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A unified approach to the summation and integration formulas for q-hypergeometric functions II

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Abstract. As a continuation of a previous report, the summation formulas for the bilateral ${}_2\psi_2$ series and the balanced ${}_3\psi_3$ series are obtained from a Pearson-type difference equation on a q-linear lattice without the benefit of a single transformation formula. The corresponding formulas for Ramanujan-type integrals also are derived. For the sake of completeness, the previously-found Barnes-type integrals for the same lattice and same coefficient functions are stated.

1. Introduction

This is a continuation of the work that the authors started in [11] on the question of how to derive the summation formulas for basic hypergeometric functions as well as the corresponding integration formulas of two different types (for an exhaustive list, see [5]) without the benefit of any transformation formula. The key to the whole approach is a Pearson-type difference equation of order one:

$$\frac{\Delta}{\nabla x_1(s)}\big[\rho(s)\sigma(s)\big] = \rho(s)\tau(s), \tag{1.1}$$

which is a difference-analogue of the Pearson equation in differential equations:

$$\frac{d}{dx}\big[\rho(x)\sigma(x)\big] = \rho(x)\tau(x) \tag{1.2}$$

where $\sigma(x)$ and $\tau(x)$ in (1.2) are polynomials of degrees at most 2 and 1, respectively. In (1.1), $x(s)$ represents a lattice in the variable s that changes in units of length 1 and $x_1(s) = x(s+1/2)$. The forward and backward difference operators Δ and ∇, respectively, are defined by

$$\Delta\big[f(x(s))\big] = f(x(s+1)) - f(x(s)), \quad \nabla\big[f(x(s))\big] = \Delta\big[f(x(s-1))\big]. \tag{1.3}$$

The coefficient functions $\sigma(s)$ and $\tau(s)$ are, as in (1.2), polynomials of degrees at most 2 and 1, respectively, in $x(s)$. It was shown in [3], [9], and [10] (see also [7] and [12]) that a solution of (1.1) for $\rho(s)$ enables one to solve the second-order hypergeometric-type difference equation:

$$\tilde{\sigma}(x(s))\frac{\nabla}{\nabla x_1(s)}\left[\frac{\Delta y(s)}{\Delta x(s)}\right] + \frac{\tilde{\tau}(x(s))}{2}\left[\frac{\Delta y(s)}{\Delta x(s)} + \frac{\nabla y(s)}{\nabla x(s)}\right] + \lambda y(s) = 0 \tag{1.4}$$

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where the relations between the coefficient functions $\sigma(s)$, $\tilde{\sigma}(x(s))$, $\tau(s)$, and $\tilde{\tau}(x(s))$ are given by

$$\tau(s) = \tilde{\tau}(x(s)),$$
$$\sigma(s) = \tilde{\sigma}(x(s)) - \frac{1}{2}\tilde{\tau}(x(s))\nabla x_1(s), \tag{1.5}$$

and λ is a parameter independent of s. If $\rho(s)$ satisfies (1.1), then (1.4) can be rewritten in the self-adjoint form:

$$\frac{\nabla}{\nabla x_1(s)}\left[\rho(s+1)\sigma(s+1)\frac{\Delta y(s)}{\Delta x(s)}\right] + \lambda\rho(s)y(s) = 0. \tag{1.6}$$

An important characterization theorem, proved in [3], is that (1.4) is truly hypergeometric, meaning that the successive difference derivatives of $y(s)$ also satisfy an equation of the same type as (1.4) if and only if $x(s)$ has the form

$$x(s) = \begin{cases} C_1 q^{-s} + C_2 q^s + C_3, & \text{if } q \neq 1, \\ C_1' s^2 + C_2' s, & \text{if } q = 1, \end{cases} \tag{1.7}$$

where C_1, C_2, C_3, C_1', C_2' are arbitrary constants such that $C_1 C_2 \neq 0 \neq C_1' C_2'$. If C_1 or C_2 is 0, then the lattice is called q-linear, otherwise it is q-quadratic. When $q = 1$, $x(s)$ is linear if $C_1' = 0$ and quadratic if $C_1' \neq 0$. To show how the first expression reduces to the second in the limit $q \to 1$, we first rewrite it in the form $A(q^{-s}-1) + B(q^s - 1)$, assuming, of course, that $x(0) = 0$, factor it as $(q^{-s}-1)(A - Bq^s)$, replace A, B by $C_2'(1-q)^{-2}$ and $-C_1'(1-q)^{-2}$, respectively, and then take the limit.

In [11], we are concerned with solutions of (1.1) when $x(s)$ is q-linear and, $\sigma(s)$ and $\tau(s)$ are polynomials of degrees 2 and 1, respectively, in $x(s)$. We showed how to derive the q-binomial formula, Ramanujan's ${}_1\psi_1$ summation formula, and his corresponding integration formula in a very elementary manner. We also obtained the two summation formulas for the ${}_2\phi_1$ series as well as their integral counterparts.

The bilateral basic hypergeometric series ${}_r\psi_r$ is an infinite series with r numerator and r denominator parameters:

$${}_r\psi_r\left[\begin{matrix} a_1, a_2, \ldots a_r \\ b_1, b_2, \ldots b_r \end{matrix}; q, z\right] = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots a_r\,;q)_n}{(b_1, b_2, \ldots b_r\,;q)_n} z^n, \tag{1.8}$$

which converges absolutely in the annulus

$$\left|\frac{b_1 b_2 \cdots b_r}{a_1 a_2 \cdots a_r}\right| < |z| < 1. \tag{1.9}$$

The q-shifted factorials on the right-hand side of (1.8) are defined by

$$(a\,;q)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n = 1, 2, \ldots, \end{cases} \tag{1.10}$$

$$(a\,;q)_\infty = \lim_{n\to\infty}(a\,;q)_n, \qquad |q| < 1, \tag{1.11}$$

$$(a_1, a_2, \ldots a_r\,;q)_n = (a_1\,;q)_n(a_2\,;q)_n\cdots(a_r\,;q)_n, \tag{1.12}$$

$$(a\,;q)_{-n} = \frac{(-q/a)^n q^{\binom{n}{2}}}{(q/a\,;q)_n}, \quad n = 0, 1, 2, \ldots. \tag{1.13}$$

If one of the parameters in the denominator of (1.8), say, b_r is q, then every term from $n = -\infty$ to $n = -1$ vanishes, and we obtain a ${}_r\phi_{r-1}$ series:

$$ {}_r\phi_{r-1}\left[\begin{matrix} a_1, a_2, \dots a_r \\ b_1, b_2, \dots b_{r-1} \end{matrix} ; q, z\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots a_r \,;\, q)_n}{(q, b_1, \dots b_{r-1} \,;\, q)_n} z^n, \tag{1.14} $$

which converges absolutely inside the unit circle $|z| = 1$. For more detailed information about these series, see [5].

