METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 8, No. 2, pp. 301–320, June 2001

006

THE ROLE OF THE DISTANCE FUNCTION IN SOME SINGULAR PERTURBATION PROBLEM

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0. Introduction. This paper deals with the study of solutions to a class of nonlinear singularly perturbed problems of the form

$$(0.1)\qquad \begin{cases} -\varepsilon^2\Delta u + u = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 \text{ or } \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 2$, $\varepsilon > 0$, $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$ or $p > 1$ if $N = 2$ and ν is the unit outward normal at the boundary of Ω.

A solution of the Dirichlet problem can be interpretated as a steady state of the corresponding reaction-diffusion equation $u_t = \varepsilon^2\Delta u - u + u^p$, which arises in a numbers of problems, such as dynamic population and pattern formation theories and chemical reactor theory. The Neumann problem is known as the stationary equation of Keller-Segal system in chemotaxis. It can also be seen as the limiting stationary equation of the Gierer-Heinhardt system in biological pattern formation.

Neumann problem

In the pioneering papers [29], [31] and [32] Lin, Ni and Takagi established the existence of least energy solutions and showed that for ε small enough the least energy solution has only one local maximum point x_ε which belongs to $\partial\Omega$. Moreover the limit point $x_0 = \lim_{\varepsilon\to 0} x_\varepsilon$ satisfies $H(x_0) = \max_{x\in\partial\Omega} H(x)$, where H denotes the mean curvature of x at $\partial\Omega$. In [33] Ni and Takagi constructed boundary spike solutions for axially symmetric domains. In [39] Wei studied the general domain case and proved that for single boundary spike solutions the boundary spike must approach a critical point of the mean curvature. He also proved that for any nondegenerate critical point of the mean curvature one can construct bondary spike solutions whose spike approaches such a point.

In [22] Gui constructed multiple boundary spike layer solutions at multiple local maximum points. In [44] Wei and Winter constructed multiple boundary spike layer solutions at multiple nondegenerate critical points of H. In [24] the authors proved that for any fixed integers K there exist boundary $K-$peaks solutions at a local minimum point of H.

In [40] and in [41] Wei proved the existence of single interior spike solutions of (0.1) under some restricted geometric conditions on Ω. In [42] and [20] the authors constructed single interior spike solutions by using the distance function $\mathrm{dist}(x, \partial\Omega)$. More precisely in [42] Wei proved that for any local maximum point x_0 of the distance function there exists a family of solutions with a single maximum point which approaches x_0. In [35]the author proved the existence of a symmetric single interior spike solution in symmetric domains, by using a degree argument.

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In [23] Gui constructed multiple interior peak solutions. It was shown that for any fixed positive integer K there exists a solution of (0.1) which has exactly K maximum points $x_\varepsilon^1,\dots,x_\varepsilon^K$ such that $\mathcal{D}_K(x_\varepsilon^1,\dots,x_\varepsilon^K)$ converges to $\max\{\mathcal{D}_K(x^1,\dots,x^K)\mid x^i\in\Omega,\ i=1,\dots,K\}$ as ε tends to zero, where

$$\mathcal{D}_K(x^1,\dots,x^K)=\min\left\{\mathrm{dist}(x^i,\partial\Omega),\ \frac{|x^j-x^l|}{2}\mid i,j,l=1\dots,K,\ j\neq l\right\}.$$

Cerami and Wei in [9] and Yan in [37] some multiplicity results are obtained by using Ljusternik-Schnirelman category. In [25] Kowalczyk proved that any "nondegenerate stationary lattice" supports a multiple spike layer solution.

We would like to point out that Bates and Fusco in [5] got similar results for the Cahn-Hilliard equation. By using a "quasi-invariant" manifold approach they established the existence of a stationary solution with many nuclei and they also gave a criteria for the asymptotic location of those nuclei as $\varepsilon\to 0$ in terms of the geometry of the domain.

Dirichlet problem

Multiplicity results about the Dirichlet problem were firstly obtained by Benci, Cerami and Passaseo in [2] and [3], by using the Ljusternik-Schnirelman category. Successively Ni and Wei in [34] established the existence of a least energy solution. They proved that as $\varepsilon\to 0$ the least energy soluton has exactly one local maximum point and this local maximum point tends to a point which attains the global maximum of the distance function $\mathrm{dist}(x,\partial\Omega)$. In [39] Wei proved that for any local maximum x_0 of the distance function there exists a family of solutions with a single global maximum point which approaches x_0. In [16] Del Pino, Felmer and Wei proved the existence of single-peaked solutions at any "suitable" critical point of the distance function. In [28] Li and Nirenberg proved another result which involves the critical points of the distance function. More precisely they show that if the Brower degree $deg(\nabla\mathrm{dist}(\cdot,\partial\Omega),V,0)\neq 0$ where V is a suitable subset of Ω, then there exist a family of solutions with a unique local maximum point which converges to a critical point of the distance function.

In [7] and [8] Cao, Dancer, Noussair and Yan constructed $K-$peak solutions with the peaks near the local maximum points or saddle points $x_1,\dots,x_K$ of $\mathrm{dist}(\cdots,\partial\Omega)$, provided $\mathrm{dist}(x_i,\partial\Omega)=\mathrm{dist}(x_j,\partial\Omega)$ for any i and j. In [17] Del Pino, Felmer and Wei used a variational method to construct a $K-$peak solution with its peaks close to some local maximum points $x_1,\dots,x_K$ of $\mathrm{dist}(\cdots,\partial\Omega)$, provided $\max_i\mathrm{dist}(x_i,\partial\Omega)$ is small when compared with the distance between $x_1,\dots,x_K$. In [15] Dancer and Wei proved the existence of two-peak solutions. Concerning the effect of the domain topology on the existence of multipeak solution Dancer and Yan in [13] proved that if the homology of the domain is nontrivial, then for any positive integer K problem (0.1) has at least one $K-$peak solution. In [14] Dancer and Yan assumed that the distance function has K isolated compact connected critical sets $T_1,\dots,T_K$ satisfying $\mathrm{dist}(x,\partial\Omega)=c_j=constant$ for all $x\in T_j$, $\min\limits_{i\neq j}\mathrm{d}(T_j,T_i)>2\max\limits_{1\le j\le K}\mathrm{d}(T_j,\partial\Omega)$ and the critical group of each critical set T_i is nontrivial. They constructed a solution which has exactly one local maximum point in a small neighbourhood of T_i for $i=1,\dots,K$. Moreover they proved that if Ω is strictly convex problem (0.1) does not have any $K-$peak solution.

Main results

In this paper we describe some results obtained by Grossi, Wei and the author in [20] and in [21].

In [20] the authors proved that any critical point "topologically non trivial" x_0 of the distance function generates a family of single interior spike solutions.

THEOREM 0.1. *Let* x_0 *be a critical point of* $\text{dist}\,(\cdot, \partial\Omega)$. *Assume* $c = \text{dist}\,(x_0, \partial\Omega)$ *is a critical value topologically nontrivial (see Definition (2.4)). Then for* ε *small enough there exists a family of solutions* u_ε *of (0.1), whose maximum point tends to a critical point* x_0' *of the distance function with* $\text{dist}\,(x_0', \partial\Omega) = \text{dist}\,(x_0, \partial\Omega)$.

Moreover they proved that the peak of any single solution must converge to a critical point of the distance function.

THEOREM 0.2. *Let* u_ε *be a solution of (0.1) with exactly one local interior maximum point* x_ε. *If* $x_0 = \lim_{\varepsilon \to 0} x_\varepsilon \in \Omega$ *then* x_0 *is a critical point of* $\text{d}_{\partial\Omega}$.

In [21] the authors proved that any critical point "topologically non trivial" of the function $\mathcal{D}_K$ generates a $K-$peaks solution.

THEOREM 0.3. *Let* $X_0 = (x_0^1, \ldots, x_0^K)$ *be a critical point of* $\mathcal{D}_K$. *Assume* $\mathcal{D}_K(X_0) > 0$ *is a critical value topologically nontrivial (see Definition (2.4)). Then for* ε *small enough there exists a family of solutions* u_ε *of (0.1), with Neumann boundary condition, whose* K *maximum points* $x_\varepsilon^1, \ldots, x_\varepsilon^K$ *tend to a point* $\hat{X}_0 = (\hat{x}_0^1, \ldots, \hat{x}_0^K)$ *such that* $\mathcal{D}_K(\hat{X}_0) = \mathcal{D}_K(X_0)$, $\hat{x}_0^i \in \Omega$, $\hat{x}_0^i \neq x_0^j$ *for* $i \neq j$ *and* $\hat{X}_0$ *is a critical point of* $\mathcal{D}_K$.

Moreover they proved that the K peaks of any single solution must converge to a critical point of the function $\mathcal{D}_K$.

THEOREM 0.4. *Let* u_ε *be a solution of (0.1), with Neumann boundary condition, with exactly* K *local interior maximum points* $x_\varepsilon^1, \ldots, x_\varepsilon^K$ *and let* $x_0^i = \lim_{\varepsilon \to 0} x_\varepsilon^i$ *for* $i = 1, \ldots, K$. *If* $x_0^i \in \Omega$ *then* $x_0^i \neq x_0^j$ *for* $i \neq j$ *and* $(x_0^1, \ldots, x_0^K)$ *is a critical point of* $\mathcal{D}_K$.

The method used to prove the results relies on an idea of Bahri (see [1]).

