

THE EINSTEIN-DIRAC-MAXWELL EQUATIONS – BLACK HOLE SOLUTIONS

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1. Introduction. We are interested in studying how different force fields interact with gravity, at a “fundamental” level, but not at the level of Quantum Field Theory. This is because there exists no theory of quantum gravity, and no understanding of “Planck-scale” physics; that is, physics at “Planck-lengths”, where the Planck-length is given by

$$\left(\frac{Gh}{c^3}\right)^{1/2} \approx 10^{-33} \text{ cm},$$

where G is Newton’s gravitational constant, h is Planck’s constant, and c denotes the speed of light.

Before discussing our results, we think that it is worthwhile to see what is the difficulty in making a theory of quantum gravity. In order to understand why a solution to the problem of reconciling gravity and quantum mechanics has been so elusive, we must consider the implications of the Heisenberg Uncertainty Principle (HUP) at small distance scales.

Recall that the HUP states that “the more precisely a spatial measurement is made, the less precisely the momentum (or the energy) of the system being measured, is known”. When the spatial measurement $\Delta x \approx 10^{-13}$ cm, there are large uncertainties in the energy E . These are realized as “fluctuations” at small distances. Since $E = mc^2$, the energy of these fluctuations can give rise to “virtual” particles and anti-particles, which arise out of the vacuum for short times before annihilating each other, and give rise to a “sea of virtual particles”. When $\Delta x \approx 10^{-33}$ cm (Planck length), the energy fluctuations are quite large over small distances, and general relativistic effects become important. In fact, from a theorem of Schoen and Yau [12], singularities must form, and these are believed to be black holes. Thus space-time becomes very curved at small distance scales; physicists say that space-time is “foamy” (not smooth). This invalidates the usual computational techniques of Quantum Field Theory, where small curvature is needed – the calculations break down at high energies. This suggests that at high energies, General Relativity, or Quantum Mechanics, (or both) must be modified. The unsolved problem is, how is this to be done?

In order to set the background for our discussion, we shall briefly review what we consider to be some of the most important results concerning the coupling of gravity to other fields. The first such result is due to Reissner, and Nordström (1918, 1919) whereby they coupled gravity to electromagnetism, and this led to the celebrated Reissner-Nordström solution, about which we shall have more to say in the next sections. In the 1920’s, Einstein, de Sitter, Friedmann, and others coupled gravity to perfect fluids, in order to study problems in Cosmology, the large scale structure

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of the Universe. In the 1930's Oppenheimer, Tolman, Snyder, Volkoff, Landau, and others also coupled gravity to perfect fluids in order to study problems connected with stellar formation and collapse. (All of the above results are classical and can be found in most standard textbooks on General Relativity; see e.g. [1].)

In the more modern era, the first important ideas stem from the 1988 paper of R. Bartnik and J. McKinnon, [2]. These authors coupled Einstein's equations to an $SU(2)$ Yang-Mills field (that is, gravity to the weak nuclear force), and they found, numerically, particle-like solutions. In the papers [15, 16], Smoller, Wasserman, and Yau rigorously proved the existence of both particle-like and black-hole solutions for the $SU(2)$ Einstein/Yang-Mills equations. Also in recent years, Smoller and Temple [13, 14] coupled gravity to perfect fluids and they constructed a theory of shock-waves in General Relativity, with applications to Cosmology and stellar structure. In the present paper, we couple gravity to both spinors and electromagnetism; i.e., we couple Einstein's equations to both the Dirac equation and Maxwell's equations and we investigate the existence of black-hole solutions; (in the works [4, 5], we consider particle-like solutions of these equations). We shall show here that in all cases considered, black-hole solutions do not exist; for complete details, see [6, 7], and also [8, 9] for related work.

2. Background. In this section we shall give a short discussion of Einstein's equations of General Relativity, Maxwell's equations of electromagnetism, and Dirac's equation of relativistic Quantum Mechanics. For more details, the reader should consult the standard textbooks, e.g. [1, 11].

