Methods and Applications of Analysis
6 (1) 1999, pp. 55–66

ISSN 1073–2772

On quasi-hypergeometric functions

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Dedicated to Richard Askey on the occasion of his 65th birthday

Abstract. We define quasi-hypergeometric functions of regular singular type and show that they are characterized by certain fractional differential equations on the one hand and by certain difference-differential equations on the other. Two examples of quasi-hypergeometric functions are given, namely quasi-algebraic functions and partition functions appearing in fractional exclusion statistics.

1. Introduction

L. Euler studied the Lambert series

$$F(x) = \sum_{n=0}^{\infty} \binom{\alpha + \beta n}{n} x^n,$$

which is intimately related to the transcendental equation

$$y - 1 = xy^{\beta},$$

[4, 13]. Recently, this kind of function has been given considerable attention by physicists. They play an important part in conformal field theory and fractional exclusion statistics. There is a pioneering work by B. Sutherland connecting them with fractional exclusion statistics and Calogero-Sutherland models [14–16]. The second author has extended some of these results to fractional exclusion statistics of multispecies of particles [10, 11], which is based on the results in [8, 18]. This corresponds exactly to an extension of transcendental functions of the above type to multivariable ones.

In this note, we would like to generalize and give a mathematical background for these functions which we call "quasi-hypergeometric functions". These functions appear as an extension of general hypergeometric functions. The latter satisfy a holonomic system of differential equations of Barnes-Mellin type by means of b-functions [1–3]. A modern observation also has been discussed in relation to toric analysis by Gelfand et al. [5, 6].

However, the quasi-hypergeometric functions $F(x_1, \dots, x_n)$ which we define here do not satisfy differential equations. We first present the system of fractional differential equations with respect to $x_1, \dots, x_n$ which $F(x_1, \dots, x_n)$ satisfy. Next, we show that $F(x_1, \dots, x_n)$ also satisfies a kind of difference-differential equations with respect to $x_1, \dots, x_n$ and other extra parameters $\alpha_1, \dots, \alpha_r$; $\alpha'_1, \dots, \alpha'_s$ (analog of contiguous relations for hypergeometric functions).

Received March 30, 1998, revised June 9, 1998.
1991 *Mathematics Subject Classification*: 33E30, 32B30.
Key words and phrases: fractional exclusion statistics, quasi-hypergeometric functions.

We can characterize these functions as the unique solutions to these functional equations.

2. System of fractional differential equations

Let α, $\beta \in \mathbf{C}$, $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\sigma_j > 0$ be given. We define a fractional derivative operator of order $-\beta$,

$$P_\sigma(\alpha,\beta)f(x) = \frac{1}{\Gamma(\beta)} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1} f(t^{\sigma_1}x_1, \ldots, t^{\sigma_n}x_n)dt$$

for a smooth function in a neighbourhood $\mathcal{U}$ of the origin of $\mathbf{C}^n$ (see [17]).

We assume that $\mathcal{U}$ is a Reinhaldt domain, i.e., $x = (x_1, \ldots, x_n) \in \mathcal{U}$ implies that $(\rho_1 x_1, \ldots, \rho_n x_n) \in \mathcal{U}$ for arbitrary complex numbers ρ_j, such that $|\rho_j| \leq 1$.

If α and β are positive, $P_\sigma(\alpha,\beta)$ is a well-defined operator, otherwise we may define it as a finite part of integrals at $t = 0$ or $t = 1$ in the sense of Hadamard [7].

The operator $P_\sigma(\alpha,\beta)$ is an operator reminiscent of the one which Humbert and Agarwal [9] defined for the function of Mittag-Leffler:

$$E_\beta(x) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(1+n\beta)} x^n.$$

$P_\sigma(\alpha,\beta)$ satisfies the following basic properties.

Proposition 1. (i) *For two arbitrary triples* (α,β,σ) *and* (α',β',σ'), $P_\sigma(\alpha,\beta)$ $P_{\sigma'}(\alpha',\beta')$ *commute with each other, i.e.,*

$$P_\sigma(\alpha,\beta) \cdot P_{\sigma'}(\alpha',\beta') = P_{\sigma'}(\alpha',\beta') \cdot P_\sigma(\alpha,\beta).$$

(ii) $P_\sigma(\alpha,0)$ *is the identity operator.*

Furthermore, if β *is a negative integer, say* $\beta = -m$, $m = 1,2,3,\ldots,$ *then* $P_\sigma(\alpha,-m)$ *reduces to a differential operator of order* m,

$$P_\sigma(\alpha,-m)f(x) = \prod_{k=1}^{m} \Big(\alpha - k + \sum_{j=1}^{n} \sigma_j x_j \frac{\partial}{\partial x_j}\Big) f(x).$$

For example,

$$P_\sigma(\alpha,-1)f(x) = \Big(\alpha - 1 + \sum_{j=1}^{n} \sigma_j x_j \frac{\partial}{\partial x_j}\Big) f(x),$$

$$P_\sigma(\alpha,-2)f(x) = \Big[(\alpha-1)(\alpha-2) + \sum_{j=1}^{n} (2(\alpha-1)\sigma_j + \sigma_j^2) x_j \frac{\partial}{\partial x_j} + \sum_{j,k=1}^{n} \sigma_j \sigma_k x_j x_k \frac{\partial^2}{\partial x_j \partial x_k}\Big] f(x).$$

(iii)

$$P_\sigma(\alpha+\beta,-\beta) \cdot P_\sigma(\alpha,\beta) = P_\sigma(\alpha,\beta) \cdot P_\sigma(\alpha+\beta,-\beta) = 1$$

so that $P_\sigma(\alpha+\beta,-\beta)$ *can be regarded as the inverse of* $P_\sigma(\alpha,\beta)$.