In this paper, we once again shall restrict ourselves to a q-linear lattice and generate summation and integration formulas for the ${}_2\psi_2$ and ${}_3\psi_3$ series. In Section 2, we shall derive the ${}_2\psi_2$ summation formula that will be used in Section 3 to obtain a Ramanujan-type integral. For the ${}_3\psi_3$ series, which will be considered in Section 4, we will need to relax the restriction on $\tau(s)$ by allowing it to have a simple pole in addition to the polynomial component. This procedure appears to destroy the hypergeometric character of (1.4) but, nonetheless, leads to a richer variety of solutions of (1.1) as well as a wider class of solutions of an equation corresponding to (1.4), namely, biorthogonal rational functions; see [8]. In Section 5, we shall conclude the paper by deriving a formula that evaluates a Ramanujan-type integral corresponding to the ${}_3\psi_3$ formula.

The summation and integration formulas that originate from the q-quadratic lattices will be dealt with in a future and final paper in this series.

2. The q-linear lattice $x(s) = q^s$: the ${}_2\psi_2$ summation formula

For the coefficient functions $\sigma(s)$ and $\tau(s)$, let us take

$$ \begin{aligned} \sigma(s) &= (1 - cq^{s-1})(1 - dq^{s-1}), \\ \sigma(s) + \tau(s)\nabla x_1(s) &= zq^{-1}(1 - aq^s)(1 - bq^s). \end{aligned} \tag{2.1} $$

Then

$$ \tau(s)\nabla x_1(s) = (z/q - 1) + [c + d - z(a + b)]q^{s-1} - cd(1 - abzq/cd)q^{2s-2}. \tag{2.2} $$

As was the case for the ${}_2\phi_1$ summation formulas in [11], partial cancellations occur in (2.2) in the two cases: $z = q$ and $z = cd/abq$. Unlike the ${}_2\phi_1$ case, however, there is no essential difference between these cases for a truly bilateral ${}_2\psi_2$ series, as we shall see shortly. So, let us take $z = cd/abq$. Then we get

$$ \tau(s)\nabla x_1(s) = \frac{cdq^{s-2}}{ab}\big[(1 - abq^2/cd)(1 - aq^s)q^{-s} - b(1 - aq/c)(1 - aq/d)\big]. \tag{2.3} $$

By (1.1), $\rho(s)$ satisfies the equation

$$ \frac{\rho(s+1)}{\rho(s)} = \frac{(1 - aq^s)(1 - bq^s)}{(1 - cq^s)(1 - dq^s)}(cd/abq^2) $$

whose solution is

$$ \rho(s) = A\,\frac{(a, b\,;\, q)_s}{(c, d\,;\, q)_s}\,(cd/abq^2)^s \tag{2.4} $$

where A is either a constant or a unit-periodic function of s which need not be an integer. The defining relation for $(a\,;\,q)_s$ in that case is $(a\,;\,q)_s = (a\,;\,q)_\infty/(aq^s\,;\,q)_\infty$. We shall assume throughout this paper that q is real and $0 < q < 1$. Since our interest in this section is a summation formula, we will take $s = s_0 + k$ where $s_0 \in \mathbb{C}$ and $k \in \mathbb{Z}$. If we choose

$$A = \frac{(c, d\,;\,q)_{s_0}}{(a, b\,;\,q)_{s_0}}(cd/abq)^{-s_0}, \tag{2.5}$$

then we get

$$\rho(s) = \rho(s_0 + k) = \alpha^{-1}\frac{(\alpha a, \alpha b\,;\,q)_k}{(\alpha c, \alpha d\,;\,q)_k}(cd/abq^2)^k \tag{2.6}$$

where $\alpha = q^{s_0}$. Our aim is to evaluate the sum

$$\begin{aligned}\sum_{s=-\infty}^{\infty}\rho(s)q^s &= \sum_{k=-\infty}^{\infty}\rho(s_0+k)q^{s_0+k} \\ &= {}_2\psi_2\left[\begin{matrix}\alpha a, \alpha b\\ \alpha c, \alpha d\end{matrix}\,;\,q, cd/abq\right] = f_1(a)\ .\end{aligned} \tag{2.7}$$

By (1.13),

$${}_2\psi_2\left[\begin{matrix}\alpha a, \alpha b\\ \alpha c, \alpha d\end{matrix}\,;\,q, cd/abq\right] = {}_2\psi_2\left[\begin{matrix}q/\alpha c, q/\alpha d\\ q/\alpha a, q/\alpha b\end{matrix}\,;\,q, q\right], \tag{2.8}$$

which justifies our earlier statement that there is no essential difference between the cases $z = q$ and $z = cd/abq$. Using (1.1), (2.3), and (2.7), we find that

$$\begin{aligned}&(1 - cd/abq^2)(1 - 1/\alpha a)f_1(aq) - (1 - c/aq)(1 - d/aq)f_1(a) \\ &\qquad = a^{-1}\Big[\lim_{k\to\infty}\sigma(s_0+k)\rho(s_0+k) - \lim_{\ell\to\infty}\sigma(s_0-\ell)\rho(s_0-\ell)\Big].\end{aligned} \tag{2.9}$$

By (2.1) and (2.6),

$$\lim_{k\to\infty}\sigma(s_0+k)\rho(s_0+k) = \alpha^{-1}\frac{(\alpha a, \alpha b\,;\,q)_\infty}{(\alpha c, \alpha d\,;\,q)_\infty}\lim_{k\to\infty}(cd/abq^2)^k = 0, \tag{2.10}$$

if

$$\left|\frac{cd}{ab}\right| < q^2 < 1. \tag{2.11}$$

Note that, by (1.9), the series in (2.7) converges in the region $|cd/ab| < q < 1$, so the condition (2.11) is more restrictive than what seems to be necessary. We shall, however, be able to relax this restriction in the end by appealing to an analytic continuation. We also have

$$\lim_{\ell\to\infty}\sigma(s_0-\ell)\rho(s_0-\ell) = \alpha cdq^{-2}\frac{(q/\alpha c, q/\alpha d\,;\,q)_\infty}{(q/\alpha a, q/\alpha b\,;\,q)_\infty}. \tag{2.12}$$

From (2.9), (2.10), and (2.12), we now obtain the nonhomogeneous recurrence relation

$$f_1(a) - \frac{(1 - cd/abq^2)(1 - 1/\alpha a)}{(1 - c/aq)(1 - d/aq)}f_1(aq) = \frac{\alpha a}{(1 - aq/c)(1 - aq/d)}\frac{(q/\alpha c, q/\alpha d\,;\,q)_\infty}{(q/\alpha a, q/\alpha b\,;\,q)_\infty}. \tag{2.13}$$