Firstly for ε small enough we reduce the problem of finding a single-peak or a multi-peak solution for (0.1) to that of finding a critical point for a function K_ε defined in a finite dimensional domain.

Secondly we compute the asymptotic expansion of the function K_ε, in order to point out the connection between K_ε and function $\mathcal{D}_K$. Such an expansion allows us to prove that any "topologically nontrivial" critical point of the function $\mathcal{D}_K$ generates a $K-$peak solution.

Finally we compute the asymptotic expansion of the function ∇K_ε, in order to point out the connection between ∇K_ε and $\nabla \mathcal{D}_K$. Such an expansion allows us to prove that the K peaks of any single solution must converge to a critical point of the function $\mathcal{D}_K$..

We would like to emphasize that $\mathcal{D}_K$ is a Lipschitz continuous function which may be not smooth. So a suitable notion of critical points for non-smooth functions is needed. The generalized gradient introduced by Clarke (see [11]) becomes our main tool. The new idea in [20] and in [21] is to evaluate the gradient of K_ε in terms of the generalized gradient of Clarke of the function $\mathcal{D}_K$. By this result, we were able to get some new results and also to clarify many results that were previously known.

The paper is organized as follows. In Section 1 we recall some properties of the generalized gradient of Clarke. In Section 2 we introduce the notion of "topologically nontrivial" critical values for locally Lipschitz continuous function. In Section 3 we study the distance function and the function $\mathcal{D}_K$ and we give a criteria to localize critical points of $\mathcal{D}_K$. In Section 4 we recall some results obtained by Ni and Wei in [34]. In Section 5 we study the one-peak solutions. In Section 6 we study the multi-peak solutions. In Section 7 we give some examples.

1. The generalized gradient. Let D be a smooth bounded domain of $\mathbb{R}^N$. Let $f : D \longrightarrow \mathbb{R}$ be a Lipschitz continuous function. We recall the following definition due to Clarke (see [11]).

Definition 1.1. *The generalized gradient of f at $x \in D$ is the set:*

$$\partial f(x) = \left\{\alpha \in \mathbb{R}^N \mid f^o(x,v) \geq \alpha \cdot v \ \forall \ v \in \mathbb{R}^N \right\}$$

where the generalized directional derivative $f^o(x,v)$ is defined by

$$f^o(x;v) = \limsup_{\substack{h\to 0\\ \lambda\to 0+}} \frac{f(x+h+\lambda v) - f(x+h)}{\lambda}.$$

If f is continuously differentiable at x then $\partial f(x) = \{\nabla f(x)\}$. If f is only differentiable at x, $\partial f(x)$ can contain points other than $\nabla f(x)$. For example, if $f(x) = x^2 \sin \frac{1}{x}$ then it is easy to show that $f^o(0;v) = |v|$ and so $\partial f(0) = [-1,1]$, which contains the derivative $f'(0) = 0$.

Definition 1.2. *The function f is said to be regular at $x \in D$ provided that for any $v \in \mathbb{R}^N$ there exists the usual one-sided directional derivative $f'(x;v) = \lim\limits_{t\to 0^+} \frac{f(x+tv)-f(x)}{t}$ and $f'(x;v) = f^o(x;v)$.*

By ([11], Proposition 2.2.4) and ([11], (b) of Proposition 2.3.6) we deduce

Proposition 1.3. *If $\partial f(x)$ reduces to a singleton $\{\alpha\}$ then f is differentiable at x and $\nabla f(x) = \alpha$. Conversely, if f is differentiable and regular at x then $\partial f(x) = \{\nabla f(x)\}$.*

It is useful to point out the following property (see [11], Proposition 2.1.5).

Remark 1.4. *Let x_n and α_n be sequences in $\mathbb{R}^N$ such that $x_n \in D$ and $\alpha_n \in \partial f(x_n)$. Suppose that x_n converges to x and α_n converges to α. Then $\alpha \in \partial f(x)$.*

Now let us suppose $x = (x_1, x_2)$. We denote by $\partial_1 f(x_1,x_2)$ the (partial) generalized gradient of $f(\cdot, x_2)$ at x_1 and by $\partial_2 f(x_1,x_2)$ that of $f(x_1,\cdot)$ at x_2. The following result holds (see [11], Proposition 2.3.15).

Remark 1.5. *If f is regular at (x_1,x_2) then*

$$\partial f(x_1,x_2) \subset \partial_1 f(x_1,x_2) \times \partial_2 f(x_1,x_2).$$

Let us recall another useful result. Assume that $\{f_i\}_{i\in\mathcal{I}}$ is a finite collection of Lipschitz continuous functions defined on D. The function

$$f(x) = \min\{f_i(x) \mid i \in \mathcal{I}\}$$

is easily seen to be a Lipschitz continuous function. For any $x \in D$ we let $\mathcal{I}(x)$ denote the set of indices i for which $f(x) = f_i(x)$ (i.e. the indices at which the minimum defining f is attained). Then the following result holds (see [11], Proposition (2.3.12)).

PROPOSITION 1.6. *If f_i is regular at x for any $i \in \mathcal{I}(x)$ then f is regular at x and*

$$\partial f(x) = co\left\{\partial f_i(x) \mid i \in \mathcal{I}(x)\right\}.$$

Finally we give the definition of a critical point for a nonsmooth function.

DEFINITION 1.7. *A point x_0 in D is said to be a critical point of f if $0 \in \partial f(x_0)$. A real number c is said to be a critical value of f if there exists a critical point x_0 of f such that $f(x_0) = c$.*

By Definition (1.1) we easily deduce that if x_0 is a minimum point or a maximum point for a Lipschitz continuous function f then $0 \in \partial f(x_0)$.

2. Critical values topologically nontrivial. In this section we recall a result of the critical point theory. The following one is given by Ramos in [36] and it is a jointed version of the classical linking theorem and the local saddle point proved in [30]. Although it concerns C^1–function, it is possible to extend such a result to Lipschitz continuous function, by using deformation lemma proved by Chang in [10].

We consider three compact subsets ∂Q, Q and A of D such that

$$\partial Q \subset Q \text{ and } Q \cap A = \emptyset. \tag{2.1}$$

∂Q is not necessarily the topological boundary of Q and A can be the empty set.
We define the class:

$$\Gamma = \left\{\gamma \in C^0([0,1] \times Q, D \setminus A) \mid \gamma_0 \equiv Id,\ \gamma_{t|_{\partial Q}} \equiv Id\ \forall\ t \in [0,1]\right\},$$

where Id is the identity map. We note that $\Gamma \neq \emptyset$ because $Id \in \Gamma$.

DEFINITION 2.1. *Let S be a subset of D. We say that S links Q via ∂Q by homotopy in $D \setminus A$ if*

$$S \cap \partial Q = \emptyset \quad \text{and} \quad \gamma_1(Q) \cap S \neq \emptyset\ \forall \gamma \in \Gamma. \tag{2.2}$$

It is useful to point out the following fact.

REMARK 2.2. *Assume ∂Q_1, Q_1, A_1 and S_1 and ∂Q_2, Q_2, A_2 and S_2 are two families of subset of D which satisfy (2.1) and (2.2). Then $\partial Q = (\partial Q_1 \times Q_2) \cup (Q_1 \times \partial Q_2)$, $Q = Q_1 \times Q_2$, $A = (A_1 \times S_2) \cup (S_1 \times A_2)$ and $S = S_1 \times S_2$ are subsets of $D \times D$ which satisfy (2.1) and (2.2).*

The following result holds.

THEOREM 2.3. *Let $f : D \longrightarrow \mathbb{R}$ be a Lipschitz continuous function. Assume S links Q via ∂Q by homotopy in $D \setminus A$ and*

$$\max_{\partial Q} f < \min_S f \leq \max_Q f < \min_A f. \tag{2.3}$$

Let

$$c = \inf_{\gamma \in \Gamma} \max_{u \in Q} f\big(\gamma_1(u)\big). \tag{2.4}$$

If $c \in \mathbb{R}$ and the set $\{x \in D \text{ s.t. } c-\varepsilon \leq f(x) \leq c+\varepsilon\}$ is complete for some $\varepsilon > 0$ then c is a critical value of f.

If $A = \emptyset$ we get the classical linking theorem. The "local saddle point" of [30] is a particular case of the previous theorem when $A \neq \emptyset$.

In the following definition we introduce the notion of critical values of a Lipschitz continuous function $f : D \longrightarrow \mathbb{R}$ which are "stable" with respect to suitable perturbations (see [20], Definition (1.7) and [21], Definition (1.11))

Definition 2.4. *We say that c is a critical value topologically nontrivial of f if there exists a family of subsets ∂Q_δ, Q_δ, A_δ and S_δ of D which satisfy (2.1), (2.2) and (2.3), with the properties*

$$\max_{\partial Q_\delta} f < \min_{S_\delta} f \leq c \leq \max_{Q_\delta} f < \min_{A_\delta} f \tag{2.5}$$

and

$$\lim_{\delta\to 0} \min_{S_\delta} f = \lim_{\delta\to 0} \max_{Q_\delta} = c. \tag{2.6}$$

We point out that if we assume that the sets $\{x \in D \text{ s.t. } c'-\varepsilon \leq f(x) \leq c'+\varepsilon\}$ are complete for any c' close enough to c and for some $\varepsilon > 0$ then by Theorem (2.3) we deduce that c is a critical value of f.

3. The distance function and the function $\mathcal{D}_K$..

Let Ω be a smooth open bounded domain of $\mathbb{R}^N$.