(iv) *For a monomial* $x_1^{\nu_1}\cdots x_n^{\nu_n}$, *we have*

$$P_\sigma(\alpha,\beta)(x_1^{\nu_1}\cdots x_n^{\nu_n}f(x)) = x_1^{\nu_1}\cdots x_n^{\nu_n}P_\sigma\left(\alpha+\sum_{j=1}^n \sigma_j\nu_j,\beta\right)f(x).$$

(v)

$$\frac{\partial}{\partial x_k}P_\sigma(\alpha,\beta)f(x) = P_\sigma(\alpha+\sigma_k,\beta)\frac{\partial}{\partial x_k}f(x).$$

(vi)

$$P_\sigma(\alpha,\beta)x_1^{\nu_1}\cdots x_n^{\nu_n} = \frac{\Gamma(\alpha+\sum_{j=1}^n\sigma_j\nu_j)}{\Gamma(\alpha+\beta+\sum_{j=1}^n\sigma_j\nu_j)}x_1^{\nu_1}\cdots x_n^{\nu_n}.$$

The proof of Proposition 1 is almost immediate except for (iii), which follows from the following lemma.

Lemma 1.

$$P_\sigma(\alpha,\beta)\cdot P_\sigma(\alpha',\beta')f(x) = \int_0^1 g(t)f(t^{\sigma_1}x_1,\ldots,t^{\sigma_n}x_n)\frac{dt}{t}$$

where $g(s)$ *denotes the Gauss hypergeometric function,*

$$g(s) = \frac{1}{\Gamma(\beta+\beta')}(1-s)^{\beta+\beta'-1}s^{\alpha-\beta'}F\Big(\beta',\alpha'+\beta'-\alpha,\beta+\beta'|\tfrac{s-1}{s}\Big).$$

In particular, the following co-cycle property holds:

$$\begin{aligned}P_\sigma(\alpha+\beta',\beta)\cdot P_\sigma(\alpha,\beta') &= P_\sigma(\alpha,\beta')\cdot P_\sigma(\alpha+\beta',\beta) \\ &= P_\sigma(\alpha,\beta+\beta'),\end{aligned} \tag{1}$$

which implies (iii) in Proposition 1 if $\beta+\beta'=0$.

We now define a system of fractional differential equations (E) as follows.

Let $\alpha_1,\ldots,\alpha_r,\alpha_1',\ldots,\alpha_s'$ be $r+s$ complex numbers, and $\beta_i=(\beta_{ij})_{j=1}^n\in\mathbf{R}_+^n$ $(1\le i\le r)$, $\beta_i'=(\beta_{ij}')_{j=1}^n\in\mathbf{R}_+^n$ $(1\le i\le s)$ be $r+s$ tuples of n-dimensional vectors with non-negative components.

We assume that the following relations hold. For each j,

$$\sum_{i=1}^s\beta_{ij}' = \sum_{i=1}^r\beta_{ij}+1. \tag{2}$$

This condition assures that the function $F(x)$ has *tempered growth* along a radial direction at the singularities, i.e., it has only a *regular singularity.*

We consider the following system of fractional differential equations for a function $F=F(x)$ depending on the variables $x_1,\ldots,x_n$:

$$\text{(E)}\qquad \frac{\partial}{\partial x_j}F = \prod_{i=1}^s P_{\beta_i'}(\alpha_i'+\beta_{ij}',-\beta_{ij}')\cdot\prod_{i=1}^r P_{\beta_i}(\alpha_i,\beta_{ij})F \tag{3}$$

for $1\le j\le n$.

As is seen from (1), this system satisfies the compatibility condition

$$\begin{aligned}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_k}F\right) &= \frac{\partial}{\partial x_j}\prod_{i=1}^{r} P_{\beta_i}(\alpha_i,\beta_{ik})\prod_{i=1}^{s} P_{\beta'_i}(\alpha'_i+\beta'_{ik},-\beta'_{ik})F \\ &= \prod_{i=1}^{r} P_{\beta_i}(\alpha_i+\beta_{ij},\beta_{ik})\prod_{i=1}^{s} P_{\beta'_i}(\alpha'_i+\beta'_{ik}+\beta'_{ij},-\beta'_{ik}) \\ &\quad\times\prod_{i=1}^{r} P_{\beta_i}(\alpha_i,\beta_{ij})\prod_{i=1}^{s} P_{\beta'_i}(\alpha'_i+\beta'_{ij},-\beta'_{ij})F \\ &= \prod_{i=1}^{r} P_{\beta_i}(\alpha_i,\beta_{ij}+\beta_{ik})\cdot P_{\beta'_i}(\alpha'_i+\beta'_{ij}+\beta'_{ik},-\beta'_{ij}-\beta'_{ik})F \\ &= \frac{\partial}{\partial x_k}(\frac{\partial}{\partial x_j}F)\end{aligned}$$

because of symmetry.

