\eta-(\eta+d(x))}^{x+\eta+(\eta+d(x))}\frac{d\xi}{r(\xi)h(\xi)},$$

$$\int_{x-d(x)}^{x+d(x)}\frac{d\xi}{r(\xi)h(\xi)}\ge\int_{x+\eta-(d(x)-\eta)}^{x+\eta+(d(x)-\eta)}\frac{d\xi}{r(\xi)h(\xi)}.$$

The first inequality implies $d(x+\eta) \le d(x)+\eta$, and from the second one, we obtain $d(x+\eta) \ge d(x)-\eta$. Hence, $-\eta \le d(x+\eta)-d(x) \le \eta$. The case $\eta \in [-d(x),0]$ can be treated in a similar way.

To check (5.1), set $t = x+\eta$ in (5.3). Then we can write (5.3) in the form

$$|d(t)-d(x)| \le |x-t| \quad \text{for} \quad |t-x| \le \varepsilon d(x). \tag{5.4}$$

In (5.4), set $|t-x| \le \varepsilon d(x),\ \varepsilon \in [0,1]$. Then

$$\left|\frac{d(t)}{d(x)} - 1\right| \le \frac{|t-x|}{d(x)} \le \varepsilon \quad \text{for} \quad |t-x| \le \varepsilon d(x).$$

The latter inequality is equivalent to (5.1). Now let us verify, for example, the second equality of (5.2) (the first one can be checked in a similar way). Assume the contrary. Then there is a sequence $\{x_n\}_{n=1}^\infty$ such that $x_n \to \infty$ as $n \to \infty$, and $x_n - d(x_n) \le c < \infty$. Relations (2.11), (2.10), and (2.7) lead to a contradiction:

$$1 = \int_{x_n-d(x_n)}^{x_n+d(x_n)} \frac{d\xi}{r(\xi)h(\xi)} \ge \int_c^{x_n} \frac{d\xi}{r(\xi)h(\xi)} \ge \frac{1}{2}\int_c^{x_n} \frac{d\xi}{r(\xi)\rho(\xi)} \to \infty \quad \text{as} \quad n \to \infty.$$

□

Corollary. *For $t \in [x-d(x)/2, x+d(x)/2]$, $x \in R$, one has the inequalities*

$$2^{-1}d(x) \le d(t) \le 2d(x). \tag{5.5}$$

Proof. In (5.1), set $\varepsilon = 1/2$, and we obtain (5.5). □

Lemma 5.2. *Let $\gamma \in (0,\infty)$. For every $x \in R$, the equations in $d \ge 0$*

$$\gamma = \int_{x-d}^{x} \frac{d\xi}{r(\xi)h(\xi)}, \quad \gamma = \int_x^{x+d} \frac{d\xi}{r(\xi)h(\xi)} \tag{5.6}$$

have unique finite positive solutions. Denote these solutions by $d_1(x,\gamma)$ and $d_2(x,\gamma)$, respectively. Then

$$\overline{\lim_{x\to-\infty}}\,(x+d_2(x,\gamma)) = -\infty, \quad \underline{\lim}_{x\to\infty}\,(x-d_1(x,\gamma)) = \infty. \tag{5.7}$$

Proof. Consider the functions

$$\Phi_1(d) = \int_{x-d}^{x} \frac{d\xi}{r(\xi)h(\xi)}, \quad \Phi_2(d) = \int_x^{x+d} \frac{d\xi}{r(\xi)h(\xi)}.$$

Clearly, $\Phi_1(0) = \Phi_2(0) = 0$, and $\Phi_1(d)$ and $\Phi_2(d)$ are monotone increasing and continuous for $d \ge 0$. Let us verify that $\Phi_1(\infty) = \Phi_2(\infty) = \infty$. Indeed, by (2.10) and (2.7), we obtain

$$\Phi_2(\infty) = \int_x^{\infty} \frac{d\xi}{r(\xi)h(\xi)} \ge \frac{1}{2}\int_x^{\infty} \frac{d\xi}{r(\xi)\rho(\xi)} = \infty,$$

$$\Phi_1(\infty) = \int_{-\infty}^{x} \frac{d\xi}{r(\xi)h(\xi)} \ge \frac{1}{2}\int_{-\infty}^{x} \frac{d\xi}{r(\xi)\rho(\xi)} = \infty.$$