Definition 3.1. *Let $\mathrm{d}_{\partial\Omega} : \Omega \longrightarrow \mathbb{R}$ be the distance function defined by* $\mathrm{d}_{\partial\Omega}(x) = \mathrm{dist}(x, \partial\Omega) = \min_{y\in\partial\Omega} |x-y|$.

It is well known that $\mathrm{d}_{\partial\Omega}$ is a Lipschitz continuous function. By using (see [11], Corollary 2, p. 87) we can compute the generalized gradient of the distance function.

Remark 3.2. *For any $x \in \Omega$ we have*

$$\partial \mathrm{d}_{\partial\Omega}(x) = \Big\{ \int_{\partial\Omega} \nu^{(i)}(y) d\mu_x(y) \ | \ d\mu_x(y) \text{ is a bounded Borel measure on } \partial\Omega, \ \int_{\partial\Omega} d\mu_x(y) = 1, \ \mathrm{supp}\big(d\mu_x(y)\big) \subset \Pi_{\partial\Omega}(x) \Big\}, \tag{3.1}$$

where

$$\Pi_{\partial\Omega}(x) = \{y \in \partial\Omega \ | \ |y-x| = \mathrm{d}_{\partial\Omega}(x)\} \tag{3.2}$$

and $\nu^{(i)}(y)$ denotes the unit inward normal at the point y of $\partial\Omega$.

By ([11], Corollary 2, p. 87) we deduce that the distance function is regular at any $x \in \Omega$. Therefore by Proposition (1.3) we get

Remark 3.3. $\mathrm{d}_{\partial\Omega}$ *is differentiable at x if and only if $\Pi_{\partial\Omega}(x)$ reduces to a singleton $\{\pi(x)\}$ and $\nabla \mathrm{d}_{\partial\Omega}(x) = \nu^{(i)}\big(\pi(x)\big)$, where $\nu^{(i)}\big(\pi(x)\big)$ denotes the unit inward normal at $\pi(x)$.*

Finally since Ω is smooth we have the following property.

PROPOSITION 3.4. *There exists a neighbourhood $\mathcal{U}$ of the boundary of Ω such that $0 \notin \partial \mathrm{d}_{\partial\Omega}(x)$ for any $x \in \mathcal{U} \cap \Omega$.*

Now let us introduce the function $\mathcal{D}_K$ which will play a crucial role in the next sections.

DEFINITION 3.5. *Let $K \geq 1$ be an integer. Set $\Omega^K = \Omega \times \ldots \times \Omega$. Let $\mathcal{D}_K : \Omega^K \longrightarrow \mathbb{R}$ be defined by*

$$\mathcal{D}_K(X) = \min\left\{\mathrm{d}_{\partial\Omega}(x^i),\ \frac{|x^j - x^l|}{2} \mid i,j,l = 1\ldots,K,\ j \neq l\right\}. \tag{3.3}$$

Let us point out that

$$\mathcal{D}_1(x) = \mathrm{d}_{\partial\Omega}(x) \qquad \forall\, x \in \Omega.$$

Set

$$\mathcal{M}_K(\Omega) = \left\{X = (x^1, \ldots, x^K) \in \Omega^K \;\middle|\; x^i \neq x^j,\ i \neq j,\ i,j = 1, \ldots, K\right\}. \tag{3.4}$$

By using the regularity of the distance function and Proposition (1.6) we can compute the generalized gradient of $\mathcal{D}_K$.

LEMMA 3.6. *For any $X \in \mathcal{M}_K(\Omega)$ we have that $\beta(X) \in \partial\mathcal{D}_K(X)$ if and only if*

$$\beta(X) = \left(a_1\alpha(x^1) + \frac{1}{2}\sum_{\substack{j=1\\ j\neq 1}}^{K} b_{1j}\frac{x^1 - x^j}{|x^1 - x^j|}, \ldots, a_K\alpha(x^K) + \frac{1}{2}\sum_{\substack{j=1\\ j\neq K}}^{K} b_{1j}\frac{x^K - x^j}{|x^K - x^j|}\right),$$

with $\alpha(x^i) \in \partial\mathrm{d}_{\partial\Omega}(x^i)$, $a_i, b_{jl} \geq 0$, $b_{jl} = b_{lj}$, $\sum\limits_{i=1}^{K} a_i + \frac{1}{2}\sum\limits_{\substack{j,l=1\\ l\neq j}}^{K} b_{jl} = 1$.

In particular by Lemma (3.6) we deduce that if $x^1, \ldots, x^K$ are K different critical points of the distance function then $X = (x^1, \ldots, x^K)$ is a critical point of $\mathcal{D}_K$.

Next results generalizes Proposition (3.4). More precisely we prove that there is not any critical point of $\mathcal{D}_K$ close to the boundary of $\mathcal{M}_K(\Omega)$.

PROPOSITION 3.7. *There exists a neighbourhood $\mathcal{U}$ of the boundary of $\mathcal{M}_K(\Omega)$ such that $0 \notin \partial\mathcal{D}_K(X)$ for any $X \in \mathcal{U} \cap \mathcal{M}_K(\Omega)$.*

Proof. We prove that if X_ε is a sequence in $\mathcal{M}_K(\Omega)$ such that $\lim\limits_{\varepsilon\to 0} X_\varepsilon = X_0$ and $X_0 \in \partial\mathcal{M}_K(\Omega)$, then there exists $\varepsilon_0 > 0$ and $C > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$

$$|\beta_\varepsilon(X_\varepsilon)| \geq C > 0 \quad \forall\, \beta_\varepsilon(X_\varepsilon) \in \partial\mathcal{D}_K(X_\varepsilon).$$

We proceed by induction on the number K.

Let $K = 1$ and let x_ε be a sequence in Ω such that $x_0 = \lim\limits_{\varepsilon} x_\varepsilon \in \partial\Omega$. By Remark (3.3) and Remark (3.4) it follows that for ε small enough $\partial\mathcal{D}_1(x_\varepsilon) = \{\nu^{(i)}(\pi(x_\varepsilon))\}$ and the claim follows.

Suppose the claim to be true for any integer $1 \leq H \leq K-1$. Let us prove the claim for K.

Let X_ε be a sequence in $\mathcal{M}_K(\Omega)$ such that $\lim\limits_{\varepsilon\to 0} X_\varepsilon = X_0$ and $X_0 \in \partial\mathcal{M}_K(\Omega)$. Then we have either

(i) $\exists\, i,j \in \{1,\dots,K\}$ *s.t.* $x_0^i \neq x_0^j$,

or

(ii) $x_0^1 = \ldots = x_0^K \in \partial\Omega$,

or

(iii) $x_0^1 = \ldots = x_0^K \in \Omega$.

By using Lemma (3.6) and inductive assumptions the claim easily follows. □

Next results allows us to localize some special critical points of the function $\mathcal{D}_K$.

PROPOSITION 3.8. *Let $(x^1,\dots,x^K) \in \mathcal{M}_K(\Omega)$ be a critical point of $\mathcal{D}_K$. Assume that for any integer $1 \le H \le K-1$ and for any set of indices $\{i_1,\dots,i_H\} \subset \{1,\dots,K\}$ $(x^{i_1},\dots,x^{i_H})$ is not a critical point of $\mathcal{D}_H$. Then $\mathrm{d}_{\partial\Omega}(x^i) = \frac{|x^l-x^h|}{2}$ for any i,l,h and $0 \in co\{\alpha(x^i) \mid \alpha(x^i) \in \partial \mathrm{d}_{\partial\Omega}(x^i),\ i=1,\dots,K\}$.*

Proof. We argue by contradiction. Then we have either

(i) $\exists\, i,j \in \{1,\dots,K\}$ *s.t.* $\mathcal{D}_K(X) < \frac{|x^i-x^j|}{2}$,

or

(ii) $\forall\, l,h \in \{1,\dots,K\} \quad \mathcal{D}_K(X) = \frac{|x^l-x^h|}{2}$ and $\exists\, i \in \{1,\dots,K\}$ s.t. $\mathcal{D}_K(X) < \mathrm{d}_{\partial\Omega}(x^i)$.

By using Lemma (3.6) a contradiction arises in both cases. □

In particular by Proposition (3.8) and by Remark (3.3) we deduce the following characterization of the critical points of $\mathcal{D}_2$.

COROLLARY 3.9. *Let $(x^1,x^2) \in \mathcal{M}_2(\Omega)$ be a critical point of $\mathcal{D}_2$ such that the distance function is differentiable at x^1 and x^2. Then $\mathrm{d}_{\partial\Omega}(x^1) = \mathrm{d}_{\partial\Omega}(x^2) = \frac{|x^1-x^2|}{2}$ and $\nu^{(i)}\left(\pi(x^1)\right) = -\nu^{(i)}\left(\pi(x^2)\right) = \frac{x^2-x^1}{|x^2-x^1|}$.*

4. Some preliminary results. Let us introduce the ground state solution U. We recall the following results (see, for example, [6], [19] and [26]).

THEOREM 4.1. *The equation:*

$$\begin{cases} -\Delta u + u = u^p & \text{in } \mathbb{R}^N \\ u(x) \to 0 & \text{for } |x| \to +\infty \end{cases} \tag{4.1}$$

possesses a unique non trivial regular solution U with the following properties:

(i) $U(x) > 0 \quad \forall\, x \in \mathbb{R}^N$,

(ii) U is spherically symmetric, i.e. $U(x) = U(r)$ where $r = |x|$, and U decreases with respect to r,

(iii) $U \in \mathcal{C}^2(\mathbb{R}^N)$,

(iv) U together with its derivatives up to order 2 have exponential decay at infinity, that is there exist $C > 0$ and $\delta > 0$ such that $|D^\alpha U(x)| \le Ce^{-\delta|x|}\ \forall\, x \in \mathbb{R}^N$ and $|\alpha| \le 2$.

