

RATIONAL INTERPOLATION OF FUNCTIONS ON THE UNIT CIRCLE

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Dedicated to Dick Askey on the occasion of his sixty-fifth birthday

ABSTRACT. We establish various results on the convergence of a sequence of rational functions that interpolate a function on the unit circle. In particular, we extend Walsh's equi-convergence theorem and the Walsh-Sharma theorem on L_2 convergence (a special case of a theorem of Lozinski) for the interpolating rational functions.

1. Introduction

Let \mathbf{T} be the unit circle in the complex plane \mathbf{C} , and let \mathbf{D} and $\overline{\mathbf{D}}$ denote the open and closed unit disks, respectively. Also, let $z_{n,k} = e^{i\theta_{n,k}}$, $0 \leq \theta_{n,0} < \theta_{n,1} < \dots < \theta_{n,n} < 2\pi$, and for a function f defined on \mathbf{T} , let $L_n(f; \cdot) \in \mathcal{P}_n$, the set of polynomials of degree at most n , such that $L_n(f; z_{n,k}) = f(z_{n,k})$ for $k = 0, 1, \dots, n$. As early as in 1884, Méray showed that if $\{z_{n,k} : k = 0, 1, \dots, n\}$ are the $n + 1$ st roots of unity, i.e., $\theta_{n,k} = 2k\pi/(n + 1)$, $k = 0, 1, \dots, n$, then the polynomials $L_n(f; \cdot)$ do not necessarily converge to f . Indeed, Méray took the function $f_M(z) = 1/z$. So, $L_n(f_M; z) = z^n$, $n = 1, 2, 3, \dots$. These polynomials do not approach the function $f_M(z)$ for all $z \in \mathbf{D}$, as $n \rightarrow \infty$, but approach the limit zero. Méray's example was examined later by Walsh [24] who pointed out that the singularity of $f_M(z) = 1/z$ at $z = 0$ caused the sequence $\{L_n(f_M; \cdot)\}$ to fail to approach $f_M(z)$ for $z \in \mathbf{D}$. Walsh also furnished the following lovely companion to Méray's example.

Walsh's Theorem (1932, [24, Theorem 1]). *Let f be continuous (or more generally Riemann integrable) on \mathbf{T} , and let $\{z_{n,k} : k = 0, 1, \dots, n\}$ be the $n + 1$ st roots of unity. Then the sequence $\{L_n(f; \cdot)\}$ converges to the limit*

$$f_{\mathbf{D}}(z) = \frac{1}{2\pi i} \int_{\mathbf{T}} \frac{f(t)dt}{t - z}$$

for $z \in \mathbf{D}$, uniformly on any compact subset of \mathbf{D} .

It is worthwhile to recall some steps in the historical development prior to Walsh's Theorem. Runge (1904, [19]) proved that if $\{z_{n,k} : k = 0, 1, \dots, n\}$ are the $n + 1$ st roots of unity and if f is analytic on $\overline{\mathbf{D}}$, then the sequence $\{L_n(f; z)\}$ converges to $f(z)$ on $\overline{\mathbf{D}}$. After that, Fejér (1918, [9]) showed that $\{L_n(f; z)\}$ converges to $f(z)$ for $z \in \mathbf{D}$ if $f(z)$ is assumed only to be continuous on $\overline{\mathbf{D}}$ and analytic in \mathbf{D} . See also [11].

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There were further developments after Walsh's Theorem. When the points of interpolation are the roots of unity, Lozinski (1940, [15]) proved, under Fejér's assumption, the L_p ($p > 0$) convergence of $\{L_n(f; \cdot)\}$ on the unit circle, i.e.,

$$\lim_{n \rightarrow \infty} \int_{\mathbf{T}} |f(z) - L_n(f; z)|^p |dz| = 0, \quad p > 0. \quad (1)$$

This result, when $p = 2$, was rediscovered later by Walsh and Sharma (1964, [26]). Some more recent results on interpolation at the roots of unity can be found in the work of Saff and Walsh [21], Cavaretta, et al. [6, 7], and Sharma and Vertesi [22].

Walsh gave an example to show that when the points of interpolation $\{z_{n,k}\}$ are not the roots of unity, $\{L_n(f; z)\}$ may fail to converge in \mathbf{D} even though $\{z_{n,k}\}$ are sufficiently dense on \mathbf{T} .

Walsh's Example ([25, pp. 293–294]). Let $z_{n,k}$ be the roots of

$$\left(\frac{1 - \alpha z}{\alpha - z}\right)^{n+1} = 1, \quad \alpha > 1.$$

Let $f_W(z) = 1/(\zeta + \beta)$, $0 < \beta < 1$, where $\zeta = (1 - \alpha z)/(\alpha - z)$. Then

$$L_n(f_W; z) = \frac{1}{\zeta + \beta} + \frac{(\alpha + \beta)^n (\zeta^{n+1} - 1)}{[(-1)^n + \beta^{n+1}](\zeta + \beta)(\zeta - \alpha)^n},$$

which converges for z in only a part of the unit disk $\overline{\mathbf{D}}$.

Walsh's example shows that the distribution of the points of interpolation $\{z_{n,k}\}$ on \mathbf{T} should be taken into account when we study the convergence of interpolating polynomials $\{L_n(f; \cdot)\}$. Closely related to this observation, the following result is now well known (cf. [25, Chapter 7]).

The Uniform Distribution Theorem. *A necessary and sufficient condition for*

$$\lim_{n \rightarrow \infty} L_n(f; z) = f(z),$$

uniformly for $z \in \overline{\mathbf{D}}$, for every f analytic on $\overline{\mathbf{D}}$, is that the sequence of points $\{z_{n,k} : k = 0, 1, \dots, n\}$ is uniformly distributed on \mathbf{T} , i.e.,

$$\lim_{n \rightarrow \infty} \left| \prod_{k=0}^n (z - z_{n,k}) \right|^{1/n} = |z|, \quad |z| > 1.$$

Now, let us take a closer look at the points of interpolation $\{z_{n,k}\}$ in Walsh's example, which satisfy, for $|z| > 1$,

$$\lim_{n \rightarrow \infty} \left| \prod_{k=0}^n (z - z_{n,k}) \right|^{1/n} = \lim_{n \rightarrow \infty} \left| \frac{(1 - \alpha z)^{n+1} - (\alpha - z)^{n+1}}{(-\alpha)^{n+1} - (-1)^{n+1}} \right|^{1/n} = \left| z - \frac{1}{\alpha} \right|.$$