This implies the first part of the assertion. (5.7) can be checked similarly to (5.2). □

Lemma 5.3. *Let (2.3) $\in S$. Then*

$$G(x,t) \le ch(t)\exp\left(-\frac{1}{c}\left|\int_x^t \frac{d\xi}{r(\xi)h(\xi)}\right|\right), \quad x,t \in R. \tag{5.8}$$

Proof. Since (2.3) $\in S$, we have $m = 1-\varepsilon$, $\varepsilon \in (0,1]$ (see (2.14)). By (2.5) and (2.10),

$$\frac{v'(\xi)}{v(\xi)} = \frac{1+r(\xi)\rho'(\xi)}{2r(\xi)\rho(\xi)} \geq \frac{\varepsilon}{2r(\xi)\rho(\xi)} \geq \frac{\varepsilon}{4r(\xi)h(\xi)}, \quad \xi \in R.$$

Therefore,

$$\frac{v(t)}{v(x)} \geq \exp\left(\frac{\varepsilon}{4}\int_x^t \frac{d\xi}{r(\xi)h(\xi)}\right) \quad \text{for} \quad t \geq x, \quad x,t \in R. \tag{5.9}$$

Then, from (2.4), (2.6), (2.10), and (5.9) for $t \geq x$, it follows that

$$G(x,t) = v(x)u(t) = \rho(t)\frac{v(x)}{v(t)} \leq 2h(t)\exp\left(-\frac{\varepsilon}{4}\left|\int_x^t \frac{d\xi}{r(\xi)h(\xi)}\right|\right).$$

This gives (5.8). The case $t \leq x$ can treated in a similar way. □

6. Proof of the assertions on solvability of the Dirichlet problem

In this section, we prove Theorems 3.1 and 3.3.

Proof of Theorem 3.1 *for* $s \in [1,\infty)$.
Necessity. Assume the contrary: for any $f(x) \in L_s(R)$, one has (1.2), but $\mathcal{D}_s = \infty$. Then there is a sequence $\{x_n\}_{n=1}^\infty$ such that

$$|x_n| \to \infty \text{ as } n \to \infty, \text{ and } h(x_n)d(x_n)^{1/s'} \geq n^2, \quad n = 1,2,\ldots. \tag{6.1}$$

For $n = 1,2,\ldots$, set

$$f_n(t) = \begin{cases} \dfrac{1}{n^2 d(t)^{1/s}}\exp\left(-\left|\displaystyle\int_{x_n}^t \frac{d\xi}{d(\xi)}\right|\right) & \text{if } |t-x_n| \leq \dfrac{d(x_n)}{2}, \\ 0 & \text{if } |t-x_n| > \dfrac{d(x_n)}{2}. \end{cases}$$

By (5.2), one can choose $\{x_n\}_{n=1}^\infty$ in such a way that for $n \neq m$, one would have $(\operatorname{supp} f_n)\cap(\operatorname{supp} f_m) = \emptyset$. Now set $f_0(t) = \sum\limits_{n=1}^\infty f_n(t)$ and verify that for $f_0(t) \in L_s(R)$,

$$\begin{aligned}\int_{-\infty}^\infty |f_0(t)|^s dt &= \sum_{n=1}^\infty \int_{\Delta_n} |f_n(t)|^s dt = \sum_{n=1}^\infty \frac{1}{n^{2s}}\left\{\int_{x_n-\frac{d(x_n)}{2}}^{x_n} \frac{1}{d(t)}\exp\left(-s\int_t^{x_n}\frac{d\xi}{d(\xi)}\right)dt\right. \\ &\qquad \left. + \int_{x_n}^{x_n+\frac{d(x_n)}{2}} \frac{1}{d(t)}\exp\left(-s\int_{x_n}^t \frac{d\xi}{d(\xi)}\right)dt\right\} \\ &\leq \frac{2}{s}\sum_{n=1}^\infty \frac{1}{n^{2s}} \leq \frac{2}{s}\sum_{n=1}^\infty \frac{1}{n^2} \leq \frac{\pi^2}{3}.\end{aligned}$$