3. Quasi-hypergeometric functions

By using the parameters in the preceding section, we consider the following power series in x at the origin.

$$F\left(\begin{matrix}\{\alpha'_1;\beta'_1\},\ldots,\{\alpha'_s;\beta'_s\}\\ \{\alpha_1;\beta_1\},\ldots,\{\alpha_r;\beta_r\}\end{matrix}\;\middle|\; x\right) = \sum_{\nu_1,\ldots,\nu_n\geq 0}\frac{\prod_{i=1}^{s}\Gamma(\alpha'_i+\sum_{j=1}^{n}\beta'_{ij}\nu_j)}{\prod_{i=1}^{r}\Gamma(\alpha_i+\sum_{j=1}^{n}\beta_{ij}\nu_j)\nu_1!\cdots\nu_n!}x_1^{\nu_1}\cdots x_n^{\nu_n}. \quad (4)$$

We first remark that the following lemma holds by Stirling's formula:

Lemma 2. *We fix a, $b\in\mathbf{R}$, $k=1,2,3,\ldots$. Then, for a large positive number t, there exists a positive constant C_0 such that*

$$\frac{\Gamma(a+t)}{\Gamma(\frac{b+t}{k})^k}\leq C_0 t^{a-b+\frac{1}{2}(k-1)}k^t.$$

As a consequence of this lemma, we have

Lemma 3. *There exists a positive constant C_1 such that*

$$\left|\frac{\prod_{i=1}^{s}\Gamma(\alpha'_i+\sum_{j=1}^{n}\beta'_{ij}\nu_j)}{\prod_{i=1}^{r}\Gamma(\alpha_i+\sum_{j=1}^{n}\beta_{ij}\nu_j)\nu_1!\cdots\nu_n!}\right| \leq C_1\left(\sum_{j=1}^{n}b'_j\nu_j\right)^{\alpha'_{1,2,\ldots,s}-\alpha_{1,2,\ldots,r}+\frac{1}{2}(-n+s-1)}\cdot(r+n)^{b'_1\nu_1+\cdots+b'_n\nu_n}$$

where $\alpha_{1,2,\ldots,r}$, $\alpha'_{1,2,\ldots,s}$ and b_j, b'_j denote the sums $\alpha_1+\cdots+\alpha_r$, $\alpha'_1+\cdots+\alpha'_s$, $\sum_{i=1}^{r}\beta_{ij}$ and $\sum_{i=1}^{s}\beta'_{ij}$, respectively, such that $b'_j=b_j+1$.

Proof. We assume that $\nu_1,\ldots,\nu_n$ are so large that $\alpha'_i+\sum_{j=1}^{n}\beta'_{ij}\nu_j>1$.

We first note the inequality

$$\begin{aligned}\Big|&\frac{\prod_{i=1}^s \Gamma(\alpha'_i+\sum_{j=1}^n \beta'_{ij}\nu_j)}{\prod_{i=1}^r \Gamma(\alpha'_{1,2,\dots,s}+\sum_{j=1}^n b'_j\nu_j)}\Big| \\ &= \int_{1\geq t_1+\cdots+t_{s-1},t_j\geq 0} t_1^{\alpha'_1+\sum_{j=1}^n \beta'_{1j}\nu_j-1}\cdots t_{s-1}^{\alpha'_{s-1}+\sum_{j=1}^n \beta'_{s-1,j}\nu_j-1} \\ &\qquad \times (1-t_1-\cdots-t_{s-1})^{\alpha'_s+\sum_{j=1}^n \beta'_{sj}\nu_j-1}dt_1\wedge\cdots\wedge dt_{s-1} \\ &\leq \frac{1}{(s-1)!}\end{aligned}$$

since the integrand on the right-hand side is smaller than 1. □

On the other hand, by the log convexity of the Gamma function $\Gamma(x)$ for $x>0$, we have

$$\prod_{i=1}^r \Gamma\Big(\alpha_i+\sum_{j=1}^n \beta_{ij}\nu_j\Big)\nu_1!\cdots\nu_n! \geq \Gamma\Big(\frac{\alpha_{1,2,\dots,r}+n+\sum_{i=1}^r\sum_{j=1}^n b'_j\nu_j}{n+r}\Big)^{n+r}.$$

These two inequalities imply Lemma 3 from Lemma 2.

As a consequence of Lemma 3, the series (4) converges in the polydisc D defined by

$$|x_1|<(r+n)^{-b'_1},\dots,|x_n|<(r+n)^{-b'_n},$$

so that the function (4) defines a holomorphic function at the origin.

Furthermore, we have

Theorem 1. *The function F satisfies the equations* (E) *and can be characterized as the unique solution to* (E) *which is holomorphic at the origin and*

$$F(0)=\frac{\prod_{i=1}^s \Gamma(\alpha'_i)}{\prod_{i=1}^r \Gamma(\alpha_i)}. \tag{5}$$

Proof. Assume that the holomorphic function at the origin

$$F(x)=\sum_{\nu_1\geq 0,\dots,\nu_n\geq 0} a_{\nu_1,\dots,\nu_n}x_1^{\nu_1}\cdots x_n^{\nu_n} \tag{6}$$

satisfies the equations (E). We fix j. From (vi) in Proposition 1, we have

$$P_{\beta'_i}(\alpha'_i+\beta'_{ij},-\beta'_{ij})x_1^{\nu_1}\cdots x_n^{\nu_n} = \frac{\Gamma(\alpha'_i+\beta'_{ij}+\sum_{k=1}^n \beta'_{ik}\nu_k)}{\Gamma(\alpha'_i+\sum_{k=1}^n \beta'_{ik}\nu_k)}x_1^{\nu_1}\cdots x_n^{\nu_n}$$

and

$$P_{\beta_i}(\alpha_i,\beta_{ij})x_1^{\nu_1}\cdots x_n^{\nu_n} = \frac{\Gamma(\alpha_i+\sum_{k=1}^n \beta_{ik}\nu_k)}{\Gamma(\alpha_i+\beta_{ij}+\sum_{k=1}^n \beta_{ik}\nu_k)}x_1^{\nu_1}\cdots x_n^{\nu_n}\ .$$