We begin with General Relativity (GR). The subject of GR is based on Einstein's three important hypotheses:

- (E_1) The gravitational field is described by the metric g_{ij} in 4-d space-time; g_{ij} is assumed to be symmetric: $g_{ij} = g_{ji}$, $i, j = 0, 1, 2, 3$.
- (E_2) At each point, the 4×4 symmetric matrix $[g_{ij}]$ can be diagonalized as $g_{ij} = \text{diag}(1, -1, -1, -1)$.
- (E_3) The equations of GR should be independent of the choice of coordinate system.

The hypothesis (E_1) is Einstein's great insight, whereby he "geometrizes" the gravitational field. (E_2) means that there should exist "inertial frames" at each point, so that Special Relativity is included in GR, and (E_3) implies that the gravitational field equations should be tensor equations.

The metric $g_{ij} = g_{ij}(x)$, $i, j = 0, \dots, 3$, $x = (x^0, x^1, x^2, x^3)$, $x^0 = ct$, is the metric tensor defined on 4-d space-time. *Einstein's equations* are ten (tensor) equations for the metric (gravitational field) and take the form

$$(2.1) \quad R_{ij} - \frac{1}{2} R g_{ij} = \sigma T_{ij} .$$

The left-hand side $G_{ij} \equiv R_{ij} - \frac{1}{2} R g_{ij}$ is the *Einstein tensor*, and is a *geometric* object, while T_{ij} the energy-momentum tensor, represents the source of the gravitational field and encodes the distribution of matter. The word "matter" in general relativity refers to everything which can produce a gravitational field. The tensor T_{ij} is required to satisfy the relation $T_{j;i}^i = 0$, (it's covariant divergence vanishes), and this in turn expresses the laws of conservation of energy and momentum. The quantities which comprise the Einstein tensor G_{ij} are given as follows: First from the metric tensor

g_{ij} , we form the *Levi-Civita connection* $\Gamma_{i,j}^k$:

$$\Gamma_{ij}^k = \frac{1}{2} g^{k\ell} \left(\frac{\partial g_{\ell j}}{\partial x^i} + \frac{\partial g_{i\ell}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^\ell} \right),$$

where $[g^{k\ell}] = [g_{k\ell}]^{-1}$, and summation convention is employed; namely an index which appears as both a subscript and superscript is to be summed from 0 to 3. Having Γ_{ij}^k , we then construct the *Riemann curvature tensor* $R_{qk\ell}^i$:

$$R_{qk\ell}^i = \frac{\partial \Gamma_{q\ell}^i}{\partial x^k} - \frac{\partial \Gamma_{qk}^i}{\partial x^\ell} + \Gamma_{pk}^i \Gamma_{q\ell}^p - \Gamma_{p\ell}^i \Gamma_{qk}^p.$$

Finally, we can explain the terms R_{ij} and R in E_{ij} ; namely $R_{ij} = R_{isj}^s$ is the *Ricci tensor*, and the scalar $R = g^{ij} R_{ij}$, is called the *scalar curvature*.

The quantity σ is a universal constant defined by

$$\sigma = \frac{8\pi G}{c^4},$$

where G is the gravitational constant, and c is the speed of light. From these definitions, one immediately sees the enormous complexity of the equations (2.1) for the unknown quantities g_{ij} . For this reason, one seeks solutions which are highly symmetric, and in what follows, we shall only consider solutions which are spherically symmetric; i.e. the metric has the form

$$(2.2) \quad ds^2 = T^{-2}(r) dt^2 - A^{-1}(r) dr^2 - r^2 d\Omega^2,$$

where $d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta \varphi^2$ is the standard metric on the 2-sphere, (r, ϑ, φ) are the usual polar coordinates, and t denotes time. Such metrics turn out to be quite interesting mathematically, in addition to having physical interest.