In order to solve this equation, we shall first try to get rid of the non-homogeneous term on the right-hand side. If we replace α by α^{-1} and denote

$$f_2(a) = {}_2\psi_2\left[\begin{matrix}a/\alpha, b/\alpha\\ c/\alpha, d/\alpha\end{matrix}\,;\,q, cd/abq\right], \tag{2.14}$$

then we have

$$f_2(a) - \frac{(1-cd/abq^2)(1-\alpha/a)}{(1-c/aq)(1-d/aq)} f_2(aq) = \frac{\alpha^{-1}a}{(1-aq/c)(1-aq/d)} \frac{(\alpha q/c, \alpha q/d\,;q)_\infty}{(\alpha q/a, \alpha q/b\,;q)_\infty}. \tag{2.15}$$

Let us introduce a third function:

$$f(a) = f_1(a) - \alpha^2 \frac{(\alpha q/a, \alpha q/b, q/\alpha c, q/\alpha d\,;q)_\infty}{(q/a\alpha, q/b\alpha, \alpha q/c, \alpha q/d\,;q)_\infty} f_2(a). \tag{2.16}$$

The motivation of this step is clear: multiplying (2.15) by the factor of $f_2(a)$ in (2.16) makes the right-hand side equal to that of (2.13). So we obtain a homogeneous recurrence formula which, on replacing a by a/q, gives

$$f(a) = \frac{(1-c/a)(1-d/a)}{(1-cd/abq)(1-q/\alpha a)} f(a/q). \tag{2.17}$$

Iterating it $n-1$ times and then taking the limit $n \to \infty$, we get

$$f(a) = \frac{(c/a, d/a\,;q)_\infty}{(cd/abq, q/\alpha a\,;q)_\infty}. \lim_{n\to\infty} f(aq^{-n}). \tag{2.18}$$

It is a simple exercise to show that the limit exists. But, $f(a)$ is symmetric in a, b by definition, so (2.18) must have the same property. Relabelling $f(a)$ by $f(a,b)$, we then have

$$f(a,b) = \frac{(c/a, d/a, c/b, d/b\,;q)_\infty}{(cd/abq, q/\alpha a, q/\alpha b\,;q)_\infty} \lambda(c,d) \tag{2.19}$$

where $\lambda(c,d) = \lim_{m,n\to\infty} f(aq^{-n}, bq^{-m})$ must be independent of a and b. Setting $a = \alpha^{-1}$ and $b = \alpha$, we find that

$$\begin{aligned} \lambda(c,d) &= \frac{(q, q/\alpha^2, cd/q\,;q)_\infty}{(\alpha c, \alpha d, c/\alpha, d/\alpha\,;q)_\infty} \left\{ {}_2\phi_1\left[\begin{matrix} q/\alpha c, q/\alpha d \\ q/\alpha^2 \end{matrix}; q, q\right] \right. \\ &\qquad \left. - \alpha^2 \frac{(q\alpha^2, q/\alpha c, q/\alpha d\,;q)_\infty}{(q/\alpha^2, \alpha q/c, \alpha q/d\,;q)_\infty} {}_2\phi_1\left[\begin{matrix} \alpha q/c, \alpha q/d \\ q\alpha^2 \end{matrix}; q, q\right] \right\} \\ &= \frac{(q, \alpha^2, q/\alpha^2, cd/q, q^2/cd\,;q)_\infty}{(\alpha c, \alpha d, c/\alpha, d/\alpha, \alpha q/c, \alpha q/d\,;q)_\infty}, \end{aligned} \tag{2.20}$$

by [5, II.23]; see also [11]. Use of (2.20) in (2.19) leads to the desired summation formula

$$\begin{aligned} &{}_2\psi_2\left[\begin{matrix} \alpha a, \alpha b \\ \alpha c, \alpha d \end{matrix}; q, cd/abq\right] - \alpha^2 \frac{(\alpha q/a, \alpha q/b, q/\alpha c, q/\alpha d\,;q)_\infty}{(q/\alpha a, q/\alpha b, \alpha q/c, \alpha q/d\,;q)_\infty} {}_2\psi_2\left[\begin{matrix} a/\alpha, b/\alpha \\ c/\alpha, d/\alpha \end{matrix}; q, cd/abq\right] \\ &\quad = \frac{(q, \alpha^2, q/\alpha^2, cd/q, q^2/cd, c/a, d/a, c/b, d/b\,;q)_\infty}{(\alpha c, \alpha d, q/a\alpha, q/b\alpha, c/\alpha, d/\alpha, \alpha q/c, \alpha q/d, cd/abq\,;q)_\infty}. \end{aligned} \tag{2.21}$$

Since both sides are analytic in the region $|cd/ab| < q < 1$, we now may replace the annulus (2.11) by the wider region. Each of the ${}_2\psi_2$ series above can be transformed to a ${}_2\psi_2$ series with argument q; see [5, Ex. 5.20 (ii)]. Then, by taking $\alpha = i$ and relabelling the parameters, one can show that (2.21) is equivalent to Askey's formula [1, 3.13].

It is instructive to note that by replacing α by $i\alpha$ and using (2.7), we can rewrite (2.21) in the form of a q-integral:

$$\int_0^\infty \left[\frac{(i\alpha qt/a, i\alpha qt/b\,;q)_\infty}{(i\alpha qt/c, i\alpha qt/d\,;q)_\infty}\alpha + \frac{(-iqt/\alpha a, -iqt/\alpha b\,;q)_\infty}{(-iqt/\alpha c, -iqt/\alpha d\,;q)_\infty}\alpha^{-1}\right] d_qt$$
$$= \frac{(1-q)}{\alpha}\frac{(q, -\alpha^2, -q/\alpha^2, cd/q, q^2/cd, c/a, d/a, c/b, d/b\,;q)_\infty}{(i\alpha c, -iq/\alpha c, i\alpha d, -iq/\alpha d, i\alpha q/c, -ic/\alpha, i\alpha q/d, -id/\alpha, cd/abq\,;q)_\infty}. \tag{2.22}$$

In addition, the expression on the left-hand side of (2.22) can be written as a single bilateral integral if we adopt the notation

$$\int_{-\infty}^\infty f(t)d_q^{(\alpha)}t = (1-q)\left\{\lim_{M\to\infty}\sum_{s=s_0-M}^{s_0+M} f(q^s)q^s + \lim_{N\to\infty}\sum_{s=-s_0-N}^{-s_0+N} f(-q^s)q^s\right\} \tag{2.23}$$

where $\alpha = q^{s_0}$, $s_0 \in \mathbb{C}$ such that it does not produce any singularities in the summands. The usual bilateral q-integral given in [5, (1.11.5)] is then a special case of (2.23) with $\alpha = 1$. The summation formula (2.21) then reads

$$\int_{-\infty}^\infty \frac{(iqt/a, iqt/b\,;q)_\infty}{(iqt/c, iqt/d\,;q)_\infty} d_q^{(\alpha)}t$$
$$= \frac{(1-q)}{\alpha}\frac{(q, -\alpha^2, -q/\alpha^2, cd/q, q^2/cd, c/a, d/a, c/b, d/b\,;q)_\infty}{(i\alpha c, -iq/\alpha c, i\alpha d, -iq/\alpha d, i\alpha q/c, -ic/\alpha, i\alpha q/d, -id/\alpha, cd/abq\,;q)_\infty} \tag{2.24}$$

where $|cd/abq| < 1$, $\alpha \neq 0$.