The equations (E) give the recurrence relations with respect to $\nu_1,\nu_2,\dots,\nu_n$ as

$$\begin{aligned}(\nu_j+1)a_{\nu_1,\dots,\nu_j+1,\dots,\nu_n} &= \prod_{i=1}^r \frac{\Gamma(\alpha_i+\sum_{k=1}^n \beta_{ik}\nu_k)}{\Gamma(\alpha_i+\beta_{ij}+\sum_{k=1}^n \beta_{ik}\nu_k)}\cdot \\ &\quad\times\prod_{i=1}^s \frac{\Gamma(\alpha'_i+\beta'_{ij}+\sum_{k=1}^n \beta'_{ik}\nu_k)}{\Gamma(\alpha'_i+\sum_{k=1}^n \beta'_{ik}\nu_k)}\cdot a_{\nu_1,\dots,\nu_j,\dots,\nu_n}.\end{aligned}$$

These relations determine uniquely the coefficients $a_{\nu_1,\ldots,\nu_j,\ldots,\nu_n}$ except for a constant factor. If $a_{0,\ldots,0}$ equals (5), then $F(x)$ coincides with

$$F\left(\begin{matrix}\{\alpha'_1;\beta'_1\},\ldots,\{\alpha'_s;\beta'_s\}\\ \{\alpha_1;\beta_1\},\ldots,\{\alpha_r;\beta_r\}\end{matrix}\;\middle|\; x\right).$$

Thus the theorem has been proved. □

As a function of $\alpha_1,\ldots,\alpha_r,\alpha'_1,\ldots,\alpha'_s$ and x, the function

$$F\left(\begin{matrix}\{\alpha'_1;\beta'_1\},\ldots,\{\alpha'_s;\beta'_s\}\\ \{\alpha_1;\beta_1\},\ldots,\{\alpha_r;\beta_r\}\end{matrix}\;\middle|\; x\right)$$

is meromophic in $\alpha_1,\ldots,\alpha_r,\alpha'_1,\ldots,\alpha'_s$ in $\mathbf{C}^{r+s}$ and holomorphic in x in the polydisc D.

When β_{ij} and β'_{ij} are integers, the functions (4) are nothing more than general hypergeometric functions of Barnes-Mellin type.

4. System of difference-differential equations

In the preceding section, we have assumed that the parameters β_{ij} are positive. This restriction is sometimes too restrictive.

In this section, we do not impose this condition on β_{ij}.

We consider a function $F = F(x;\alpha;\alpha')$ depending on the $(n+r+s)$ variables, $x = (x_1,\ldots,x_n)$, $\alpha = (\alpha_1,\ldots,\alpha_r)$, and $\alpha' = (\alpha'_1,\ldots,\alpha'_s)$.

We denote by T_{α_i}, $T_{\alpha'_i}$ the shift operators deriving from the displacements $\alpha_i \to \alpha_i + 1$, $\alpha'_i \to \alpha'_i + 1$,

$$\begin{aligned}
T_{\alpha_i} f(x;\alpha_1,\ldots,\alpha_i,\ldots,\alpha_r;\alpha') &= f(x;\alpha_1,\ldots,\alpha_i+1,\ldots,\alpha_r;\alpha'),\\
T_{\alpha'_i} f(x;\alpha_1,\ldots,\alpha_i,\ldots,\alpha_r;\alpha') &= f(x;\alpha;\alpha'_1,\ldots,\alpha'_i+1,\ldots,\alpha'_s),
\end{aligned}$$

and also by $T^a_{\alpha_i}$, $T^a_{\alpha'_i}$ the shift operators of the displacements $\alpha_i \to \alpha_i + a$, $\alpha'_i \to \alpha'_i + a$, respectively.

We consider the system of difference-differential equations (E*):

$$(\mathrm{E}^*)\quad \left\{\begin{aligned}
F &= \Big(\alpha_i + \sum_{k=1}^n \beta_{ik} x_k \frac{\partial}{\partial x_k}\Big) T_{\alpha_i} F, && 1 \le i \le r, && (7)\\
T_{\alpha'_i} F &= \Big(\alpha'_i + \sum_{k=1}^n \beta'_{ik} x_k \frac{\partial}{\partial x_k}\Big) F, && 1 \le i \le s, && (8)\\
\frac{\partial}{\partial x_j} F &= T^{\beta_{1j}}_{\alpha_1} \cdots T^{\beta_{rj}}_{\alpha_r} \cdot T^{\beta'_{1j}}_{\alpha'_1} \cdots T^{\beta'_{sj}}_{\alpha'_s} F, && 1 \le j \le n. && (9)
\end{aligned}\right.$$

Then we have the following theorem.