Now consider the problem of finding the gravitational field exterior to a ball in \mathbb{R}^3 ; that is, there is no matter exterior to the ball. This actually models well our solar system, where we think of the Sun as the 3-ball, and we ignore the masses of the planets. In this case the Einstein equations become $R_{ij} - \frac{1}{2} R g_{ij} = 0$, and we seek spherically symmetric solutions. This problem was solved by K. Schwarzschild in 1916, and its solution is called the *Schwarzschild metric*:

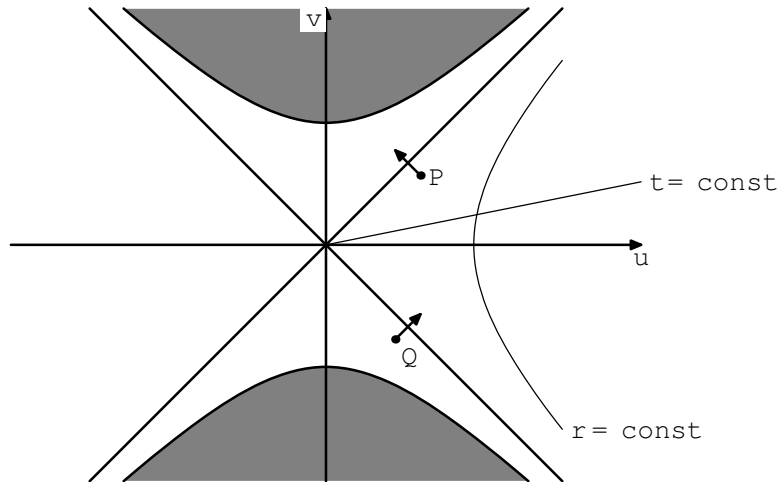
$$ds^2 = \left(1 - \frac{2m}{r} \right) c^2 dt^2 - \left(1 - \frac{2m}{r} \right)^{-1} dr^2 - d\Omega^2.$$

Here $m = GM/c^2$ and M is the mass of the 3-ball. Since $2m$ has dimension of length, it is called the Schwarzschild radius. One sees immediately that the Schwarzschild metric is singular at $r = 2m$; namely $g_{00} = 0$ and $g_{11} = \infty$. For $r < 2m$, the signs of the metric components change: $g_{00} > 0$ and $g_{11} < 0$. So we must reconsider the physical meaning of t and r inside the Schwarzschild radius.

To investigate the mathematical and physical nature of the Schwarzschild radius, we introduce Kruskal coordinates $(r, t) \rightarrow (u, v)$, and we seek a transformed metric of the form

$$ds^2 = f(u, v)(dv^2 - du^2) + r^2 d\Omega^2.$$

This yields the following in the region $r > 2m$, (see Figure 2.1)

FIG. 2.1. *Kruskal Coordinates*

$$\begin{aligned}
 u &= \sqrt{\frac{r}{2m} - 1} e^{r/4m} \cosh \frac{ct}{4m} \\
 v &= \sqrt{\frac{r}{2m} - 1} e^{r/4m} \sinh \frac{ct}{4m} \\
 f &= \frac{32m^3}{r} e^{r/4m},
 \end{aligned}$$

with similar expressions in the region $r < 2m$.

We can use Kruskal coordinates to study light rays traveling radially inward towards the Schwarzschild radius, starting at a point P outside of the Schwarzschild radius; i.e., in the region $r > 2m$; see Figure 1. In u/v coordinates, the light ray leaves at the point P in $r > 2m$, t finite, and travels inward towards the Schwarzschild radius $r = 2m$, as $t \rightarrow \infty$. It crosses the line $t = \infty$ into the interior of the Schwarzschild sphere. Incoming light is thus in effect, totally absorbed by the Schwarzschild sphere.

We can also study light emitted from inside the Schwarzschild sphere starting at a point Q in the region $r < 2m$; again see Figure 1. The trajectory starts at some $r < 2m$ and finite t , travels through increasing r but decreasing t , and crosses the Schwarzschild radius $t = -\infty$ to the exterior of the Schwarzschild sphere, where its evolution is normal. Thus, light emerging from the Schwarzschild sphere must have been traveling since before $t = -\infty$; in effect since before the beginning of time. It is questionable whether such light is physically observable. If not, the Schwarzschild sphere has the physical properties of a black-hole: it absorbs all light and emits none.

In general, for a metric of the form (2.2), we define a *black-hole solution* to be a solution of Einstein's equations which satisfies

$$(2.3) \quad A(\rho) = 0, \quad \text{and} \quad A(r) > 0 \quad \text{if} \quad r > \rho,$$

for some $\rho > 0$; ρ is the radius of the black-hole and is referred to as the "event horizon".