They are not uniformly distributed unless $\alpha = \infty$. How do we recover a function f analytic on $\overline{\mathbf{D}}$ if we are given the values of f at the points $\{z_{n,k}\}$ as in Walsh's example? According to the Uniform Distribution Theorem, the interpolating polynomials do not necessarily converge to f on $\overline{\mathbf{D}}$; we must use interpolating functions that are more general than polynomials. Indeed, in Walsh's example, we just need to use rational functions of the form

$$\frac{p_n(z)}{(\alpha - z)^{n+1}}, \quad p_n \in \mathcal{P}_n$$

that interpolate f at $\{z_{n,k} : k = 0, 1, \dots, n\}$. Then, by a simple transformation $z = (1 - \alpha\zeta)/(\alpha - \zeta)$, Runge's result mentioned earlier would imply that this sequence of interpolating rational functions converges to f on $\overline{\mathbf{D}}$. It is the purpose of this paper to explore this direction further and hence extend Walsh's Theorem to interpolation on the unit circle \mathbf{T} by rational functions.

2. More Notation

Let α_n , $n = 0, 1, \dots$, be given (not necessarily distinct) points in \mathbf{D} , and let

$$w_n(z) := \prod_{k=0}^n (z - \alpha_k) \quad \text{and} \quad w_n^*(z) := \prod_{k=0}^n (1 - \overline{\alpha_k}z).$$

We now define the space of rational functions for each n . Let

$$\mathcal{R}_n := \left\{ \frac{p(z)}{w_n^*(z)} : p \in \mathcal{P}_n \right\},$$

and let

$$\mathcal{R}_n^* := \left\{ \overline{r(1/\bar{z})} : r \in \mathcal{R}_n \right\}.$$

Then we see that, when we restrict z to \mathbf{T} , we have $\mathcal{R}_n^* = \{\bar{r} : r \in \mathcal{R}_n\}$. In general, an element in \mathcal{R}_n^* has the form

$$\frac{zp(z)}{w_n(z)} \quad \text{for some } p \in \mathcal{P}_n.$$

Define

$$\mathcal{R} = \bigcup_{n=0}^{\infty} \mathcal{R}_n \quad \text{and} \quad \mathcal{R}^* = \bigcup_{n=0}^{\infty} \mathcal{R}_n^*.$$

We will write $B_n(z) = w_n(z)/w_n^*(z)$. From now on, we will let the points of interpolation $\{z_{n,k} : k = 0, 1, \dots, n\}$ be given as the roots of

$$w_n(z) = w_n^*(z) \quad \text{or, equivalently,} \quad B_n(z) = 1.$$

It is easy to see that $z_{n,k} \in \mathbf{T}$ for all $k = 0, 1, \dots, n$ and $n = 1, 2, \dots$. Moreover, the $z_{n,k}$ s are distinct as implied by formula (9) to be given later. For a function f defined on \mathbf{T} , let $R_n(f; \cdot)$ denote the unique rational function from \mathcal{R}_n that interpolates f at $\{z_{n,k} : k = 0, 1, \dots, n\}$. Then, it is easy to verify that

$$R_n(f; z) = \sum_{k=0}^n \frac{f(z_{n,k})(B_n(z) - 1)}{B_n'(z_{n,k})(z - z_{n,k})}.$$

We will use $C(\mathbf{T})$ to denote the space of functions continuous on \mathbf{T} equipped with the sup-norm on \mathbf{T} . Let $A(\mathbf{D})$ denote the disk algebra, that is, the set of functions continuous on $\overline{\mathbf{D}}$ and analytic in \mathbf{D} .

Finally, let

$$\sigma_n := \sum_{k=0}^n (1 - |\alpha_k|) \quad \text{and} \quad \delta_n := \min_{0 \leq k \leq n} (1 - |\alpha_k|).$$

3. Main results

We will establish various convergence results of the sequence $\{R_n(f; \cdot)\}$ according to different assumptions on f . First, we give a simple and natural extension of Walsh's Theorem to rational interpolation.

Theorem 1. *Assume that $\lim_{n \rightarrow \infty} \sigma_n = \infty$. If $f \in C(\mathbf{T})$, then*

$$\lim_{n \rightarrow \infty} R_n(f; z) = f_{\mathbf{D}}(z)$$

for $z \in \mathbf{D}$, and the convergence is uniform on any compact subset of \mathbf{D} .

We remark that the requirement $\lim_{n \rightarrow \infty} \sigma_n = \infty$ is equivalent to saying $\lim_{n \rightarrow \infty} B_n(z) = 0$ for $z \in \mathbf{D}$ (see, e.g., [25, §10.1]). It is also a necessary and sufficient condition for the denseness of certain rational functions in $C(\mathbf{T})$ (cf. [1, 12]); see Lemma 3 for a more precise statement.

Our proof of Theorem 1 depends on the following result of weak-star convergence of a sequence of discrete measures supported at the points of interpolation. Let $d\delta_z$ denote the unit measure supported at the single point z . For $n = 0, 1, 2, \dots$, define

$$d\nu_n = \sum_{k=0}^n \frac{d\delta_{z_{n,k}}}{|B'_n(z_{n,k})|}.$$

Theorem 2. *If $\lim_{n \rightarrow \infty} \sigma_n = \infty$, then $\{d\nu_n\}$ converges in the weak-star topology to the uniform measure on \mathbf{T} , $d\theta/(2\pi)$. More precisely,*

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{f(z_{n,k})}{|B'_n(z_{n,k})|} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta, \quad (2)$$

for every $f \in C(\mathbf{T})$.

Note that when $\alpha_n = 0$ for all $n = 0, 1, 2, \dots$, we have $B_n(z) = z^{n+1}$ and $z_{n,k}$ are the $n+1$ st roots of unity, and $|B'_n(z)| = n+1$ for $z \in \mathbf{T}$. So, equation (2) becomes the well-known equality

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{f(e^{2i\pi/(n+1)})}{n+1} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta.$$

In general, one should compare Theorem 2 with the quadrature formulas for the Poisson integrals as obtained recently by Bultheel et al. [3]. It is essentially (though not exactly) equivalent to the convergence result in [3] (via suitable transformations and modifications). We will give a proof of Theorem 2 that is independent of [3] to indicate a different and direct approach.