Theorem 2. *The function* (4) *satisfies the equations* (E*). *It is characterized as the unique solution to* (E*) *which satisfies the initial condition*

$$F(0;\alpha;\alpha') = \frac{\prod_{i=1}^s \Gamma(\alpha'_i)}{\prod_{i=1}^r \Gamma(\alpha_i)}.$$

Proof. The function (4) satisfies (7) and (8) because of the equalities

$$T_{\alpha_i}\Gamma\Big(\alpha_i+\sum_{k=1}^n\beta_{ik}\nu_k\Big)=\Big(\alpha_i+\sum_{k=1}^n\beta_{ik}\nu_k\Big)\Gamma\Big(\alpha_i+\sum_{k=1}^n\beta_{ik}\nu_k\Big),$$
$$T_{\alpha'_i}\Gamma\Big(\alpha'_i+\sum_{k=1}^n\beta'_{ik}\nu_k\Big)=\Big(\alpha'_i+\sum_{k=1}^n\beta'_{ik}\nu_k\Big)\Gamma\Big(\alpha'_i+\sum_{k=1}^n\beta'_{ik}\nu_k\Big).$$

As for (9), we have

$$\begin{aligned}\frac{\partial}{\partial x_j}F(x)&=\sum_{\nu_1,\dots,\nu_n\ge 0}\frac{\prod_{i=1}^s\Gamma\Big(\alpha'_i+\beta'_{ij}+\sum_{k=1}^n\beta'_{ik}\nu_k\Big)}{\prod_{i=1}^r\Gamma\Big(\alpha_i+\beta_{ij}+\sum_{k=1}^n\beta_{ik}\nu_k\Big)\nu_1!\cdots\nu_n!}x_1^{\nu_1}\cdots x_n^{\nu_n}\\&=\text{ the right-hand side of (9).}\end{aligned}$$

Conversely, assume that $F(x)$ has the expansion (6) at the origin $x=0$ such that $a_{\nu_1,\dots,\nu_n}=a_{\nu_1,\dots,\nu_n}(\alpha,\alpha')$ depend on α, α' meromorphically. From (9), we have the recurrence relations

$$\begin{aligned}&(\nu_j+1)a_{\nu_1,\dots,\nu_j+1,\dots,\nu_n}(\alpha,\alpha')\\&\qquad=a_{\nu_1,\dots,\nu_j,\dots,\nu_n}(\alpha_1+\beta_{1j},\dots,\alpha_r+\beta_{rj};\alpha'_1+\beta'_{1j},\dots,\alpha'_s+\beta'_{sj}),\end{aligned}$$

so that $a_{\nu_1,\dots,\nu_j,\dots,\nu_n}(\alpha,\alpha')$ are uniquely determined from $a_{0,\dots,0}(\alpha,\alpha')$.

The last one satisfies the difference equations from (7) and (8):

$$T_{\alpha_i}a_{0,\dots,0}(\alpha,\alpha')=\alpha_i^{-1}a_{0,\dots,0}(\alpha,\alpha'),\quad T_{\alpha'_i}a_{0,\dots,0}(\alpha,\alpha')=\alpha'_i a_{0,\dots,0}(\alpha,\alpha').$$

A general solution to these can be expressed as

$$a_{0,\dots,0}(\alpha,\alpha')=\frac{\prod_{i=1}^s\Gamma(\alpha'_i)}{\prod_{i=1}^r\Gamma(\alpha_i)}\cdot H(\alpha,\alpha') \tag{10}$$

where $H(\alpha,\alpha')$ denotes an arbitrary periodic function with the periods 1 relative to each variable α_i, α'_i.

In particular, if one takes $H(\alpha,\alpha')=1$, $F(x)$ coincides with

$$F\left(\begin{matrix}\{\alpha'_1;\beta'_1\},\dots,\{\alpha'_s;\beta'_s\}\\\{\alpha_1;\beta_1\},\dots,\{\alpha_r;\beta_r\}\end{matrix}\;\middle|\;x\right).\qquad\square$$

We now fix a system of integers $\mathbf{l}=(l_1,\dots,l_r)$ and $\mathbf{l}'=(l'_1,\dots,l'_s)$. We can take as $H(\alpha,\alpha')$ the periodic function

$$H(\alpha,\alpha')=\exp\Big[2\pi i\Big(\sum_{\mu=1}^r l_\mu\alpha_\mu+\sum_{\mu=1}^s l'_\mu\alpha'_\mu\Big)\Big], \tag{11}$$

then we have the solution $F(x)$ to (E*) which has the expression

$$F(x)=\exp\Big[2\pi i\Big(\sum_{\mu=1}^r l_\mu\alpha_\mu+\sum_{\mu=1}^s l'_\mu\alpha'_\mu\Big)\Big]F\left(\begin{matrix}\{\alpha'_1;\beta'_1\},\dots,\{\alpha'_s;\beta'_s\}\\\{\alpha_1;\beta_1\},\dots,\{\alpha_r;\beta_r\}\end{matrix}\;\middle|\;x^*\right)$$

where $x^*=(x^*_1,\dots,x^*_n)$ denotes the point such that

$$x^*_1=x_1\exp\Big[2\pi i\Big(\sum_{\mu=1}^r l_\mu\beta_{\mu 1}+\sum_{\mu=1}^s l'_\mu\beta'_{\mu 1}\Big)\Big],\dots,x^*_n=x_n\exp\Big[2\pi i\Big(\sum_{\mu=1}^r l_\mu\beta_{\mu n}+\sum_{\mu=1}^s l'_\mu\beta'_{\mu n}\Big)\Big].$$

We shall abbreviate these solutions as $F_{\mathbf{l}\mathbf{l}'}(x)$,

Since an arbitrary periodic function $H(\alpha, \alpha')$ has a Fourier expansion by using a sequence (11), we can conclude the following.

Proposition 2. *Every solution to* (E*) *is a linear combination of a countable number of the solutions* $F_{\mathbf{l}\mathbf{l}'}(x)$.