We now consider, in a coordinate invariant way, Maxwell's equations for an electromagnetic field. First we let¹

¹Throughout this paper we use summation convention.

$$\mathcal{A} = A_i dx^i$$

be a 1-form, called the electromagnetic potential (connection), and the 2-form $F = d\mathcal{A}$ is the associated electromagnetic field; in coordinates

$$F = F_{ij} dx^i \wedge dx^j, \quad F_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}.$$

Maxwell's energy momentum tensor is

$$T_{ij} = \frac{1}{4\pi} \left[g^{\ell m} F_{i\ell} F_{jm} - \frac{1}{4} F_{\ell m} F^{\ell m} g_{ij} \right],$$

where as usual, we always use the metric to raise the indices; thus $F^{\ell m} = g^{\ell i} g^{mj} F_{ij}$. Maxwell's equations are the pair of equations

$$\begin{aligned} dF &= 0 && \text{(automatic)} \\ d^*F &= 0, \end{aligned}$$

where “*” denotes the Hodge star operator, mapping 2-forms into themselves, defined by

$$(*F)_{k\ell} = \frac{1}{2} \sqrt{|g|} \varepsilon_{ijkl} F^{ij},$$

where $g = \det(g_{ij})$, and ε_{ijkl} is the completely anti-symmetric symbol defined as $\varepsilon_{ijkl} = \text{sgn}(i, j, k, \ell)$. It is important to notice that the *-operator depends on the metric (g_{ij}). The question we now ask is the following: Find the solution of the Einstein/Maxwell equations outside of a *charged* ball in \mathbb{R}^3 . That is, we seek the spherically symmetric static, exterior (gravitational and electromagnetic) field of a charged distribution of matter. This problem was solved in 1918-1920 by Reissner and Nordström, and the gravitational field is described by the metric

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{q}{r^2} \right) dt^2 - \left(1 - \frac{2m}{r} + \frac{q}{r^2} \right)^{-1} dr^2 - r^2 d\Omega^2,$$

where $m = GM/c^2$, $q = 2\pi GQ^2/c^4$; M and Q denote the mass and charge respectively, of the ball. Notice that if Q is sufficiently large, the metric coefficient $A(r) = 1 - \frac{2m}{r} + \frac{q}{r^2}$ is never zero; otherwise there are two possibilities (see Figure 2.2). Case (a) is called a non-extreme Reissner-Nordström solution and Case (b) is called an extreme Reissner-Nordström solution; these black-hole solutions will come up again in the next sections.

We now turn to the Dirac equation. The Dirac equation brings in Quantum Mechanics and particles into our work, and it also describes the intrinsic “spin” of the particle (fermion). The Dirac equation can be written as

$$(2.4) \quad (G - m) \Psi = 0$$

where G is the Dirac operator, and Ψ is the wave function of a fermion (proton, anti-proton, electron, positron, neutrino, etc.) having (rest) mass m . Ψ is a complex 4-vector and is called a *spinor*. The Dirac operator G takes the form

$$G = iG^j(x) \frac{\partial}{\partial x^j} + B(x),$$

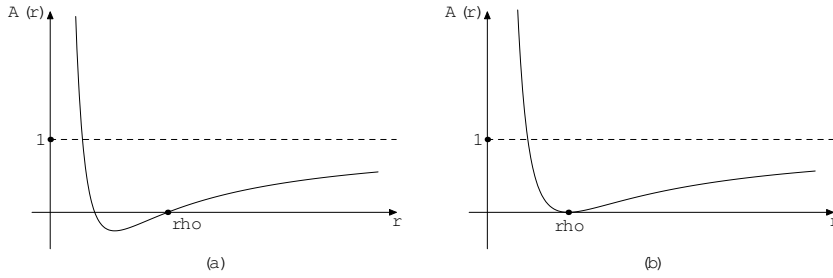


FIG. 2.2. *The Reissner-Nordström Black Hole*

where the 4×4 matrices G^j are called Dirac matrices, and B is a 4×4 matrix. The Dirac matrices and the Lorentzian metric g^{jk} are related by the anti-commutation relations

$$(2.5) \quad g^{jk}I = \frac{1}{2}\{G^j, G^k\} = \frac{1}{2}(G^jG^k + G^kG^j).$$