(v) there exists $\beta > 0$ such that $\lim\limits_{r\to\infty} r^{\frac{n-1}{2}} e^r U(r) = \beta > 0$.

Let us introduce some notation. Set $\Omega_\varepsilon = \{y \mid \varepsilon y \in \Omega\}$ and for $x \in \Omega$ $\Omega_{\varepsilon,x} = \{y \mid \varepsilon y + x \in \Omega\}$. Of course solving problem (0.1) is equivalent to solve the rescaled problem

$$\begin{cases} -\Delta u + u = u^p & \text{in } \Omega_\varepsilon \\ u = 0 \text{ or } \frac{\partial u}{\partial \nu} = 0 & \text{in } \partial\Omega_\varepsilon. \end{cases} \tag{4.2}$$

We set $\mathcal{P}_{\Omega_{\varepsilon,x}} U$ to be the unique solution of the problem

$$\begin{cases} -\Delta u + u = U^p & \text{in } \Omega_{\varepsilon,x} \\ u = 0 \text{ or } \frac{\partial u}{\partial \nu} = 0 & \text{in } \partial\Omega_{\varepsilon,x}. \end{cases} \tag{4.3}$$

$\mathcal{P}_{\Omega_{\varepsilon,x}} U$ is the projection of the ground state U into $\mathrm{H}^1_0(\Omega_{\varepsilon,x})$ in the Dirichlet case or into $\mathrm{H}^1(\Omega_{\varepsilon,x})$ in the Neumann case. The idea of projections has been introduced in [1].

Set

$$\varphi_{\varepsilon,x}(z) = U(y) - \mathcal{P}_{\Omega_{\varepsilon,x}} U(y) \quad \text{with} \quad z = \varepsilon y + x,\ x \in \Omega,\ z \in \Omega.$$

The following estimate plays a fondamental role (see [41], Section 2 and [34], Section 4).

Lemma 4.2. *For* $x \in \Omega$ *set*

$$\psi_\varepsilon(x) = -\varepsilon \log\big(\varphi_{\varepsilon,x}(x)\big) \qquad \textit{in the Dirichlet case,} \tag{4.4}$$

or

$$\psi_\varepsilon(x) = -\varepsilon \log\big(-\varphi_{\varepsilon,x}(x)\big) \qquad \textit{in the Neumann case,} \tag{4.5}$$

Then

$$\lim_{\varepsilon \to 0} \psi_\varepsilon(x) = 2\mathrm{d}_{\partial\Omega}(x) \quad \textit{uniformly in } \Omega. \tag{4.6}$$

By Lemma (4.2) and by (v) of Theorem (4.1) we easily deduce that

Lemma 4.3. *Let for* $X \in \mathcal{M}_K(\Omega)$

$$\Phi_\varepsilon(X) = -\varepsilon \log\left[-\sum_{i=1}^{K} \varphi_{\varepsilon,x^i}(x^i) + \sum_{\substack{j,l=1 \\ j \neq l}}^{K} U\left(\frac{|x^j - x^l|}{\varepsilon}\right)\right]. \tag{4.7}$$

Then in the Neumann case

$$\lim_{\varepsilon \to 0} \Phi_\varepsilon(X) = 2\mathcal{D}_K(X) \qquad \textit{uniformly in } \mathcal{M}_K(\Omega).$$

5. Existence of one-peak solutions. Let H_ε be the Hilbert space

$$\mathrm{H}_\varepsilon = \mathrm{H}^2(\Omega_\varepsilon) \cap \mathrm{H}^1_0(\Omega_\varepsilon) \quad \text{in the Dirichlet case}$$

or

$$\mathrm{H}_\varepsilon = \left\{ u \in \mathrm{H}^2(\Omega_\varepsilon) \;\middle|\; \frac{\partial u}{\partial \nu_\varepsilon} = 0 \text{ on } \partial\Omega_\varepsilon \right\} \quad \text{in the Neumann case}$$

Define

$$\mathcal{S}_\varepsilon(u) = \Delta u - u + (u^+)^p \quad \text{for} \quad u \in \mathrm{H}_\varepsilon .$$

Then solving equation (0.1) or equation (4.2) is equivalent to solve the following one

$$\mathcal{S}_\varepsilon(u) = 0, \quad u \in \mathrm{H}_\varepsilon .$$

Let us consider the linearized operator $\mathcal{L}_\varepsilon : \mathrm{H}_\varepsilon \longrightarrow \mathrm{L}^2(\Omega_\varepsilon)$ given by

$$\mathcal{L}_\varepsilon(v) = \Delta v - v + p\mathcal{P}_{\Omega_{\varepsilon,x}} U^{p-1} v.$$

It is easy to see that the cokernel of $\mathcal{L}_\varepsilon$ coincides with its kernel. Choose approximate cokernel and kernel as

$$\mathcal{K}_{\varepsilon,x} = \operatorname{span} \left\{ \frac{\partial \mathcal{P}_{\Omega_{\varepsilon,x}} U}{\partial x_i} \mid i = 1, \ldots, N \right\} \subset \mathrm{H}^2(\Omega_\varepsilon),$$

$$\mathcal{C}_{\varepsilon,x} = \operatorname{span} \left\{ \frac{\partial \mathcal{P}_{\Omega_{\varepsilon,x}} U}{\partial x_i} \mid i = 1, \ldots, N \right\} \subset \mathrm{L}^2(\Omega_\varepsilon).$$

Now we state the following lemmas, which allow us to reduce problem (4.2) to a finite dimensional problem.

Lemma 5.1. *For any compact set $K \subset \Omega$ there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $x \in K$ there exists a unique $\Phi_{\varepsilon,x} \in \mathcal{K}^\perp_{\varepsilon,x}$ such that*

$$\mathcal{S}_\varepsilon(\mathcal{P}_{\Omega_{\varepsilon,x}} U + \Phi_{\varepsilon,x}) \in \mathcal{C}_{\varepsilon,x}.$$

Moreover $\Phi_{\varepsilon,x}$ is C^1 in x and

$$\|\Phi_{\varepsilon,x}\|_{\mathrm{H}^2(\Omega_\varepsilon)} \leq C e^{-(1+\sigma)\frac{\mathrm{d}_{\partial\Omega}}{\varepsilon}}, \tag{5.1}$$

where C is a positive constant and $\sigma = \min\{1, p-1\}$.

Proof. The proof relies on a contraction mapping argument. The claim can be proved by collecting some results obtained in [41] and [42]. □

Now we define the function $K_\varepsilon : \Omega \longrightarrow \mathbb{R}$

$$K_\varepsilon(x) = J_\varepsilon(\mathcal{P}_{\Omega_{\varepsilon,x}} U + \Phi_{\varepsilon,x}), \tag{5.2}$$

where the "rescaled" energy functional $J_\varepsilon : \mathrm{H}^1(\Omega_\varepsilon) \longrightarrow \mathbb{R}$ is defined by

$$J_\varepsilon(u) = \left[\frac{1}{2} \int\limits_{\Omega_\varepsilon} \left(|\nabla u|^2 + u^2 \right) - \frac{1}{p+1} \int\limits_{\Omega_\varepsilon} \left(u^+ \right)^{p+1} \right]. \tag{5.3}$$

Now we evaluate the asymptotic expansion of K_ε.

PROPOSITION 5.2. *x_ε is a critical point of K_ε if and only if $u_\varepsilon = \mathcal{P}_{\Omega_{\varepsilon,x_\varepsilon}} U + \Phi_{\varepsilon,x_\varepsilon}$ is a solution of 4.2. Moreover the following estimates hold uniformly on compact sets of Ω*

$$K_\varepsilon(x) = A + \frac{1}{2}\gamma e^{-\frac{\psi_\varepsilon(x)}{\varepsilon}} + o\left(e^{-\frac{\psi_\varepsilon(x)}{\varepsilon}}\right) \quad \textit{in the Dirichlet case} \tag{5.4}$$

or

$$K_\varepsilon(x) = A - \frac{1}{2}\gamma e^{-\frac{\psi_\varepsilon(x)}{\varepsilon}} + o\left(e^{-\frac{\psi_\varepsilon(x)}{\varepsilon}}\right) \quad \textit{in the Neumann case,} \tag{5.5}$$

where

$$A = \frac{1}{2}\int\limits_{\mathbb{R}^N} \left(|\nabla U|^2 + U^2\right) - \frac{1}{p+1}\int\limits_{\mathbb{R}^N} U^{p+1}, \ \gamma = \int\limits_{\mathbb{R}^N} U^p(y) e^{-y_1} dy.$$

Proof. See [20], [23], [41] and [42]. ◻

The next results play a crucial role in connecting the topological structure of the sublevels of the distance function with the topological structure of the sublevels of the function K_ε.