Since $f_0(t) \in L_s(R)$, we have $y(x) = (Gf_0)(x) \to 0$ as $|x| \to \infty$. Hence, $y(x_n) \to 0$ as $n \to \infty$. Let us verify that this is not true and thus a contradiction. First note that by (5.5), for $t \in [x_n - d(x_n)/2, x_n + d(x_n)/2]$, one has the estimates

$$\left|\int_{x_n}^t \frac{d\xi}{d(\xi)}\right| = \left|\int_{x_n}^t \frac{d(x_n)}{d(\xi)}\,\frac{d\xi}{d(x_n)}\right| \leq \frac{2|t-x_n|}{d(x_n)} \leq 1, \quad n = 1,2,\ldots. \tag{6.2}$$

Therefore, using (5.5), (6.2), (2.12), and (2.10) we get

$$\begin{aligned} y(x_n) &= \int_{-\infty}^{\infty} G(x_n,t) f_0(t)dt \\ &\ge u(x_n)\int_{x_n-\frac{d(x_n)}{2}}^{x_n} v(t)\frac{1}{n^2}\left(\frac{1}{d(t)}\right)^{1/s}\exp\left(-\left|\int_{x_n}^{t}\frac{d\xi}{d(\xi)}\right|\right)dt \\ &\quad + v(x_n)\int_{x_n}^{x_n+\frac{d(x_n)}{2}} u(t)\frac{1}{n^2}\left(\frac{1}{d(t)}\right)^{1/s}\exp\left(-\left|\int_{x_n}^{t}\frac{d\xi}{d(\xi)}\right|\right)dt \\ &\ge \frac{1}{9}\exp(-1)\frac{\rho(x_n)}{n^2}\int_{x_n-\frac{d(x_n)}{2}}^{x_n+\frac{d(x_n)}{2}}\left(\frac{d(x_n)}{d(t)}\right)^{1/s}\left(\frac{1}{d(x_n)}\right)^{1/s}dt \\ &\ge \frac{1}{36}\exp(-1)\frac{h(x_n)d(x_n)^{1/s'}}{n^2} \ge \frac{\exp(-1)}{36}, \end{aligned}$$

a contradiction. Hence $\mathcal{D}_s < \infty$.

Sufficiency. Let us apply estimate (5.8) and Hölder's inequality to (1.4):

$$\begin{aligned} |y(x)| &\le c\int_{-\infty}^{\infty} h(t)\exp\left(-\frac{1}{c}\left|\int_x^t \frac{d\xi}{r(\xi)h(\xi)}\right|\right)|f(t)|dt \\ &\le c\left\{\int_{-\infty}^{\infty} h(t)^{s'}\exp\left(-\frac{1}{c}\left|\int_x^t \frac{d\xi}{r(\xi)h(\xi)}\right|\right)dt\right\}^{1/s'} \\ &\quad\times\left\{\int_{-\infty}^{\infty}\exp\left(-\frac{1}{c}\left|\int_x^t \frac{d\xi}{r(\xi)h(\xi)}\right|\right)|f(t)|^s dt\right\}^{1/s}. \end{aligned} \tag{6.3}$$

Let us estimate the first factor. Let $\{\Delta_n\}_{n\in N'}$ be segments from an $R(x)$-covering (see Lemma 2.5 and Definition 2.1). Then, by (2.11), one has inequalities, for $t \in \Delta_n$,

$$\begin{aligned} \int_x^t \frac{d\xi}{r(\xi)h(\xi)} &= \sum_{k=1}^{n-1}\int_{\Delta_k}\frac{d\xi}{r(\xi)h(\xi)} + \int_{\Delta_n^-}^{t}\frac{d\xi}{r(\xi)h(\xi)} \ge (n-1) \quad \text{if } n \ge 1, \\ \int_t^x \frac{d\xi}{r(\xi)h(\xi)} &= \sum_{k=1}^{|n|-1}\int_{\Delta_k}\frac{d\xi}{r(\xi)h(\xi)} + \int_t^{\Delta_n^+}\frac{d\xi}{r(\xi)h(\xi)} \ge (|n|-1) \quad \text{if } n \le -1. \end{aligned} \tag{6.4}$$