3. A Ramanujan-type integral for ${}_2\psi_2$

Suppose $\alpha \neq q^k$, $k = 0, \pm 1, \pm 2, \ldots$. Let us consider the integral

$$I(a,b,c,d)$$
$$:= \int_{-\infty}^\infty \left[\frac{(q^{1-s}/\alpha a, q^{1-s}/\alpha b\,;q)_\infty}{(q^{1-s}/\alpha c, q^{1-s}/\alpha d\,;q)_\infty}q^{-s} - \alpha^2\frac{(\alpha q^{s+1}/a, \alpha q^{s+1}/b\,;q)_\infty}{(\alpha q^{s+1}/c, \alpha q^{s+1}/d\,;q)_\infty}q^s\right]\omega(s)\,ds \tag{3.1}$$

where $\omega(s \pm 1) = \omega(s)$ and the integrand has no singularities on the real line. It can be shown that with appropriately chosen $\omega(s)$, the integral exists provided $|cd/ab| < q < 1$, which is the same condition as required for the existence of the bilateral series in the previous section.

Suppose $f(x)$ is continuous on $\mathbb{R}$, $\int_{-\infty}^\infty f(x)dx$ exists, and the bilateral sum $\sum_{n=-\infty}^\infty f(x+n)$ converges uniformly for $x \in [0,1]$. Then

$$\int_{-\infty}^\infty f(x)dx = \int_0^1 \sum_{n=-\infty}^\infty f(x+n)\,dx. \tag{3.2}$$

This is the theorem that was used in [6], [10] to produce some basic bilateral integrals of Ramanujan-type. Using (3.2) in (3.1), we find that

$$I(a,b,c,d) = \frac{(q, c/a, c/b, d/a, d/b, cd/q, q^2/cd\,;\,q)_\infty}{(cd/abq\,;\,q)_\infty} \times \int_0^1 \frac{(\alpha^2 q^{2s}, q^{1-2s}/\alpha^2\,;\,q)_\infty \omega(s) q^{-s} ds}{(\alpha c q^s, \alpha d q^s, \alpha q^{s+1}/c, \alpha q^{s+1}/d, q^{1-s}/\alpha c, q^{1-s}/\alpha d, c q^{-s}/\alpha, d q^{-s}/\alpha\,;\,q)_\infty}. \tag{3.3}$$

Note that this formula, along with (3.1), has the same formal appearance as the q-integral in (2.22) if we transform the integrals in (3.1) by taking $q^{-s} = t$ in the first, and $q^s = t$ in the second. Also, choosing

$$\omega(s) = \frac{(\alpha c q^s, \alpha d q^s, \alpha q^{s+1}/c, \alpha q^{s+1}/d, q^{1-s}/\alpha c, q^{1-s}/\alpha d, c q^{-s}/\alpha, d q^{-s}/\alpha\,;\,q)_\infty}{(\alpha^2 q^{2s}, q^{1-2s}/\alpha^2\,;\,q)_\infty} q^s \tag{3.4}$$

right-hand side of (3.3) yields the value 1 for the integral, leading to the integration formula

$$\begin{aligned} &\int_{-\infty}^{\infty} \Big[(q^{1-s}/\alpha a, q^{1-s}/\alpha b, \alpha q^{s+1}/c, \alpha q^{s+1}/d\,;\,q)_\infty \\ &\qquad - \alpha^2 q^{2s} (\alpha q^{s+1}/a, \alpha q^{s+1}/b, q^{1-s}/c\alpha, q^{1-s}/d\alpha\,;\,q)_\infty \Big] \\ &\qquad \times \frac{(c\alpha q^s, d\alpha q^s, c q^{-s}/\alpha, d q^{-s}/\alpha\,;\,q)_\infty}{(\alpha^2 q^{2s}, q^{1-2s}/\alpha^2\,;\,q)_\infty} ds \\ &= \frac{(q, c/a, d/a, c/b, d/b, cd/q, q^2/cd\,;\,q)_\infty}{(cd/abq\,;\,q)_\infty}, \end{aligned} \tag{3.5}$$

provided, of course, the integrand on the left-hand side has no poles on the real line and $|cd/abq| < 1$. In particular, setting $\alpha = i$, transforming q^{-s} to t in the first integral and q^s to t in the second, and then replacing a, b by q/a, q/b, respectively, we obtain

$$\begin{aligned} &\operatorname{Re} \int_0^\infty \frac{(iat, ibt, ict, idt, -ic/t, -id/t, -iq/ct, -iq/dt\,;\,q)_\infty}{(-t^2, -t^{-2}\,;\,q)_\infty} \frac{(1+t^2)}{t} dt \\ &\qquad = \frac{\log(q^{-1})}{2} \frac{(q, ac/q, ad/q, bc/q, bd/q, cd/q, q^2/cd\,;\,q)_\infty}{(abcd/q^3\,;\,q)_\infty}, \end{aligned} \tag{3.6}$$

provided $|abcd/q^3| < 1$, a, b, c, d is assumed to be real.

In closing this section, we would like to point out that the Barnes-type integral corresponding to the ${}_2\psi_2$ sum is the result

$$\begin{aligned} &\frac{1}{2\pi i} \int_C \frac{(\alpha c q^s, q^{1-s}/\alpha c, d q^{s+1}/\alpha, \alpha q^{-s}/d\,;\,q)_\infty}{(c q^s, d q^s, a q^{-s}, b q^{-s}\,;\,q)_\infty} ds \\ &\qquad = \frac{(\alpha, q/\alpha, \alpha c/d, qd/\alpha c, abcd\,;\,q)_\infty}{(q, ac, ad, bc, bd\,;\,q)_\infty \log(q^{-1})} \end{aligned} \tag{3.7}$$

where C is the part of the imaginary axis from $-iT$ to iT, $T = \pi/\log(q^{-1})$, which was proved in [10]. This is equivalent to an integral due to Askey and Roy [2].