LEMMA 5.3. *Let $x_1^\varepsilon, x_2^\varepsilon$ be sequences in Ω be such that $\lim\limits_{\varepsilon\to0} x_1^\varepsilon = x_1 \in \Omega$, $\lim\limits_{\varepsilon\to0} x_2^\varepsilon = x_2 \in \Omega$ and $\mathrm{d}_{\partial\Omega}(x_1) < \mathrm{d}_{\partial\Omega}(x_2)$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$*

$$K_\varepsilon(x_1^\varepsilon) > K_\varepsilon(x_2^\varepsilon) \quad \textit{in the Dirichlet case} \tag{5.6}$$

or

$$K_\varepsilon(x_1^\varepsilon) < K_\varepsilon(x_2^\varepsilon) \quad \textit{in the Neumann case.} \tag{5.7}$$

Proof. We prove (5.7). The proof of (5.6) is the same. By the expansion of K_ε given in (5.5) of Proposition (5.2) we have

$$\begin{aligned} K_\varepsilon(x_2^\varepsilon) - K_\varepsilon(x_1^\varepsilon) &= \frac{1}{2}\gamma\left(e^{-\frac{\psi_\varepsilon(x_1^\varepsilon)}{\varepsilon}} - e^{-\frac{\psi_\varepsilon(x_2^\varepsilon)}{\varepsilon}}\right) \\ &+ o\left(e^{-\frac{\psi_\varepsilon(x_1^\varepsilon)}{\varepsilon}}\right) + o\left(e^{-\frac{\psi_\varepsilon(x_2^\varepsilon)}{\varepsilon}}\right). \end{aligned} \tag{5.8}$$

Since $\mathrm{d}_{\partial\Omega}(x_1) < \mathrm{d}_{\partial\Omega}(x_2)$, by Lemma (4.2) we deduce that for ε small enough $\psi_\varepsilon(x_1^\varepsilon) < \psi_\varepsilon(x_2^\varepsilon)$. Then by (5.8) we get

$$e^{\frac{\psi_\varepsilon(x_1^\varepsilon)}{\varepsilon}}\left[K_\varepsilon(x_2^\varepsilon) - K_\varepsilon(x_1^\varepsilon)\right] = \frac{1}{2}\gamma\left[1 - e^{-\frac{\psi_\varepsilon(x_2^\varepsilon) - \psi_\varepsilon(x_1^\varepsilon)}{\varepsilon}}\right] + o(1)$$

and the claim follows. ◻

LEMMA 5.4. *Let C_1, C_2 be two compact subsets of Ω. If*

$$\min_{x\in C_1} \mathrm{d}_{\partial\Omega}(x) > \max_{x\in C_2} \mathrm{d}_{\partial\Omega}(x)$$

then there exists $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_0)$

$$\min_{x\in C_1}(-K_\varepsilon)(x) > \max_{x\in C_2}(-K_\varepsilon)(x) \quad \textit{in the Dirichlet case,} \tag{5.9}$$

or

$$\min_{x\in C_1} K_\varepsilon(x) > \max_{x\in C_2} K_\varepsilon(x) \quad \textit{in the Neumann case.} \tag{5.10}$$

Now we prove that a suitable critical point of the distance function generates a critical point of K_ε.

Theorem 5.5. *Let* c *be a critical value topologically nontrivial of the distance function (see Definition (2.4)). Then there exists a sequence* (x_ε) *of critical points of* K_ε *such that* $\lim_{\varepsilon\to 0} x_\varepsilon = x_0$ *and* $\mathrm{d}_{\partial\Omega}(x_0) = c$.

Proof. We prove the claim in Neumann case. In the Dirichlet case we consider the function $-K_\varepsilon$ and we argue in the same way. By Definition (2.4) there exist a family of subsets ∂Q_δ, Q_δ, A_δ and S_δ of Ω which satisfy (2.1), (2.2), (2.5) and (2.6) , that is:

$$\max_{x\in\partial Q_\delta} \mathrm{d}_{\partial\Omega}(x) < \min_{x\in S_\delta} \mathrm{d}_{\partial\Omega}(x) \le c \le \max_{x\in Q_\delta} \mathrm{d}_{\partial\Omega}(x) < \min_{x\in A_\delta} \mathrm{d}_{\partial\Omega}(x). \tag{5.11}$$

Then by (5.9) of Lemma (5.4) there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$

$$a_{\varepsilon,\delta} = \max_{x\in\partial Q_\delta} K_\varepsilon(x) < \min_{x\in S_\delta} K_\varepsilon(x) \le \max_{x\in Q_\delta} K_\varepsilon(x) < \min_{x\in A_\delta} K_\varepsilon(x) = b_{\varepsilon,\delta}. \tag{5.12}$$

It is not difficult to prove that for ε and δ small enough the set $\{x \in \Omega \mid a_{\varepsilon,\delta} \le K_\varepsilon(x) \le b_{\varepsilon,\delta}\}$ is complete. Now by (5.12) and Theorem (2.3) there exists $x_{\varepsilon,\delta}$ critical point of K_ε in Ω such that:

$$\min_{x\in S_\delta} K_\varepsilon(x) \le K_\varepsilon(x_{\varepsilon,\delta}) \le \max_{x\in Q_\delta} K_\varepsilon(x). \tag{5.13}$$

Up to a subsequence we can assume that $x_{\varepsilon,\delta}$ goes to x_0 as ε and δ go to 0. It is easy to show that $\mathrm{d}_{\partial\Omega}(x_0) = c > 0$. Therefore the claim is proved. □

Finally we want to show that a family of critical points of K_ε converges to a critical point of the distance function. Firstly we have to compute the asymptotic expansion of the gradient of K_ε.

Proposition 5.6. *Let* x_ε *be a sequence in* Ω *such that* $\lim_{\varepsilon\to 0} x_\varepsilon = x_0 \in \Omega$. *Then*

$$\nabla K_\varepsilon(x_\varepsilon) = -\frac{1}{\varepsilon}\gamma\alpha(x_0)e^{-\frac{\psi_\varepsilon(x_\varepsilon)}{\varepsilon}} + o\left(\frac{1}{\varepsilon}e^{-\frac{\psi_\varepsilon(x_\varepsilon)}{\varepsilon}}\right), \quad \textit{in the Dirichlet case,} \tag{5.14}$$

or

$$\nabla K_\varepsilon(x_\varepsilon) = \frac{1}{\varepsilon}\gamma\alpha(x_0)e^{-\frac{\psi_\varepsilon(x_\varepsilon)}{\varepsilon}} + o\left(\frac{1}{\varepsilon}e^{-\frac{\psi_\varepsilon(x_\varepsilon)}{\varepsilon}}\right), \quad \textit{in the Neumann case,} \tag{5.15}$$

where $\alpha(x_0) \in \partial\mathrm{d}_{\partial\Omega}(x_0)$ *(see (3.1)) and* γ *is a positive constant (see Proposition 5.2).*

Proof. See Lemma (4.1) of [20]. □

THEOREM 5.7. *Let x_ε be a critical point of K_ε such that $x_0 = \lim_{\varepsilon\to 0} x_\varepsilon \in \Omega$. Then x_0 is a critical point of the distance function.*

Proof. Since x_ε is a critical point of K_ε by Proposition (5.6) we get

$$0 = \nabla K_\varepsilon(x_\varepsilon) = \frac{1}{\varepsilon}\gamma\alpha(x_0)e^{-\frac{\psi_\varepsilon(x_\varepsilon)}{\varepsilon}} + o\left(\frac{1}{\varepsilon}e^{-\frac{\psi_\varepsilon(x_\varepsilon)}{\varepsilon}}\right), \tag{5.16}$$

where $\alpha(x_0) \in \partial \mathrm{d}_{\partial\Omega}(x_0)$ and γ is a positive constant. By (5.16) we deduce

$$\gamma\alpha(x_0) + o(1) = 0,$$

which implies $\alpha(x_0) = 0$. Then x_0 is a critical point of the distance function since $0 \in \partial \mathrm{d}_{\partial\Omega}(x_0)$ (see Definition (1.7)). □

Proof of Theorem (0.1). It follow by Proposition (5.2) and Theorem (5.5). □

Proof of Theorem (0.2). It follow by Proposition (5.2) and Theorem (5.7). □

6. Existence of multi-peak solutions. Let H_ε be the Hilbert space

$$\mathrm{H}_\varepsilon = \left\{ u \in \mathrm{H}^2(\Omega_\varepsilon) \;\middle|\; \frac{\partial u}{\partial \nu_\varepsilon} = 0 \text{ on } \partial\Omega_\varepsilon \right\} \quad \text{in the Neumann case}$$

Define

$$\mathcal{S}_\varepsilon(u) = \Delta u - u + (u^+)^p \quad \text{for} \quad u \in \mathrm{H}_\varepsilon.$$

Then solving equation (0.1) or equation (4.2) is equivalent to solve the following one

$$\mathcal{S}_\varepsilon(u) = 0, \quad u \in \mathrm{H}_\varepsilon.$$

Fix $X = (x^1, \ldots, x^K) \in \mathcal{M}_K(\Omega)$. Let us consider the linearized operator $\mathcal{L}_\varepsilon : \mathrm{H}_\varepsilon \longrightarrow \mathrm{L}^2(\Omega_\varepsilon)$ given by

$$\mathcal{L}_\varepsilon(v) = \Delta v - v + p\left(\sum_1^K \mathcal{P}_{\Omega_{\varepsilon,x^i}} U\right)^{p-1} v.$$

It is easy to see that the cokernel of $\mathcal{L}_\varepsilon$ coincides with its kernel. Choose approximate cokernel and kernel as

$$\mathcal{K}_{\varepsilon,X} = \operatorname{span}\left\{ \frac{\partial \mathcal{P}_{\Omega_{\varepsilon,x^i}} U}{\partial x^i_j} \mid i = 1, \ldots, K,\ j = 1, \ldots, N \right\} \subset \mathrm{H}_\varepsilon,$$

$$\mathcal{C}_{\varepsilon,X} = \operatorname{span}\left\{ \frac{\partial \mathcal{P}_{\Omega_{\varepsilon,x^i}} U}{\partial x^i_j} \mid i = 1, \ldots, K,\ j = 1, \ldots, N \right\} \subset \mathrm{L}^2(\Omega_\varepsilon).$$

Now we state the following lemmas, which allow us to reduce problem (4.2) to a finite dimensional problem.