Throughout, we use the notation $\sum_n{}' = \sum_{n\in N'}$, $N' = \pm 1, \pm 2, \dots$. By Lemmas 2.5, 2.4, and estimates (6.4), we have

$$\begin{aligned} \int_{-\infty}^{\infty} h(s)^{s'}\exp\left(-\frac{1}{c}\left|\int_x^t \frac{d\xi}{r(\xi)h(\xi)}\right|\right)dt &= \sum_n{}'\int_{\Delta_n} h(t)^{s'}\exp\left(-\frac{1}{c}\left|\int_x^t \frac{d\xi}{r(\xi)h(\xi)}\right|\right)dt \\ \le \sum_n{}'\left(\int_{\Delta_n} h(t)^{s'}dt\right)\exp\left(-\frac{|n|-1}{c}\right) &\le c\sum_n{}'\left(h(x_n)^{s'}d(x_n)\right)\exp\left(-\frac{|n|}{c}\right) \\ \le c\Big(\sum_{n=0}^{\infty}\exp\left(-\frac{n}{c}\right)\Big)\mathcal{D}_s^{s'} \le c < \infty. \end{aligned}$$

Hence,

$$|y(x)|^s \le c\int_{-\infty}^{\infty}\exp\left(-\frac{1}{c}\left|\int_x^t \frac{d\xi}{r(\xi)h(\xi)}\right|\right)|f(t)|^s dt, \quad x \in R. \tag{6.5}$$

Fix $\gamma \in (0,\infty)$ and denote $\omega(x,\gamma) = [x - d_1(x,\gamma), x + d_2(x,\gamma)]$ where $d_1(x,\gamma),\ d_2(x,\gamma)$ are the solutions of (5.6), respectively. Then from (6.5), it follows that

$$\begin{aligned}|y(x)|^s &\le c\left\{\int_{t\notin\omega(x,\gamma)} \exp\left(-\frac{1}{c}\left|\int_x^t \frac{d\xi}{r(\xi)h(\xi)}\right|\right)|f(t)|^s dt\right.\\ &\quad \left.+\int_{t\in\omega(x,\gamma)} \exp\left(-\frac{1}{c}\left|\int_x^t \frac{d\xi}{r(\xi)h(\xi)}\right|\right)|f(t)|^s dt\right\}\\ &\le c\left\{\exp\left(-\frac{\gamma}{c}\right)\int_{t\notin\omega(x,\gamma)}|f(t)|^s dt + \int_{x-d_1(x,\gamma)}^{\infty}|f(t)|^s dt\right\}\\ &\le c\exp\left(-\frac{\gamma}{c}\right)\|f\|_s^s + c\int_{x-d_1(x,\gamma)}^{\infty}|f(t)|^s dt.\end{aligned}$$

From the latter estimate, by (5.7), we get

$$0 \le \overline{\lim_{x\to\infty}}|y(x)|^s \le c\exp\left(-\frac{\gamma}{c}\right)\|f\|_s^s. \tag{6.6}$$

In (6.6), γ is an arbitrary number. Letting γ tend to infinity, we obtain $\overline{\lim\limits_{x\to\infty}}\, y(x) = 0$. Since $0 \le \underline{\lim\limits_{x\to\infty}}\, |y(x)| \le \overline{\lim\limits_{x\to\infty}}\, y(x) = 0$, we finally get $\lim\limits_{x\to\infty} y(x) = 0$. The case $x \to -\infty$ can be treated in a similar way. □