4. The q-linear lattice $x(s)=q^{-s}$: the ${}_3\psi_3$ formula

Let us take

$$\sigma(s) = fgq^{-2}(1-q^{1-s}/f)(1-q^{1-s}/g),$$
$$\sigma(s)+\tau(s)\nabla x_1(s) = \frac{(1-aq^s)(1-bq^s)(1-cq^s)}{1-hq^s}q^{-2s}, \tag{4.1}$$

such that there is the balance condition

$$abcq^2 = fgh. \tag{4.2}$$

This corresponds to a Pearson-type equation for the rational functions on q-linear grid, [8]–[10]. From (4.1) and (4.2), it follows that

$$\tau(s)\nabla x_1(s)\, q^s = -\frac{fg(1-cq/f)(1-cq/g)}{cq^2} + \frac{fg(1-h/a)(1-h/b)}{cq^2}\;\frac{1-cq^s}{1-hq^s}, \tag{4.3}$$

which shows that $\tau(s)$ has a pole at $s=\log h/\log(q^{-1})$. Since the right-hand side must be symmetric in a, b, c, this particular form is a reflection of our intention to set up a recurrence in the parameter c. From (1.1) and (4.1), we get

$$\frac{\rho(s+1)}{\rho(s)} = \frac{(1-aq^s)(1-bq^s)(1-cq^s)}{(1-fq^s)(1-gq^s)(1-hq^s)}q^2,$$

whose solution is

$$\rho(s) = A\frac{(fq^s, gq^s, hq^s\,;\,q)_\infty}{(aq^s, bq^s, cq^s\,;\,q)_\infty}q^{2s} \tag{4.4}$$

where A is a constant (but it can be any unit periodic function of s). In particular, for $s_0\in\mathbb{C}$, $k=0,\pm1,\pm2,\dots$,

$$\rho(s_0+k) = A\frac{(fq^{s_0}, gq^{s_0}, hq^{s_0}\,;\,q)_\infty}{(aq^{s_0}, bq^{s_0}, cq^{s_0}\,;\,q)_\infty}q^{2s_0}\,\frac{(aq^{s_0}, bq^{s_0}, cq^{s_0}\,;\,q)_k}{(fq^{s_0}, gq^{s_0}, hq^{s_0}\,;\,q)_k}q^{2k}. \tag{4.5}$$

Let

$$q^{s_0}=\alpha \quad\text{and}\quad A=\alpha^{-1}\frac{(\alpha a, \alpha b, \alpha c\,;\,q)_\infty}{(\alpha f, \alpha g, \alpha h\,;\,q)_\infty}, \tag{4.6}$$

so that

$$\rho(s)q^{-s} = \rho(s_0+k)q^{-s_0-k} = \frac{(\alpha a, \alpha b, \alpha c\,;\,q)_k}{(\alpha f, \alpha g, \alpha h\,;\,q)_k}q^k. \tag{4.7}$$

Let us denote

$$\begin{aligned} u_1(c) &:= \lim_{\substack{M\to\infty\\ N\to\infty}} \sum_{s=s_0-M}^{s_0+N} \rho(s)q^{-s} \qquad (4.8)\\ &= \sum_{k=-\infty}^{\infty} \frac{(\alpha a, \alpha b, \alpha c\,;\,q)_k}{(\alpha f, \alpha g, \alpha h\,;\,q)_k}q^k \\ &= {}_3\psi_3\left[\begin{matrix}\alpha a, \alpha b, \alpha c\\ \alpha f, \alpha g, \alpha h\end{matrix}\,;\,q, q\right]. \end{aligned}$$

Then, using (4.3) and (4.7), and after summing (1.1), we obtain the formula

$$\frac{(1-h/a)(1-h/b)(1-\alpha c)}{(1-\alpha h)} u_1(cq) - (1-cq/f)(1-cq/g)u_1(c) \\ = \frac{cq^2}{fg}\Big[\lim_{k\to\infty} \rho(s_0+k)\sigma(s_0+k) - \lim_{\ell\to\infty} \rho(s_0-\ell)\sigma(s_0-\ell)\Big]. \quad (4.9)$$

From (4.1) and (4.7), it follows that

$$\lim_{k\to\infty} \rho(s_0+k)\sigma(s_0+k) = \alpha^{-1}\frac{(\alpha a, \alpha b, \alpha c\,;q)_\infty}{(\alpha f, \alpha g, \alpha h\,;q)_\infty} \quad (4.10)$$

and

$$\lim_{\ell\to\infty} \rho(s_0-\ell)\sigma(s_0-\ell) = \frac{\alpha fg}{q^2}\frac{(q/\alpha f, q/\alpha g, q/\alpha h\,;q)_\infty}{(q/\alpha a, q/\alpha b, q/\alpha c\,;q)_\infty}. \quad (4.11)$$

Thus we have a nonhomogeneous recurrence formula

$$u_1(cq) - \frac{(1-cq/f)(1-cq/g)(1-\alpha h)}{(1-h/a)(1-h/b)(1-\alpha c)} u_1(c) \\ = \frac{cq^2}{\alpha(1-h/a)(1-h/b)fg}\frac{(\alpha a, \alpha b, q\alpha c\,;q)_\infty}{(\alpha f, \alpha g, q\alpha h\,;q)_\infty} \\ - \frac{\alpha h}{(1-h/a)(1-h/b)}\frac{(q/\alpha f, q/\alpha g, 1/\alpha h\,;q)_\infty}{(q/\alpha a, q/\alpha b, 1/\alpha c\,;q)_\infty}. \quad (4.12)$$

Unlike the situation in Section 2, we now have two nonhomogeneous terms on the right-hand side, both of which cannot be transformed away. So we will try to get rid of one of them. But which one? To address this question, we recall that we are trying to iterate (4.12) in the parameter c which appears in the first term in $(q\alpha c\,;q)_\infty$ and $(q\alpha h\,;q)_\infty$, while it appears in the second term in $(1/\alpha h\,;q)_\infty$ and $(1/\alpha c\,;q)_\infty$. It is the second term that will be impossible to iterate, so the logical thing to do is to eliminate this term. Denoting $u_2(c) := u_1(c\,;\alpha^{-1})$, i.e.,

$$u_2(c) = {}_3\psi_3\left[\begin{matrix} a/\alpha, b/\alpha, c/\alpha \\ f/\alpha, g/\alpha, h/\alpha \end{matrix}\,;q,q\right], \quad (4.13)$$

so that

$$u_2(cq) - \frac{(1-cq/f)(1-cq/g)(1-h/\alpha)}{(1-h/a)(1-h/b)(1-c/\alpha)} u_2(c) \\ = \frac{\alpha cq^2}{fg(1-h/a)(1-h/b)}\frac{(a/\alpha, b/\alpha, cq/\alpha\,;q)_\infty}{(f/\alpha, g/\alpha, hq/\alpha\,;q)_\infty} \\ - \frac{h}{\alpha(1-h/a)(1-h/b)}\frac{(\alpha q/f, \alpha q/g, \alpha/h\,;q)_\infty}{(\alpha q/a, \alpha q/b, \alpha/c\,;q)_\infty}. \quad (4.14)$$