LEMMA 6.1. *For any compact set $C \subset \mathcal{M}_K(\Omega)$ there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $X \in C$ there exists a unique $\Phi_{\varepsilon,X} \in \mathcal{K}^\perp_{\varepsilon,X}$ such that*

$$\mathcal{S}_\varepsilon\Big(\sum_{i=1}^K \mathcal{P}_{\Omega_{\varepsilon,x^i}} U + \Phi_{\varepsilon,X}\Big) \in \mathcal{C}_{\varepsilon,X}.$$

Moreover $\Phi_{\varepsilon,X}$ *is* C^1 *in* X *and*

$$\|\Phi_{\varepsilon,X}\|_{\mathrm{H}^2(\Omega_\varepsilon)} \le Ce^{-(1+\sigma)\frac{\mathcal{D}_K(X)}{\varepsilon}}, \tag{6.1}$$

where C *is a positive constant,* $\sigma = \min\{1, p-1\}$ *and* $\mathcal{D}_K$ *is defined in (3.3).*

Proof. The proof relies on a contraction mapping argument. The claim can be proved by collecting some results obtained in [9] and [23]. $\square$

We now define the function $K_\varepsilon : \mathcal{M}_K(\Omega) \longrightarrow \mathbb{R}$ by

$$K_\varepsilon(x^1, \ldots, x^K) = J_\varepsilon\Big(\sum_{i=1}^{K} \mathcal{P}_{\Omega_{\varepsilon,x^i}} U + \Phi_{\varepsilon,X}\Big), \tag{6.2}$$

where the "rescaled" energy functional $J_\varepsilon : \mathrm{H}^1(\Omega_\varepsilon) \longrightarrow \mathbb{R}$ is defined in (5.3).

Firstly we compute the asymptotic expansion of K_ε.

Proposition 6.2. *$X_\varepsilon = (x^1_\varepsilon, \ldots, x^K_\varepsilon)$ is a critical point of K_ε if and only if $u_\varepsilon = \sum\limits_{i=1}^{K} \mathcal{P}_{\Omega_{\varepsilon,x^i_\varepsilon}} U + \Phi_{\varepsilon,X_\varepsilon}$ is a solution of (4.2). Moreover the following estimate holds uniformly on compact sets of $\mathcal{M}_K(\Omega)$*

$$K_\varepsilon(x) = KA - \frac{1}{2}\gamma e^{-\frac{\Phi_\varepsilon(X)}{\varepsilon}} + o\left(e^{-\frac{\psi_\varepsilon(x)}{\varepsilon}}\right), \tag{6.3}$$

where

$$A = \frac{1}{2}\int\limits_{\mathbb{R}^N} \left(|\nabla U|^2 + U^2\right) - \frac{1}{p+1}\int\limits_{\mathbb{R}^N} U^{p+1}, \ \gamma = \int\limits_{\mathbb{R}^N} U^p(y)e^{-y_1}dy.$$

Proof. See [9] and [23]). $\square$

The next results play a crucial role in connecting the topological structure of the sublevels of the function $\mathcal{D}_K$ with the topological structure of the sublevels of the function K_ε.

Lemma 6.3. *Let $X^\varepsilon_1, X^\varepsilon_2$ be sequences in $\mathcal{M}_K(\Omega)$ such that $\lim\limits_{\varepsilon\to 0} X^\varepsilon_1 = X_1 \in \mathcal{M}_K(\Omega)$, $\lim\limits_{\varepsilon\to 0} X^\varepsilon_2 = X_2 \in \mathcal{M}_K(\Omega)$ and $\mathcal{D}_K(X_1) < \mathcal{D}_K(X_2)$. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$*

$$K_\varepsilon(X^\varepsilon_1) < K_\varepsilon(X^\varepsilon_2). \tag{6.4}$$

Proof. We argue as in the proof of Lemma (5.3) using asymptotic expansion (6.3).

Lemma 6.4. *Let C_1, C_2 be two compact subsets of $\mathcal{M}_K(\Omega)$. If*

$$\min_{X\in C_1} \mathcal{D}_K(X) > \max_{X\in C_2} \mathcal{D}_K(X)$$

then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$

$$\min_{X\in C_1} K_\varepsilon(X) > \max_{X\in C_2} K_\varepsilon(X). \tag{6.5}$$

Now we prove that a suitable critical point of the function $\mathcal{D}_K$ generates a critical point of K_ε.

THEOREM 6.5. *Let c be a critical value topologically nontrivial of the function $\mathcal{D}_K$ (see Definition (2.4)). Then there exists a sequence (X_ε) of critical points of K_ε such that $\lim_{\varepsilon\to 0} X_\varepsilon = X_0$, $\mathcal{D}_K(X_0) = c$ and $X_0 \in \mathcal{M}_K(\Omega)$.*

Proof. By definition (2.4) there exist a family of ∂Q_δ, Q_δ, A_δ and S_δ of $\mathcal{M}_K(\Omega)$, which satisfy (2.1), (2.2), (2.3) and (2.6), namely:

$$\max_{X\in\partial Q_\delta} \mathcal{D}_K(X) < \min_{X\in S_\delta} \mathcal{D}_K(X) \le c \le \max_{X\in Q_\delta} \mathcal{D}_K(X) < \min_{X\in A_\delta} \mathcal{D}_K(X) \tag{6.6}$$

and

$$\lim_{\delta\to 0} \min_{X\in S_\delta} \mathcal{D}_K(X) = \lim_{\delta\to 0} \max_{X\in Q_\delta} \mathcal{D}_K(X) = c. \tag{6.7}$$

Then by (6.5) of Lemma (6.4) for any δ small enough there exists $\varepsilon_0(\delta) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$

$$\max_{X\in\partial Q_\delta} K_\varepsilon(X) < \min_{X\in S_\delta} K_\varepsilon(X) \le \max_{X\in Q_\delta} K_\varepsilon(X) < \min_{X\in A_\delta} K_\varepsilon(X). \tag{6.8}$$

Finally we want to show that a family of critical points of K_ε converges to a critical point of the function $\mathcal{D}_K$. Firstly we have to compute the asymptotic expansion of the gradient of K_ε.

First of all we have to compute the expansion of the gradient of K_ε.

PROPOSITION 6.6. *For any $X \in \mathcal{M}_K(\Omega)$*

$$\nabla K_\varepsilon(X) = \frac{\gamma}{\varepsilon}\beta_\varepsilon(X) e^{-\frac{\Phi_\varepsilon(X)}{\varepsilon}} + o(e^{-\frac{\Phi_\varepsilon(X)}{\varepsilon}}), \tag{6.9}$$

where $\beta_\varepsilon(X) \in \partial\mathcal{D}_K(X)$ (see Lemma sottodifffik)) and γ is a positive constant (see Theorem (6.2)).

Proof. See Lemma (5.1) of [21]. ☐

THEOREM 6.7. *Let $X_\varepsilon = (x_\varepsilon^1, \ldots, x_\varepsilon^K)$ be a critical point of K_ε such that for $i = 1, \ldots, K$ $x_0^i = \lim_{\varepsilon\to 0} x_\varepsilon^i \in \Omega$. Then $X_0 = (x_0^1, \ldots, x_0^K) \in \mathcal{M}_K(\Omega)$ and X_0 is a critical point of the function $\mathcal{D}_K$.*

Proof. First of all we prove that $(x_0^1, \ldots, x_0^K) \in \mathcal{M}_K(\Omega)$, namely $x_0^i \ne x_0^j$ if $i \ne j$ (see [21], Theorem (6.1)).