Now let \mathcal{H} be a space-like hypersurface with a future directed normal vector field ν , and define an inner product on solutions of the Dirac equation by

$$(\Phi, \Psi) = \int_{\mathcal{H}} \bar{\Phi}G^i\Psi \nu_j d\mu ,$$

where

$$\bar{\Phi} = \Phi^* \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

is called the *adjoint spinor*, $*$ denotes complex conjugation, and μ is the invariant measure on \mathcal{H} induced by the metric. This inner product is positive definite and independent of \mathcal{H} because of *current conservation*,

$$\nabla_i \bar{\Phi}G^i\Psi = 0.$$

From a direct generalization of $\bar{\Psi}\gamma^0\Psi$ in M_1^4 (Minkowski space), where $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\bar{\Psi}G^j\Psi\nu_j$ is interpreted as a *probability density* and for non-black-hole solutions, we normalize solutions of the Dirac equation by the requirement

$$(\Psi, \Psi) = 1.$$

For a black-hole solution with event horizon ρ , we demand that (cf. [6,7]),

$$(2.6) \quad \int_{\{t=\text{const.}, r>r_0\}} \bar{\Psi}G^j\Psi\nu_j d\mu < \infty, \text{ for all } r_0 > \rho.$$

Now a result in [3] allows us to choose the Dirac matrices G^j to be any 4×4 matrices which are Hermitian with respect to the inner product

$$\langle \Phi, \Psi \rangle = \int_{\mathbb{R}^4} \bar{\Phi}\Psi \sqrt{|g|}d^4x ,$$

and satisfy (2.5). Here $B(x)$ is defined by

$$B(x) = G^j(x)E_j(x) + G^j(x)A_j(x),$$

where $A_i dx^i$ is the em potential,

$$E_j = \frac{i}{2} \rho(\partial_j \rho) - \frac{i}{16} \text{Tr}(G^m \nabla_j G^m) G_m G_n + \frac{i}{8} \text{Tr}(\rho G_j \nabla_m G^m) \rho$$

is the spin connection, and

$$\rho = \frac{i}{4} \sqrt{|g|} \varepsilon_{ijkl} G^i G^j G^k G^l.$$

In this framework, the *Einstein-Dirac-Maxwell* (EDM) equations for one particle are

$$\begin{aligned} R_{ij} - \frac{1}{2} R g_{ij} &= \sigma T_{ij} \\ (G - m)\Psi &= 0 \\ \nabla_k F^{jk} &= 4\pi e \bar{\Psi} G^j \Psi, \end{aligned}$$

where T_{ij} is the Dirac energy-momentum tensor, and where e denotes the charge of the Dirac particles. These are 18 PDE's for the 18 unknowns $g_{ij}(10)$, $A_i(4)$, and $\Psi(4)$; where Ψ is complex. In this generality, the equations are hopelessly complicated, and are impossible to analyze. We thus specialize by imposing certain symmetry conditions; see [5, 6] for complete details.

For $j = \frac{1}{2}, \frac{3}{2}, \dots$ we consider a static, spherically symmetric system of $(2j + 1)$ Dirac particles, each having angular momentum j . (In the language of atomic physics, we consider the completely filled shell of states with angular momentum j . Classically, one can think of this multi-particle system as several Dirac particles rotating around a common center such that their total angular momentum is zero.) Since the system of fermions is spherically symmetric, we can obtain a consistent set of equations by assuming that both the gravitational and em fields are spherically symmetric. This allows us to separate out the angular dependence and reduce the problem to a system of nonlinear ODE's. Thus taking the metric in the form (2.2) and the em potential in the Coulomb gauge, $\mathcal{A} = \phi(r) dt$, we can take for the Dirac matrices the following:

$$\begin{aligned} G^t &= T\gamma^0 \\ G^r &= \sqrt{A}(\gamma^1 \cos \vartheta + \gamma^2 \sin \vartheta \cos \varphi + \gamma^3 \sin \vartheta \sin \varphi) \\ G^\vartheta &= \frac{1}{r}(-\gamma^1 \sin \vartheta + \gamma^2 \cos \vartheta \cos \varphi + \gamma^3 \sin \vartheta \sin \varphi) \\ G^\varphi &= \frac{1}{r \sin \vartheta}(-\gamma^2 \sin \varphi + \gamma^3 \cos \varphi) \end{aligned}$$

where

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \quad i = 1, 2, 3,$$

and $\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices (the standard basis for $su(2)$).