Walsh's Theorem holds for all f that are Riemann integrable on \mathbf{T} , while in Theorem 1 we only managed to obtain the case when f is assumed continuous on \mathbf{T} . With more restrictive conditions on the rate in which the poles are allowed to approach the unit circle \mathbf{T} , we do have an extension of Walsh's theorem for all Riemann integrable functions as follows.

Theorem 3. *Assume that*

$$\lim_{n \rightarrow \infty} \sigma_n \delta_n = \infty. \quad (3)$$

If f is Riemann integrable on \mathbf{T} , then

$$\lim_{n \rightarrow \infty} \sum_{k=0}^n \frac{f(z_{n,k})}{|B'_n(z_{n,k})|} = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta \quad (4)$$

and

$$\lim_{n \rightarrow \infty} R_n(f; z) = f_{\mathbf{D}}(z) \quad (5)$$

for $z \in \mathbf{D}$, and the convergence is uniform on any compact subset of \mathbf{D} .

Condition (3) still allows α_n to approach \mathbf{T} but is more restrictive on its rate of convergence. For example, $\alpha_n = 1 - 1/n^\delta$, $n = 1, 2, \dots$, with $\delta \in [0, 1/2)$ satisfy (3). Since $\sigma_n > \sigma_n \delta_n$, (3) implies $\lim_{n \rightarrow \infty} \sigma_n = \infty$, which is what we assumed in Theorem 1. It is not clear whether (3) can be replaced by the weaker condition $\lim_{n \rightarrow \infty} \sigma_n = \infty$ in the assumption of Theorem 3.

Next, we present the L_2 convergence of $\{R_n(f; \cdot)\}$ for $f \in A(\mathbf{D})$, which generalizes the result of Walsh and Sharma [26] and that of Lozinski [15] for $p = 2$ in (1).

Theorem 4. *If $\lim_{n \rightarrow \infty} \sigma_n = \infty$, then*

$$\lim_{n \rightarrow \infty} \int_{\mathbf{T}} |R_n(f; z) - f(z)|^2 |dz| = 0,$$

for every $f \in A(\mathbf{D})$.

Similar to the polynomial case, this implies a rational version of Fejer's result mentioned in the Introduction.

The truths of Theorems 1 and 4 not only reveal a clear analogy between polynomial and rational interpolations on the unit circle \mathbf{T} , but also implicitly indicate how the extension from polynomial cases to rational cases can be carried out. For example, the well-known result of Walsh, the equi-convergence theorem [25, Theorem 1, p. 153], is extended for rational interpolation in the following result.

Theorem 5. *Assume that the set $\{\alpha_0, \alpha_1, \dots\}$ has no limit point on \mathbf{T} and its closure $\text{cls}(\{\alpha_0, \alpha_1, \dots\})$ does not separate the plane \mathbf{C} . If*

$$\lim_{n \rightarrow \infty} |B_n(z)|^{1/n} = \psi(z) \neq \text{constant} \quad (6)$$

holds locally uniformly on $\mathbf{C} \setminus (\mathbf{D} \cup \overline{\{1/\overline{\alpha_0}, 1/\overline{\alpha_1}, \dots\}})$, then

$$\lim_{n \rightarrow \infty} [R_n(f; z) - r_n(f; z)] = 0 \quad \text{for } \psi(z) < \rho^2$$

where f is assumed analytic for $\psi(z) < \rho$ ($\rho > 1$) and $r_n(f; z)$ is the least-square approximation to f out of \mathcal{R}_n , i.e.,

$$\int_{\mathbf{T}} |f(z) - r_n(f; z)|^2 |dz| = \min_{r \in \mathcal{R}_n} \int_{\mathbf{T}} |f(z) - r(z)|^2 |dz|.$$

This result is analogous to one proved by Saff and Sharma in [20] for a different rational system in which poles are the roots of $z^n - r^n = 0$ for some $r > 1$ (and so the poles are not from a single sequence as we have assumed in this paper).

In the assumption of Theorem 5, the required range of z implied by (6) can be made smaller. We have used the stronger assumption for simplicity. When $\{\alpha_n\}_{n=0}^\infty$ is cyclic, i.e.,

$$\alpha_{km+r} = \alpha_r, \quad r = 0, 1, \dots, m-1, \quad k = 1, 2, \dots,$$

for some $m \geq 1$, then (6) is satisfied as

$$\lim_{n \rightarrow \infty} |B_n(z)|^{1/n} = \left| \prod_{k=0}^{m-1} \frac{z - \alpha_k}{1 - \overline{\alpha_k} z} \right|^{1/m}.$$

We refer to Walsh's discussion in [25, §§9.4, 9.5, in particular, p. 236] for more properties of $\psi(z)$. Here we only mention the following fact that will be needed in our proof: The locus $\gamma(\rho_1) := \{z : \psi(z) = \rho_1\}$ is rectifiable and, for $1 < \rho_1 < \rho$, $\gamma(\rho_1)$, lies in $\{z : \psi(z) < \rho\}$ and is exterior to \mathbf{T} .

Note that for f analytic on $\overline{\mathbf{D}}$, the least square approximation to f out of \mathcal{P}_n is the same as the n th Taylor polynomial of f about $z = 0$. As all our theorems imply the corresponding known results for polynomial interpolation, Theorem 5 will be given the following form when we let $\alpha_n = 0$ for all $n = 0, 1, 2, \dots$.

Corollary 6. *If f is analytic for $|z| < \rho$ ($\rho > 1$), then*

$$\lim_{n \rightarrow \infty} [L_n(f; z) - s_n(f; z)] = 0 \quad \text{for } |z| < \rho^2$$

where $L_n(f; z)$ is the polynomial interpolating f at the $n + 1$ st roots of unity, and $s_n(f; z)$ is the Taylor polynomial of degree n of f about $z = 0$.

This is the celebrated Walsh's equi-convergence theorem mentioned above. This theorem has been much extended in recent years; for the latest results, see [4, 5, 17, 18] and the bibliographies therein.

4. Auxiliary results

We collect some known results and prove some new ones that are needed in the proofs of the theorems.