Proof of Theorem 3.1 *for* $s=\infty$. *Necessity.* Let $y(x) = (Gf)(x) \to 0$ as $|x| \to \infty$ for any $f(x) \in L_\infty(R)$. Choose $f_0(x) \equiv 1$. Then, using (2.10) and (2.13), we obtain the inequality which implies (3.2′):

$$\begin{aligned}y(x) &= \int_{-\infty}^{\infty} G(x,t)f_0(t)dt = \int_{-\infty}^{\infty} G(x,t)dt\\ &\ge u(x)\int_{x-d(x)}^{x} v(t)dt + v(x)\int_x^{x+d(x)} u(t)dt \ge \frac{2u(x)v(x)d(x)}{9} \ge \frac{h(x)d(x)}{9}.\end{aligned}$$

Sufficiency. First note that if $a_n \to 0$ as $n\to\infty$ and n_0 is fixed, then

$$\lim_{n\to\infty}\left(\sum_{i=n-n_0}^{n+n_0} a_i\right) = 0. \tag{6.7}$$

Suppose that $\{\Delta_n\}_{n\in N'}$ from the $R(0)$-covering. Introduce the function $k(x) = n$ if $x \in [x_n - d(x_n), x_n + d(x_n))$ for $n \in N'$. Clearly $|k(x)| \to \infty$ as $|x| \to \infty$. Let n_0 be any positive integer. Below we use (5.8), Definition 2.1, (6.4), and (2.13):

$$\begin{aligned}\int_{-\infty}^{\infty} G(x,t)dt &\le c\int_{-\infty}^{\infty} h(t)\exp\left(-\frac{1}{c}\left|\int_x^t \frac{d\xi}{r(\xi)h(\xi)}\right|\right)\\ &= c\sum_{n\in N'}\int_{\Delta_n} h(t)\exp\left(-\frac{1}{c}\left|\int_x^t \frac{d\xi}{r(\xi)h(\xi)}\right|\right)dt\\ &\le c\sum_{i=k(x)-n_0}^{k(x)+n_0} h(x_i)d(x_i) + c\sum_{|i-k(x)|>n_0} h(x_i)d(x_i)\exp\left(-\frac{|i-k(x)|}{c}\right).\end{aligned}$$

From (3.2′), it follows that $h(x)d(x) \le c_1$ for $x \in R$. Take $\varepsilon > 0$ and choose $n_0 = n_0(\varepsilon)$ such that $2c_1 \exp\left(-\frac{n_0}{c}\right) \exp\left(\frac{1}{c}\right) < \varepsilon$. Then we can continue the above estimate:

$$\begin{aligned}\int_{-\infty}^{\infty} G(x,t)dt &\le c \sum_{i=k(x)-n_0}^{k(x)+n_0} h(x_i)d(x_i) + c_1 \exp\left(-\frac{n_0}{c}\right) \sum_{i=0}^{\infty} \exp\left(-\frac{i}{c}\right) \\ &\le c \sum_{i=k(x)-n_0}^{k(x)+n_0} h(x_i)d(x_i) + \varepsilon.\end{aligned}$$

By (5.2) and (6.7), this implies

$$0 \le \varliminf_{x\to\infty} \int_{-\infty}^{\infty} G(x,t)dt \le \varlimsup_{x\to\infty} \int_{-\ iy}^{\infty} G(x,t)dt \le c \lim_{x\to\infty} \sum_{i=k(x)-n_0}^{k(x)+n_0} h(x_i)d(x_i) + \varepsilon = \varepsilon.$$

Since ε is arbitrary, letting it tend to zero, we get

$$\lim_{|x|\to\infty} \int_{-\infty}^{\infty} G(x,t)dt = 0. \tag{6.8}$$

(For $x \to -\infty$, one can check (6.8) in a similar way.) Therefore,

$$\begin{aligned}0 &\le \varliminf_{|x|\to\infty} |y(x)| \le \varlimsup_{|x|\to\infty} |y(x)| \le \varlimsup_{|x|\to\infty} \int_{-\infty}^{\infty} G(x,t)|f(t)|dt \\ &\le \lim_{|x|\to\infty} \left(\int_{-\infty}^{\infty} G(x,t)dt\right) \cdot \|f\|_\infty = 0.\end{aligned}$$