By using a suitable ansatz and taking the wave functions in the form $\Psi = e^{i\omega t} f(r)$, we can reduce the complex 4-spinors to real two spinors

$$\Phi = (\alpha, \beta), \quad \alpha, \beta \text{ real.}$$

The Dirac equations become

$$(2.7) \quad \sqrt{A} \alpha' = \frac{2j+1}{2r} \alpha - [(\omega - e\phi)T + m]\beta$$

$$(2.8) \quad \sqrt{A} \beta' = [(\omega - e\phi - m)T - m] \alpha - \frac{2j+1}{2r} \beta,$$

and the normalization condition (2.6) is the requirement that

$$(2.9) \quad \int_{r_0}^{\infty} (\alpha^2 + \beta^2) \frac{\sqrt{T}}{A} dr < \infty,$$

for every $r_0 > \rho$ where $r = \rho$ defines the event horizon.

The full EDM equations consist of (2.7), (2.8), together with the Einstein equations

$$(2.10) \quad rA' = 1 - A - 2(2j+1)(\omega - e\phi) T^2 (\alpha^2 + \beta^2) - r^2 AT^2 (\phi')^2$$

$$(2.11) \quad 2rA \frac{T'}{T} = 1 - A - 2(2j+1)(\omega - e\phi) T^2 (\alpha^2 + \beta^2) + 2 \frac{(2j+1)^2}{r} T \alpha\beta \\ + 2(2j+1) mT (\alpha^2 + \beta^2) + r^2 AT^2 (\phi')^2,$$

and Maxwell's equation

$$(2.12) \quad r^2 A \phi'' = -(2j+1) e (\alpha^2 + \beta^2) - \left(2rA + r^2 A \frac{T'}{T} + \frac{r^2}{2} A' \right) \phi'.$$

Notice that the EDM equations are invariant under the gauge transformations

$$\phi \rightarrow \phi + k, \quad \omega \rightarrow \omega + ek, \quad k \in \mathbb{R}.$$

Finally, in addition to (2.9), it is required that the following conditions hold:

$$(2.13) \quad \lim_{r \rightarrow \infty} r (1 - A(r)) < \infty,$$

(finite total mass),

$$(2.14) \quad \lim_{r \rightarrow \infty} T(r) = 1$$

(asymptotic flatness),

$$(2.15) \quad \lim_{r \rightarrow \infty} \phi(r) = 0$$

(em potential vanishes at infinity).

We make the following 3 assumptions on the regularity properties of the event horizon $r = \rho$:

- (I) The volume element $\sqrt{\det |g|}$ is smooth and non-zero on the horizon: $T^{-2} A^{-1}$ and $T^2 A \in C^\infty([\rho, \infty))$.
- (II) By definition, the strength of the em field is the scalar $F_{ij} F^{ij} = -2(\phi')^2 AT^2$. We assume this to be bounded near $r = \rho$, so that in view of (I), we have

$$|\phi'(r)| < c_1, \quad \rho < r < \rho + \varepsilon_1.$$

(III) $A(r)$ satisfies a power law near $r = \rho$,

$$A(r) = c(r - \rho)^s + O(r - \rho)^{s+1}, \quad s > 0.$$

We remark that if (I) or (II) were violated, then an observer freely falling into the black-hole would encounter strong forces when crossing the event horizon; such an effect is seen to be false when one passes to Kruskal coordinates. Hypothesis (III) is a technical condition which includes all known physically relevant horizons; $s = 1$ for Schwarzschild and Reissner-Nordström black-holes, and $s = 2$ for the extreme Reissner-Nordström black-hole. This condition could probably be weakened.

Here is our main result.