Lemma 1. *For $k = 0, 1, \dots, n$, we have $z_{n,k} B'_n(z_{n,k}) = |B'_n(z_{n,k})|$.*

Proof. A straightforward calculation (cf. (i) of [14, Lemma 1]) shows that

$$\frac{z B'_n(z)}{B_n(z)} = |B'_n(z)| \quad \text{for all } z \in \mathbf{T}.$$

Letting $z = z_{n,k}$ yields the desired equality. \square

Lemma 2. *For $n = 0, 1, 2, \dots$, we have*

$$\sum_{k=0}^n \frac{1}{|B'_n(z_{n,k})|} = \frac{1 - |B_n(0)|^2}{|1 - B_n(0)|^2}.$$

Proof. Let $f_I(z) := 1 - \overline{B_n(0)}B_n(z)$. Then $f_I \in \mathcal{R}_n$ and

$$f(z_{n,k}) = 1 - \overline{B_n(0)}B_n(z_{n,k}) = 1 - \overline{B_n(0)}, \quad k = 0, 1, \dots, n.$$

So, by the uniqueness of the interpolating rational function,

$$f_I(z) = R_n(f_I; z) = \sum_{k=0}^n \frac{(1 - \overline{B_n(0)})(B_n(z) - 1)}{B'_n(z_{n,k})(z - z_{n,k})}.$$

Hence, we have the identity

$$\sum_{k=0}^n \frac{1}{B'_n(z_{n,k})(z - z_{n,k})} = \frac{1 - \overline{B_n(0)}B_n(z)}{(1 - \overline{B_n(0)})(B_n(z) - 1)}.$$

Letting $z = 0$ gives us

$$\sum_{k=0}^n \frac{1}{z_{n,k}B'_n(z_{n,k})} = \frac{1 - |B_n(0)|^2}{|1 - B_n(0)|^2},$$

which, by Lemma 1, is the identity we need to verify. \square

Lemma 3. *The linear span of $\mathcal{R} \cup \mathcal{R}^*$ is dense in $C(\mathbf{T})$ if and only if $\lim_{n \rightarrow \infty} \sigma_n = \infty$.*

Proof. This follows from an application of a result in [1, Addendum A, §2, p. 244]. It can be proved by the same method. \square

Lemma 4. *For $k = 0, 1, \dots, n-1$, there exist numbers $\xi_k, \zeta_k \in (\theta_{n,k}, \theta_{n,k+1})$ such that*

$$\theta_{n,k+1} - \theta_{n,k} = \frac{2\pi}{|B'_n(e^{i\xi_k})|} \quad (7)$$

and

$$2\pi - (\theta_{n,k+1} - \theta_{n,k})|B'_n(z_{n,k})| = \frac{2\pi\gamma_n''(\zeta_k)(\theta_{n,k+1} - \zeta_k)}{\gamma_n'(\zeta_k)} \quad (8)$$

where $\gamma_n(\theta)$ is defined as a continuous function satisfying

$$B_n(e^{i\theta}) = e^{i\gamma_n(\theta)}, \quad \theta \in [0, 2\pi].$$

It turns out that, such a function γ_n is uniquely determined (up to an additional constant) and continuously differentiable on $[0, 2\pi]$. Indeed, we have (see, for example, [13])

$$\gamma_n'(\theta) = |B'_n(e^{i\theta})| = \sum_{k=0}^n \frac{1 - |\alpha_k|^2}{|e^{i\theta} - \alpha_k|^2} = \sum_{k=0}^n \frac{1 - r_k^2}{1 - 2r_k \cos(\theta - \omega_k) + r_k^2} \quad (9)$$

where $\alpha_k = r_k e^{i\omega_k}$ with $r_k > 0$, $k = 0, 1, \dots$. From (9), we can estimate γ' as follows:

$$\gamma'(\theta) = |B'_n(e^{i\theta})| \geq \sum_{k=0}^n \frac{1 - r_k}{1 + r_k} > \frac{1}{2} \sum_{k=0}^n (1 - r_k) = \frac{1}{2} \sigma_n. \quad (10)$$

Proof of Lemma 4. Note that $\gamma_n(\theta_{n,k+1}) - \gamma_n(\theta_{n,k}) = 2\pi$. Now, by the mean value theorem, we can write, for some $\xi_k \in (\theta_{n,k}, \theta_{n,k+1})$,

$$2\pi = \gamma_n(\theta_{n,k+1}) - \gamma_n(\theta_{n,k}) = \gamma_n'(\xi_k)(\theta_{n,k+1} - \theta_{n,k}),$$

which, by (9), implies (7).

To verify (8), let

$$g(\theta) := -\gamma_n(\theta_{n,k+1}) + \gamma_n(\theta) + (\theta_{n,k+1} - \theta)\gamma_n'(\theta).$$

Then,

$$g(\theta_{n,k+1}) - g(\theta_{n,k}) = 2\pi - (\theta_{n,k+1} - \theta_{n,k})|B_n'(z_{n,k})|$$

and

$$g'(\theta) = \gamma_n''(\theta)(\theta_{n,k+1} - \theta).$$

Then, by the generalized Cauchy mean value theorem, there exists a number $\zeta_k \in (\theta_{n,k}, \theta_{n,k+1})$ such that

$$\frac{g(\theta_{n,k+1}) - g(\theta_{n,k})}{\gamma_n(\theta_{n,k+1}) - \gamma_n(\theta_{n,k})} = \frac{g'(\zeta_k)}{\gamma_n'(\zeta_k)}.$$

Now, equation (8) follows. \square

Lemma 5. For $\zeta, z \in \mathbf{C}$, we have

$$\sum_{j=0}^n \frac{B_n(z)}{B_n'(\alpha_j)(z - \alpha_j)(1 - \bar{\zeta}\alpha_j)} = \begin{cases} \frac{zB_n'(z)}{B_n(z)}, & \text{if } z = \frac{1}{\bar{\zeta}}, \\ \frac{1 - \overline{B_n(\zeta)}B_n(z)}{1 - \bar{\zeta}z}, & \text{otherwise.} \end{cases}$$

Proof. First, assume $\zeta \in \mathbf{D}$. Then, note that

$$\frac{1 - \overline{B_n(\zeta)}B_n(z)}{1 - \bar{\zeta}z}$$

is the rational function from \mathcal{R}_n that interpolates $f_\zeta(z) = 1/(1 - \bar{\zeta}z)$ at $z = \alpha_0, \alpha_1, \dots, \alpha_n$. Thus, by Lagrange's interpolation formula, we have

$$\frac{1 - \overline{B_n(\zeta)}B_n(z)}{1 - \bar{\zeta}z} = \sum_{k=0}^n \frac{B_n(z)}{B_n'(z_{n,k})(z - \alpha_k)(1 - \bar{\zeta}\alpha_k)}. \quad (11)$$

Observe that both sides of (11) are rational functions of the same type in z and, after taking the complex conjugate of both sides in (11), in ζ . So, in general, the equality in (11) holds as long as $\bar{\zeta}z \neq 1$.