Then, defining

$$u(c) = u_1(c) - \alpha^2\frac{(\alpha q/a, \alpha q/b, \alpha q/c, q/\alpha f, q/\alpha g, q/\alpha h\,;q)_\infty}{(\alpha q/f, \alpha q/g, \alpha q/h, q/\alpha a, q/\alpha b, q/\alpha c\,;q)_\infty}\, u_2(c), \quad (4.15)$$

we find from (4.14), on replacing c by c/q, h by h/q, and using (4.2), that

$$u(c) - \frac{(1-f/c)(1-g/c)(1-q/\alpha h)}{(1-aq/h)(1-bq/h)(1-q/\alpha c)} u(c/q) = \frac{q/\alpha h}{(1-aq/h)(1-bq/h)} \frac{(\alpha a, \alpha b, \alpha c\,;q)_\infty}{(\alpha f, \alpha g, \alpha h\,;q)_\infty} p(c) \quad (4.16)$$

where

$$p(c) = 1 - \alpha^4 \frac{(a/\alpha, \alpha q/a, b/\alpha, \alpha q/b, c/\alpha, \alpha q/c, \alpha f, q/\alpha f, \alpha g, q/\alpha g, \alpha h, q/\alpha h\,;q)_\infty}{(\alpha a, q/\alpha a, \alpha b, q/\alpha b, \alpha c, q/\alpha c, f/\alpha, \alpha q/f, g/\alpha, \alpha q/g, h/\alpha, \alpha q/h\,;q)_\infty}. \quad (4.17)$$

It can be easily verified that $p(cq) = p(c)$ and that, by symmetry, similar relations hold in all six parameters a, b, c, f, g, h.

Iterating (4.16), we obtain

$$u(c) = \frac{(f/c, g/c, q/\alpha h\,;q)_2}{(aq/h, bq/h, q/\alpha c\,;q)_2} u(cq^{-2}) + \frac{(q/\alpha h)p(c)}{(1-aq/h)(1-bq/h)} \frac{(\alpha a, \alpha b, \alpha c\,;q)_\infty}{(\alpha f, \alpha g, \alpha h\,;q)_\infty} \sum_{k=0}^{1} \frac{(f/c, g/c\,;q)_k}{(aq^2/h, bq^2/h\,;q)_k} (cq/h)^k. \quad (4.18)$$

By induction, it can be proved easily that

$$u(c) = \frac{(f/c, g/c, q/\alpha h\,;q)_n}{(aq/h, bq/h, q/\alpha c\,;q)_n} u(cq^{-n}) + \frac{(q/\alpha h)p(c)}{(1-aq/h)(1-bq/h)} \frac{(\alpha a, \alpha b, \alpha c\,;q)_\infty}{(\alpha f, \alpha g, \alpha h\,;q)_\infty} \sum_{k=0}^{n-1} \frac{(f/c, g/c\,;q)_k}{(aq^2/h, bq^2/h\,;q)_k} (cq/h)^k. \quad (4.19)$$

It can be shown that

$$\lim_{n\to\infty} u(cq^{-n}) = {}_2\psi_2 \left[\begin{matrix}\alpha a, \alpha b \\ \alpha f, \alpha g\end{matrix}\,;q, cq/h\right] - \alpha^2 \frac{(\alpha q/a, \alpha q/b, q/\alpha f, q/\alpha g\,;q)_\infty}{(q/\alpha a, q/\alpha b, \alpha q/f, \alpha q/g\,;q)_\infty}\, {}_2\psi_2 \left[\begin{matrix}a/\alpha, b/\alpha \\ f/\alpha, g/\alpha\end{matrix}\,;q, cq/h\right] \quad (4.20)$$

which, by (2.22), equals

$$\frac{(q, \alpha^2, q/\alpha^2, fg/q, q^2/fg, f/a, g/a, f/b, g/b\,;q)_\infty}{(f/\alpha, \alpha q/f, g/\alpha, \alpha q/g, q/\alpha a, q/\alpha b, \alpha f, \alpha g, cq/h\,;q)_\infty}.$$

Hence, by taking the limit $n \to \infty$ in (4.19), we obtain the formula

$$\begin{aligned} &{}_3\psi_3 \left[\begin{matrix}\alpha a, \alpha b, \alpha c \\ \alpha f, \alpha g, \alpha h\end{matrix}\,;q, q\right] - \alpha^2 \frac{(\alpha q/a, \alpha q/b, \alpha q/c, q/\alpha f, q/\alpha g, q/\alpha h\,;q)_\infty}{(q/a\alpha, q/b\alpha, q/c\alpha, \alpha q/f, \alpha q/g, \alpha q/h\,;q)_\infty}\, {}_3\psi_3 \left[\begin{matrix}a/\alpha, b/\alpha, c/\alpha \\ f/\alpha, g/\alpha, h/\alpha\end{matrix}\,;q, q\right] \\ &= \frac{(q, \alpha^2, q/\alpha^2, fg/q, q^2/fg, q/\alpha h, f/a, g/a, f/b, g/b, f/c, g/c\,;q)_\infty}{(f/\alpha, \alpha q/f, g/\alpha, \alpha q/g, q/\alpha a, q/\alpha b, q/\alpha c, \alpha f, \alpha g, aq/h, bq/h, cq/h\,;q)_\infty} \\ &\quad + \frac{(q/\alpha h)p(c)}{(1-aq/h)(1-bq/h)} \frac{(\alpha a, \alpha b, \alpha c\,;q)_\infty}{(\alpha f, \alpha g, \alpha h\,;q)_\infty}\, {}_3\phi_2 \left[\begin{matrix}q, f/c, g/c \\ aq^2/h, bq^2/h\end{matrix}\,;q, cq/h\right]. \end{aligned} \quad (4.21)$$

Note that if we set $\alpha = q/h$, then this reduces to

$$
{}_3\phi_2\left[\begin{matrix} aq/h, bq/h, cq/h \\ fq/h, gq/h \end{matrix}; q, q\right] = \frac{(aq^2/h, bq^2/h, cq/h\,;\,q)_\infty}{(q, fq/h, gq/h\,;\,q)_\infty}\,{}_3\phi_2\left[\begin{matrix} q, f/c, g/c \\ aq^2/h, bq^2/h \end{matrix}; q, cq/h\right], \quad (4.22)
$$

which is a transformation formula between a balanced ${}_3\phi_2$ series and an unbalanced one and, indeed, a special case of [5, (III.10)]. Substituting this into (4.21), we obtain a more symmetric formula:

$$
\begin{aligned}
&{}_3\psi_3\left[\begin{matrix} \alpha a, \alpha b, \alpha c \\ \alpha f, \alpha g, \alpha h \end{matrix}; q, q\right] \\
&\quad - \alpha^2 \frac{(\alpha q/a, \alpha q/b, \alpha q/c, q/\alpha f, q/\alpha g, q/\alpha h\,;\,q)_\infty}{(q/a\alpha, q/b\alpha, q/c\alpha, \alpha q/f, \alpha g/g, \alpha q/h\,;\,q)_\infty}\,{}_3\psi_3\left[\begin{matrix} a/\alpha, b/\alpha, c/\alpha \\ f/\alpha, g/\alpha, h/\alpha \end{matrix}; q, q\right] \\
&\qquad = \frac{(q, \alpha^2, q/\alpha^2, fg/q, q^2/fg, q/\alpha h, f/a, f/b, f/c, g/a, g/b, g/c\,;\,q)_\infty}{(f/\alpha, \alpha q/f, g/\alpha, \alpha q/g, q/\alpha a, q/\alpha b, q/\alpha c, \alpha f, \alpha g, aq/h, bq/h, cq/h\,;\,q)_\infty} \\
&\qquad\quad + \frac{(\alpha a, \alpha b, \alpha c, q, fq/h, gq/h\,;\,q)_\infty}{(\alpha f, \alpha g, \alpha h, aq/h, bq/h, cq/h\,;\,q)_\infty}\,\frac{qp(c)}{\alpha h}\,{}_3\phi_2\left[\begin{matrix} aq/h, bq/h, cq/h \\ fq/h, gq/h \end{matrix}; q, q\right]
\end{aligned} \quad (4.23)
$$

where $p(c)$ is defined by (4.17).

If we now set $\alpha = q/f$, we obtain

$$
\begin{aligned}
&{}_3\phi_2\left[\begin{matrix} aq/f, bq/f, cq/f \\ gq/f, hq/f \end{matrix}; q, q\right] - \frac{(aq/f, bq/f, cq/f, fq/h, gq/h\,;\,q)_\infty}{(aq/h, bq/h, cq/h, gq/f, hq/f\,;\,q)_\infty}\frac{f}{h} \\
&\qquad \times {}_3\phi_2\left[\begin{matrix} aq/h, bq/h, cq/h \\ fq/h, gq/h \end{matrix}; q, q\right] = \frac{(g/a, g/b, g/c, f/h\,;\,q)_\infty}{(aq/h, bq/h, cq/h, gq/f\,;\,q)_\infty},
\end{aligned} \quad (4.24)
$$

which is simply the non-terminating q-Saalschütz formula [5, (II.24)].

5. A Ramanujan-type integral for ${}_3\psi_3$

As in Section 3, we will assume that $\alpha \neq q^k$, $k = 0, \pm 1, \pm 2, \ldots$. In fact, to ensure convergence of all the integrals and sums we shall consider, we may assume, without any loss of generality, that Im $\alpha \neq 0$, and all other parameters are real. Let us then consider the integral

$$
\begin{aligned}
J_\alpha(a, b, c, ; f, g, h) = \int_{-\infty}^{\infty} \Bigg[&\frac{(q^{1-s}/a\alpha, q^{1-s}/b\alpha, q^{1-s}/c\alpha\,;\,q)_\infty}{(q^{1-s}/\alpha f, q^{1-s}/\alpha g, q^{1-s}/\alpha h\,;\,q)_\infty} q^{-s} \\
&- \alpha^2 \frac{(\alpha q^{s+1}/a, \alpha q^{s+1}/b, \alpha q^{s+1}/c\,;\,q)_\infty}{(\alpha q^{s+1}/f, \alpha q^{s+1}/g,\ \alpha q^{s+1}/h\,;\,q)_\infty} q^s \Bigg] \omega(s)\, ds
\end{aligned} \quad (5.1)
$$

where the balance condition (4.2) is assumed to hold.

Of course, we may allow the integrals to have some simple poles, but we must then interpret the integrals appropriately in a principal-value sense. These questions were considered in detail in [9], so the interested readers may look up that reference to see how one could deal with the same questions here. For our purposes, we shall take the easier route by making the assumptions indicated above.

Using (3.2) and (4.23), we find, in a manner similar to our treatment of the ${}_2\psi_2$ integral in Section 3, that

$$\begin{aligned}
J_\alpha(a,b,c\,;f,g,h) &= \frac{(q,f/a,g/a,f/b,g/b,f/c,g/c,fg/q,q^2/fg\,;q)_\infty}{(aq/h,bq/h,cq/h\,;q)_\infty} \\
&\quad\times\int_0^1 \frac{(\alpha^2q^{2s},q^{1-2s}/\alpha^2\,;q)_\infty\omega(s)q^{-s}ds}{(\alpha q^{s+1}/f,fq^{-s}/\alpha,\alpha q^{s+1}/g,gq^{-s}/\alpha,\alpha fq^s,q^{1-s}/\alpha f,\alpha gq^s,q^{1-s}/\alpha g\,;q)_\infty} \\
&\quad+\frac{q}{\alpha h}\frac{(q,fq/h,gq/h\,;q)_\infty}{(aq/h,bq/h,cq/h\,;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}aq/h,bq/h,cq/h\\ fq/h,gq/h\end{matrix};q,q\right] \\
&\quad\times\int_0^1\frac{(\alpha aq^s,\alpha bq^s,\alpha cq^s,q^{1-s}/a\alpha,q^{1-s}/b\alpha,q^{1-s}/c\alpha\,;q)_\infty}{(\alpha fq^s,\alpha gq^s,\alpha hq^s,q^{1-s}/\alpha f,q^{1-s}/\alpha g,q^{1-s}/\alpha h\,;q)_\infty}q^{-2s}\omega(s) \\
&\qquad\times\left[1-\alpha^4q^{4s}\frac{(aq^{-s}/\alpha,bq^{-s}/\alpha,cq^{-s}/\alpha,q^{1-s}/\alpha f,q^{1-s}/\alpha g,q^{1-s}/\alpha h\,;q)_\infty}{(q^{1-s}/\alpha a,q^{1-s}/\alpha b,q^{1-s}/\alpha c,fq^{-s}/\alpha,gq^{-s}/\alpha,hq^{-s}/\alpha\,;q)_\infty}\right. \\
&\qquad\quad\left.\times\frac{(\alpha q^{s+1}/a,\alpha q^{s+1}/b,\alpha q^{s+1}/c,\alpha fq^s,\alpha gq^s,\alpha hq^s\,;q)_\infty}{(\alpha aq^s,\alpha bq^s,\alpha cq^s,\alpha q^{s+1}/f,\alpha q^{s+1}/g,\alpha q^{s+1}/h\,;q)_\infty}\right]ds. \qquad (5.2)
\end{aligned}$$