THEOREM. *Every black-hole solution of the EDM equations satisfying Assumptions (I) - (III) is either*

a) *a non-extreme Reissner-Nordström solutions with $\alpha(r) \equiv 0 \equiv \beta(r)$, for all $r \geq \rho$, or*

b) *$s = 2$ and we find numerically that either the normalization condition (2.9) fails, or the solution is not regular everywhere outside the event horizon.*

Thus the only black-hole solutions of the EDM equations are Reissner-Nordström solutions: the spinors vanish identically, and so are not normalizable. Thus these “quantum effects” dis-allow (stationary) black-holes. The result indicates too that the Dirac particles must either disappear inside the event horizon, or tend to infinity; namely there is zero probability for the particles to remain in a finite region outside of the black hole.

We now give some ideas of the proof; the reader should consult [6] for the full details. We assume that the EDM equations admit a solution outside of the event horizon $r = \rho$, with $(\alpha(r), \beta(r)) \not\equiv (0, 0)$, and we shall obtain a contradiction. We begin with the case $s < 2$.

LEMMA 1. *If $s < 2$, then there exist constants $c > 0$ and $\varepsilon > 0$ such that*

$$(2.16) \quad c \leq \alpha^2(r) + \beta^2(r) \leq \frac{1}{c}, \quad \rho < r < \rho + \varepsilon.$$

Proof. From the Dirac equations (2.7), (2.8) we have

$$\sqrt{A}(\alpha^2 + \beta^2)' = 2(\alpha, \beta) \begin{pmatrix} \frac{2j+1}{2r} & -m \\ -m & \frac{2j+1}{2r} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \leq \sqrt{4m^2 + \frac{(2j+1)^2}{r^2}}(\alpha^2 + \beta^2).$$

By uniqueness, $(\alpha^2 + \beta^2)(r) > 0$ on $\rho < r < \rho + \varepsilon$ for any $\varepsilon > 0$. Dividing by $\sqrt{A}(\alpha^2 + \beta^2)$, integrating from $r > \rho$ to $\rho + \varepsilon$, and noting that $A^{-1/2}$ is integrable near ρ gives

$$\begin{aligned} \left| \log \frac{(\alpha^2 + \beta^2)(\rho + \varepsilon)}{(\alpha^2 + \beta^2)(r)} \right| &\leq \int_r^{\rho + \varepsilon} \left(4m^2 + \frac{(2j+1)^2}{t^2} \right)^{1/2} A^{-1/2}(t) dt \\ &\leq \int_\rho^{\rho + \varepsilon} \left(4m^2 + \frac{(2j+1)^2}{t^2} \right)^{1/2} A^{-1/2}(t) dt < \infty, \end{aligned}$$

which implies the result.

PROPOSITION 2. *$s < 2$ cannot hold.*

Proof. From Einstein’s equations (2.10), (2.11), we can write

$$r(AT^2)' = -4(2j + 1)(\omega - e\phi)(\alpha^2 + \beta^2)T^4 + \left[2\frac{(2j + 1)^2}{r}\alpha\beta + 2(2j + 1)m(\alpha^2 - \beta^2) \right] T^3.$$

By (I), the lhs is bounded, and $T \rightarrow \infty$ as $r \searrow \rho$. Thus, in view of (2.16), we must have

$$(2.17) \quad \lim_{r \searrow \rho} (\omega - e\phi(r)) = 0.$$

We next write Maxwell’s equation (2.12) in the form

$$(2.18) \quad \phi'' = -\frac{1}{A} \frac{(2j + 1)e}{r^2} [\alpha^2 + \beta^2] - \left(\frac{1}{r^2 t \sqrt{A}} [r^2 T \sqrt{A}]' \phi' \right).$$

Since the term () is bounded near $r = \rho$, and $s \geq 1$ implies A^{-1} is not integrable, we would conclude if $s \geq 1$ that ϕ' is unbounded, thereby violating (II). It follows that we must have $s < 1$.

Now if we integrate (2.18), we obtain, for r near ρ ,

$$\phi'(r) = c_1 (r - \rho)^{-s+1} + c_2 + O((r - \rho)^{-s+2}).$$

Integrating again, and using (2.17) gives

$$\phi(r) = c_1 (r - \rho)^{-s+2} + c_2 (r - \rho) + \frac{\omega}{e} + O((r - \rho)^{-s+3}) ,$$

and thus $(\omega - e\phi) = O(r - \rho)$. Now from (2.10)

$$rA' = (1 - A) - 2(2j + 1)(\omega - e\phi)[T^2(\alpha^2 + \beta^2)] - [r^2(AT^2)(\phi')^2] = O(1) - O((r - \rho)^{1-s}) - O(1)$$

so that the rhs is bounded, but $s < 1$ implies that the lhs blows up. This contradiction shows that $s < 2$ cannot hold.