Finally, the case when $z = 1/\bar{\zeta}$ can be handled by first taking the limit as $z \rightarrow 1/\bar{\zeta}$ in (11) and then write ζ as $1/\bar{z}$. \square

We will need the system of functions from \mathcal{R} given by

$$\varphi_n(z) = \frac{\sqrt{1 - |\alpha_n|^2}}{z - \alpha_n} B_n(z), \quad n = 0, 1, 2, \dots$$

This system was probably introduced (with a constant multiplier of modulus 1) first by Takenaka in [23] and Malmquist in [16]. See also [8] and [25, p. 224]. It is orthogonal with respect to the uniform measure $d\theta/(2\pi)$ on the unit circle \mathbf{T} (with $z = e^{i\theta}$). We collect some properties of $\{\varphi_n\}$ in the following lemma.

Lemma 6. (i) For $n = 0, 1, \dots$, $\{\varphi_k(z)\}_{k=0}^n \subseteq \mathcal{R}_n$ and, $j, k = 0, 1, 2, \dots$,

$$\frac{1}{2\pi} \int_{z \in \mathbf{T}} \varphi_j(z) \overline{\varphi_k(z)} |dz| = \begin{cases} 0, & j \neq k, \\ 1, & j = k. \end{cases} \quad (12)$$

(ii) (Christoffel-Darboux formula for $\{\varphi_n\}$) For all $\zeta, z \in \mathbf{C}$, we have

$$\sum_{k=0}^n \overline{\varphi_k(\zeta)} \varphi_k(z) = \frac{1 - \overline{B_n(\zeta)} B_n(z)}{1 - \overline{\zeta} z}. \quad (13)$$

Proof. A proof of equation (12) can be found in [25, p. 227]. The identity (13) is well known in the literature (see [8]). It is a special case of the more general Christoffel-Darboux formula established for rational functions orthogonal on the unit circle. See, for example, [2]. \square

Our next result is interesting in its own right. It generalizes a relation first observed by Walsh and Sharma [26, Formula (16)] on polynomials.

Lemma 7. For $p, q = 0, 1, \dots, n$, we have

$$\frac{1}{2\pi} \int_{\mathbf{T}} \frac{B_n(z) - 1}{z - z_{n,p}} \overline{\left(\frac{B_n(z) - 1}{z - z_{n,q}} \right)} |dz| = \begin{cases} 0, & p \neq q, \\ |B'_n(z_{n,p})|, & p = q. \end{cases}$$

Proof. Let

$$I(p, q) := \frac{1}{2\pi} \int_{\mathbf{T}} \frac{B_n(z) - 1}{z - z_{n,p}} \overline{\left(\frac{B_n(z) - 1}{z - z_{n,q}} \right)} |dz|.$$

We claim that

$$I(p, q) = z_{n,q} \overline{z_{n,p}} \sum_{j=0}^n \frac{B_n(z_{n,p})}{B'_n(\alpha_j)(z_{n,p} - \alpha_j)(1 - \overline{z_{n,q}} \alpha_j)}. \quad (14)$$

Assuming the truth of (14), we see that Lemma 5 implies $I(p, q) = 0$ when $p \neq q$, and Lemmas 1 and 5 infer that $I(p, q) = |B'_n(z_{n,p})|$. Therefore, the lemma follows from (14).

Now, let us verify (14). We need the orthonormal basis $\{\varphi_k\}_{k=0}^n$ in \mathcal{R}_n introduced above. Since

$$\frac{B_n(z) - 1}{z - z_{n,p}}, \quad \frac{B_n(z) - 1}{z - z_{n,q}} \in \mathcal{R}_n,$$

we can write

$$\frac{B_n(z) - 1}{z - z_{n,p}} = \sum_{k=0}^n a_k \varphi_k(z) \quad \text{and} \quad \frac{B_n(z) - 1}{z - z_{n,q}} = \sum_{k=0}^n b_k \varphi_k(z)$$

for some $\{a_k\}_{k=0}^n$ and $\{b_k\}_{k=0}^n$. Then, for $p, q = 0, 1, \dots, n$, by (12) in Lemma 6, we have $I(p, q) = \sum_{k=0}^n a_k \overline{b_k}$. Now,

$$\begin{aligned} \overline{a_k} &= \frac{1}{2\pi} \int_{\mathbf{T}} \frac{B_n(z) - 1}{z - z_{n,p}} \overline{\varphi_k(z)} |dz| = \frac{1}{2\pi i} \int_{\mathbf{T}} \frac{(1 - B_n(z)) \varphi_k(z)}{B_n(z)(1 - \overline{z_{n,p}} z)} dz \\ &= \sum_{j=0}^n \frac{\varphi_k(\alpha_j)}{B'_n(\alpha_j)(1 - \overline{z_{n,p}} \alpha_j)}, \end{aligned}$$

by the residue theorem. Similarly,

$$\overline{b}_k = \sum_{j=0}^n \frac{\varphi_k(\alpha_j)}{B'_n(\alpha_j)(1 - \overline{z_{n,q}}\alpha_j)}.$$

Hence,

$$\begin{aligned} I(p, q) &= \sum_{k=0}^n \sum_{j_1=0}^n \frac{\overline{\varphi_k(\alpha_{j_1})}}{B'_n(\alpha_{j_1})(1 - z_{n,p}\overline{\alpha_{j_1}})} \sum_{j_2=0}^n \frac{\varphi_k(\alpha_{j_2})}{B'_n(\alpha_{j_2})(1 - \overline{z_{n,q}}\alpha_{j_2})} \\ &= \sum_{j_1, j_2=0}^n \frac{1}{\overline{B'_n(\alpha_{j_1})}B'_n(\alpha_{j_2})(1 - z_{n,p}\overline{\alpha_{j_1}})(1 - \overline{z_{n,q}}\alpha_{j_2})} \sum_{k=0}^n \overline{\varphi_k(\alpha_{j_1})}\varphi_k(\alpha_{j_2}) \\ &= \sum_{j_1, j_2=0}^n \frac{1}{\overline{B'_n(\alpha_{j_1})}B'_n(\alpha_{j_2})(1 - z_{n,p}\overline{\alpha_{j_1}})(1 - \overline{z_{n,q}}\alpha_{j_2})} \frac{1 - \overline{B_n(\alpha_{j_1})}B_n(\alpha_{j_2})}{1 - \overline{\alpha_{j_1}}\alpha_{j_2}}, \end{aligned}$$