5. Examples

Example 1. Let α_1, $\alpha_2 \in \mathbf{R}$ and β_1, $\beta_2 \in \mathbf{R}_+$ be given such that $\beta_1 + \beta_2 = 1$. The function

$$F = F(\{\alpha_1; \beta_1\}, \{\alpha_2; \beta_2\} \mid x) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + \beta_1 n)\Gamma(\alpha_2 + \beta_2 n)}{n!} x^n$$

converges in the disc $|x| < \beta_1^{-\beta_1} \beta_2^{-\beta_2}$ and satisfies the equation (E):

$$\frac{dF}{dx} = P_{\beta_1}(\alpha_1 + \beta_1, -\beta_1) P_{\beta_2}(\alpha_2 + \beta_2, -\beta_2) F. \tag{12}$$

The function $F(\{\alpha_1; \beta_1\}, \{\alpha_2\ \beta_2\}|x)$ is the unique solution to (E) which is holomorphic at the origin and such that $F(0) = \Gamma(\alpha_1)\Gamma(\alpha_2)$. It also satisfies the equation (E*):

$$T_{\alpha_1} F = (\alpha_1 + \beta_1 x \frac{d}{dx}) F, \quad T_{\alpha_2} F = (\alpha_2 + \beta_2 x \frac{d}{dx}) F, \quad \frac{d}{dx} F = T_{\alpha_1}^{\beta_1} T_{\alpha_2}^{\beta_2} F. \tag{13}$$

The equations (E*) also are satisfied by the functions

$$F_{l_1, l_2}(x) = \exp[2\pi i(l_1\alpha_1 + l_2\alpha_2)] F(\{\alpha_1; \beta_1\}, \{\alpha_2; \beta_2\} \mid \exp[2\pi i(l_1\beta_1 + l_2\beta_2)] x)$$

for all $(l_1, l_2) \in \mathbf{Z}^2$.

F_{l_1, l_2} does not satisfy the equation (E) but instead satisfies

$$\frac{dF}{dx} = \exp[2\pi i(l_1\beta_1 + l_2\beta_2)] P_{\beta_1}(\alpha_1 + \beta_1, -\beta_1) P_{\beta_2}(\alpha_2 + \beta_2, -\beta_2) F. \tag{14}$$

It is characterized as the unique solution to (13), which is holomorphic at the origin and $F(0) = \exp[2\pi i(l_1\alpha_1 + l_2\alpha_2)]\Gamma(\alpha_1)\Gamma(\alpha_2)$.

On the other hand, by using the equalities

$$\frac{\Gamma(\alpha_1 + \beta_1 n)\Gamma(\alpha_2 + \beta_2 n)}{\Gamma(\alpha_1 + \alpha_2 + n)} = \int_0^\infty u^{\alpha_1 + \beta_1 n - 1} (1 + u)^{-\alpha_1 - \alpha_2 - n} du$$

and the binomial expansion

$$\sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + \alpha_2 + n)}{n!} x^n = \Gamma(\alpha_1 + \alpha_2)(1 - x)^{-\alpha_1 - \alpha_2},$$

we get the integral expression for $F(x)$ given by

$$F(x) = \Gamma(\alpha_1 + \alpha_2) \int_0^\infty u^{\alpha_1 - 1} (1 + u - u^{\beta_1} x)^{-\alpha_1 - \alpha_2} du \tag{15}$$

for $|x| < \beta_1^{-\beta_1} \beta_2^{-\beta_2}$. We simply denote the number $\beta_1^{-\beta_1} \beta_2^{-\beta_2}$ by c.

At $x = 0$, the quasi-algebraic equation

$$1 + u - x u^{\beta_1} = 0$$

has the two particular solutions $u = u_+(x)$, $u_-(x)$

$$u_+(x) = -1 + e^{\pi i \beta_1} x + \cdots, \quad u_-(x) = -1 + e^{-\pi i \beta_1} x + \cdots,$$

whose coefficients are complex conjugates of each other. When x increases and approaches c, then $u_{\pm}(x)$ move in the upper (lower) half plane and approach the positive number β_1/β_2. One sees that the function (15) has a singularity of the braid type at the point $x = c$.

Example 2. Let α_1, α_2, β_1, $\beta_2 \in \mathbf{R}$ be given such that $\beta_1 = \beta_2 + 1$.

Consider the function

$$F = F\left(\begin{matrix}\{\alpha_1;\beta_1\}\\ \{\alpha_2;\beta_2\}\end{matrix}\,\middle|\, x\right) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1+\beta_1 n)}{\Gamma(\alpha_2+\beta_2 n)n!}x^n$$

in the following two cases.

Case (i) $\beta_1 > 1$, $\beta_2 > 0$. F converges for $|x| < c$, $c = \beta_1^{-\beta_1}(\beta_1 - 1)^{\beta_1 - 1}$, and is the unique solution to

$$\text{(E)} \qquad \frac{dF}{dx} = P_{\beta_1}(\alpha_1+\beta_1, -\beta_1)P_{\beta_2}(\alpha_2,\beta_2)F,$$

such that $F(0) = \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_2)}$. F also satisfies the equations (E*):

$$T_{\alpha_1}F = (\alpha_1 + \beta_1 x\frac{d}{dx})F, \quad F = (\alpha_2+\beta_2 x\frac{d}{dx})T_{\alpha_2}F, \quad \frac{d}{dx}F = T_{\alpha_1}^{\beta_1}T_{\alpha_2}^{\beta_2}F. \tag{16}$$

F has the integral expression

$$F = -\frac{\Gamma(\alpha_1+1-\alpha_2)}{2\pi i}\int_{\mathcal{L}} w^{\alpha_1-1}(1-w+xw^{\beta_1})^{-\alpha_1-1+\alpha_2}dw \tag{17}$$

for $|x| < c$. Assume for simplicity that $0 < x < c$. Then the path of integration $\mathcal{L}$ is constructed as follows. There exist two positive solutions w_1, w_2 to the equation

$$1 - w + xw^{\beta_1} = 0$$

such that $1 < w_1 < w_2$.