Taking

$$\omega(s)=\frac{(\alpha q^{s+1}/f,fq^{-s}/\alpha,\alpha q^{s+1}/g,gq^{-s}/\alpha,\alpha fq^s,q^{1-s}/\alpha f,\alpha gq^s,q^{1-s}/\alpha g\,;q)_\infty}{(\alpha^2q^{2s},q^{1-2s}/\alpha^2\,;q)_\infty}q^s, \qquad (5.3)$$

which is obviously a unit-periodic function, we obtain the formula

$$\begin{aligned}
&\int_{-\infty}^{\infty}\left[\frac{(q^{1-s}/a\alpha,q^{1-s}/b\alpha,q^{1-s}/c\alpha,\alpha q^{s+1}/f,\alpha q^{s+1}/g\,;q)_\infty}{(q^{1-s}/\alpha h\,;q)_\infty}q^{-s}\right. \\
&\qquad\left.-\alpha^2\frac{(\alpha q^{s+1}/a,\alpha q^{s+1}/b,\alpha q^{s+1}/c,q^{1-s}/\alpha f,q^{1-s}/\alpha g\,;q)_\infty}{(\alpha q^{s+1}/h\,;q)_\infty}q^s\right] \\
&\qquad\quad\times\frac{(fq^{-s}/\alpha,gq^{-s}/\alpha,\alpha fq^s,\alpha gq^s\,;q)_\infty}{(\alpha^2q^{2s},q^{1-2s}/\alpha^2\,;q)_\infty}q^sds \\
&=\frac{(q,f/a,g/a,f/b,g/b,f/c,g/c,fg/q,q^2/fg\,;q)_\infty}{(aq/h,bq/h,cq/h\,;q)_\infty} \\
&\quad+\frac{q}{\alpha h}\,\frac{(q,fq/h,gq/h\,;q)_\infty}{(aq/h,bq/h,cq/h\,;q)_\infty}\,{}_3\phi_2\left[\begin{matrix}aq/h,bq/h,cq/h\\ fq/h,gq/h\end{matrix};q,q\right] \\
&\quad\times\int_0^1\frac{(\alpha aq^s,\alpha bq^s,\alpha cq^s,\alpha q^{s+1}/f,\alpha q^{s+1}/g\,;q)_\infty}{(\alpha^2q^{2s},q^{1-2s}/\alpha^2,\alpha hq^s,q^{1-s}/\alpha h\,;q)_\infty}q^{-s} \\
&\qquad\times(q^{1-s}/\alpha a,q^{1-s}/\alpha b,q^{1-s}/\alpha c,fq^{-s}/\alpha,gq^{-s}/\alpha\,;q)_\infty \\
&\qquad\times\left[1-\alpha^4q^{4s}\frac{(aq^{-s}/\alpha,bq^{-s}/\alpha,cq^{-s}/\alpha,q^{1-s}/\alpha f,q^{1-s}/\alpha g,q^{1-s}/\alpha h\,;q)_\infty}{(q^{1-s}/\alpha a,q^{1-s}/\alpha b,q^{1-s}/\alpha c,fq^{-s}/\alpha,gq^{-s}/\alpha,hq^{-s}/\alpha\,;q)_\infty}\right. \\
&\qquad\quad\left.\times\frac{(\alpha q^{s+1}/a,\alpha q^{s+1}/b,\alpha q^{s+1}/c,\alpha fq^s,\alpha gq^s,\alpha hq^s\,;q)_\infty}{(\alpha aq^s,\alpha bq^s,\alpha cq^s,aq^{s+1}/f,\alpha q^{s+1}/g,\alpha q^{s+1}/h\,;q)_\infty}\right]ds. \qquad (5.4)
\end{aligned}$$

We may try to simplify this formula a bit by taking special values of α, say, $\alpha = i$, but because of the integral on the right-hand side, the simplification does not lead to any substantial improvement. Nevertheless, this is an example of a Ramanujan-type balanced integral that arises out of a q-linear lattice. It is balanced in the sense that there is an overall balance between the parameters in the denominator and numerator of the integrand expressed by (4.2).

It is interesting, however, that the Barnes-type integral corresponding to the same lattice and same $\sigma(s)$ and $\tau(s)$ yields a much simpler result:

$$\frac{1}{2\pi i}\int_C \frac{(\alpha f q^{s+1}, q^{-s}/\alpha f, g q^s/\alpha, \alpha q^{1-s}/g, h q^s\,;q)_\infty}{(q^{1-s}/f, q^{1-s}/g, a q^s, b q^s, c q^s\,;q)_\infty} ds = \frac{(\alpha, q/\alpha, \alpha f/g, qg/\alpha f, h/a, h/b, h/c\,;q)_\infty}{\log(q^{-1})(q, aq/f, bq/f, cq/f, aq/g, bq/g, cq/g\,;q)_\infty} \tag{5.5}$$

where a, b, c, f, g, h are still related by the balance condition (4.2), α is another parameter such that $\alpha f \neq 0$, $\alpha/g \neq 0$, and C is the same contour described in Section 3. This formula is Gasper's [4] extension of Askey and Roy's formula (3.7). For evaluation of the integral (5.5) by using a Pearson-type equation, see [10].

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References

1. R. Askey, *A q-extension of Cauchy's form of the beta integral*, Quart. J. Math. **32** (1981), 255–266.
2. R. Askey and R. Roy, *More q-beta integrals*, Rocky Mountain J. Math. **16** (1986), 365–372.
3. N. M. Atakishiyev, M. Rahman, and S. K. Suslov, *On the classical orthogonal polynomials*, Constr. Approx. **11** (1995), 181–226.
4. G. Gasper, *q-Extensions of Barnes', Cauchy's, and Euler's beta integrals*. In: Topics in Mathematical Analysis (T. M. Rassias, ed.), World Scientific Pub. Co., London, 1989, pp.294–314.
5. G. Gasper and M. Rahman, *Basic Hypergeometric Series*, Cambridge University Press, 1990.
6. M. E. H. Ismail and M. Rahman, *Some basic bilateral sums and integrals*, Pacific J. Math. **170** (1995), 497–515.
7. A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*, Nauka, Moscow, 1985 (in Russian); English translation, Springer-Verlag, 1991.
8. M. Rahman and S. K. Suslov, *Classical biorthogonal rational functions*. In: Methods of Approximation Theory in Complex Analysis and Mathematical Physics (A. A. Gonchar and E. B. Saff, eds.) Lecture Notes in Mathematics, Vol. 1550, Springer-Verlag, Berlin, 1993, pp.131–146.
9. ———, *The Pearson equation and the beta integrals*, SIAM J. Math. Anal. **25** (1994), 646–693.
10. ———, *Barnes and Ramanujan-type integrals on the q-linear lattice*, SIAM J. Math. Anal. **25** (1994), 1002–1022.
11. ———, *A unified approach to the summation and integration formulas for q-hypergeometric functions*, I. Journal of Statistical Planning and Inference **54** (1996), 101–118.

12. S. K. Suslov, *The theory of difference analogues of special functions of hypergeometric type*, Russian Mathematical Surveys **44** (1989), London Math. Soc. (in English), 227–278.
13. ———, *Multiparameter Ramanujan-type q-beta integrals*, The Ramanujan Journal **2** (1998), 30–49.

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