by (13) in Lemma 6. So,

$$\begin{aligned} I(p, q) &= \sum_{j_1, j_2=0}^n \frac{1}{\overline{B'_n(\alpha_{j_1})}B'_n(\alpha_{j_2})(1 - z_{n,p}\overline{\alpha_{j_1}})(1 - \overline{z_{n,q}}\alpha_{j_2})} \frac{1}{1 - \overline{\alpha_{j_1}}\alpha_{j_2}} \\ &= \sum_{j_1=0}^n \frac{1}{\overline{B'_n(\alpha_{j_1})}(1 - z_{n,p}\overline{\alpha_{j_1}})} \left(\sum_{j_2=0}^n \frac{1}{B'_n(\alpha_{j_2})(1 - \overline{z_{n,q}}\alpha_{j_2})(1 - \overline{\alpha_{j_1}}\alpha_{j_2})} \right). \end{aligned}$$

Now, note that

$$\sum_{j_2=0}^n \frac{1}{B'_n(\alpha_{j_2})(1 - \overline{z_{n,q}}\alpha_{j_2})(1 - \overline{\alpha_{j_1}}\alpha_{j_2})} = z_{n,q} \sum_{j_2=0}^n \frac{B_n(z_{n,q})}{B'_n(\alpha_{j_2})(z_{n,q} - \alpha_{j_2})(1 - \overline{\alpha_{j_1}}\alpha_{j_2})},$$

which, by using Lemma 5 with $z = z_{n,q}$ and $\zeta = \alpha_{j_1}$, is equal to

$$z_{n,q} \frac{1 - \overline{B_n(\alpha_{j_1})}B_n(z_{n,q})}{1 - \overline{\alpha_{j_1}}z_{n,q}} = \frac{z_{n,q}}{1 - \overline{\alpha_{j_1}}z_{n,q}}.$$

Thus,

$$\begin{aligned} I(p, q) &= \sum_{j_1=0}^n \frac{1}{\overline{B'_n(\alpha_{j_1})}(1 - z_{n,p}\overline{\alpha_{j_1}})} \frac{z_{n,q}}{1 - \overline{\alpha_{j_1}}z_{n,q}} \\ &= z_{n,q} \overline{z_{n,p}} \sum_{j=0}^n \frac{B_n(z_{n,p})}{B'_n(\alpha_j)(z_{n,p} - \alpha_j)(1 - \overline{z_{n,q}}\alpha_j)}, \end{aligned}$$

which is (14). This completes the proof. \square

5. Proofs of the main theorems

We prove Theorem 2 first since our proof of Theorem 1 is based on it.

Proof of Theorem 2. We first claim that

$$\lim_{n \rightarrow \infty} \int_{\mathbf{T}} r \, d\nu_n = \frac{1}{2\pi} \int_{\mathbf{T}} r(z) \, |dz| \quad (15)$$

for every $r \in \mathcal{R} \cup \mathcal{R}^*$. This is a consequence of the following identity.

$$\sum_{k=0}^n \frac{f(z_{n,k})}{|B'_n(z_{n,k})|} = \frac{1}{2\pi(1-B_n(0))} \int_{\mathbf{T}} f(z) \, |dz| \quad (16)$$

for every $f \in \mathcal{R}_n$. Indeed, for $f \in \mathcal{R}_n$, we have $f(z) = R_n(f; z)$. So, by Lemma 1,

$$f(0) = R_n(f; 0) = \sum_{k=0}^n \frac{f(z_{n,k})(1-B_n(0))}{z_{n,k} B'_n(z_{n,k})} = \sum_{k=0}^n \frac{f(z_{n,k})(1-B_n(0))}{|B'_n(z_{n,k})|}.$$

This, on writing $f(0) = \int_{\mathbf{T}} f(z) |dz|$, implies equation (16). Since $\mathcal{R}_n \subset \mathcal{R}_{n+1}$, we let $n \rightarrow \infty$ in (16) to obtain (15) for $r \in \mathcal{R}$ by using the fact that $\lim_{n \rightarrow \infty} B_n(0) = 0$. Next, by taking the complex conjugate of both sides of (15), we see that (15) holds for $f \in \mathcal{R}_n^*$ as well.

Now, from Lemma 2, it follows that

$$\int_{\mathbf{T}} d\nu_n = \frac{1 - |B_n(0)|^2}{|1 - B_n(0)|^2} \leq \frac{2}{1 - |B_n(0)|} \leq \frac{2}{1 - |\alpha_0|},$$

for $n = 0, 1, 2, \dots$. Thus $\{d\nu_n\}$ is compact in the weak-star topology. Let $d\nu$ be any weak-star limit of $\{d\nu_n\}$ and

$$\lim_{\substack{n \rightarrow \infty \\ n \in \Gamma}} \int_{\mathbf{T}} f \, d\nu_n = \int_{\mathbf{T}} f \, d\nu,$$

for every $f \in C(\mathbf{T})$. In view of (15), this means

$$\int_{\mathbf{T}} r \, d\nu = \frac{1}{2\pi} \int_{\mathbf{T}} r(z) \, |dz|$$

for every $r \in \mathcal{R} \cup \mathcal{R}^*$. Since, by Lemma 3, the linear span of $\mathcal{R} \cup \mathcal{R}^*$ is dense in $C(\mathbf{T})$, then $d\nu(\theta) = d\theta/(2\pi)$. Therefore, the whole sequence converges to $d\theta/(2\pi)$. \square

Proof of Theorem 1. Let $f \in C(\mathbf{T})$. By using Lemma 1, we can write

$$R_n(f; z) = \sum_{k=0}^n \frac{f(z_{n,k})(B_n(z) - 1)}{B'_n(z_{n,k})(z - z_{n,k})} = (1 - B_n(z)) \sum_{k=0}^n \frac{z_{n,k} f(z_{n,k})}{z_{n,k} - z} \frac{1}{|B'_n(z_{n,k})|}.$$