□

Proof of Theorem 3.3. From (2.6), (2.18), (2.21), and (2.22), it follows that

$$G(x,t) \le c \exp\left(-\frac{|t-x|}{c}\right), \quad x,t \in R. \tag{6.9}$$

Let $s \in [1,\infty)$. From (6.9) and Hölder's inequality, we get

$$\begin{aligned}|y(x)| &\le \int_{-\infty}^{\infty} G(x,t)|f(t)|dt \\ &\le c\left(\int_{-\infty}^{\infty} G(x,t)dt\right)^{1/s'} \left(\int_{-\infty}^{\infty} G(x,t)|f(t)|^s dt\right)^{1/s} \\ &\le c\left(\int_{-\infty}^{\infty} \exp\left(-\frac{|t-x|}{c}\right)dt\right)^{1/s'} \left(\int_{-\infty}^{\infty} \exp\left(-\frac{|t-x|}{c}\right)|f(t)|^s dt\right)^{1/s} \\ &\le c\left(\int_{-\infty}^{\infty} \exp\left(-\frac{|t-x|}{c}\right)|f(t)|^s dt\right)^{1/s}.\end{aligned}$$

Hence,

$$|y(x)|^s \le c \int_{-\infty}^{\infty} \exp\left(-\frac{|t-x|}{c}\right)|f(t)|^s dt, \quad x \in R. \tag{6.10}$$

Let $A \in (0, \infty)$. Then, from (6.10), it follows that

$$|y(x)|^s \le c\left\{\int_{|t-x|>A} \exp\left(-\frac{|t-x|}{c}\right)|f(t)|^s dt + \int_{|t-x|\le A} \exp\left(-\frac{|t-x|}{c}\right)|f(t)|^s dt\right\} \tag{6.11}$$

$$\le c\exp\left(-\frac{A}{c}\right)\|f\|_s^s + c\int_{|t-x|\le A} |f(t)|^s dt.$$

In (6.11), letting x tend to infinity, we obtain

$$0 \le \overline{\lim_{x\to\infty}}\, |y(x)|^s \le c\exp\left(-\frac{A}{c}\right)\|f\|_s^s. \tag{6.12}$$

Since A is arbitrary, we deduce from (6.12) as $A \to \infty$ that

$$0 \le \underline{\lim}_{x\to\infty}\, |y(x)|^s \le \overline{\lim_{x\to\infty}}\, |y(x)|^s = 0.$$

Thus, $y(x) \to 0$ as $x \to \infty$. Similarly, one can check that $y(x) \to 0$ as $x \to -\infty$.

Now, let $s = \infty$. Necessity of our assertion follows from the proof of necessity of the hypotheses of Theorem 3.1, inequalities (2.20) and (2.21), and Lemma 2.7. To prove sufficiency, note that from (2.6), (2.18), (2.21), and (2.22), it also follows that

$$G(x,t) \le c\sqrt{\tilde{d}(x)}\exp\left(-\frac{|t-x|}{c}\right), \quad t, x \in R. \tag{6.13}$$

By Lemma 2.7, we obtain $\tilde{d}(x) \to 0$ as $|x| \to \infty$. The conclusion of the theorem follows from (6.13) and the following obvious inequalities:

$$|y(x)| \le \int_{-\infty}^{\infty} G(x,t)|f(t)|dt \le c\sqrt{\tilde{d}(x)}\left(\int_{-\infty}^{\infty}\exp\left(-\frac{|t-c|}{c}\right)dt\right)\|f\|_\infty$$
$$\le c\sqrt{\tilde{d}(x)}\|f\|_\infty.$$

□

7. Proof of the theorem on estimating the solution of the Dirichlet problem in the uniform metric

In this section, we prove Theorem 3.2.