We construct a path $\mathcal{L}$ starting from 0 in the lower half plane, crossing the interval $[w_1, w_2]$ and going to 0 in the upper half plane.

When x tends to 0, w_1 approaches 1, and the integral (17) is holomorphic in x at $x = 0$. On the other hand, when x approaches c, then w_1, w_2 approach each other. Therefore, the integral (17) is no longer holomorphic at $x = c$. The function $F(x)$ then has a singularity of braid type there. In particular, if $\alpha_1 = \alpha_2$, then F reduces to

$$F(x) = \sum_{n=0}^{\infty}\binom{\alpha_1+\beta_1 n}{n}x^n = \frac{w_1^{\alpha_1}}{\beta_1+(1-\beta_1)w_1}$$

which is a well-known formula [13].

Case (ii) $1 > \beta_1 > 0$, $0 > \beta_2 > -1$.

By using the Gauss identity $\Gamma(\lambda)\Gamma(1-\lambda) = \pi/\sin\pi\lambda$, $F(x)$ can be written as

$$F(x) = \sum_{n=0}^{\infty}\frac{\Gamma(\alpha_1+\beta_1 n)\Gamma(1-\alpha_2-\beta_2 n)}{n!}\frac{\sin\pi(\alpha_2+\beta_2 n)}{\pi}x^n\ .$$

Hence, $F(x)$ can be written as $F = F_+ - F_-$ where

$$F_+(x) = \frac{\exp[\pi i\alpha_2]}{2\pi i} F\big(\{\alpha_1;\beta_1\},\{1-\alpha_2;-\beta_2\} \mid \exp[\pi i\beta_2]x\big),$$
$$F_-(x) = \frac{\exp[-\pi i\alpha_2]}{2\pi i} F\big(\{\alpha_1;\beta_1\},\{1-\alpha_2;-\beta_2\} \mid \exp[-\pi i\beta_2]x\big).$$

Each of them satisfies the equations of the type (14) which are different from each other. But both of them satisfy the same equations (E*) and (16).

Example 3.

$$F = \sum_{n=0}^{\infty} \frac{\prod_{k=1}^{s}\Gamma(\alpha'_k+\beta'_k n)}{\prod_{k=1}^{r}\Gamma(\alpha_k+\beta_k n)n!} x^n$$

for $\beta_k,\, \beta'_k > 0$ and with the relation $\beta_1 + \cdots \beta_r + 1 = \beta'_1 + \cdots + \beta'_s$, is a general one in the single variable case. In a way similar to Examples 1 and 2, one can show that the above series is convergent for $|x| < c$ for $c = \beta_1^{\beta_1}\cdots\beta_r^{\beta_r}\cdot{\beta'_1}^{-\beta'_1}\cdots{\beta'_s}^{-\beta'_s}$.

F satisfies the equations

$$\text{(E)} \qquad \tfrac{d}{dx}F = \prod_{k=1}^{r} P_{\beta_k}(\alpha_k,\beta_k) \prod_{k=1}^{s} P_{\beta'_k}(\alpha'_k+\beta'_k,-\beta'_k)F,$$

$$\text{(E*)} \qquad \begin{cases} F = (\alpha_k+\beta_k x\tfrac{d}{dx})T_{\alpha_k}F, & 1\le k\le r,\\ T_{\alpha'_k}F = (\alpha'_k+\beta'_k x\tfrac{d}{dx})F, & 1\le k\le s,\\ \tfrac{d}{dx}F = T_{\alpha_1}^{\beta_1}\cdots T_{\alpha_r}^{\beta_r}T_{\alpha'_1}^{\beta'_1}\cdots T_{\alpha'_s}^{\beta'_s}F. \end{cases}$$

We will show in a subsequent article that F has a singularity at $x = c$ and has a power series expansion near c, namely,

$$F(x) = (c-x)^{\delta}[a_0 + a_1(c-x) + a_2(c-x)^2 + \cdots] + \text{ (a holomorphic function)}$$

where δ denotes $\delta = \alpha_{1,2,\ldots,r} - \alpha'_{1,2,\ldots,s} + \frac{1}{2}(s-r-1)$.

In view of (iii) in Proposition 1, (E) is equivalent to

$$\prod_{k=1}^{r} P_{\beta_k}(\alpha_k+\beta_k,-\beta_k)\frac{d}{dx}F = \prod_{k=1}^{s} P_{\beta'_k}(\alpha'_k+\beta'_k,-\beta'_k)F\ .$$

When $\beta_k = \beta'_k = 1$ for all k, s must be equal to $r+1$. $F(x)$ reduces to the hypergeometric function of higher order [2, 3]

$${}_rF_{r+1}\left(\begin{matrix}\alpha'_1,\ldots,\alpha'_{r+1}\\ \alpha_1,\ldots,\alpha_r\end{matrix}\ \middle|\ x\right).$$

(E) reduces to the ordinary differential equation

$$\prod_{k=1}^{r}(\alpha_k + x\frac{d}{dx})\frac{d}{dx}F = \prod_{k=1}^{r+1}(\alpha'_k + x\frac{d}{dx})F\ .$$