For each fixed $z \in \mathbf{D}$, we have $\lim_{n \rightarrow \infty} B_n(z) = 0$ and that

$$\frac{tf(t)}{t-z}$$

is continuous for $t \in [0, 2\pi]$. So, using Theorem 2, we obtain

$$\lim_{n \rightarrow \infty} R_n(f; z) = \frac{1}{2\pi} \int_{\mathbf{T}} \frac{tf(t)}{t-z} \, |dt| = \frac{1}{2\pi} \int_{\mathbf{T}} \frac{f(t)}{t-z} \, dt,$$

which is $f_{\mathbf{D}}(z)$. \square

Proof of Theorem 3. We show that

$$\left| \frac{1}{|B'_n(z_{n,k})|} - \frac{\theta_{n,k+1} - \theta_{n,k}}{2\pi} \right| \leq \frac{8\pi}{|B'_n(z_{n,k})|\sigma_n\delta_n}. \quad (17)$$

Dividing both sides of (8) by $2\pi|B'_n(z_{n,k})|$, we have

$$\left| \frac{1}{|B'_n(z_{n,k})|} - \frac{\theta_{n,k+1} - \theta_{n,k}}{2\pi} \right| = \frac{|\gamma''(\zeta_k)(\theta_{n,k+1} - \zeta_k)|}{|B'_n(z_{n,k})|\gamma'(\zeta_k)}. \quad (18)$$

Now, note that

$$\left| \frac{\gamma''_n(\zeta_k)}{\gamma'_n(\zeta_k)} \right| \leq \frac{\sum_{j=0}^n c_j \frac{2r_j |\sin(\zeta_k - \omega_j)|}{1 - 2r_j \cos(\zeta_k - \omega_j) + r_j^2}}{\sum_{j=0}^n c_j}$$

with

$$c_j := \frac{1 - r_j^2}{1 - 2r_j \cos(\zeta_k - \omega_j) + r_j^2}, \quad j = 0, 1, \dots, n.$$

Note that, $c_j > 0$, $j = 0, 1, \dots, n$, and, for $0 \leq r < 1$,

$$\max_{\theta \in [0, 2\pi]} \frac{2r |\sin(\theta)|}{1 - 2r \cos(\theta) + r^2} = \frac{2r}{1 - r^2}.$$

So, we have

$$\left| \frac{\gamma''_n(\zeta_k)}{\gamma'_n(\zeta_k)} \right| \leq \max_{0 \leq j \leq n} \frac{2r_j}{1 - r_j^2} < \frac{2}{\delta_n}. \quad (19)$$

On the other hand, by (7) in Lemma 4 and (10), we have

$$|\theta_{n,k+1} - \zeta_k| \leq |\theta_{n,k+1} - \theta_{n,k}| = \frac{2\pi}{|B'_n(e^{i\xi_k})|} \leq \frac{2\pi}{\frac{1}{2}\sigma_n} = \frac{4\pi}{\sigma_n}. \quad (20)$$

Using (19) and (20) in (18), we obtain (17).

To prove (4), we use (17) and compare the sum

$$\sum_{k=0}^n \frac{f(z_{n,k})}{|B'_n(z_{n,k})|}$$

with

$$\frac{1}{2\pi} \sum_{k=0}^n f(e^{i\theta_{n,k}})(\theta_{n,k+1} - \theta_{n,k}),$$

a Riemann sum of f on \mathbf{T} . We have, with M_f denoting an upper bound of f on \mathbf{T} , by using (17) and Lemma 2,

$$\begin{aligned} & \left| \sum_{k=0}^n \frac{f(z_{n,k})}{|B'_n(z_{n,k})|} - \frac{1}{2\pi} \sum_{k=0}^n f(e^{i\theta_{n,k}})(\theta_{n,k+1} - \theta_{n,k}) \right| \\ & \leq \sum_{k=0}^n |f(z_{n,k})| \left| \frac{1}{|B'_n(z_{n,k})|} - \frac{\theta_{n,k+1} - \theta_{n,k}}{2\pi} \right| \\ & \leq M_f \sum_{k=0}^n \frac{8\pi}{|B'_n(z_{n,k})| \sigma_n \delta_n} = \frac{8\pi M_f (1 - |B_n(0)|^2)}{\sigma_n \delta_n (|1 - B_n(0)|^2)}, \end{aligned}$$

which goes to the limit zero by the assumption (3). This proves (4).

The proof of (5) is obtained from (4) in the same way as that of Theorem 1 is obtained from Theorem 2. \square

Proof of Theorem 4. Since $f \in A(\mathbf{D})$, by Mergelyan's theorem ([10, Theorem 1, p. 97]), there exists a sequence of polynomials (the arithmetic means of the partial sums of the Taylor series of f about $z = 0$) that converges uniformly to f on $\overline{\mathbf{D}}$. By using the argument in [1, Addendum A, pp. 243–246], we can see that each monomial z^k ($k \geq 0$) can be approximated by rational functions from \mathcal{R} as closely as we please. Therefore, for $\varepsilon > 0$, there exists a function $r \in \mathcal{R}$ such that

$$\max_{z \in \overline{\mathbf{D}}} |f(z) - r(z)| < \varepsilon.$$

Note that $R_n(r; z) = r(z)$ for all n large enough. Now we have, for n large enough,

$$\begin{aligned} & \int_{\mathbf{T}} |R_n(f; z) - f(z)|^2 |dz| = \int_{\mathbf{T}} |R_n(f - r; z) + r(z) - f(z)|^2 |dz| \\ & \leq 2 \left\{ \int_{\mathbf{T}} |R_n(f - r; z)|^2 |dz| + \int_{\mathbf{T}} |r(z) - f(z)|^2 |dz| \right\} \\ & \leq 2 \int_{\mathbf{T}} |R_n(f - r; z)|^2 |dz| + 4\pi\varepsilon^2. \end{aligned}$$

The integral in the last expression can be expanded as

$$\sum_{p,q=0}^n \frac{f(z_{n,p}) - r(z_{n,p})}{B'_n(z_{n,p})} \overline{\left[\frac{f(z_{n,q}) - r(z_{n,q})}{B'_n(z_{n,q})} \right]} \int_{\mathbf{T}} \frac{B_n(z) - 1}{z - z_{n,p}} \overline{\left(\frac{B_n(z) - 1}{z - z_{n,q}} \right)} |dz|,$$

which, according to Lemma 7, is equal to

$$\sum_{k=0}^n \frac{2\pi |f(z_{n,k}) - r(z_{n,k})|^2}{|B'_n(z_{n,k})|}.$$

Now, combining the above estimates and using Lemma 2, we have

$$\int_{\mathbf{T}} |R_n(f; z) - f(z)|^2 |dz| \leq \frac{4\pi\varepsilon^2 (1 - |B_n(0)|^2)}{|1 - B_n(0)|^2} + 4\pi\varepsilon^2 \leq \frac{12\pi\varepsilon^2}{1 - |B_n(0)|}.$$