Secondly we show that X_0 is a critical point of the function $\mathcal{D}_K$. Since X_ε is a critical point of K_ε by Proposition (6.6) we get

$$0 = \nabla K_\varepsilon(X_\varepsilon) = \frac{1}{\varepsilon}\gamma\beta(X_\varepsilon) e^{-\frac{\Phi_\varepsilon(X_\varepsilon)}{\varepsilon}} + o\left(\frac{1}{\varepsilon} e^{-\frac{\Phi_\varepsilon(X_\varepsilon)}{\varepsilon}}\right), \tag{6.10}$$

where $\beta(X_\varepsilon) \in \partial\mathcal{D}_K(X_\varepsilon)$ and γ is a positive constant. By (6.10) we deduce

$$\beta(X_\varepsilon) + o(1) = 0. \tag{6.11}$$

Let $X_0 = \lim_{\varepsilon\to 0} X_\varepsilon$. By using Remark (1.4) we get $\lim_{\varepsilon\to 0} \beta(X_\varepsilon) = \beta(X_0) \in \partial\mathcal{D}_K(X_0)$ and by (6.11) we deduce that $\beta(X_0) = 0$. Then X_0 is a critical point of the function $\mathcal{D}_K$ since $0 \in \partial\mathcal{D}_K(X_0)$ (see Definition (1.7)). ☐

Proof of Theorem (0.3). It follow by Proposition (6.2) and Theorem (6.5). ☐

Proof of Theorem (0.4). It follow by Proposition (6.2) and Theorem (6.7). ☐

7. Examples.

Example 7.1. *(A domain with one hole) Let* $\Omega = \Sigma \setminus \overline{\sigma}$ *where* $\sigma \subset \Sigma$ *are open sets. Assume* $\max_{\Omega} \mathrm{d}_{\partial\Omega} > \frac{1}{2}\text{dist }(\partial\sigma, \partial\Sigma)$. *Then* $c_1 = \mathrm{d}_{\partial\Omega}(x_1) = \max_{\Omega} \mathrm{d}_{\partial\Omega}$ *and* $c_2 = \mathrm{d}_{\partial\Omega}(x_2) = \frac{1}{2}\text{dist }(\partial\sigma, \partial\Sigma)$ *are two critical values topologically nontrivial of the distance function.*

Proof. The existence of c_1 is trivial. Let us prove the existence of c_2. Let $y_0 \in \partial\Sigma$ and $z_0 \in \partial\sigma$ such that $|y_0 - z_0| = \text{dist }(\partial\sigma, \partial\Sigma)$. Set $x_0 = \frac{y_0+z_0}{2}$. Then $\mathrm{d}_{\partial\Omega}(x_0) = \frac{1}{2}|y_0 - z_0|$. Let:

$$S = \{x \in \Sigma \mid \text{dist }(x, \partial\sigma) = \mathrm{d}_{\partial\Omega}(x_0)\}$$

and

$$Q = \{ty_0 + (1-t)z_0 \mid t \in [\delta, 1-\delta]\} \quad \text{for some } \delta > 0.$$

Then it is easy to prove that the sets Q and S satisfies assumptions (2.1), (2.2) and (2.3):

$$\max_{x \in \partial Q} \mathrm{d}_{\partial\Omega}(x) < \min_{x \in S} \mathrm{d}_{\partial\Omega}(x) = \mathrm{d}_{\partial\Omega}(x_0) = \max_{x \in Q} \mathrm{d}_{\partial\Omega}(x).$$

That proves that $\mathrm{d}_{\partial\Omega}(x_0)$ is a critical value topologically nontrivial of the distance function in the sense of Definition (2.4). □

Example 7.2. *(A domain with two holes) Let* $\Omega = \Sigma \setminus (\overline{\sigma}_1 \cup \overline{\sigma}_2)$ *where* $\sigma_i \subset \Sigma$ *are open sets,* σ_1 *and* σ_2 *are strictly convex and* $\sigma_1 \cap \sigma_2 = \emptyset$. *Assume*

$$\text{dist }(\partial\sigma_1, \partial\Sigma) < \text{dist }(\partial\sigma_2, \partial\Sigma) < \text{dist }(\partial\sigma_1, \partial\sigma_2) \tag{7.1}$$

Then $c_1 = \mathrm{d}_{\partial\Omega}(x_1) = \max_{\Omega} \mathrm{d}_{\partial\Omega}$, $c_2 = \mathrm{d}_{\partial\Omega}(x_2) = \frac{1}{2}\text{dist }(\partial\sigma_1, \partial\Sigma)$, $c_3 = \mathrm{d}_{\partial\Omega}(x_3) = \frac{1}{2}\text{dist }(\partial\sigma_2, \partial\Sigma)$ *and* $c_4 = \mathrm{d}_{\partial\Omega}(x_4) = \frac{1}{2}\text{dist }(\partial\sigma_1, \partial\sigma_2)$ *are four critical values topologically nontrivial of the distance function.*

Proof. The existence of c_1 is trivial. First of all we prove the existence of c_2 and c_3. Let $i = 1, 2$. Let $y_0^i \in \partial\Sigma$ and $z_0^i \in \partial\sigma_i$ such that $|y_0^i - z_0^i| = \text{dist }(\partial\sigma_i, \partial\Sigma)$. Set $x_0^i = \frac{y_0^i + z_0^i}{2}$. Then $\mathrm{d}_{\partial\Omega}(x_0^i) = \frac{1}{2}|y_0^i - z_0^i|$. Let:

$$S_i = \left\{x \in \Sigma \mid \text{dist }(x, \partial\sigma_i) = \mathrm{d}_{\partial\Omega}(x_0^i)\right\}$$

and

$$Q_i = \left\{ty_0^i + (1-t)z_0^i \mid t \in [\delta, 1-\delta]\right\} \quad \text{for some } \delta > 0.$$

We point out that (7.1) ensures that $\mathrm{d}_{\partial\Omega}(x) = \text{dist }(x, \partial\sigma_i) \quad \forall\, x \in S_i$. Then it is easy to prove that the sets Q_i and S_i satisfies assumptions (2.1), (2.2) and (2.3):

$$\max_{x \in \partial Q_i} \mathrm{d}_{\partial\Omega}(x) < \min_{x \in S_i} \mathrm{d}_{\partial\Omega}(x) = \mathrm{d}_{\partial\Omega}(x_0^i) = \max_{x \in Q_i} \mathrm{d}_{\partial\Omega}(x).$$

That proves that $\mathrm{d}_{\partial\Omega}(x_0^i)$ is a critical value topologically nontrivial of the distance function in the sense of Definition (2.4).

Now we prove the existence of c_4. Since σ_1 and σ_2 are strictly convex there exist exactly two points $z_1 \in \partial\sigma_1$ and $z_2 \in \partial\sigma_2$ such that $|z_1 - z_2| = \operatorname{dist}(\partial\sigma_1, \partial\sigma_2)$. Set $x_0 = \frac{z_1+z_2}{2}$. Then $\mathrm{d}_{\partial\Omega}(x_0) = \frac{1}{2}|z_1 - z_2|$. Let for some $\delta > 0$

$$Q = \{tz_1 + (1-t)z_2 \mid t \in [\delta, 1-\delta]\},$$

$$S = \{\text{hyperplane perpendicular to } Q \text{ crossing the point } x_0\} \cap B(x_0, \delta).$$

and

$$A = \{\text{hyperplane perpendicular to } Q \text{ crossing the point } x_0\} \cap \partial B(x_0, \delta).$$

Then it is easy to prove that the sets Q, S and A satisfies assumptions (2.1), (2.2) and (2.3):

$$\max_{x\in\partial Q} \mathrm{d}_{\partial\Omega}(x) < \min_{x\in S} \mathrm{d}_{\partial\Omega}(x) = \mathrm{d}_{\partial\Omega}(x_0) = \max_{x\in Q} \mathrm{d}_{\partial\Omega}(x) < \min_{x\in A} \mathrm{d}_{\partial\Omega}(x).$$

That proves that $\mathrm{d}_{\partial\Omega}(x_0)$ is a critical value topologically nontrivial of the distance function in the sense of Definition (2.4). □

If the domain has a lot of holes the existence of many critical values topologically nontrivial of the distance function strongly depends on the geometry of the holes.

EXAMPLE 7.3. *(A domain with k handles) Let Ω be a domain with k handles. Then there exist at least $2k+1$ distinct critical values topologically nontrivial of the distance function: $k+1$ local maxima of $\mathrm{d}_{\partial\Omega}$ and k local saddle levels.*

Note that we can have more than a critical point at the same level.

EXAMPLE 7.4. *Let Ω be the dumbell. Then d is a critical value topologically non trivial of $\mathcal{D}_2$. Moreover one can choose the dumbell so that $(0, d, 0, -d)$ is the unique critical point of $\mathcal{D}_2$ at level d.*

We prove that the point $(0, d, 0, -d)$ is a "local saddle point" of $\mathcal{D}_2$. Fix $\varepsilon > 0$ and set

$$\begin{aligned}
Q_\varepsilon &= \{(0, x_2^1) \in \Omega || \; |x_2^1 - d| \le \varepsilon\} \times \{(0, x_2^2) \in \Omega || \; |x_2^2 + d| \le \varepsilon\} \\
\partial Q_\varepsilon &= \big(\{(0, d \pm \varepsilon)\} \times \{(0, x_2^2) \in \Omega || \; |x_2^2 + d| \le \varepsilon\}\big) \\
&\quad \cup \{(0, x_2^1) \in \Omega || \; |x_2^1 - d| \le \varepsilon\} \times \{(0, -d \pm \varepsilon)\}
\end{aligned}$$

For $\delta > 0$ and $\rho > 0$ set

$$\begin{aligned}
C_\delta &= \{x \in \Omega \mid \mathrm{d}_{\partial\Omega}(x) = d + \delta\} \\
S_\delta &= \Big(B\Big((0, d), \rho\Big) \cap C_\delta\Big) \times \Big(B\Big((0, -d), \rho\Big) \cap C_\delta\Big) \\
A_\delta &= \Big(\partial B\Big((0, d), \rho\Big) \cap C_\delta\Big) \times \Big(B\Big((0, -d), \rho\Big) \cap C_\delta\Big) \\
&\quad \cup \Big(B\Big((0, d), \rho\Big) \cap C_\delta\Big) \times \Big(\partial B\Big((0, -d), \rho\Big) \cap C_\delta\Big).
\end{aligned}$$

Then by using Remark (2.2) it is easy to check that if we choose δ and ρ small enough ∂Q_δ, Q_δ, A_δ and S_δ are subsets of $\Omega \times \Omega$ which satisfy (2.1), (2.2) and (2.3) and

$$\max_{\partial Q_\delta} \mathcal{D}_2 = d - \varepsilon < \min_{S_\delta} \mathcal{D}_2 = d - \delta < d = \max_{Q_\delta} \mathcal{D}_2 < \min_{A_\delta} \mathcal{D}_2 = d + \delta.$$

Moreover $\lim_{\delta\to 0}\min_{S_\delta}\mathcal{D}_2 = d$. By Lemma (3.7) we deduce that the sets $\{X \in \mathcal{M}_K(\Omega)\ s.t.\ c \le \mathcal{D}_K(X)\}$ are complete for any $c > 0$. Therefore d is a critical value of $\mathcal{D}_2$.