Example 4. We fix $\lambda_0, \lambda_1, \ldots, \lambda_n \in \mathbf{R}$. The quasi-algebraic equation with respect to y given by

$$y^{\lambda_0} + x_1 y^{\lambda_1} + \cdots + x_n y^{\lambda_n} - 1 = 0$$

has a holomorphic solution in $(x_1, \dots, x_n)$ at the origin such that $y = 1$ for $x = 0$. Then, for an arbitrary $\rho \in \mathbf{C}$, y^ρ has the expansion in x,

$$y^\rho = \frac{\rho}{\lambda_0} \sum_{\nu_1 \geq 0, \dots, \nu_n \geq 0} (-1)^{|\nu|} \frac{\Gamma(A_\nu)}{\Gamma(A_\nu - |\nu| + 1)\nu_1! \cdots \nu_n!} x_1^{\nu_1} \cdots x_n^{\nu_n} \tag{18}$$

where $A_\nu = \frac{1}{\lambda_0}(\rho + \lambda_1 \nu_1 + \cdots + \lambda_n \nu_n)$ and $|\nu| = \nu_1 + \cdots + \nu_n$, i.e.,

$$y^\rho = \frac{\rho}{\lambda_0} F\left(\begin{matrix} \{\frac{\rho}{\lambda_0}; (\frac{\lambda_1}{\lambda_0}, \dots, \frac{\lambda_n}{\lambda_0})\} \\ \{\frac{\rho}{\lambda_0} + 1; (\frac{\lambda_1}{\lambda_0} - 1, \dots, \frac{\lambda_n}{\lambda_0} - 1)\} \end{matrix} \;\middle|\; x_1, \dots, x_n \right).$$

In fact, assume that y^ρ has an expansion

$$y^\rho = 1 + \sum_{\nu_1 \geq 0, \dots, \nu_n \geq 0, |\nu| > 0} a_{\nu_1, \dots, \nu_n} x_1^{\nu_1} \cdots x_n^{\nu_n} \ .$$

Then, by the Cauchy integral formula, we have for a small positive number ϵ,

$$a_{\nu_1, \dots, \nu_n} = \left(\frac{1}{2\pi i} \right)^n \int_{|x_1| = \epsilon, \dots, |x_n| = \epsilon} y^\rho x_1^{-\nu_1 - 1} \cdots x_n^{-\nu_n - 1} dx_1 \wedge \cdots \wedge dx_n \ .$$

The change of variables of integration, $(x_1, \dots, x_n) \to (x_1, \dots, x_{n-1}, y)$, gives

$$a_{\nu_1, \dots, \nu_n} = \left(\frac{1}{2\pi i} \right)^n \int_{|x_1| = \epsilon, \dots, |x_{n-1}| = \epsilon, |y - 1| = \epsilon} y^\rho x_1^{-\nu_1 - 1} \cdots x_n^{-\nu_n - 1} \times \frac{\partial x_n}{\partial y} dx_1 \wedge \cdots \wedge dx_{n-1} \wedge dy$$

for $x_n = (1 - y^{\lambda_0} - x_1 y^{\lambda_1} - \cdots - x_{n-1} y^{\lambda_{n-1}})/y^{\lambda_n}$.

Using the binomial expansion $y^\rho = \sum_{l=0}^\infty \binom{\rho}{l} (y-1)^l$, $a_{\nu_1, \dots, \nu_n}$ is evaluated by residue calculus as in (18).

It has been known since H. Mellin that if we put $\lambda_0 = n + 1$, $\lambda_1 = n, \dots, \lambda_n = 1$, then y reduces to a general algebraic function corresponding to the singularity of the A-type root system (see [3, 12], etc.).

Example 5. Consider the function

$$F(x) = \sum_{\nu_1, \dots, \nu_n \geq 0} \frac{\prod_{i=1}^n \Gamma(\alpha_i + \sum_{j=1}^n \beta'_{ij} \nu_j)}{\prod_{i=1}^n \Gamma(\alpha_i + \sum_{j=1}^n \beta_{ij} \nu_j) \nu_1! \cdots \nu_n!} x_1^{\nu_1} \cdots x_n^{\nu_n} \tag{19}$$

where we assume $\beta'_{ij} = \beta_{ij} = -g_{ij}$ for $i \neq j$ and $\beta'_{ii} = \beta_{ii} + 1 = 1 - g_{ii}$ for suitable real numbers g_{ij}.

This function has been investigated in recent papers on statistical mechanics [10, 11] by the second author. It is described simply by using the function $w_1^{\alpha_1} \cdots w_n^{\alpha_n}$ depending on $x_1, \dots, x_n$, which is derived from a system of the quasi-algebraic equations

$$w_i = 1 + x_i w_i^{1 - g_{ii}} w_1^{-g_{i1}} \cdots w_n^{-g_{in}} \quad 1 \leq i \leq n. \tag{20}$$

These are the fundamental equations discovered by Wu [18] for describing mutual fractional exclusion statistics following Haldane [8] which is an extension of an earlier work by Sutherland [15] in the one variable case. The equations (20) can be solved explicitly as a power series expansion in $x_1, \dots, x_n$ by the Lagrange inversion formula in the multivariable case, see [11] for details.

It seems an interesting problem to study the singularities and the monodromy property of $F(x)$ when $F(x)$ is analytically continued.

Recently Prof. V. S. Retakh pointed out to us that quasi-hypergeometric functions are very similar to the GG-functions defined by Gelfand and Graev [19]. They seem to obtain an equivalent form of our equations (E*), although we have not yet shown this. They define GG-functions as a wider class of functions which are not necessarily of regular singular type. For geometric reasons, we only consider here quasi-hypergeometric functions of regular singular type.

Acknowledgment. We would like to acknowledge a valuable comment by A. Gyoja.

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