Thus, by letting $n \rightarrow \infty$, we obtain

$$0 \leq \limsup_{n \rightarrow \infty} \int_{\mathbf{T}} |R_n(f; z) - f(z)|^2 |dz| \leq 12\pi\varepsilon^2.$$

This implies the desired result. \square

Proof of Theorem 5. By Hermite's formula, for each $n = 1, 2, \dots$,

$$R_n(f; z) = \frac{1}{2\pi} \int_{\gamma} \frac{f(\zeta)(B_n(\zeta) - B_n(z))}{(B_n(\zeta) - 1)(\zeta - z)} d\zeta \quad (21)$$

for all $z \in \text{Int}(\gamma)$, the interior of γ where γ is a Jordan curve in $\{z : \psi(z) < \rho\}$ that surrounds $\{z_{n,0}, z_{n,1}, \dots, z_{n,n}\} \subseteq \mathbf{T}$. On the other hand, by [25, Lemmas I and II on p. 225, Theorem 2 on p. 227], the least square approximation $r_n(f; z)$ is the same as the rational function from \mathcal{R}_n that interpolates f at $\alpha_0, \alpha_1, \dots, \alpha_n$. (For repeated points, r_n interpolates f and its derivatives.) Again, by Hermite's formula,

$$r_n(f; z) = \frac{1}{2\pi} \int_{\gamma} \frac{f(\zeta)(B_n(\zeta) - B_n(z))}{B_n(\zeta)(\zeta - z)} d\zeta, \quad z \in \text{Int}(\gamma), \quad (22)$$

where γ is as in (21). Therefore, by (21) and (22), we obtain, for $z \in \text{Int}(\gamma)$,

$$R_n(f; z) - r_n(f; z) = \frac{1}{2\pi} \int_{\gamma} \frac{f(\zeta)(B_n(\zeta) - B_n(z))}{(\zeta - z)} \frac{d\zeta}{(B_n(\zeta) - 1)B_n(\zeta)}.$$

Assume $1 < \rho_1 < \rho_2 < \rho$. Let $\gamma(r) = \{z : \psi(z) = r\}$, $r > 1$. (For the properties of $\gamma(r)$, see the remarks after the statement of Theorem 5 in Section 3.) Then, for $z \in \text{Int}(\gamma(\rho_1))$ and $\varepsilon \in (0, \rho - 1)$, when n is large enough, we have

$$|R_n(f; z) - r_n(f; z)| \leq \frac{1}{2\pi} \frac{\max_{\zeta \in \gamma(\rho_2)} |f(\zeta)| ((\rho_2 + \varepsilon)^n + (\rho_1 + \varepsilon)^n) l(\gamma(\rho_2))}{\text{dist}(\gamma(\rho_1), \gamma(\rho_2)) ((\rho_2 - \varepsilon)^n - 1) (\rho_2 - \varepsilon)^n}$$

where $l(\gamma(\rho_2))$ denotes the length of $\gamma(\rho_2)$ and $\text{dist}(\gamma(\rho_1), \gamma(\rho_2))$ is the distance between $\gamma(\rho_1)$ and $\gamma(\rho_2)$. It follows that

$$\limsup_{n \rightarrow \infty} \max_{z \in \gamma(\rho_1)} |R_n(f; z) - r_n(f; z)|^{1/n} \leq \frac{1}{\rho_2}.$$

Letting $\rho_2 \nearrow \rho$ first and then letting $\rho_1 \nearrow \rho$ yield

$$\limsup_{n \rightarrow \infty} \max_{z \in \gamma(\rho)} |R_n(f; z) - r_n(f; z)|^{1/n} \leq \frac{1}{\rho}. \quad (23)$$

Now, from the fact that

$$\frac{R_n(f; z) - r_n(f; z)}{B_n(z)}$$

is analytic in $\mathbf{C} \setminus \mathbf{D}$ (including ∞), by using the maximum modulus principle, we have

$$\max_{z \in \gamma(\rho_2)} \left| \frac{R_n(f; z) - r_n(f; z)}{B_n(z)} \right| \leq \max_{z \in \gamma(\rho_1)} \left| \frac{R_n(f; z) - r_n(f; z)}{B_n(z)} \right|.$$

Thus, for $1 < \rho_1 < \rho_2$,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \max_{z \in \gamma(\rho_2)} \left| \frac{R_n(f; z) - r_n(f; z)}{B_n(z)} \right|^{1/n} \\ &= \limsup_{n \rightarrow \infty} \max_{z \in \gamma(\rho_2)} \frac{|R_n(f; z) - r_n(f; z)|^{1/n}}{\rho_2} \\ &\leq \max_{z \in \gamma(\rho_1)} \left| \frac{R_n(f; z) - r_n(f; z)}{B_n(z)} \right|^{1/n} \\ &= \limsup_{n \rightarrow \infty} \max_{z \in \gamma(\rho_1)} \frac{|R_n(f; z) - r_n(f; z)|^{1/n}}{\rho_1}, \end{aligned}$$

and so,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \max_{z \in \gamma(\rho_2)} |R_n(f; z) - r_n(f; z)|^{1/n} \\ & \leq \frac{\rho_2}{\rho_1} \limsup_{n \rightarrow \infty} \max_{z \in \gamma(\rho_1)} |R_n(f; z) - r_n(f; z)|^{1/n}. \end{aligned}$$

This and (23) imply that, for $1 < \rho_* < \rho$,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \max_{z \in \gamma(\rho_*^2)} |R_n(f; z) - r_n(f; z)|^{1/n} \\ & \leq \rho_* \limsup_{n \rightarrow \infty} \max_{z \in \gamma(\rho_*)} |R_n(f; z) - r_n(f; z)|^{1/n} \\ & \leq \rho_* \limsup_{n \rightarrow \infty} \max_{z \in \gamma(\rho)} |R_n(f; z) - r_n(f; z)|^{1/n} \\ & \leq \frac{\rho_*}{\rho} < 1, \end{aligned}$$

which implies that $R_n(f; z) - r_n(f; z)$ converges to the limit zero locally and uniformly in $\text{Int}(\gamma(\rho^2))$ at a geometric rate. This finishes the proof of Theorem 5. \square

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