Proof of assertion (a). Let $s \in [1, \infty)$. Introduce a function:

$$f_x(t) = \begin{cases} 1 & \text{if } t \in [x - d(x), x + d(x)] \\ 0 & \text{if } t \notin [x - d(x), x + d(x)]. \end{cases}$$

Now note that if $t \in [x - d(x), x + d(x)]$, then from Lemma 2.4, it follows that

$$y(t) = \int_{-\infty}^{\infty} G(x,t) f_x(\xi)d\xi = u(t)\int_{x-d(x)}^{t} v(\xi)d\xi + v(t)\int_t^{x+d(x)} u(\xi)d\xi$$
$$\ge \frac{2}{9}u(x)v(x)d(x) \ge \frac{2}{81}\rho(x)d(x) \ge \frac{h(x)d(x)}{81}.$$

Therefore, if (3.3) holds, then

$$c(s) \geq \sup_{f \in L_s} \frac{\|y\|_{C(R)}}{\|f\|_s} = \sup_{f \in L_s} \frac{\|Gf\|_{C(R)}}{\|f\|_s} \geq \sup_{x \in R} \frac{\|Gf_x\|_{C(R)}}{\|f_x\|_s}$$
$$\geq \frac{1}{81} \sup_{x \in R} \frac{h(x)d(x)}{(2d(x))^{1/s}} = \frac{1}{162}\mathcal{D}_s.$$

If $s = \infty$, then for $f_0(x) \equiv 1$, as shown in §6, we have

$$y(x) = \int_{-\infty}^{\infty} G(x,t) f_0(t) dt \geq \frac{h(x)d(x)}{9}.$$

Therefore, if (3.3) holds, then

$$c \geq \frac{\|y\|_{C(R)}}{\|f_0\|_\infty} \geq \sup_{x \in R} (h(x)d(x)).$$

□

Proof of assertion (b). If (3.3) holds for some $s \in [1, \infty)$, then $\mathcal{D}_s < \infty$. Since (1.1) $\in S$, Theorem 3.1 implies (3.1). Conversely, if (3.1) holds, then $\mathcal{D}_s < \infty$. Since (1.1) $\in S$, we can repeat the proof of sufficiency in Theorem 3.1 and obtain (6.5). This immediately gives (3.3). □

Proof of assertion (c). Necessity is proved in (a). Since the inversion problem (1.1) is regular in L_p, by Theorem 2.3, we conclude that $B < \infty$. Since (1.1) $\in S$, by (2.17), we also have $H < \infty$, whence

$$\|y\|_{C(R)} \leq \sup_{x \in R} \int_{-\infty}^{\infty} G(x,t)|f(t)|dt \leq H\|f\|_\infty \leq cB\|f\|_\infty.$$

□

Acknowledgment. The first author is supported by the Israel Academy of Sciences under Grant 421/95. The second author is supported by the Israel Academy of Sciences under Grant 505/95.

References

1. N. Chernyavskaya and L. Shuster, *Regularity of the inversion problem for the Sturm-Liouville equation in the spaces L_p*, preprint, AMSPPS # 1130-34-003 (1997).
2. ———, *Estimates for the Green function of a general Sturm-Liouville operator and their applications*, preprint, AMSPPS # 130-34-002 (1997).
3. E.B. Davies and E.M. Harrell, *Conformally flat Riemannian metrics, Schrödinger operators and semiclassical approximation*, J. Diff. Eq. **66** (1987), 165–188.
4. K. Mynbaev and M. Otelbaev, *Weighted Functional Spaces and Differential Operator Spectrum,* Nauka, Moscow, 1988.
5. L. Shuster, *A priori properties of solutions of a Sturm-Liouville equation and A.M. Molchanov's criterion*, Math. Notes **50**(1991), 746–751.
6. E.C. Titchmarsh, *The Theory of Functions*, Oxford, 1932.

Department of Mathematics and Computer Science, Ben-Gurion University of the Negev, P.O.B. 653, Beer-Sheva, 84105, Israel

Department of Mathematics and Computer Science, Bar-Ilan University, Ramat-Gan, 52900, Israel