PROPOSITION 3. $s > 2$ cannot hold.

Proof. In [6], we prove the following estimates

$$(2.19) \quad \lim_{r \searrow \rho} (r - \rho)^{-\frac{s}{2}} (\alpha^2(r) + \beta^2(r)) = 0,$$

and

$$(2.20) \quad \lim_{r \searrow \rho} \phi'(r)^2 = \frac{1}{\rho^2} \lim_{r \searrow \rho} A^{-1}(r)T^{-2}(r) > 0.$$

Then (2.19) shows that near $r = \rho$,

$$(\omega - e\phi)(r) = c + d(r - \rho) + o(r - \rho), \quad d \neq 0,$$

and

$$[(\omega - e\phi)T]' = (r - \rho)^{-s} [d + c(r - \rho)^{-1}],$$

so that as $r \searrow \rho$,

$$[(\omega - e\phi)T] \longrightarrow \infty \quad \text{monotonically.}$$

Now the Dirac equations (2.7), (2.8) can be written as

$$\begin{aligned} \sqrt{A} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}' &= \begin{pmatrix} \frac{2j+1}{2r} & -[(\omega - e\phi)T + m] \\ [(\omega - e\phi)T - m] & \frac{-2j+1}{2r} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &\approx \begin{pmatrix} a & -b \\ b & -a \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \end{aligned}$$

for r near ρ , where $b \nearrow \infty$. The eigenvalues of this last matrix satisfy $\lambda^2 \approx a^2 - b^2 \rightarrow -\infty$ as $r \searrow \rho$. This suggests that the vector (α, β) spins around the origin faster and faster as $r \searrow \rho$, and that $(\alpha, \beta)(r) \not\rightarrow (0, 0)$. In fact, we prove that

$$\underline{\lim}_{r \searrow \rho} (\alpha^2(r) + \beta^2(r)) > 0,$$

but this contradicts (2.19). Thus Proposition 3 holds.

The remaining case is $s = 2$. In this case the metric and electric field behave near the horizon like the extreme Reissner-Nordström solution. Physically, this means that the electric charge of the black-hole is so large that the electric repulsion balances the gravitational attraction and prevents the Dirac particles from entering the black hole. Obviously, this is not the physical situation which one can expect, say, from the gravitational collapse of a star. On the other hand, extreme Reissner-Nordström black-holes have zero temperature [10, 17], and can be considered as end-states of black-holes emitting Hawking radiation. It is thus interesting to see whether the case $s = 2$ can admit normalizable solutions of the EDM equations. Considering the Dirac equation in an extreme Reissner-Nordström black-hole background field, it was proved in [6] that the normalization condition (2.6) is violated. What we want to know here is whether the influence of the spinors on the gravitational and em fields can make the normalization condition finite. Clearly this is a very difficult mathematical problem. We have made numerical investigations, and from these we conclude that the solutions of the EDM equations either develop singularities at some finite r , or the normalization condition fails. Thus in the case $s = 2$, our numerical investigations show that there are no normalizable solutions of the EDM equations.

Our results show that taking quantization and spin into account implies a breakdown of the classical situation; namely, there cannot exist normalizable black-hole solutions of the EDM equations. Applied to the gravitational collapse of a “cloud” of spin-1/2 -particles to a black hole, our result indicates that the Dirac particles must either disappear into the black-hole, or escape to infinity.

We remark that these results have been extended to the axisymmetric case including the Kerr-Newman rotating black hole [9]. In addition, we have recently proved that there are no normalizable solutions of the Einstein-Dirac- $SU(2)$ Yang/Mills equations; this result holds under a more general condition than (III), and applies for every $s > 0$.

We close with two facts. First, our results are basically a consequence of the Heisenberg Uncertainty Principle, together with the specific form of the Dirac current. Second, it is essential for our results that the particles have spin; spin is an important effect which must be considered in studying gravitational collapse.

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