Finally by using Remark (3.4) and Remark (3.3) one can construct a dumbell in such a way the distance function is differentiable at any x with $\mathrm{d}_{\partial\Omega}(x) = d$ and by using Corollary (3.9) one can check that $(0, d, 0, -d)$ is the unique critical point of $\mathcal{D}_2$ at level d. $\square$

Remark 7.5. *We note that in the dumbell the points* $(a, r, a, -r)$ *and* $(b, R, b, -R)$ *are two local maximum points of the function* $\mathcal{D}_2$ *at different levels* $\mathcal{D}_2(a, r, a, -r) = r$ *and* $\mathcal{D}_2(b, R, b, -R) = R$.

However we point out that such points are not isolated critical points of $\mathcal{D}_2$ at levels r and R, respectively. In fact if x^1 is a point close enough to the point (a, r), which belongs to the sphere centered at $(a, 0)$ with radius r and x^2 is the point diametrically opposite, it is easy to check that (x^1, x^2) is a local maximum point of the function $\mathcal{D}_2$ at level r.

REFERENCES

[1] A. Bahri, *Critical point at infinity in some variational problems*, Pitman Research Notes Math., 182 Longman House, Harlow (1989).

[2] V. Benci, G. Cerami, *The effect of the domain topology on the number of positive solutions of nonlinear elliptic problem*, Arch. Rat. Mech. Anal., 114 (1991), pp. 79–93.

[3] V. Benci, G. Cerami, D. Passaseo, *On the number of the positive solutions of some nonlinear elliptic problems*, Nonlinear Analysis: A Tribute in Honour of Giovanni Prodi, S.N.S. (1991), pp. 93–107.

[4] A. Bahri, Y.Y.Li, O.Rey, *On a variational problem with lack of compactness: the topological effect of the critical points at infinity*, Calculus of Variat., 3 (1995), pp. 67–93.

[5] P.W.Bates, G.Fusco, *Equilibria with many nuclei for the Cahn-Hilliard equation*, J. Differential Equations, 160 (2000), pp. 283–356.

[6] H. Berestycki, P.L. Lions, *Nonlinear scalar field equations I: existence of a ground state*, Arch. Rat. Mech. Anal., 82 (1983), pp. 313–346.

[7] D.Cao, E.N.Dancer, E.Noussair, S.Yan, *On the existence and profile of multi-peaked solutions to singularly perturbed semilinear Dirichlet problems*, Disc. Cont. Dyn. Syst., 2 (1996), pp. 221–236.

[8] D.Cao,E.Noussair, *Multi-peak solutions for a singularly perturbed semilinear elliptic problem*, preprint.

[9] G. Cerami, J. Wei, *Multplicity of multiple interior peaks solutions for some singularly perturbed Neumann problems*, Intern. Math. Res. Notes, 12 (1998), pp. 601–626.

[10] K.C. Chang, *Variational methods for non-differentiable functionals and their applications to partial differential equations*, J. Math. Anal. Appl., 80 (1981), pp. 102–129.

[11] F.H. Clarke, *Optimization and nonsmooth analysis*, John Wiley and Sons (1983).

[12] E.N.Dancer, S.Yan, *Multipeak solutions for a singularly perturbed Neumann problem*, Pacific Journal Math., to appear.

[13] E.N.Dancer, S.Yan, *A singularly perturbed elliptic problem in bounded domains with nontrivial topology*, Adv. Diff. Equat., 4 (1999), pp. 347–368.

[14] E.N.Dancer, S.Yan, *Effect of the domain geometry on the existence of multipeak solutions for an elliptic problem*, Topol. Meth. Nonlin. Anal., to appear.

[15] E.N.Dancer, J.Wei, *On the effect of domain topology in a singular perturbation problem*, Topol. Meth. Nonlin. Anal., 11 (1998), pp. 227–248.

[16] M. Del Pino, P. L. Felmer, J. Wei, *On the role of distance function in some singular perturbation problems*, Comm. in P.D.E., 25 (2000), pp. 155–177.

[17] M. Del Pino, P. L. Felmer, J. Wei, *Multiple-peak solutions for some singular perturbation problems*, submitted.

[18] A.Floer, A.Weinstein, *Nonspreading wave packets for the cubic Scröedinger equation with a bounded potential*, J. Funct. Anal., 69 (1986), pp. 397–408.

[19] B. Gidas, W. N. Ni, L. Nirenberg, *Symmetry of positive solutions of nonlinear elliptic equations in* $\mathbb{R}^N$, Mathematical analysis and applications, Part A, Adv. Math. Suppl. Studies, 7A, Acad. Press, New York, 1981.

[20] M.Grossi, A.Pistoia, *On the effect of critical points of distance function in superlinear elliptic problems*, Advances in Diff. Equat., to appear.

[21] M.Grossi, A.Pistoia, J.Wei, *Multipeak solutions for a semilinear Neumann problem via nonsmooth analysis*, Calculus of Variat, to appear.

[22] C.Gui, *Multipeak solutions for a semilinear Neumann problem*, Duke Math. J., 84 (1996), pp. 739–769.

[23] C.Gui, J. Wei, *Multiple interior peak solutions for a for some singularly perturbed Neumann problem*, Journ. Diff. Equat., to appear.

[24] C.Gui, J. Wei, M.Winter, *Multiple boundary peak solutions for some singularly perturbed Neumann problems*, Ann. Inst. H.Poincarè, to appear.

[25] M.Kowalczyk, *Multiple spike layers in the shadow Gierer-Meinhardt system: existence of equilibria and quasi invariant manifold*, Duke Math. J., 98 (1999), pp. 59–111.

[26] M. K. Kwong, *Uniqueness of positive solutions of* $\Delta u - u + u^p = 0$ *in* $\mathbb{R}^n$, Arch. Rat. Mech. Anal., 105 (1989), pp. 243–266.

[27] Y.Y. Li, *On a singularly perturbed equation with Neumann boundary condition*, Comm. P.D.E., 23 (1998), pp. 487–545.

[28] Y.Y. Li, L. Nirenberg, *The Dirichlet problem for singularly perturbed elliptic equation*, Comm. Pure Appl. Math., 51 (1998), pp. 1445–1490.

[29] C. Lin, W.M. Ni, I. Takagi, *Large amplitude stationary solutions to a chemotaxis system*, J. Diff. Eqns., 72 (1988), pp. 1–27.

[30] A. Marino, A.M. Micheletti, A. Pistoia, *Some variational results on semilinear problems with asymptotically nonsimmetric behaviour*, Nonlinear Analysis "A Tribute in honor of G.Prodi, S.N.S. Pisa (1991), pp. 243–256.

[31] W.M. Ni, I. Takagi, *On the shape of least energy solutions to a semilinear neumann problem*, Comm. Pure Math. Appl., 41 (1991), pp. 819–851.

[32] W.M. Ni, I. Takagi, *Locating the peaks of least energy solutions to a semilinear Neumann problem*, Duke Math. Journ., 70 (1992), pp. 247–281.

[33] W.M. Ni, I. Takagi, *Point-condensation generated by a reaction-diffusion system in axially symmetric domains*, Japan J.Industr.Appl. Math., 12 (1995), pp. 327–365.

[34] W.M. Ni, J. Wei, *On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems*, Comm. Pure Math. Appl., 48 (1995), pp. 731–768.

[35] A.Pistoia, *On the existence of symmetric solutions to some singularly perturbed semilinear elliptic problems*, Asympt. Anal., 22 (2000), pp. 149–161.

[36] M. Ramos, *Teoremas de enlace na teoria dos pontos criticos*, Textos de Matematica, Dep. Matematica, Univ. de Lisboa, (1993).

[37] S.Yan, *On the number of interior multipeak solutions for singularly perturbed Neumann problems*, preprint.

[38] Z.Q. Wang, *On the existence of multiple single-peaked solution for a semilinear Neumann problem*, Arch. Rat. Mech. Anal., 120 (1992), pp. 375–399.

[39] J. Wei, *On the boundary spike layer solutions to a singularly perturbed Neumann problem*, J. Diff. Equat., 134 (1997), pp. 104–133.

[40] J. Wei, *On the interior spike solutions for some singular perturbation problems*, Proc. Roy. Soc. Edinb., 128 (1998), pp. 849–874.

[41] J. Wei, *On the construction of interior spike layer solutions to a singularly perturbed semilinear Neumann problem*, Tohoku Math. J., 50 (1998), pp. 159–178.

[42] J. Wei, *On the construction of interior spike layer solutions to a singularly perturbed semilinear Neumann problem*, Partial Differential Equations: Theory and Numerical solution, CRC Press, (1998), pp. 336–349.

[43] J. Wei, *Conditions for two-peaked solutions of a singularly perturbed elliptic equation*, Manuscript. Math., 96 (1998), pp. 113–131.

[44] J. Wei, M.Winter, *Multiple boundary spike solutions for a wide class of singular perturbation problems*, J. London Math. Soc., 59 (1999), pp. 82–101.